On Hankel Determinant of the Inverse of ǩ-Bounded Turning Functions

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Candidate of Master of Science in Mathematics at the National University of Modern Languages, do hereby declare that the thesis On Hankel Determinant of the Inverse of $\check{q}$-Bounded Turning Functions submitted by me in partial fulfillment of MS Math degree, is my original work and has not been submitted or published earlier. I also solemnly declare that it shall not, in the future, be submitted by me for obtaining any other degree from this or any other university or institution. I also understand that if evidence of plagiarism is found in my thesis/dissertation at any stage, even after the award of a degree, the work may be canceled and the degree revoked.

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ABSTRACT

Title:

On Hankel Determinant of the Inverse of q̌-Bounded Turning Functions

The aim of this thesis is to investigate the Hankel determinants of the inverse functions of a subclass of univalent functions known as q̌-bounded turning functions. This class, which generalizes classical bounded turning functions by incorporating the parameter q̌, has attracted attention due to its connections with fractional analysis and q̌-calculus. In this work, we focus on estimating the third Hankel determinant for the inverse functions associated with this class. By leveraging the analytical properties of q̌-Carathéodory functions and the relationships between a function's coefficients and those of its inverse, we derive an exact inequality for the determinant. Using tools from q̌-calculus, subordination theory, and coefficient bounds, we obtain new sharp bounds that explicitly illustrate how the parameter q̌ influences the determinant's value. The results not only generalize known outcomes for classical bounded turning functions but also provide new insights into the analytic structure and geometric behavior of inverse q̌-bounded turning functions in the open unit disc. Furthermore, we discuss geometric properties of the inverses and explore applications to extremal problems, thereby extending the understanding of coefficient problems in Geometric Function Theory. A minor graphical analysis is also performed to validate the new results against the classical literature.

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LIST OF SYMBOLS

M	-	Open unit disc
A	-	Class of Analytic functions
S	-	Class of Univalent functions
P	-	Class of Carathéodory functions
S^*	-	Class of Starlike functions
C	-	Class of Convex functions
K	-	Class of Close to Convex functions
C^*	-	Class of Qausi Convex functions
H	-	Hankel determinant
$D_{\check{q}}$	-	ǭ-derivative operator

ACKNOWLEDGMENT

I begin this recognition with the verse, "You Alone we Worship; You Alone we ask for Help," in the name of Allah, the Most Gracious and Merciful. I am incredibly appreciative to Allah, the Most Wise, for all of His gifts that have helped me along my academic path. I’m reminded of the Prophet Muhammad’s (peace be upon him) Hadith, which states, "Seek knowledge from the cradle to the grave."

I want to sincerely thank my family for their unwavering support and my teachers for their priceless advice, because it is thanks to your combined inspiration and encouragement that I am able to stand here today, gratefully acknowledging the opportunities and blessings Allah has given me.

May Allah accept our efforts and lead us to the wise and noble path.

DEDICATION

This thesis is dedicated to my parents, whose unending love and sacrifices, and my teachers, whose knowledge and direction have illuminated my path and enabled me to accomplish this.

CHAPTER 1

INTRODUCTION AND LITERATURE REVIEW

1.1 Introduction

The concept of Geometric Function Theory(GFT) was introduced in the beginning of the 20^{th} century. Even if real analysis concern the same ideas, complex analysis has been more affected by the geometry of functions. GFT offers a wide range of applications in science and Mathematics. Bernard Riemann found the Riemann Mapping Theorem result in 1851, for detail see [1]. As the keystone of GFT, this theorem is fundamental. The univalent function theory's basis was built in the nineteenth century along the major tasks of Weierstrass, Riemann, and Cauchy. Subsequently, in 1907, Koebe proposed functions that are univalent and analytically defined on the open unit disk M, see [2]. Researchers have studied these functions comprehensively.

Within complex analysis, GFT is the subfield dedicated to the geometric study of holomorphic functions. However, Krzyz *et al.* [3] studied constraints such as starlike functions on their initial coefficients of the inverse of α-order, and Kapoor and Mishra [4] built on their findings. Additionally, Ali [5] examined the sharp bounds for coefficients of the inverse function when the function belongs to the extremely starlike function class. GFT focuses on the geometric transformation aspects of functions, such as how they distort shapes, preserve angles, map domains, and affect the geometry of the complex plane, rather than just evaluating functions regarding power series or differential equations.

Understanding the geometric behavior of conformal and univalent (one-to-one) mappings, particularly in domains like the unit disk, is the basic concept of GFT. This area of mathematics integrates methods from differential geometry, topology, complex analysis, and potential theory. The development of GFT started in the early 1900s, when mathematicians such as Henri Poincaré and Felix Klein developing concepts related to conformal mappings. Ludwig Bieberbach is renowned for the Bieberbach Conjecture, which Louis de Branges was established in 1985. The Loewner differential equation, being an integral part of modern GFT and stochastic processes, was introduced by Charles Loewner. Coefficient bounds are part of many undergraduate courses and subclasses of the coefficient inequalities of GFT. Functions in this framework are divided into various subfamilies that are members of the analytic normalized family of class A functions. The German mathematician Ludwig Bieberbach introduced Bieberbach's theorem in 1916, which was a remarkable result of that period. The theorem holds true solely to the class S, which consist of univalent functions. For the function of class S, or univalent functions, he computed the second coefficient $\hat{\alpha}_2$. Bieberbach conjecture, which resulted in substantial progress in the field as well as being regularly sought in attempts to prove it, is based on this theorem.

For a function of class S, the known function conjecture ξ, stated that if $\xi \in S$, then the function's coefficients ξ meet the relation $|c_m| \leq m$ for $m \in \{2,3,4,\dots\}$. If a function ξ was either the Koebe Function or one of it rotations, he demonstrated $|c_2| \leq 2$ with equality. The bieberbach conjecture is simple to articulate; it posed a significant challenge to mathematicians for many decades, see [6].

Although many of the mathematicians have attempted to verify this hypothesis many times, it has proven to be an obstacle to overcome. Karl Loewner, a mathematician, demonstrated in 1923 that $|\breve{c}_3| \leq 3$, see [7]. This proof made it possible for others to demonstrate this outcome in broader context. Before Gangadharan *et al.* [8] confirmed Bieberbach conjecture for $m = 4$ for the first time in 1955, that is $|\breve{c}_4| \leq 4$, more then 30 years had passed with no advancement.

In 1985, mathematician Louis-de-Branges successfully demonstrated the broad form of the Bieberbach conjecture; for detail see [9]. He created an elaborate, time-consuming yet precise evidence for this theory. In 1933 Feketo-Szego developed the Feketo-Szego inequality, which is related to the Bieberbach conjecture and mostly utilized in complex analysis. It involves polynomial coefficients with specific features, for detail see [10].

The inverse of a class of analytic functions called bounded turning functions, which are defined by normalization $h(0) = 0$ and $h'(0) = 1$. These functions satisfy the condition $Re(h'(k)) > 0$ within the open unit disc M.

Although previous work had established sharp bounds for $H_{3,1}(h)$, the exact bound for $H_{3,1}(h^{-1})$ remained unresolved. The authors systematically determine exact relations among the coefficients of the function and use them to compute $H_{3,1}(h^{-1})$ explicitly. This is achieved by expressing the derivative $h'(k)$ as a Carathéodory function$p(k)$, such that $h'(k) = p(k)$ and $Re(p(k) > 0$, for detail see [11]. This approach leads to a precise evaluation of sharp inverse function's upper bound for the third Hankel determinant.

1.2 Literature Review

Let suppose

$$h(k) = k + \sum_{n=2}^{\infty} c_n k^n \tag{1.1}$$

is a univalent function such that $M = \{k : |k| < 1\}$. The relationship between geometry functions and complex analysis is perhaps one of the most interesting areas of complex function theory. A function from the analytic class A can be expressed in series form given in (1.1). The function $h(k)$ is holomorphic in a field which is complex if it is differentiable at every point in the given area. A function with complex value can be differentiated at h_0, if it has derivative at h_0, such as

$$h'(k) = \lim_{k \to k_0} \frac{h(k) - h(k_0)}{k - k_0}. \tag{1.2}$$

A function is analytic at k_0 if it is differentiable at each point in its locality. Some of the advantages of all orders of theory of complex cunction are that k_0 must have a derivative and that h has a Taylor series expansion, that is converges in some open unit disk, which is centered on k_0.

Koebe [2] created the univalent function theory, elegant and intricate in the complex analysis in 1907. The class S was named on the set of functions in the disk M that are analytic and univalent and satisfy normalization criteria. This analysis's major focus will be on the subclasses of S functions $h(k) = k + c_2k^2 + c_3k^3 + ...$, univalent and analytic in M. It contains every univalent function that has been normalized by these conditions $h(0) = 0$ and $h'(0) = 1$.

The foundation of the concept of the Univalent functions is the relationship between the GFT and the analytic structure of complex functions. In a domain, a single-valued function is called a univalent function if it has various values for different points, that is $h(k_1)-h(k_2)=0$, if $k_1=k_2$. Pommerenke [12] introduced the Hankel determinant in 1967 of univalent functions. The Hankel determinant of a few analytic functions was inspected by Noonan and Thomas [13]. For each h in S, the inverse h^{-1} is provided by

$$h^{-1}(e)=e+\sum_{n=2}^{\infty}t_n e^n, |e|<r_0; (r_0(h)\geq\frac{1}{4}). \tag{1.3}$$

Libera found a correlation between h and h^{-1} coefficients for each h, when $h(M)$ is a region that is convex. Conversely, Kapoor and Mishra expanded upon the discoveries made by Krzyz, who examined first-order coefficient constraints for inverse Starlike functions. For k, the v^{th}-Hankel determinant of order u with v, $u\in N=\{1,2,3,...\}$, was characterized by Pommerenke given by

$$H_{v,u}(h)=\begin{vmatrix} a_u & a_{u+1} & \cdots & a_{u+v-1} \\ a_{u+1} & a_{u+2} & \cdots & a_{u+v} \\ \vdots & \vdots & \ddots & \vdots \\ a_{u+v-1} & a_{u+v} & \cdots & a_{u+2v-2} \end{vmatrix}. \tag{1.4}$$

Research on estimating maximum values regarding Hankel determinant of the third order derived for $v=3$ and $u=1$ in (1.3) has been conducted recently, as follows

$$H_{3,1}(h)=\begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}, \tag{1.5}$$

which numerous writers have addressed, see [14, 15]. The sharp bounds of $|H_{3,1}(h)|$ includes a number of well-known subfamilies, including convex, bounded turning, analytic, and starlike functions are represented by the R, T, S^* and K correspondingly meeting the Analytic requirements Re $(\frac{h(k)}{k})>0$, Re $(h^{'}(k))>0$, Re $\left\{\frac{kh'(k)}{h(k)}\right\}>0$, and Re$\left\{1+\frac{kh''(k)}{h'(k)}\right\}>0$ in the unit disc M, were provided by Kowalczyk and derived the bounds as 4, $\frac{1}{4},\frac{4}{9},\frac{4}{135}$, for detail see [16–20]. Numerous writers have obtained additional result on the sharp limits of Hankel's third Determinant for different subclass of functions that are analytic, see [21]. Rath calculated the third Hankel determinant's sharp bound very recently, for the starlike function's inverse with regard to symmetric points.

The class R, referred to as the class of bounded turning functions, is an essential subject in the GFT, for detail see [22]. To fully comprehend the central themes of our recent research, we must go over a few fundamental notions, for detail see [23]. These are analytic functions $h(k)$ normalized by $h(0) = 0$ and $h'(0) = 1$, in accordance with the criterion, transfer the unit disk M into a domain whose boundary has finite turning $\mathrm{Re}(h'(k)) > 0$ for all $k \in M$, see [24]. The functions of bounded turning meet Suffridge's inequality and have a number of important characteristics, including being univalent (one-to-one) in M which is $|h'(k)| \leq \frac{1+|k|}{1-|k|}$, $k \in M$, Chichra's inequality $|h''(k)| \leq \frac{2}{(1-|k|)^2}$, $k \in M$, and the Noshiro-Warschawski inequality $|a_n| \leq n$, $n \geq 2$ for their Maclaurin series expansion $h(k) = k + \sum_{n=2}^{\infty} a_n k^n$. The class P is strongly related to starlike and convex functions, which are significant subclasses of univalent functions, and is the biggest known subclass of univalent functions for which the Bieberbach conjecture remains true. Many mathematicians have investigated estimates on the coefficients of bounded turning functions and other functionals in detail, since they have applications in diverse fields such as GFT, differential equations, and potential theory. Because of its intriguing characteristics and links to other significant function classes, bounded turning functions are still a subject of active study in GFT.

In GFT, Hankel determinants have attracted a lot of attention, especially because of their fundamental properties and some findings concerning the third Hankel determinant. Hankel derivations symbol $H_{v,u}(k)$ for an analytical function h, are determined by a determinant that contains the coefficients of the function expansion in the Taylor series. Particularly when considering univalent functions, these determinants are essential to comprehending the characteristics of analytic functions. Important findings about the coefficients and behavior of Hankel determinants have been established by foundational studies, like those by Pommerenke, providing the framework for additional research in this field. Lately, studies have concentrated on the third Hankel determinant, $H_{3,1}(h)$, which, for functions of bounded rotation, has been demonstrated to have sharp boundaries. They show that the third Hankel determinant can be well determined to some classes of analytic functions, which gives the relations between the coefficients of these functions. Additional studies have addressed the way such determinantal inequalities affect classes of entire or other holomorphic functions, including the Lemniscate of Bernoulli, and have found more involved links between geometrical properties and analytical behavior. Overall, the third Hankel determinant and studies of Hankel determinants in general are prolific areas of research that increase our comprehension of analytic functions in GFT.

The role of Hankel determinants in the study of the coefficients of analytic functions has been established through the pioneering work. The third Hankel determinant is denoted through the symbol $H_{3,1}(h)$ being particularly crucial in the understanding of the geometric properties of functions. Specifically, in the case of functions with bounded turning, recent findings (e.g., Kowalczyk and Lecko) included sharp bounds of the third Hankel determinant, demonstrating that the above upper bound may be effectively restricted in some instances. Comparative analyses of various function subclasses have produced a range of results, demonstrating how difficult it is to define precise boundaries. For starlike and convex functions, the third Hankel determinant has been studied, particular upper bounds have been derived, and their connections to the coefficients of these functions have been investigated. Furthermore, studies of the inverses of some classes, including Ozaki-type close-to-convex functions, have shed more light on how Hankel determinants behave. All things considered, the continued study of the third Hankel determinant's sharp bounds advances both the field of GFT and the comprehension of analytic functions by illuminating complex relationships between function properties and their coefficient determinants.

The study of Hankel determinants for inverse functions has gained significant attention in complex analysis. Specifically, the study focuses on how these determinants behave for limited turning functions under certain normalizing requirements. The Hankel determinants that offer important information concerning the characteristics of the function are generated using the coefficients of the analytic function. The second Hankel determinant was later dealt with in the research work that generated predominantly precise and generalizable bounds. The definition of the determinants accounts for normalized functions from which the requirements should be stipulated in the framework under analysis. These requirements are $h(0) = 0$ and $h'(0) = 1$. Special focus has been placed on the third Hankel determinant $H_{3,1}(h)$ has been examined in terms of functions with limited turning. It is important to carefully analyze the normalization requirements because research has demonstrated that the behavior of Hankel determinants for inverse functions can differ dramatically from their direct counterparts. Studies have proven, for instance, the relationship of the coefficients of the function and the inverse function's coefficients, with more accurate bounds as well as with a better understanding of the geometrical properties of the functions. Overall, the exploration of Hankel determinants of the inverse function, particularly with the normalization assumptions of the bounded turning function, is an influential area of research that contributes further to the discussion on the structure of analytic functions in GFT.

In mathematical analysis and other applications, the Hankel determinant is used widely, and its consequences. Particularly in the univalent functions and certain subclasses of this. These limitations encompass information associated with the characteristics of the analytic functions. Precise estimates of third-order Hankel determinants were determined using the bounded turning functions, the convex functions, and starlike functions, which become useful tools in the sense of determining the geometrical features of functions. New results in this direction, some of which appear in Kowalczyk and Lecko, verified that the absolute value inequality $|H_{3,1}(h)| \leq \frac{1}{4}$ is acute for the functions in the bounded turning class, advancing our understanding of geometrical effects of function coefficient interactions. Hankel determinants are found from the orders that come as v and u as fixed values. If $u = 1$ and $v = 2$, then

$$|H_{2,1}(h)| = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = |a_3 - a_2^2|, \text{ where } a_1 = 1.$$

Note that $H_{2,1}(h) = a_3 - a_2^2$ is traditional Fekete-Szego function. The ultimate possible value of the latter upper bound of every subclass $|H_2, 1(h)|$ of the family A was discussed by Islam *et al.* [25]. These are precise bounds with real-world applications that extend far beyond their theoretical significance due to the possibility of using the bounds in any number of other alternative applications, such as complex analysis, numerical analysis, and even applied fields such as fluid dynamics and engineering, where the understanding of the extent of the behavior of the analytic functions is critical. These findings provide the impetus toward more research in the nature of the Hankel determinants in other subclasses of functions along with new families of functions and the corresponding determinants, for detail see [26–28]. The role of the sharp borders in pure and applied mathematics is emphasized in this present research, which also generalises the subject matter of the GFT and extends possibilities of interdisciplinary applications. Overall, the Hankel determinants and the sharp borders study is nonetheless quite active with future breakthroughs in the realm of knowledge, see [29].

1.3 Preface

This thesis is intended to analyze and describe the Hankel determinant concerning the inverse of q̌-bounded turning functions. The work is organized into five distinct chapters, each accompanied by a concise introduction as outlined below:

Chapter 1 offers an extensive introduction literature review that delves into the fundamental concepts of q̌-bounded turning functions. This work included classes of analytic functions, Caratheodory functions, and univalent functions, along with the inverse of q̌-bounded turning functions.

Chapter 2 mainly focuses on the fundamental theory of Geometric Functions, which serves as a crucial framework of subsequent chapters. After studying the ideas behind normalized univalent functions and analytic functions, Hankel determinant, coefficient bounds, and functions of bounded turning with reference to the open unit disk, it defines the multiple basic subclasses of univalent functions. Finally, the chapters end with preliminary lemmas that will be presented in the next chapters.

Chapter 3 focuses on the sharp bounds for the inverse of the Hankel determinant of third order associated with bounded turning functions. The study improves our knowledge of how inverse functions behave in GFT. The findings provide a foundation for future research and build upon earlier findings.

Chapter 4 presented the inverse of the q̌-bounded turning function's Hankel determinant, which is the extension of the sharp bound of the inverse of the bounded turning function's third Hankel determinant. It contains the most recent findings from earlier researchers.

Chapter 5 concludes this thesis by giving a thorough overview of the work completed and the primary contributions made. In order to highlight their importance within the context of GFT, the main ideas, approaches, and analytical findings established during the study are briefly reviewed. This chapter also includes closing thoughts and suggestions for future research that might be used as a foundation for more studies in related fields.

CHAPTER 2

DEFINITIONS AND PRELIMINARY CONCEPTS

2.1 Introduction

This chapter focuses on providing a framework for ongoing investigation by covering some basic definitions and traditional outcomes. Consideration will be given to a few important functions and preparatory lemmas. In complex function theory, the connection between geometric functions and complex analysis is arguably its most fascinating aspect.

2.2 Analytic Functions and Class A

Analytic functions are complex-valued functions that are both defined and differentiable at each stage of a particular area of the complex plane, commonly known as a holomorphic function, see [30].

Definition 2.2.1. *[31] The function h is differentiable at each point in an area, then it is said to be analytic in that area. The series representation of an analytic function can be expressed as*

$$h(k) = k + \sum_{n=2}^{\infty} c_n k^n. \tag{2.1}$$

Definition 2.2.2. *[32] The well-known class A contains such functions that are analytic and normalized. Miller and Mocanu [33] defined a mathematical technique called subordination* *by using Schwartz functions. It is normalized using these condition* $h(0) = 0$ *and* $h^{'}(0) = 1$ *in open unit disk.*

2.3 Univalent Functions and Class S

A univalent function is a complex valued function that is analytic and one-to-one in a given domain of complex plane.

Definition 2.3.1. *[34] Let h be a analytic function with complex values that is defined on open unit disk M of a complex plane then any two points* $k_1, k_2, \in M$*, like* $h(k_1) = h(k_2)$ *such that* $k_1 = k_2$*, is called a univalent function.*

Definition 2.3.2. *[6] Assume that h is a function belongs to class A and injective in the open unit disk M , then* $h \in S$*. The normalized univalent function of the S class is important in GFT. Class S holds four major subclasses, such as class of starlike function* S^**, the class of Convex functions C, class of close-to-convex functions K and the class of Quasi-Convex functions* C^**. This evaluation was first made in an endeavour to brace the Bieberbach conjecture.*

These subclasses of univalent functions as described below;

2.3.1 Starlike Functions (S^*)

Definition 2.3.3. *[35, 36] Let a function h is called Starlike if it is analytic and univalent in* $M = \{h \in C : |h| < 1\}$*, and satisfy the condition*

$$Re\left(\frac{kh'(k)}{h(k)}\right) > 0, k \in M, \tag{2.2}$$

and the image $h(M)$ *is a Starlike domain with respect to origin that is, the line segment connects the origin to any point in* $(h)M$ *lies entirely in* $h(M)$*.*

The following figure represents the subclass in which all starlike functions are included.

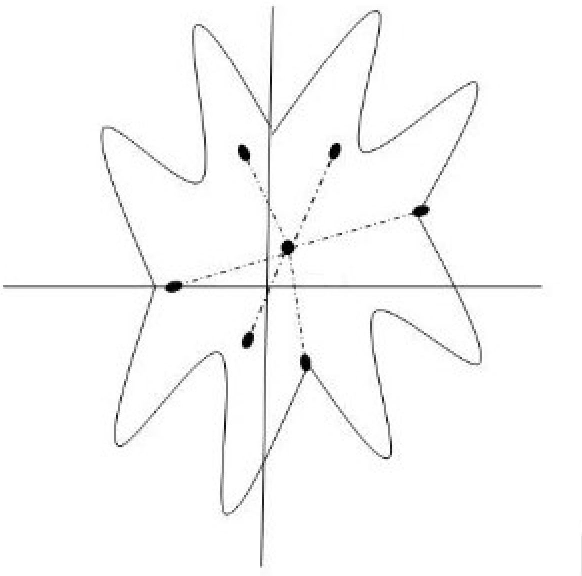

Figure 2.1: Starlike Domain

2.3.2 Convex Functions (C)

The disk M is transferred by a convex function into a convex domain M origin-centered, as presented in Figure 2.2. The subclass of S that comprises all Convex functions and is represented by C.

Definition 2.3.4. *[37] A function $h \in A$ is called Convex if it is univalent in a Domain M and the image $h(M)$ is Convex domain in a complex plane. Analytically, a function $h \in A$ is convex if and only if it satisfy this inequality*

$$R\left(1+\frac{kh''(k)}{h'(k)}\right) > 0, \textit{ for all } h \in M. \tag{2.3}$$

Definition 2.3.5. *[38] In a complex plane, when the domain M is convex, then a line segment that connects two of its points is fully included within it, such as*

$$[\nu(k_1)+(1-\nu)k_2] \in M, \tag{2.4}$$

where both k_1, k_2 belong to M with $\{0 \le k \le 1\}$.

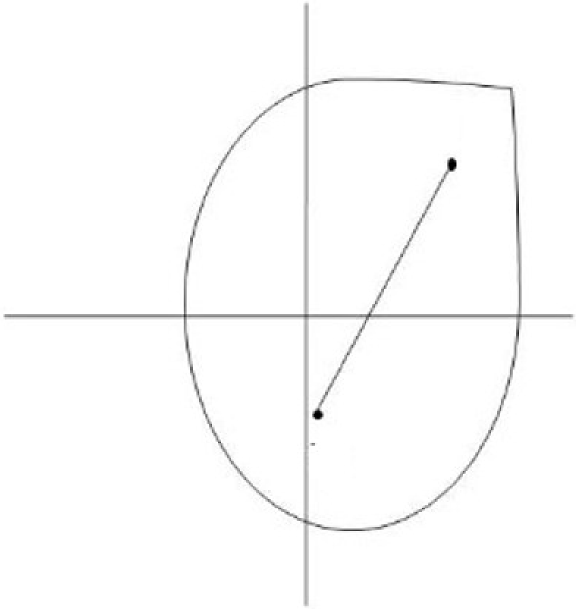

Figure 2.2: Convex Domain

2.3.3 Close-to-Convex Function (K)

Definition 2.3.6. *[39, 40] The function h is refered to as close-to-convex function if*

$$R\left(\frac{h^{'}(k)}{g^{'}(k)}\right) > 0, \tag{2.5}$$

for some convex univalent function g. All functions that are close to convex are also univalent, but not all close to convex are characterized as starlike or convex.

2.3.4 Qausi Convex Function (C^{*})

Definition 2.3.7. *[41, 42] A function h is called Qausi-convex function if*

$$R\left(\frac{h^{'}(k)}{g(k)}\right) > 0, k \in M. \tag{2.6}$$

Correspondingly, for some convex function g;

$$\frac{kh'(k)}{g(k)} \prec \frac{k+1}{1-k}. \tag{2.7}$$

The convexity is expanded by this subordination condition. If $g(k) = h(k)$, then it is determined that h is starlike.

2.4 Caratheodory Functions and Class P

These functions are fundamental to GFT because of their close relationship to univalent functions, convex functions, and Herglotz representation, which expresses any $p(k)$ as an integral over the unit circle involving a positive measure.

Definition 2.4.1. *[31] A Caratheodory function in complex analysis is a holomorphic function $p(k)$ defined in $M = \{k : |k| < 1, k \in C\}$ such that $Re(p(k)) > 0$, for all $k \in M$; all these types of functions belonging to class P. It can be expressed as*

$$p(k) = 1 + \sum_{n=1}^{\infty} c_n k^n. \tag{2.8}$$

2.5 Bounded Turning Functions

Assume that h is a univalent (one-to-one) function that is defined on the unit disk M, which is often the domain M.

Definition 2.5.1. *[43] A function h is said to have bounded turning if there is a constant $j > 0$ such that, for any two points $k_1, k_2 \in M$, and the following condition holds:*

$$\frac{1}{j}|k_1 - k_2| \leq |h(k_1 - h(k_2)| \leq |k_1 - k_2|. \tag{2.9}$$

Because of this inequality, when a distance is mapped by h, it is neither too much compressed nor too much enlarged within the domain.

To put it another way, the function prevents extremely large distance distortion, see [44, 45].

1. $|k_1 - k_2|$ indicates the distance in Euclidean terms between the two points in the domain, k_1 and k_2.

2. $|h(k_1) - h(k_2)|$ is the Euclidean separation, under the function h, between the pictures of those points.

3. The distortion's magnitude is set by the constant J. More distortion is permitted with a bigger J, whereas a smaller j denotes less distortion.

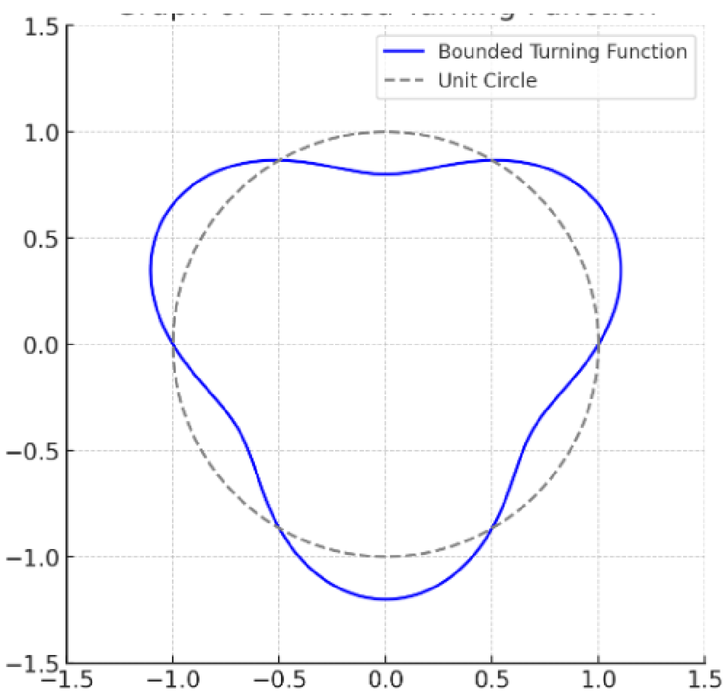

Figure 2.3: Bounded Turning Functions

2.6 Hankel determinant

Definition 2.6.1. *[46–48] The Hankel determinant is the determinant of its associated Hankel matrix. Pommerenke [49] investigated the Hankel determinant for the class of univalent functions for positive integers $v,u \in \mathbb{N} = \{1,2,3,...\}$ such that*

$$H_{v,u}(h) = \begin{vmatrix} a_u & a_{u+1} & \cdots & a_{u+v-1} \\ a_{u+1} & a_{u+2} & \cdots & a_{u+v} \\ \vdots & \vdots & \ddots & \vdots \\ a_{u+v-1} & a_{u+v} & \cdots & a_{u+2v-2} \end{vmatrix}. \tag{2.10}$$

The Hankel determinant is helpful in both the study of singularities and the analysis of integral coefficient power series. Babalola [43] initiated the Hankel determinant of third-order for the well-known disk M-convex and starlike function classes. For a convex function, the Hankel determinant is $H_{3,1}(h) \leq \frac{15}{24}$, while that of a star-like function is $H_{3,1}(h) \leq 16$.

2.7 Fekete-Szego Inequality

In complex analysis, the Fekete-Szegő inequality has a number of significant ramifications. The main goal of Fekete-Szegö problem is to determine the ideal constant μ, for any analytic function that equals to μ or such that the inequality is less than it. A well-known Fekete-Szegö inequality for univalent function h is $|c_3 - c_2^2| = H_2(1)$. It is commonly written as $|c_3 - \mu c_2^2|$ for undeniable μ, where μ could be real or complex. Fekete-Szego provided a difficult inequality that holds for, $0 \leq \mu < 1$.

Definition 2.7.1. *[50–52] A well-known finding in complex analysis is the Fekete-Szego inequality, which determines the supremum of the determinant in absolute terms for a specific class of analytic functions. It specifically pertains to functions specified in $M = \{|k| \leq 1, k \in C\}$ and normalized such that $h(0) = 0, h^{'}(0) = 1$. It gives an upper bounds with respect to the absolute value of k considering modulus $|k|$.*

2.8 The Class R

Definition 2.8.1. *[23] Let h be defined on an open unit disk $M = \{k \in C : |k| < 1\}$ normalized such that $h(0) = 0$ and $h'(0) = 1$ and satisfy the condition $Re(h'(k)) > 0$ for all $k \in M$. The functions in this class are referred to as real derivatives or functions with bounded turnings because the derivative has a strictly positive real part across the disk.*

2.9 Quantum Calculus(q̌-derivative)

The foundation of quantum calculus were laid by American mathematicaian Jackson in the early twentieth century. Jackson was the first mathematician to define the q̌-analog of the derivative and integral operator.

Definition 2.9.1. *[53, 54] In the study of GFT, the q̌-calculus represents a variation of classical calculus that includes the parameter q̌ and alters the conventional definitions of integrals, and other operators. This area of research is referred to as q̌-calculus or quantum calculus, and it deals with calculus without the requirement for limits. Special functions, orthogonal polynomials, and fractal geometry are some common applications of q̌-calculus.*

The q̌-calculus is helpful in GFT because it makes it possible to explore some mappings, deformations, and transformations that are more flexible than those covered in classical calculus. Additionally, by modeling discrete and continuous systems concurrently, the idea has significant implications for approximation theory, complex analysis, and geometric functions.

Definition 2.9.2. *[55] The extension of the classical derivative is* ǭ*-derivative operator and is especially useful in exploring* ǭ*-analogs of analytic and univalent functions. It is defined for analytic functions $h(k)$ particularly when $h(0) = 0$, and introduce a deformation parameter into a traditional parameter theory. The* ǭ*-derivative Operator is given by*

$$D_{\check{q}}h(k) = \frac{h(k) - h(\check{q}k)}{(1-\check{q})k} \text{ for } (0 < \check{q} < 1). \tag{2.11}$$

It converges to the ordinary derivative when $\check{q} \to 1^-$.

Now we define a new class using the ǭ-derivative operator as in the following.

2.10 The class $R_{\check{q}}$

Definition 2.10.1. *A function h is a member of class $R_{\check{q}}$ if it is analytic on the open unit disk M normalized, so that $h((0) = 0$ and its* ǭ*-derivative satisfy*

$$Re(D_{\check{q}}h(k)) > 0 \tag{2.12}$$

for every $k \in M$,. The classical class R can be obtained from this generalization as $\check{q} \to 1^-$.

2.11 Lemmas

Lemma 2.11.1. *[56] For functions $R(k) = \frac{1+k}{1-k}, k \in M$, equality is obtained for $p \in P$, where $|c_t| < 2$ and $t \in N$.*

Lemma 2.11.2. *For $p \in P$, we obtain $C_2 = \frac{1}{2}\left[c_1^2 + t\omega\right]$*

$c_3 = \frac{1}{4}\left[c_1^3 + 2c_1t - c_1t\omega^2 + 2t(1-|\omega|^2)\tau\right]$

$c_4 = \frac{1}{8}\left\{c_1^4 + t\omega\left[c_1^2(\omega^2 - 3\omega + 3) + 4\omega\right] - 4t(1-|\omega|^2)\left[c_1(\omega-1)\tau - (1-|\tau|^2)\xi + \omega\tau^2\right]\right\}$,

where $t = 4 - c_1^2$ for some ω, τ and ξ with $|\omega| \le 1, |\tau| \le 1, |\xi| \le 1$.

CHAPTER 3

THE SHARP BOUND OF THE THIRD HANKEL DETERMINANT FOR THE INVERSE OF BOUNDED TURNING FUNCTIONS

3.1 Introduction

This research work investigates a sharp bounds regarding the inverse function of the third-order Hankel determinant of $H_{3,1}(h)$ related to bounded turning functions, indicated by R, containing holomorphic function h in the unit disk such that $Re(h^{'}(k)) > 0$. Although several coefficient estimates exist for functions in R, finding the exact value of the inverse functions for the Hankel determinant remained unresolved, see [57]. Let A stand for the family of mappings h of this kind.

$$h(k) = k + \sum_{n=2}^{\infty} c_n k^n \tag{3.1}$$

in $M = \{k : |k| < 1\}$ is the Open unit disk, where S denotes the subgroup A class with functions that are univalent. Each h in S possesses an inverse h^{-1}, which is provided by

$$h^{-1}(e) = e + \sum_{n=2}^{\infty} c_n e^n, |e| < r_0(h); \left(r_0(h) \geq \frac{1}{4} \right). \tag{3.2}$$

Assume that P is the class of every function in M that has a positive real component.

$$p(k) = 1 + \sum_{t=1}^{\infty} c_t k^t \tag{3.3}$$

All of these functions are referred to as Caratheodory functions, for detail see [58]. To obtain our result, we employ the method that Libera and Zlotkiewicz have been using. The required sharp estimates, which apply functions with a positive real portion, are also used in the form of lemmas shown below.

3.2 Final Results

Theorem 3.2.1. *If $h \in R$ and $h^{-1}(e) = e + \sum_{n=2}^{\infty} t_n e^n$ is the inverse of h then*

$$|H_{3,1}(h^{-1})| \leq \frac{44}{135}, \tag{3.4}$$

the inequality is sharp for

$$h_1(k) = \frac{log(1+k)}{(1-k)} - k. \tag{3.5}$$

Proof. There is a holomorphic function $p \in P$ for $h \in R$, where

$$h'(k) = p(k) \tag{3.6}$$

$$h(k) = k + \sum_{n=2}^{\infty} t_n k^n \tag{3.7}$$

$$h(k) = k + a_2k^2 + a_3k^3 + a_4k^4 + \cdots$$

$$h^{'}(k) = 1 + 2a_2k + 3a_3k^2 + 4a_4k^3 + \ldots$$

$$p(k) = 1 + \sum_{t=1}^{\infty} c_t k^t \tag{3.8}$$

$$= 1 + c_1k^1 + c_2k^2 + c_3k^3 + \ldots$$

here

$$h^{'}(k) = p(k)$$

$$1 + 2a_2k + 3a_3k^2 + 4a_4k^4 + \ldots = 1 + c_1k + c_2k^2 + c_3k^3 + \ldots,$$

by comparing

$$2a_2k + 3a_3k^2 + 4a_4k^3 + \ldots = c_1k^1 + c_2k^2 + c_3k^3 + \ldots$$

$$2a_2 + 3a_3 + 4a_4 + \ldots = c_1 + c_2 + \ldots c_n + c_{n-1}$$

$$na_n = c_{n-1}$$

$$a_n = \frac{c_{n-1}}{n}. \tag{3.9}$$

Using the concept of the inverse function h, we obtain since $h \in R$.

$$h^{-1}(e) = e + \sum_{n=2}^{\infty} t_n e^n. \tag{3.10}$$

$$e = h(h^{-1}) = e + \sum_{n=2}^{\infty} t_n e^n + \sum_{n=2}^{\infty} a_n (e + \sum_{n=2}^{\infty} (t_n e^n)^n), \tag{3.11}$$

through simplification

$$(t_2 + a_2)e^2 + (t_3 + a_3 + 2a_2 t_2)e^3 + (t_4 + a_4 + 2a_2 t_3 + a_2 t_2^2 + 3a_3 t_2)e^4$$
$$+ (t_5 + 2a_2 t_4 + 2a_2 t_2 t_3 + 3a_3 t_3 + 3a_3 t_2^2 + 4a_4 t_2 + a_5)e^5 + \ldots = 0. \tag{3.12}$$

By comparing the coefficients of the same powers

$$\begin{aligned}
t_2 &= -a_2 \\
t_3 &= -a_3 + 2a_3^2 \\
t_4 &= -a_4 + 5a_2 a_3 - 5a_2^3 \\
t_5 &= -a_5 + 6a_2 a_4 - 21a_2^2 a_3 + 3a_3^2 + 14a_2^2,
\end{aligned} \tag{3.13}$$

from equation (3.9),

$$\begin{aligned}
t_2 &= \frac{-c_1}{2} \\
t_3 &= \frac{-2c_2 + 3c_1^2}{6} \\
t_4 &= \frac{-6c_3 + 20c_1 c_2 - 15c_1^3}{24} \\
t_5 &= \frac{-24c_4 + 90c_1 c_3 - 210c_1^2 c_2 + 40c_2^2 + 105c_1^4}{120}.
\end{aligned} \tag{3.14}$$

Now,

$$H_{3,1}(h^{-1}) = \begin{vmatrix} 1 & t_2 & t_3 \\ t_2 & t_3 & t_4 \\ t_3 & t_4 & t_5 \end{vmatrix}. \tag{3.15}$$

After putting the value of t_2, t_3, t_4, t_5 we get

$$= \left(\frac{-2c_2 + 3c_1^2}{6}\right)\left(\frac{-24c_4 + 90c_1 c_3 - 210c_1^2 c_2 + 40c_2^2 + 105c_1^4}{120}\right)$$

$$-\left(\frac{-6c_3+20c_1c_2-15c_1^3}{24}\right)^2$$

$$+\frac{c_1}{2}\left[-\frac{c_1}{2}\left(\frac{-24c_4+90c_1c_3-210c_1^2c_2+40c_2^2+105c_1^4}{120}\right)-\left(\frac{-2c_2+3c_1^2}{6}\right)\right.$$

$$\left.\left(\frac{-6c_3+20c_1c_2-15c_1^3}{24}\right)\right]+\left(\frac{-2c_2+3c_1^2}{6}\right)\left[-\frac{c_1}{2}\left(\frac{-6c_3+20c_1c_2-15c_1^3}{24}\right)\right.$$

$$\left.-\left(\frac{-2c_2+3c_1^2}{6}\right)\right]. \tag{3.16}$$

$$=\frac{1}{720}\left[48c_2c_4-180c_1c_2c_3+420c_1^2c_2^2-80c_2^3-210c_1^4c_2-72c_1^2c_4+180c_1^3c_3\right.$$

$$\left.-630c_1^4c_2+120c_1^2c_2^2+315c_1^6\right]$$

$$-\frac{1}{576}\left[36c_3^2-120c_1c_2c_3+90c_1^3c_3-120c_1c_2c_3+400c_1^2c_2^2\right.$$

$$\left.-300c_1^4c_2+90c_1^3c_3-300c_1^4c_2+225c_1^6\right]$$

$$+\frac{c_1}{2}\left[\frac{1}{240}\left(24c_1c_4-90c_1^2c_3+210c_1^3c_2-40c_1c_2^2-105c_1^5\right)\right.$$

$$\left.-\frac{1}{144}\left(12c_2c_3-40c_1c_2^2+30c_1^3c_2-18c_1^2c_3+60c_1^3c_2-45c_1^5\right)\right]$$

$$+\left(\frac{-2c_2+3c_1^2}{6}\right)\left[\frac{1}{48}\left(6c_1c_3-20c_1^2c_2+15c_1^4\right)\right.$$

$$\left.-\frac{1}{36}\left(4c_2^2-6c_1^2c_2-6c_1^2c_2+9c_1^4\right)\right]. \tag{3.17}$$

$$=\frac{c_2c_4}{15}-\frac{c_1c_2c_3}{4}+\frac{7c_1^2c_2^2}{12}-\frac{c_2^3}{9}-\frac{7c_1^4c_2}{24}-\frac{c_1^2c_4}{10}$$

$$+\frac{c_1^3c_3}{4}-\frac{7c_1^4c_2}{9}+\frac{c_1^2c_2^2}{6}+\frac{7c_1^6}{16}-\frac{c_3^2}{16}+\frac{5c_1c_2c_3}{24}$$

$$-\frac{5c_1^3c_3}{32}+\frac{5c_1c_2c_3}{24}-\frac{25c_1^2c^2}{36}+\frac{25c_1^4c_2}{48}-\frac{5c_1^3c_3}{32}+\frac{25c_1^4c_2}{48}-\frac{25c_1^6}{64}$$

$$+\frac{c_1}{2}\left[\frac{c_1c_4}{10}-\frac{3c_1^2c_3}{8}+\frac{7c_1^3c_2}{8}-\frac{c_1c_2^2}{6}-\frac{7c_1^5}{16}-\frac{c_2c_3}{12}+\frac{5c_1c_2^2}{18}\right.$$

$$\left.-\frac{5c_1^3c_2}{24}+\frac{c_1^2c_3}{8}-\frac{5c_1^3c_2}{12}+\frac{5c_1^5}{16}\right]+\left(\frac{-2c_2+3c_1^2}{6}\right)$$

$$\left[\frac{c_1c_3}{8}-\frac{5c_1^2c_2}{12}+\frac{5c_1^4}{16}-\frac{c_2^2}{19}+\frac{c_1^2c_2}{6}+\frac{c_1^2c_2}{6}-\frac{c_1^4}{4}\right],$$

$$=\frac{c_2c_4}{15}-\frac{c_1c_2c_3}{4}+\frac{7c_1^2c_2^2}{12}-\frac{c_2^3}{9}-\frac{7c_1^4c_2}{24}-\frac{c_1^2c_4}{10}$$

$$+\frac{c_1^3c_3}{4}-\frac{7c_1^4c_2}{8}+\frac{c_1^2c_2^2}{6}+\frac{7c_1^6}{16}-\frac{c_3^2}{16}+\frac{5c_1c_2c_3}{24}$$

$$
\begin{aligned}
&-\frac{5c_1^3c_3}{32}+\frac{5c_1c_2c_3}{24}-\frac{25c_1^2c_2^2}{36}+\frac{25c_1^4c_2}{48}-\frac{5c_1^3c_3}{32}\\
&+\frac{25c_1^4c_2}{48}-\frac{25c_1^6}{64}+\frac{c_1^2c_4}{20}+\frac{3c_1^3c_3}{16}+\frac{7c_1^4c_2}{16}\\
&-\frac{c_1^2c_2^2}{12}-\frac{7c_1^6}{32}-\frac{c_1c_2c_3}{24}+\frac{5c_1^2c_2^2}{36}-\frac{5c_1^4c_2}{24}-\frac{c_1^3c_3}{16}\\
&-\frac{5c_1^4c_2}{24}+\frac{5c_1^6}{32}-\frac{c_1c_2c_3}{24}+\frac{5c_1^2c_2^2}{36}-\frac{5c_1^4c_2}{48}+\frac{c_2^3}{27}-\frac{c_1^2c_2^2}{18}\\
&-\frac{c_1^2c_2^2}{18}+\frac{c_1^4c_2}{12}-\frac{c_1^3c_3}{16}-\frac{5c_1^4c^2}{24}+\frac{5c_1^6}{32}-\frac{c_1^2c_2^2}{18}+\frac{c_1^4c_2}{12}+\frac{c_1^4c_2}{12}-\frac{c_1^6}{8},\\
=&\frac{7c_1^6}{16}-\frac{25c_1^6}{64}-\frac{7c_1^6}{32}+\frac{5c_1^6}{32}+\frac{5c_1^6}{32}-\frac{c_1^6}{8}-\frac{7c_1^4c_2}{24}\\
&-\frac{7c_1^4c_2}{8}+\frac{25c_1^4c_2}{48}+\frac{25c_1^4c_2}{48}+\frac{7c_1^4c_2}{16}-\frac{5c_1^4c_2}{24}\\
&-\frac{c_1^4c_2}{24}-\frac{5c_1^4c_2}{48}+\frac{c_1^4c_2}{12}-\frac{5c_1^4c_2}{24}+\frac{c_1^4c_2}{12}+\frac{c_1^4c_2}{12}\\
&+\frac{7c_1^2c_2^2}{12}+\frac{c_1^2c_2^2}{6}-\frac{25c_1^2c_2^2}{36}-\frac{c_1^2c_2^2}{12}+\frac{5c_1^2c_2^2}{36}+\frac{5c_1^2c_2^2}{36}\\
&-\frac{c_1^2c_2^2}{18}-\frac{c_1^2c_2^2}{18}-\frac{c_1^2c_2^2}{18}+\frac{c_2c_4}{15}-\frac{c_1^2c_4}{10}+\frac{c_1^2c_4}{20}-\frac{c_1c_2c_3}{4}\\
&+\frac{5c_1c_2c_3}{24}+\frac{5c_1c_2c_3}{24}-\frac{c_1c_2c_3}{24}-\frac{c_1c_2c_3}{24}-\frac{c_2^3}{9}\\
&+\frac{c_2^3}{27}-\frac{c_3^2}{16}+\frac{c_1^3c_3}{4}-\frac{5c_1^3}{c_3}-\frac{5c_1^3c_3}{32}\\
&-\frac{c_1^3c_3}{16}-\frac{c_1^3c_3}{16}+\frac{3c_1^3c_3}{16},\\
&=\frac{1}{8640}\bigg[135c_1^6-540c_1^4c_2+720c_1^2c_2^2+576c_2c_4\\
&\quad -432c_1^2c_4+720c_1c_2c_3-640c_2^3-540c_3^2\bigg]. \qquad (3.18)
\end{aligned}
$$

Using Lemmma (2.11.2);

$$c_2=\frac{1}{2}\left[c_1^2+t\omega\right]$$

$$-540c_1^4c_2=-270\left[c_1^6+c_1^4t\omega\right]$$

$$-640c_2^3=-640\left[\frac{1}{2}\left(c_1^2+t\omega\right)\right]^3$$

$$-640c_2^3=-80\left[c_1^6+3c_1^4t\omega+3c_1^2t^2\omega^2+t^3\omega^3\right]$$

$$720c_1^2c_2^2=720c_1^2\left[\frac{1}{2}\left(c_1^2+t\omega\right)\right]^2$$

$$=90\left[c_1^6+3c_1^2t\omega+2c_1^2t^2\omega^2-c_1^4t\omega^2-c_1^2t^2\omega^3+2t\left(c_1^3+c_1t\omega\right)\left(1-\omega^2\right)\tau\right]$$

$$-540c_3^2 = -540\left[\frac{1}{4}\left(c_1^3+2c_1t\omega-c_1t\omega^2+2t\left(1-\omega^2\right)\tau\right)\right]^2$$

$$= -\frac{135}{4}\Big[c_1^6+4c_1^4t\omega+4c_1^2t^2\omega^2-2c_1^4t\omega^2-4c_1^2t^2\omega+c_1^2t^2\omega^4$$
$$+4t\left(c_1^3+2c_1t\omega-c_1t\omega^2\right)(1-\omega^2)\tau+4t^2(1-\omega^2)\tau^2\Big]$$

$$576c_2c_4-432c_1^2c_4 = \frac{1}{8}\Big(c_1^4+t\omega\left[c_1^2(\omega^2-3\omega+3)+4\omega\right]$$
$$-4t(1-\omega^2)\left[c_1(\omega-1)\tau+\omega\tau^2-(1-\tau^2)\xi\right]\Big)$$
$$\times\Big(576\big(\frac{1}{2}(c_1^2+t\omega)\big)-432c_1^2\Big)$$
$$=18\Big[-c_1^6-3c_1^4t\omega-c_1^2(4-3c_1^2)-c_1^4t\omega^3-4c_1^3t(1-|\omega|^2)(1-|\omega|)\tau$$
$$+4c_1^2t(1-|\omega|^2)\tau^2+2c_1^4t\omega-4c_1^3t(1-|\omega|)(1-t\tau)+6c_1^2t^2\omega^2$$
$$+2(4-3c_1^2)t^2\omega^3+2c_1^2t^2\omega^4+8t^2c_1\omega(1-|\omega|^2)(1-|\omega|)\tau$$
$$-8t^2(1-|\omega|^2)^2\omega^2\tau^2+8t^2(1-|\tau|^2)\omega\tau(1-|\omega|^2)\Big]. \tag{3.19}$$

Putting these values in (3.18), we get

$$=\frac{1}{8640}\Big[135c_1^6-270c_1^6-270c_1^4t\omega+180c_1^6+360c_1^4t\omega+180c_1^2t^2\omega^2$$
$$-18c_1^6-54c_1^4t\omega-18c_1^2(4-3c_1^2)-18c_1^4t\omega^3-18c_1^3t(1-|\omega|)(1-|\omega|^2)\tau$$
$$-72c_1^2t(1-t)\phi+72c_1^2t(1-|\omega|^2)\omega\tau^2+36c_1^4t\omega+108c_1^2t^2\omega^2$$
$$+36(4-3c_1^2)t^2\omega^3+36c_1^2t^2\omega^4+144t^2c_1\omega(1-|\omega|)\tau(1-|\omega|^2)$$
$$-144t^2\omega^2\tau^2(1-|\omega|^2)+144t^2(1-|\omega|^2)(1-|\omega|^2)\omega\tau$$
$$+90c_1^6+270c_1^4t\omega+180c_1^2t^2\omega^2-90c_1^4t\omega^2-90c_1^2t^2\omega^3$$
$$+2t(1-\omega^2)\tau(c_1^3+c_1t\omega)-80c_1^6-240c_1^4t\omega-240c_1^2t^2\omega^2-80t^3\omega^3$$
$$-\frac{135}{4}c_1^6-135c_1^4t\omega-135c_1^2t^2\omega^2+\frac{135}{2}c_1^4t\omega^2+135c_1^2t^2\omega^3-\frac{135}{4}c_1^2t^2\omega^4$$
$$-135t(c_1^3+2c_1t\omega-c_1t\omega^2)\tau(1-|\omega|^2)-(1-|\omega|^2)\tau^2 135t^2\Big].$$

$$=\frac{1}{8640}\Big[\frac{13}{4}+t\Big(-33c_1^4\omega+\frac{63}{2}c_1^4\omega^2-72c_1^2\omega^2+93c_1^2t\omega^{\text{‘}2}-18c_1^4\omega^3$$

$$+t\left(-63c_1^2\omega^3+\frac{9}{4}c_1^2\omega^4+144\omega^3-80t\omega^3\right)$$
$$+\left((-27+72\omega)c_1^3+c_1t\omega(54-9\omega)\right)(1-\omega^2)\tau$$
$$+\left(72c_1^2\omega-t(135+9\omega^2)\right)(1-|\omega|^2)\tau^2$$
$$+72\left(2t\omega-c_1^2\right)(1-|\omega|^2)(1-|\tau|^2)\xi\Bigg)\Bigg]. \tag{3.20}$$

Put $c_1=c$ and $t=4-c^2$ in (3.20), we get

$$H_{3,1}(h^{-1})=\frac{1}{8640}\Bigg[\frac{13}{4}c^6+(4-c^2)\Big(-33c^4\omega+\frac{3}{2}c^2(200-41c^2)\omega^2-18c^4\omega^3$$
$$+(4-c^2)(\frac{9}{4}c^2\omega^4-(176-17c^2)\omega^3)\Big)$$
$$+\left[(-27+72\omega)c^3+c(4-c^2)\omega(-41)\right](1-|\omega|^2)\tau$$
$$+\Big(72c^2\omega^2-(4-c^2)(135+9\omega^2)(1-|\omega|^2)\tau^2$$
$$+72(2(4-c^2)\omega-c^2)(1-|\omega|^2)(1-|\tau|^2)\xi\Big)\Bigg]. \tag{3.21}$$

The modulus of each side taken with $|\omega|=x$ such that $x\in(0,1)$, $|\tau|=y$ and y lies between 0 and 1,$c_1=c$ and c lies between 0 and 2, and $\omega\leq 1$.

$$\left|H_{3,1}(h^{-1})\right|\leq\frac{\phi(c,x,y)}{8640}, \tag{3.22}$$

$$H_{3,1}(h^{-1})=\frac{1}{8640}\Bigg[\frac{13}{4}c^6+(4-c^2)\Big(-33c^4x+\frac{3}{2}c^2(200-41c^2)x^2-18c^4x^3$$
$$+(4-c^2)\left(\frac{9}{4}c^2x^4-(176-17c^2)x^3\right)\Big)$$
$$+\left[(-27+72x)c^3+c(4-c^2)x(-41)\right](1-x^2)y$$

$$+\Big(72c^2x^2-(4-c^2)(135+9x^2)(1-x^2)y^2$$
$$+72\big(2(4-c^2)x-c^2\big)(1-y^2)(1-x^2)\xi\Big)\Bigg]. \tag{3.23}$$

here $\phi(c,x,y);R^3\to R$ means three Dimensional space to real numbers as

$$\phi(c,x,y)=\Bigg[\frac{13}{4}c^6+(4-c^2)\Big(-33c^4x+\frac{3}{2}c^2(200-41c^2)x^2-18c^4x^3$$

$$+(4-c^2)\Big(\frac{9}{4}c^2x^4-(176-17c^2)x^3)\Big)$$

$$+\big((-27+72x)c^3+c(4-c^2)x(57-98)\big)\,(1-x^2)y+(72c_2^2x^2$$

$$-(4-c^2)(135+9x^2)(1-x^2)y^2)+72(2(4-c^2)(x-c^2)(1-x^2)(1-y^2))\Bigg]. \quad (3.24)$$

Now we will maximize the region of Parallellopiped established by $[0,2]\times[0,1]\times[0,1]$, where $c\in[0,2], x\in[o,1], y\in[0,1]$.

A. First, we find eight vertices of a parallelepiped.

(i) $\phi(0,0,0)=0$

(ii) $\phi(2,0,0)=\frac{13}{4}(2)^6+0=\frac{832}{4}+0=208$

(iii) $\phi(2,1,0)=\frac{13}{4}(2)^6+0=208$

(iv) $\phi(2,1,1)=\frac{13}{4}(2)^6+0=\frac{832}{4}+0=208$

(v) $\phi(2,0,0)=\frac{13}{4}(2)^6+0=\frac{832}{4}=208.$

(vi) For $y=1$.

$$\begin{aligned}\phi(0,0,1)=\frac{13}{4}(0)^6+(4-0)\Bigg[&33(0)+\frac{3}{2}(0)^2(200-41(0)^2)(0)^2+18(0)^3\\&+(4-0)\left(\frac{9}{4}(0)^4+(176-17(0)^3)(0)^3\right)\\&+(27+72(0)+0(4-0)(0)(57+9(0)))\,(1-0)(1)\\&+(72(0)+(4-0)(135+9(0)))\,(1-0)(1)\\&+72\,(2(4-0)(0)+0)\,(1-0)(1-1)\Bigg]\\&=0+4\,[0+4(0)+0+540+72(0)]\\&=2160.\end{aligned} \quad (3.25)$$

(vii) For $x=1$.

$$\begin{aligned}\phi(0,1,0)&=\frac{13}{4}(0)^6+(4-0)\left[0+\frac{3}{2}(0)+0+4(176)+0+0+72(0)\right]\\&=0+4\,[0+4(176)]\\&=4(704)\\&=2816.\end{aligned} \quad (3.26)$$

$(viii)$ For $x=1, y=1$.

$$\begin{aligned}\phi(0,1,1) &= \frac{13}{4}(0)+4\left[33(0)+\frac{3}{2}(0)+18(0)+4(0+176)+0+72(0)+0\right]\\ &= 0+4(0+4(176))\\ &= 2816. \qquad (3.27)\end{aligned}$$

B. Now we take eight parallelepiped edges.

(i) For $x=1, y=0$ and $x=1, y=1$.

$$\begin{aligned}\phi(c,1,y) &= \frac{13}{4}c^6+(4-c^2)\left[33c^4+\frac{3}{2}c^2(200-41c^2)+18c^4+(4-c^2)\left(\frac{9}{4}c^2+176-17c^2\right)\right]\\ &= 2816-44c^2-48c^4-c^6 \leq 2816, \qquad \text{for } c\in(0,2). \qquad (3.28)\end{aligned}$$

(ii) For $c=0, y=0$.

$$\begin{aligned}\phi(0,x,0) &= \frac{13}{6}(0)+(4-0)\Bigg[33(0)+\frac{3}{2}(0)(200-41\cdot 0)x+18(0)\\ &\quad +(4-0)\Bigg(\frac{9}{4}(0)+(176-17(0))x^3+\big((27+72x)0\\ &\quad +(4-0)(135+9x^2)(1-x^2)\cdot 0+72(2(4-0)x+0)(1-x^2)(1-0)\Bigg)\Bigg]\\ &= 5312x^3+2304x \leq 2816, \text{ where } x\in(0,1). \qquad (3.29)\end{aligned}$$

(iii) For $x=0, y=1$.

$$\begin{aligned}\phi(c,0,1) &= \frac{13}{4}c^6+(4-c^2)\Big[0+0+0+(4-c^62)(0+0)+(27c^3+0)\\ &\quad +\big(0+(4-c^2)(135)\big)(1)+72(0+c^2)(0)\Big]\\ &= 2160+\frac{13}{4}c^6-27c^2(4(5-c)+5(4-c^2)+c^3)\\ &\leq 2160+\frac{13}{4}c^6 \leq 2368. \qquad (3.30)\end{aligned}$$

(iv) For $c = 0, y = 1$.

$$\begin{aligned}\phi(0,x,1) &= \frac{13}{4}\cdot 0 + (4-0)\Big[0 + \frac{3}{2}\cdot 0 + 18\cdot 0 + (4-0)(0 + (176-17)x^3) \\ &\quad + (0+0)(1-x^2) + \big(72\cdot 0 + 4(135+9x^2)\big)(1-x^2) \\ &\quad + 72(2\cdot 4\cdot x + 0)(1-x^2)\cdot 0\Big] \\ &= 4\left[(4)(159x^3) + 4(135+9x^2)(1-x^2)\right] \\ &= 4\left[636x^3 + 4(135+9x^2)(1-x^2)\right] \\ &= 2544x^3 + 16(135+9x^2)(1-x^2) \\ &\le 2816, \text{ for } x \in (0,1). \end{aligned} \tag{3.31}$$

(v) For $c = 0, y = 0$.

$$\begin{aligned}\phi(0,0,y) &= \frac{13}{4}(0) + 4\left[135(4y^2)\right] \\ &= 2160y^2 \le 2160 \text{ for } y \in (0,1). \end{aligned} \tag{3.32}$$

(vi) For $c = 0, y = 1$.

$$\begin{aligned}\phi(0,1,c) &= \frac{13}{4}(0) + (4-0)\left[0 + 0 + (4-0)(0+176) + 72(0) + 135((0)4y^2) + 72(0+0+0)\right] \\ &= 2816. \end{aligned} \tag{3.33}$$

(vii) With $c = 2,$ the conditions are evaluated for $x = (0,1)$ and $y = (0,1)$.

$$\phi(c,x,y) = 208. \tag{3.34}$$

$(viii)$ When $x = 0, y = 0$.

$$\begin{aligned}\phi(c,0,0) &= \frac{13}{4}(c)^6 + (4-c^2)\left[0+0+0+(4-c^2)(0+0) + 27c^3(1-0)y + 72(c^2)\right] \\ &= 288c^2 - 72c^4 + \frac{13}{4}c^6 \quad \text{for } c \in (0,2) \\ &\le 288c^2 + \frac{13}{4}c^6 \le 1360. \end{aligned} \tag{3.35}$$

C. Consider six faces of a Parallelopiped.

(i) For $c = 2$.

$$\phi(2,x,y) = 208, \text{ for } (x,y) \in (0,1). \tag{3.36}$$

(ii) For $c = 2$.

$$\phi(0,x,y) = 0 + (4-0)\left[0 + 704x^3 + (0 + 4(135+9x^2)(1-x^2)y^2) + 72(8x)(1-x^2)(1-y^2)\right]$$
$$= 2304x + 512^3 + 144(1+x)(15-x)(1-x^2)$$
$$\leq 2816. \tag{3.37}$$

(iii) For $x = 0$.

$$\begin{aligned}\phi(c,0,y) &= \frac{13}{4}c^6 + (4-c^2)\left[27c^3y + 135(4-c^2)y^2 + 72c^2(1-y^2)\right]\\ &= \frac{13}{4}c^6 + (4-c^2)\left[27c^3y + 540y^2 + c^2(72-207y^2)\right]\\ &\leq \frac{13}{4}c^6 + (4-c^2)\left[27c^3y + 540 + 72c^2\right]\\ &\leq 2368. \end{aligned} \tag{3.38}$$

(iv) For $x = 1$.

$$\begin{aligned}\phi(c,1,y) &= -444c^2 - 48c^4 - c^6 + 2816\\ &\leq 2816, \text{ for } c \in (0,2). \end{aligned} \tag{3.39}$$

(v) For $y = 0$.

$$\begin{aligned}\phi(c,x,0) = &\frac{13}{4}c^6 + (4-c^2)\left[33c^4x + \frac{3}{2}c^2(200-41c^2)x^2 + 18c^4x^3\right.\\ &\left. + (4-c^2)\left(\frac{9}{4}c^2x^4 + (176-17c^2)x^3\right) + 72\left(2(4-c^2)x + c^2\right)(1-x^2)\right]\\ \leq &\frac{13}{4}c^6 + (4-c^2)\left(704 + 72c^2 + 6c^4\right)\\ \leq &2816 \quad \text{for } c \in (0,2),\ x \in (0,1). \end{aligned} \tag{3.40}$$

(vi) For $y = 1$

$$\begin{aligned}\phi(c,x,1) = &\frac{13}{4}c^6 + (4-c^2)\Big[33c^4x + \frac{3}{2}c^2(200-41c^2)x^2 + 18c^4x^3\\ &+ (4-c^2)\left(\frac{9}{4}c^2x^4 + (176-c^2)x^3\right)\\ &+ \left((27+72x)c^3 + c(4-c^2)x(57+9x)(1-x^2)\right)\\ &+ \left(72cx^2 + (4-c^2)(135+9x^2)\right)(1-x^2)\Big]\\ = &g(c,x) \text{ with } c \in (0,2) \text{ and } x \in (0,1). \end{aligned} \tag{3.41}$$

Further

$$\frac{\partial g}{\partial c} = -2160c + 324c^2 + 540c^3 - 135c^4 + \frac{39c^5}{2} + 864x + 576cx - 432c^2x + 240c^3x$$
$$- 90c^4x - 198c^5x + 144x^2 + 4416cx^2 - 540c^2x^2 - 2688c^3x^2 + 180c^4x^2$$
$$+ 369c^5x^2 - 864x^3 - 3936cx^3 + 432c^2x^3 + 1824c^3x^3 + 90c^4x^3 - 210c^5x^3$$
$$- 144x^4 + 216cx^4 + 216c^2x^4 - 108c^3x^4 - 45c^4x^4 + \frac{27c^5x^4}{2},$$

and

$$\frac{\partial g}{\partial x} = 864c + 288c^2 - 144c^3 + 60c^4 - 18c^5 - 33c^6 - 4032x + 288cx + 4416c^2x$$
$$- 360c^3x - 1344c^4x + 72c^5x + 123c^6x + 8448x^2 - 2592cx^2 - 5904c^2x^2$$
$$+ 432c^3x^2 + 1368c^4x^2 + 54c^5x^2 - 105c^6x^2 - 576x^3 - 576cx^3 + 432c^2x^3$$
$$+ 288c^3x^3 - 108c^4x^3 - 36c^5x^3 + 9c^6x^3. \tag{3.42}$$

$\frac{\partial g}{\partial c} = 0$ and $\frac{\partial g}{\partial x} = 0$ with $(0,2).(0,1)$ and $(c_1, x_1) = (0.248233, 0.445259)$
Taking again partial derivatives of $\frac{\partial g}{\partial c}$ and $\frac{\partial g}{\partial x}$
we get

$$\frac{\partial^2 g}{\partial c^2} = -2160 + 648c + 1620c^2 - 540c^3 + \frac{195}{2}c^4 + 576x - 864cx + 720c^2x$$
$$- 360c^3x - 990c^4x + 4416x^2 - 1080cx^2 - 8064c^2x^2 + 720c^3x^2 + 1845c^4x^2$$
$$- 3036x^3 + 864cx^3 + 5472c^2x^3 - 180c^3x^4 + \frac{135}{2}c^4x^4,$$

$$\frac{\partial^2 g}{\partial x^2} = -4032 + 288c + 4416c^2 - 360c^3 - 1344c^4 + 72c^5 + 123c^6 + 1689x - 5184cx$$
$$- 11808c^2x + 864c^3x + 2768c^4x + 108c^5x - 210c^6x - 1728x^2 - 1728cx^2$$
$$+ 1296c^2x^2 + 864c^3x^2 - 324x^2 - 108c^5x^2 + 27c^5x^2. \tag{3.43}$$

$$\frac{\partial^2 g}{\partial c \partial x} = 864 + 576 - 432c^2 + 240c^3 - 90c^4 - 396x + 288c^2x - 1080c^3x$$
$$- 5376cx + 360c^4x. \tag{3.44}$$

Hence

$$\left(\frac{\partial^2 g}{\partial c^2}\right)\left(\frac{\partial^2 g}{\partial x^2}\right) - \left(\frac{\partial^2 g}{\partial c \partial x}\right)^2 < 0. \tag{3.45}$$

Therefore, at $y=1$, there is no critical point.

D. Now, examining the parallelepiped's interior with: $(0,1)*(0,2)*(0,1)$,

hence $\frac{\partial\phi}{\partial y}=0$ exactly when

$$y_0(c,x)=\frac{4cx(x+6)+c^3(3+2x-x^2)}{2(x-1)\left(c^2(x-23)-4(x-15)\right)}. \tag{3.46}$$

This is true for only $(c,x)\in(0,2)*(0,1)$, also $c^2(x-23)\neq 4(x-15)$.

The equation systems $\frac{\partial\phi}{\partial x}(c,x,y_0(c,x))=0$ and $\frac{\partial\phi}{\partial c}(c,x,y_0(c,x))=0$ are as follows.

$$(c\approx\pm2.1038,\ x\approx107.05),\quad (c\approx\pm1.5487,\ x\approx-0.95411),$$
$$(c\approx\pm0.88470,\ x\approx16.439),\quad (c\approx\pm2,\ x\approx-0.24168).$$

Consequently,the interior of the parallelepiped does not have a critical point for $\phi(c,x,y)$.

From A,B,C and D we have

$$\max\left[\phi(c,x,y)\ ;\ c\in(0,2),\ x\in(0,1),\ y\in(0,1)\right]=2816. \tag{3.47}$$

From (3.22) and (3.47), we get

$$|H_{3,1}(h^{-1})|\leq\frac{2816}{8640}. \tag{3.48}$$

By Simplifying

$$|H_{3.1}(h^{-1})|\leq\frac{44}{135}. \tag{3.49}$$

This is the required result. □

3.3 Conclusion

In this research work, we estimate the third-order Hankel determinant of the inverse function of h, provided that h belongs to the class of bounded turning functions. In our study, we have employed the relationship between the Carathéodory class and the coefficients of functions belonging to the class under consideration.

Considering the function's coefficient h and its inverse h^{-1}, we calculated the coefficient of h^{-1} using the Carathéodory class function's coefficient. The results gained in this study can be used to get similar results for other well-known subclasses of univalent functions.

CHAPTER 4

ON HANKEL DETERMINANT OF THE INVERSE OF q̌-BOUNDED TURNING FUNCTIONS

4.1 Introduction

The purpose of this chapter is to determine the optimal bounds for the inverse of the third Hankel determinant of q̌-bounded turning functions. In the analytic theory of functions, much interest is taken by the analysis of functions like Hankel determinants that give much information about the growth, distortion and behaviour of coefficients of univalent and related functions. The Hankel determinant plays a critical role in researching sharp estimates and coefficients inequalities and is applied to research the second-order and higher-order coefficient inequalities. The Hankel determinant is the inverse Property of the third-order functions in the theory of turning functions. This is aimed at developing an improved understanding of the analytic structure of such inverse functions, and developing the topic of GFT with the techniques of the q̌-calculus and new estimates of coefficients.

Let suppose

$$h(k) = k + \sum_{n=2}^{\infty} c_n k^n \tag{4.1}$$

is a univalent function such that $M = \{k : |k| < 1\}$. One of the most interesting topics in complex function theory is probably the relationship between the theory of geometric functions and complex analysis. A function from analytic class A can be expressed in series form as,

$$h(k) = k + \sum_{n=2}^{\infty} c_n k^n, k \in M. \tag{4.2}$$

The Hankel determinants of order v^{th} and third-order are shown in equations (1.4) and (1.5). The Class R represented by

$$R = h \in A : Re(h^{'}(k)) > 0, k \in C, \tag{4.3}$$

in the form of q^{th} derivative

$$R_q = h \in A : Re\{D_q h(k)\} > 0, k \in C. \tag{4.4}$$

The class p represented by

$$p(k) = 1 + \sum_{n=1}^{\infty} c_n k^n. \tag{4.5}$$

4.2 Final Results

Theorem 4.2.1. *If $h \in R$ and $h^{-1}(e) = e + \sum_{n=2}^{\infty} t_n e^n$ is given by the inverse of h, subsequently*

$$|H_{3,1}(h^{-1})| \leq \frac{4}{(1+\check{q}+\check{q}^2)(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4)} - \frac{16}{(1+\check{q}+\check{q}^2)^3}, \tag{4.6}$$

this inequality is sharp for $\sum_{n=1}^{\infty} \frac{\check{q}^{n-1}}{n} k^n$.

Proof. For $h \in R$, there exist such a univalent function $p \in P$,

$$h^{'}(k) = p(k) \tag{4.7}$$

$$D_{\check{q}} h(k) = p(k),$$

$$D_{\check{q}}[k + \sum_{n=2}^{\infty} a_n k^n] = 1 + \sum_{n=2}^{\infty} c_t k^t,$$

$$\sum_{n=2}^{\infty} [n]_{\check{q}} a_n k^{n-1} + 1 = \sum_{n=2}^{\infty} c_t k^t + 1,$$

$$1 + [2]_{\check{q}} a_2 k + [3]_{\check{q}} a_3 k^2 + [4]_{\check{q}} a_4 k^3 + \ldots = 1 + c_1 k^1 + c_2 k^2 + c_3 k^3 + c_4 k^4 + \ldots,$$

$$[2]_{\check{q}} a_2 k^1 + [3]_{\check{q}} a_3 k^2 + [4]_{\check{q}} a_4 k^3 + \ldots = c_1 k^1 + c_2 k^2 + c_3 k^3 + \ldots,$$

$$[2]_{\check{q}} a_2 + [3]_{\check{q}} a_3 + \ldots [n]_{\check{q}} a_n = c_1 + c_2 + \cdots + c_{n-1},$$

$$a_n = \frac{c_{n-1}(1-\check{q})}{(1-\check{q}^n)}, \tag{4.8}$$

where

$$[n]_{\check{q}} = \frac{1-\check{q}^n}{(1-\check{q})},$$
$$a_2 = \frac{c_1}{[2]_{\check{q}}} = \frac{c_1}{(1+\check{q})},$$
$$a_3 = \frac{c_2}{[3]_{\check{q}}} = \frac{c_2}{(1+\check{q}+\check{q})},$$
$$a_4 = \frac{c_3}{[4]_{\check{q}}} = \frac{c_3}{(1+\check{q}+\check{q}^2)},$$

$$a_5 = \frac{c_4}{[5]_{\check{q}}} = \frac{c_4}{(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4)}. \tag{4.9}$$

Here is the inverse function

$$e = h(h^{-1}) = e + \sum_{n=2}^{\infty} t_n e^n + \sum_{n=2}^{\infty} a_n (e + \sum_{n=2}^{\infty} (t_n e^n)^n). \tag{4.10}$$

through simplification

$$(t_2+a_2)e^2 + (t_3+a_3+2a_2t_2)e^3 + (t_4+a_2t_2^2+2a_2t_3+3a_3t_2+a_4)e^4$$
$$+(t_5+2a_2t_2t_3+2a_2t_4+3a_3t_3+4a_4t_2+3a_3t_2^2+a_5)e^5+\cdots = 0. \tag{4.11}$$

Coefficients are equated like powers

$$t_2 = -a_2,$$
$$t_3 = 2a_3^2 - a_3,$$
$$t_4 = 5a_2a_3 - 5a_3^3 - a_4,$$
$$t_5 = -a_5 + 6a_2a_4 - 21a_2^2a_3 + 3a_3^2 + 14a_2^2, \tag{4.12}$$

from equation (4.9),

$$t_2 = \frac{-c_1}{1+\check{q}}$$
$$t_3 = -\frac{c_2}{(1+\check{q}+\check{q}^2)} + \frac{2c_1^2}{1+2\check{q}+\check{q}^2}$$
$$t_4 = \frac{-c_3}{(1+\check{q}+\check{q}^2+\check{q}^3)} - \frac{5c_1^3}{(1+3\check{q}+3\check{q}^2+\check{q}^3)} + \frac{5c_1c_2}{(1+2\check{q}+2\check{q}^2+\check{q}^3)}$$
$$t_5 = \frac{-c_4}{(1+2\check{q}+3\check{q}^2+2\check{q}^3+\check{q}^4)} - \frac{21c_1^2c_2}{(1+3\check{q}+4\check{q}^2+2\check{q}^3+\check{q}^4)}$$
$$+\frac{6c_1c_2}{(1+2\check{q}+2\check{q}^2+2\check{q}^3+\check{q}^4)} + \frac{3c_2^2}{(1+2\check{q}+3\check{q}^2+2\check{q}^3+\check{q}^4)} + \frac{14c_1^4}{(1+\check{q})^4}, \tag{4.13}$$

here

$$H_{3,1}(h^{-1}) = \begin{vmatrix} 1 & t_2 & t_3 \\ t_2 & t_3 & t_4 \\ t_3 & t_4 & t_5 \end{vmatrix}. \tag{4.14}$$

After putting the values of $t_i(i = 1,2,3,,4,5)$, we obtain

$$= 1\Bigg[\Bigg(-\frac{c_4}{(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4)} + \frac{6c_1c_2}{(1+2\check{q}+2\check{q}^2+2\check{q}^3+\check{q}^4)} - \frac{21c_1^2c_2}{(1+3\check{q}+4\check{q}^2+2\check{q}^3+\check{q}^4)}$$
$$+\frac{c_1^4}{(1+q)^4} + \frac{3c_2^2}{(1+2\check{q}+3\check{q}^2+2\check{q}^3+\check{q}^4)}\Bigg)\Bigg(-\frac{c_2}{(1+\check{q}+\check{q}^2)} + \frac{2c_1^2}{(1+2\check{q}+\check{q}^2)}\Bigg)$$
$$-\Bigg(\frac{5c_1c_2}{(1+2\check{q}+2\check{q}^2+\check{q}^3)} - \frac{c_3}{(1+\check{q}+\check{q}^2+\check{q}^3)}$$
$$-\frac{5c_1^3}{(1+3\check{q}+3\check{q}^2+\check{q}^3)}\Bigg)\Bigg(-\frac{5c_1^3}{(1+3\check{q}+3\check{q}+\check{q}^3)} + \frac{5c_1c_2}{(1+2\check{q}+2\check{q}^2+2\check{q}^3+\check{q}^3)}\Bigg)\Bigg]$$
$$+\frac{c_1}{(1+\check{q})}\Bigg[\Bigg(\frac{-c_1}{1+\check{q}}\Bigg)\Bigg(\frac{-c_4}{(1+\check{q}^2+\check{q}+\check{q}^3+q^4)} + \frac{6c_1c_2}{(1+2\check{q}+2\check{q}^2+2\check{q}^3+\check{q}^4)}$$
$$-\frac{21c_1^2c_2}{(1+3\check{q}+4\check{q}^2+2\check{q}^3+\check{q}^4)} + \frac{3c_2^2}{(1+2\check{q}+3\check{q}^2+2\check{q}^3+\check{q}^4)} + \frac{c_1^4}{(1+\check{q})^4}\Bigg)$$
$$-\Bigg(\frac{-c_3}{(1+\check{q}+\check{q}^2+\check{q}^3)} - \frac{5c_1^3}{(1+3\check{q}+3\check{q}^2+\check{q}^3)} + \frac{5c_1c_2}{(1+2\check{q}+2\check{q}^2+2\check{q}^3+v^4)}\Bigg)$$
$$\Bigg(\frac{-c_2}{(1+\check{q}+\check{q}^2)} + \frac{2c_1^2}{(1+2\check{q}+\check{q}^2)}\Bigg) + \frac{2c_1^2}{(1+2\check{q}+\check{q}^2)}$$
$$-\frac{c_2}{(1+\check{q}+\check{q}^2)}\Bigg[\Bigg(\frac{-c_1}{(1+q)}\Bigg)\Bigg(\frac{-c_3}{(1+\check{q}+\check{q}^2+\check{q}^3)} + \frac{5c_1c_2}{(1+2\check{q}+2\check{q}^2+\check{q}^3)}\Bigg)$$
$$-\Bigg(\frac{-c_2}{(1+\check{q}+\check{q}^2)} + \frac{2c_1^2}{(1+2\check{q}+\check{q}^2)}\Bigg)^2\Bigg],$$

$$= \frac{c_2c_4}{(1+\check{q}+\check{q})(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4)} - \frac{6c_1c_2c_3}{(1+2\check{q}+2\check{q}^2+\check{q}^3)(1+\check{q}+\check{q}^2+\check{q}^3)}$$
$$+\frac{21c_1^2c_2^2}{(1+2\check{q}+\check{q}^2)(1+\check{q}+\check{q}^2)^2} - \frac{3c_2^3}{(1+\check{q}+\check{q}^2)^3}$$
$$-\frac{14c_1^4c_2}{(1+2\check{q}+\check{q}^2)^2(1+\check{q}+\check{q}^2)} - \frac{2c_1^2c_4}{(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4)(1+2\check{q}+\check{q}^2)}$$
$$+\frac{12c_1^3c_3}{(1+\check{q})(1+2\check{q}^2+\check{q}^2)(1+\check{q}+\check{q}^2+\check{q}^3)} - \frac{42c_1^4c_2}{(1+\check{q})^2(1+2\check{q}+\check{q}^2)(1+2\check{q}+\check{q}^2)}$$
$$+\frac{28c_1^6}{(1+2\check{q}+\check{q}^2)^2(1+2\check{q}+\check{q}^2)} + \frac{6c_1^2c_2^2}{(1+2\check{q}+\check{q}^2)(1+2\check{q}+3\check{q}^2+2\check{q}^3+\check{q}^4)}$$
$$+\frac{5c_1c_2c_3}{(1+2\check{q}+2\check{q}^2+\check{q}^3)(1+\check{q}+\check{q}^2+\check{q}^3)} - \frac{c_3^2}{(1+\check{q}+\check{q}^2+\check{q}^3)^2}$$
$$-\frac{5c_1^3c_3}{(1+\check{q})^3(1+\check{q}+\check{q}^2+\check{q}^3)} + \frac{5c_1c_2c_3}{(1+2\check{q}+2\check{q}^2+\check{q}^3)(1+\check{q}+\check{q}^2+\check{q}^3)}$$

$$
\begin{aligned}
&+\frac{25c_1^4c_2}{(1+2\check{q}+\check{q}^2)^2(1+\check{q}+\check{q}^2)}-\frac{25c_1^2c_2^2}{1+2\check{q}+\check{q}^2)(1+\check{q}+\check{q}^2)^2}\\
&+\frac{25c_1^4c_2}{(1+2\check{q}+\check{q}^2)^2(1+\check{q}+\check{q}^2)}-\frac{5c_1^3c_3}{(1+\check{q})^3(1+\check{q}+\check{q}^2)}\\
&-\frac{25c_1^6}{(1+\check{q})^6}+\frac{c_1^2c_4}{(1+2\check{q}+\check{q}^2)(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4)}\\
&+\frac{21c_1^4c_2}{(1+2\check{q}^2+\check{q}^2)^2(1+\check{q}+\check{q}^2)}-\frac{6c_1^3c_3}{(1+\check{q})^3(1+\check{q}+\check{q}^2+\check{q}^3)}\\
&-\frac{3c_1^2c_2^2}{(1+2\check{q}+\check{q}^2)(1+2\check{q}+3\check{q}^2+2\check{q}^3+\check{q}^4)}-\frac{14c_1^6}{(1+\check{q})^6}\\
&+\frac{2c_1^3c_3}{(1+\check{q})(1+\check{q}+\check{q}^2+\check{q}^3)(1+2\check{q}+\check{q}^2)}-\frac{c_1c_2c_3}{(1+2\check{q}+2\check{q}^2+\check{q}^3)(1+\check{q}+\check{q}^2+\check{q}^3)}\\
&+\frac{5c_1^2c_2^2}{(1+2\check{q}+\check{q}^2)(1+\check{q}+\check{q}^2)^2}-\frac{10c_1^4c_2}{(1+2\check{q}+\check{q}^2)(1+\check{q}+\check{q}^2)(1+2\check{q}+\check{q}^2)}\\
&-\frac{5c_1^4c_2}{(1+2\check{q}+\check{q}^2)^2(1+\check{q}+\check{q}^2)}+\frac{10c_1^6}{(1+2\frac{q}{+}\check{q}^2)^2(1+2\check{q}+\check{q}^2)}\\
&-\frac{c_1c_2c_3}{(1+\check{q}+\check{q}^2+\check{q}^3)(1+2\check{q}+2\check{q}^2+\check{q}^3)}+\frac{5c_1^2c_2^2}{(1+2\check{q}+2\check{q}^2+\check{q}^3)(1+2\check{q}+2\check{q}^2+\check{q}^3)}\\
&-\frac{5c_1^4c_2}{(1+2\check{q}+2\check{q}^2+\check{q}^3)(1+3\check{q}+3\check{q}^2+\check{q}^3)}+\frac{c_2^3}{(1+\check{q}+\check{q}^2)^3}\\
&-\frac{2c_1^2c_2^2}{(1+\check{q}+\check{q}^2)^2(1+2\check{q}+\check{q}^2)}-\frac{2c_1^2c_2^2}{(1+2\check{q}+\check{q}^2)(1+\check{q}+\check{q}^2)^2}\\
&+\frac{4c_1^4c_2}{(1+\check{q}+\check{q}^2)^2(1+2\check{q}+\check{q}^2)}+\frac{2c_1^3c_3}{(1+2\check{q}+\check{q}^2)(1+\check{q})(1+\check{q}+\check{q}^2+\check{q}^3)}\\
&-\frac{10c_1^4c_2}{(1+2\check{q}+\check{q}^2)(1+2\check{q}+\check{q}^2)(1+\check{q}+\check{q}^2)}+\frac{10c_1^6}{(1+2\check{q}+\check{q}^2)^2(1+2\check{q}+\check{q}^2)}\\
&-\frac{2c_1^2c_2^2}{(1+\check{q}+\check{q}^2)^2(1+2\check{q}+\check{q})}+\frac{4c_1^4c_2}{(1+2\check{q}+\check{q}^2)^2(1+\check{q}+\check{q}^2)}\\
&+\frac{4c_1^4c_2}{(1+\check{q}+\check{q}^2)(1+2\check{q}+\check{q}^2)^2}-\frac{8c_1^6}{(1+2\check{q}+\check{q}^2)^3}.
\end{aligned}
\tag{4.15}
$$

This simplifies to

$$
\begin{aligned}
&=\frac{c_1^6}{(1+\check{q})^6}-\frac{3c_1^4c_2}{(1+\check{q}+\check{q}^2)(1+2\check{q}+\check{q}^2)^2}\\
&\quad+\frac{3c_1^2c_2^2}{(1+2\check{q}+\check{q}^2)(1+\check{q}+\check{q}^2)^2}+\frac{c_2c_4}{(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4)(1+\check{q}+\check{q}^2)}\\
&\quad-\frac{c_1^2c_2^4}{(1+2\check{q}+\check{q}^2)(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4)}+\frac{2c_1c_2c_3}{(1+\check{q})(1+\check{q}+\check{q}^2)(1+\check{q}+\check{q}^2+\check{q}^3)}\\
&\quad-\frac{2c_2^3}{(1+\check{q}+\check{q}^2)^3}-\frac{c_3^2}{(1+\check{q}+\check{q}^2+\check{q}^3)^2}.
\end{aligned}
\tag{4.16}
$$

In the view of Lemma (2.11.2), we obtain

$$C_2 = \frac{1}{2}\left[c_1^2 + t\omega\right]$$
$$c_3 = \frac{1}{4}\left[c_1^3 + 2c_1 t - c_1 t\omega^2 + 2t(1-|\omega|)^2\tau\right]$$
$$c_4 = \frac{1}{8}c_1^4 + t\omega\big[(c_1^2(\omega^2 - 3\omega + 3) + 4\omega) - 4t(1-|\omega|^2)(c_1(|\omega|-1)|tau + \omega\tau^2$$
$$-(1-|\tau|c^2)\xi)\big]. \tag{4.17}$$

From (4.16) and (4.17), we get

$$= \frac{c_1^6}{(1+\check{q})^6} - \frac{3\big[c_1^6 + c_1^4 t\omega\big]}{2(1+2\check{q}+\check{q}^2)^2(1+\check{q}+\check{q}^2)}$$
$$+ \frac{3\big[c_1^6 + c_1^2\omega^2 t^2 + 2c_1^4\omega t\big]}{4(1+2\check{q}+\check{q}^2)(1+\check{q}+\check{q}^2)^2}$$
$$+ \frac{1}{16(1+\check{q}+\check{q}^2)(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4)}\Big[c_1^6 + c_1^2 t\omega\left(c_1^2(\omega^2 - 3\omega + 3) + 4\omega\right)$$
$$-4c_1^2 t(1-|\omega|^2)(c_1\omega\tau - c_1\tau + \tau^2\omega - (1-|\omega|^2)\xi) + c_1^4 t\tau$$
$$+t^2\omega^2(c_1^2(\omega^2 - 3\omega + 3) + 4\omega) - 4t^2\omega(1-\omega^2)(c_1(\omega - 1)\tau + \omega\tau^2 - (1-|\tau|^2)\xi\Big]$$
$$- \frac{1\big[c_1^6 + c_1^2 t\omega(4\omega + c_1^2(\omega^2 - 3\omega + 3)) - c_1^2 t(1-\omega^2)\{\omega\tau^2 + c_1\omega\tau - c_1\tau - (1-|\tau|^2)\xi\}\big]}{8(1+2\check{q}+\check{q}^2)(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4)}$$
$$+ \frac{1\big[c_1^6 + 3c_1^4 t\omega + 2c_1^2 t^2\omega^2 - c_1^2 t^2\omega^3 + 2c_1 t^2\omega(\tau - \tau\omega^2) + 2c_1^3 t(\tau - \tau\omega^2) - c_1^4 t\omega^2\big]}{4(1+2\check{q}+2\check{q}^2+\check{q}^3)(1+\check{q}+\check{q}^2+\check{q}^3)}$$
$$- \frac{1\big[c_1^6 + 3c_1^4 t\omega + 3c_1^2 t^2\omega^2 + t^3\omega^3\big]}{4(1+\check{q}+\check{q}^2)^3}$$
$$- \frac{1}{16(1+\check{q}+\check{q}^2+q^3)^2}\Big[c_1^6 + 4c_1^2 t^2\omega^2 + 4t^2(1-|\omega|^2)\tau^2 - c_1^2 t^2\omega^4$$
$$+4c_1^4 t\omega - 2c_1^4 t\omega^2 + 4c_1^3 t(1-|\omega|^2)\tau - 4c_1^2 t^2\omega^3 + 4c_1 t^2\omega(1-|\omega|^2)\tau$$
$$-4c_1^2 t^2\omega^2(1-|\omega|^2)\tau\Big], \tag{4.18}$$

for $c_1 = c$, $t = 4 - c^2$ in (4.18), we get

$$= \frac{c^6}{(1+\check{q})^6} - \frac{3\big[c^6 + c^4(4-c^2)\omega\big]}{2(1+2\check{q}+\check{q}^2)^2(1+\check{q}+\check{q}^2)}$$
$$+ \frac{3\big[c^6 + c^2\omega^2(4-c^2) + 2c^4(4-c^2)\omega\big]}{4(1+2\check{q}+\check{q}^2)(1+\check{q}+\check{q}^2)^2}$$
$$+ \frac{1}{16(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4)(1+\check{q}+\check{q}^2)}\Big[c^6 + c^2(4\omega - \omega c^2)\left(c^2(\omega^2 - 3\omega + 3) + \omega\right)$$

$$-4c^2(4-c^2)(1-|\omega|^2)(c(\omega-1)\tau+\tau\omega^2-(1-|\omega|^2)\tau)+c^4(4-c^2)\tau+(4-c^2)^2\omega^2$$
$$(c^2(\omega^2-3\omega+3)+4\omega)-4(4-c^2)^2\omega(1-|\omega|^2)c_1(\omega\tau-\tau)+\tau^2\omega^2-(1-|\tau|^2)\xi\Big]$$
$$-\frac{1}{8(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4)(1+\check{q})^2}\Big[c^6+c^2(4-c^2)\omega(c^2(\omega^2$$
$$-3\omega+3)+4\omega)-c^2(4-c^2)(1-\omega^2)(c(\omega-1)\tau+\omega^2\tau^2-(1-|\tau|)^2\xi)\Big]$$
$$+\frac{1}{4(1+2\check{q}+2\check{q}^2+\check{q}^3`)(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4)}\Big[c^6+3c^4(4-c^2)\omega+2c^2(4-c^2)^2\omega^2$$
$$-c^4(4-c^2)\omega^2-c^2(4-c^2)^2\omega^3+2c^2(4-c^2)^2\omega(1-|\omega|^2)\tau+2c^3(4-c^2)(1-|\omega|^2)\tau\Big]$$
$$-\frac{\left[c^6+3c^4(4\omega-c^2\omega)+3c^2(4-c^2)^2\omega^2+(4-c^2)^3\omega^3\right]}{4(1+\check{q}+\check{q}^2)^3}$$
$$-\frac{1}{(16)(1+\check{q}+\check{q}^2+\check{q}^3)^2}\Big[c^6+4c^2(-c^2+4)^2\omega^2-c^2(-c^2+4)^2\omega^4$$
$$+4(-c^2+4)^2(1-|\omega|^2)\tau^2+4c^4(-c^2+4)\omega-2c^4(4-c^2)\omega^2+4c^3(4-c^2)(1-\omega^2)\tau$$
$$-4c^2(4-c^2)^2\omega^3+4c(4-c^2)^2\omega(1-|\omega|^2)\tau-4c^2(4-c^2)^2\omega^2(1-|\omega|^2)\tau\Big]. \tag{4.19}$$

Applying the modulus to both sides of (4.9) with $|\omega|=x\in(0,1), |\tau|=y\in(0,1), c\in(0,2)$, we obtain

$$|H_{3,1}(h^{-1})|\leq\phi(c,x,y) \tag{4.20}$$

$$\phi(c,x,y)=\frac{c^6}{(1+\check{q})^6}-\frac{3\left[c^6+c^4(4-c^2)x\right]}{2(1+2\check{q}+\check{q}^2)^2(1+\check{q}+\check{q}^2)}$$
$$+\frac{3\left[c^6+2c^4(4-c^2)x+c^2(4-c^2)x^2\right]}{4(1+2\check{q}+\check{q}^2)(1+\check{q}+\check{q}^2)^2}$$
$$+\frac{1}{16(1+\check{q}+\check{q}^2)(1+\check{q}+\check{q}^3+\check{q}^2+\check{q}^4)}\Big[c^6+c^2(4x-c^2x)\left(4x+(x^2-3x+3)c^2\right)$$
$$-4c^2(4-c^2)(1-x^2)(c(x-1)y+yx^2-(1-x^2)y)+c^4(4-c^2)y+(4-c^2)^2x^2$$
$$(4x+c^2(x^2-3x+3))-4(4-c^2)^2x(1-x^2)(c(x-1)y+y^2x^2-(1-y^2)\Big]$$
$$-\frac{1}{8(1+\check{q})^2(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4)}\Big[c^6+c^2(4-c^2)x(4x+c^2(x^2-3x+3))$$
$$-c^2(4-c^2)(1-x^2)c(x-1)y+x^2y^2-(1-y)^2\Big]$$
$$+\frac{1}{4(1+\check{q}+\check{q}^2)(1+\check{q}+q^2+\check{q}^3)(1+\check{q})}\Big[c^6+3c^4(4-c^2)x+2c^2(4-c^2)^2x^2$$
$$-c^4(4-c^2)x^2-c^2(4-c^2)^2x^3+2c^2(4-c^2)^2x(1-x^2)y+2c^3(4-c^2)(1-x^2)y\Big]$$

$$-\frac{1\left[c^6+3c^4(4-c^2)x+3c^2(4-c^2)^2x^2+(4-c^2)^3x^3\right]}{4(1+\check{q}+\check{q}^2)^3}$$
$$-\frac{1}{16(1+\check{q}+\check{q}^2+\check{q}^3)^2}\Big[c^6+4c^2(4-c^2)^2x^2+c^2(4-c^2)^2x^4$$
$$+4(4-c^2)^2(1-x)^2y^2+4c^4(4-c^2)x-2c^4(4-c^2)x^2+4c^3(4-c^2)(1-x^2)y$$
$$-4c^2(4-c^2)^2x^3+8c(4-c^2)^2x(1-x^2)y-4c^2(4-c^2)^2x^2(1-x^2)y\Big]. \tag{4.21}$$

Now, $\phi(c,x,y)$ will be maximized within parallelepiped defined by $[0,1]\times[0,2]\times[0,1]$, with $c\in(0,2),\ y\in(0,1)$ and $x\in(0,1)$.

A. The parallelepiped Vertices will be

$$\begin{aligned}
&\phi(0,0,0)=0,\\
&\phi(2,1,0)=\phi(2,0,0)=\frac{64}{(1+\check{q})^6}-\frac{96}{(1+\check{q}+\check{q}^2)(1+2\check{q}+\check{q}^2)^2}+\frac{48}{(1+2\check{q}+\check{q}^2)(1+\check{q}+\check{q}^2)^2}\\
&\quad+\frac{4}{(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4)(1+\check{q}+\check{q}^2)}-\frac{8}{(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4)(1+\check{q})^2}\\
&\quad+\frac{16}{(1+\check{q}+\check{q}^2+\check{q}^3)(1+2\check{q}+2\check{q}^2+\check{q}^3)}-\frac{16}{(1+\check{q}+\check{q}^2)^3}-\frac{4}{(1+\check{q}+\check{q}^2+\check{q}^3)^2},\\
&\phi(0,0,1)=\frac{-4}{(1+\check{q}+\check{q}^2+\check{q}^3)^2},\\
&\phi(0,1,0)=\phi(0,1,1)=\frac{4}{(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4)(1+\check{q}+\check{q}^2)}-\frac{16}{(1+\check{q}+\check{q}^2+\check{q}^3)^3}.
\end{aligned} \tag{4.22}$$

B. Now, we are looking at the eight edges of the Parallelepiped.

(i). At $x=1,y=0$ and $x=1,y=1$, we have

$$\begin{aligned}
\phi(c,1,y)=&\frac{c^6}{(1+\check{q})^6}-\frac{3\left[c^6+c^4(4-c^2)\right]}{2(1+\check{q}+\check{q}^2)(1+2\check{q}+\check{q}^2)^2}\\
&+\frac{3\left[c^6+c^2(4-c^2)+2c^4(4-c^2)\right]}{4(1+\check{q}+\check{q}^2)^2(1+2\check{q}+\check{q}^2)}\\
&+\frac{1\left[c^6+c^4(4-c^2)+5c^4(4-c^2)+(4-c^2)(c^2+4)\right]}{16(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4)(1+\check{q}+q^2)}\\
&-\frac{1\left[c^6+c^2(4-c^2)(4-c^2))\right]}{8(1+2\check{q}+\check{q}^2)(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4)}\\
&+\frac{1\left[c^6+3c^4(4-c^2)-c^4(4x^2-c^2x^2)+2c^2(4-c^2)^2-c^2(4-c^2)\right]}{4(1+2\check{q}+2\check{q}^2+\check{q}^3)(1+\check{q}+\check{q}^2+\check{q}^3)}\\
&-\frac{1\left[c^6+3c^4(4-c^2)+3c^2(4-c^2)^2+(4-c^2)^3\right]}{4(1+\check{q}+\check{q}^2)^3}
\end{aligned}$$

$$-\frac{1\left[c^6+(4-c^2)^2 4c^2+(4-c^2)^2c^2+4c^4(4-c^2)-2c^4(4-c^2)-4c^2(4-c^2)^2\right]}{16(1+\check{q}+\check{q}^2+\check{q}^3)^2}$$

$$\leq \frac{4}{(1+\check{q}+\check{q}^2)(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4)}-\frac{16}{(1+\check{q}+\check{q}^2)^3} \text{ for } c\in(0,2). \tag{4.23}$$

(ii) At $y=0, c=0$.

$$\begin{aligned}\phi(0,x,0) &= \frac{4x^2}{(1+\check{q}+\check{q}^2)(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4)}-\frac{16}{(1+\check{q}+\check{q}^2)^3}\\ &\leq \frac{4}{(1+\check{q}+\check{q}^2)(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4)}-\frac{16}{(1+\check{q}+\check{q}^2)^4}.\end{aligned} \tag{4.24}$$

(iii) When $x=0, y=1$.

$$\begin{aligned}\phi(c,0,1) &= \frac{c^6}{(1+\check{q})^6}-\frac{3\left[c^6\right]}{2(1+2\check{q}+\check{q}^2)^2(1+\check{q}+\check{q}^2)}\\ &+\frac{3\left[c^6\right]}{4(1+2\check{q}+\check{q}^2)(1+\check{q}+\check{q}^2)^2}\\ &+\frac{1\left[c^6+c^4(4-c^2)+4c^2(4-c^2)(-c-1)\right]}{16(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4)(1+\check{q}+\check{q}^2)}\\ &-\frac{1\left[c^6-c^2(4-c^2)(-c)\right]}{8(1+2\check{q}+\check{q}^2)(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4)}\\ &+\frac{1\left[c^6+2c^3(4-c^2)\right]}{4(1+2\check{q}+2\check{q}^2+\check{q}^3)(1+\check{q}+\check{q}^2+\check{q}^3)}\\ &-\frac{1\left[c_6\right]}{4(1+\check{q}+\check{q}^2)^3}\\ &-\frac{1\left[c^6+16(4-c^2)^2+4c^3(4-c^2)\right]}{16(1+\check{q}+\check{q}^2+\check{q}^3)^2}\\ &\leq \frac{-4}{(1+\check{q}+\check{q}^2+\check{q}^3)^2}.\end{aligned} \tag{4.25}$$

(iv) When $c=0, y=1$.

$$\begin{aligned}\phi(0,x,1) &= \frac{1\left[64x^3+x^2\right]}{16(1+\check{q}+\check{q}^2)(1+\check{q}^2+\check{q}^3+\check{q}^4)}\\ &-\frac{64x^3}{4(1+\check{q}+\check{q}^2)^3}-\frac{64(1-x^2)}{16(1+\check{q}+\check{q}^2+\check{q}^3)^2}\\ &\leq \frac{4}{(1+\check{q}+\check{q}^2)(1+\check{q}+\check{q}^3+\check{q}^4)}-\frac{16}{(1+\check{q}+\check{q}^2)^2}, \text{ for } x\in(0,1).\end{aligned} \tag{4.26}$$

(v) When $(c,x)=(0,0)$.

$$\phi(0,0,y) = -\frac{4}{(1+\check{q}+\check{q}^2)^3} \leq -\frac{4}{(1+\check{q}+\check{q}^2+\check{q}^3)^2}. \tag{4.27}$$

(vi) When $(c,x)=(0,1)$.

$$\begin{aligned}\phi(0,1,y) &= \frac{4}{(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4)(1+\check{q}+\check{q}^2)}-\frac{16}{(1+\check{q}+\check{q}^2)^3}\\ &\leq \frac{4}{(1+\check{q}+\check{q}^2)(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4)}-\frac{16}{(1+\check{q}+\check{q}^2)^3}.\end{aligned} \tag{4.28}$$

(vii) At $c=2, y=1, c=2, y=0, c=2$ and $x=1, c=0, x=0$, we have

$$\begin{aligned}\phi(c,x,y)=&\frac{64}{(1+\check{q})^6}-\frac{96}{(1+2\check{q}+\check{q}^2)^2(1+\check{q}+\check{q}^2)}+\frac{48}{(1+\check{q}+q^2)^2(1+2\check{q}+\check{q}^2)}\\&+\frac{4}{(1+\check{q}+\check{q}^2)(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4)}-\frac{8}{(1+\check{q}+q^2+\check{q}^3+\check{q}^4)(1+2\check{q}+\check{q}^2)}\\&+\frac{16}{(1+\check{q}+\check{q}^2)(1+\check{q})(1+\check{q}+\check{q}^2+\check{q}^2)}\\&-\frac{16}{(1+\check{q}+\check{q}^2)^3}-\frac{4}{(1+\check{q}+\check{q}^2+\check{q}^3)^2}.\end{aligned}\tag{4.29}$$

$(viii)$. At $x=0, y=0$.

$$\begin{aligned}\phi(c,0,0)=&\frac{c^6}{(1+\check{q})^6}-\frac{3c^6}{2(1+2\check{q}+\check{q}^2)^2(1+\check{q}+\check{q}^2)}\\&+\frac{3c^6}{4(1+2\check{q}+\check{q}^2)(1+\check{q}+\check{q}^2)^2}+\frac{c^6}{16(1+\check{q}+\check{q}+\check{q}^2)(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4)}\\&-\frac{c^6}{8(1+\check{q})^2((1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4)}+\frac{c^6}{4(1+\check{q}+\check{q}^2+\check{q}^3)(1+2\check{q}+\check{q}^2)}\\&-\frac{c^6}{4(1+\check{q}+\check{q}^2)^3}-\frac{c^6}{16(1+\check{q}+\check{q}^2+\check{q}^3)^2}\\&\le\frac{4}{(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4)(1+\check{q}+\check{q}^2)}-\frac{16}{(1+\check{q}+\check{q}^2)^3}.\end{aligned}\tag{4.30}$$

C. Now we examine six parallelepiped faces.

(i) When $c=2$.

$$\begin{aligned}\phi(2,x,y)=&\frac{64}{(1+\check{q})^6}-\frac{96}{(1+2\check{q}+\check{q}^2)^2(1+\check{q}+\check{q}^2)}+\frac{48}{(1+2\check{q}+\check{q}^2)(1+\check{q}+\check{q}^2)^2}\\&+\frac{4}{(1+\check{q}+\check{q}^2)(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4)}-\frac{8}{(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4)(1+2\check{q}+\check{q}^2)}\\&+\frac{16}{(1+\check{q}+\check{q}^2)(1+\check{q}+\check{q}^2+\check{q}^3)(1+\check{q})}\\&-\frac{16}{(1+\check{q}+\check{q}^2)^3}-\frac{4}{(1+\check{q}+\check{q}^2+\check{q}^3)^2}\text{ for }(x,y)\in\{(0,1)\}.\end{aligned}\tag{4.31}$$

(ii) When $c=0$.

$$\begin{aligned}\phi(0,x,y)=&\frac{64x^3-64x(1-x^2)(x^2y^2-(1-y^2))}{16(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4)(1+\check{q}+\check{q}^2)}-\frac{64x^3}{4(1+\check{q}+\check{q}^2)^3}\\&\le\frac{4}{(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4)(1+\check{q}+\check{q}^2)}-\frac{16}{(1+\check{q}+\check{q}^2)^3}.\end{aligned}\tag{4.32}$$

(iii) When $x=0$.

$$\begin{aligned}\phi(c,0,y) &= \frac{c^6}{(1+\check{q})^6} - \frac{3c^6}{2(1+2\check{q}+\check{q}^2)^2(1+\check{q}+\check{q}^2)} \\ &+ \frac{3c^6}{4((1+\check{q}+\check{q}^2)^2(1+2\check{q}+\check{q}^2)} \\ &+ \frac{1\left[c^6+c^4(4-c^2)y-4c^2(4-c^2)(-cy-y)\right]}{16(1+\check{q}+\check{q}^2)(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4)} \\ &- \frac{1\left[c^6-c^2(4-c^2)(-cy-(1-y))\right]}{8(1+2\check{q}+\check{q}^2)(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4)} \\ &+ \frac{1\left[c^6+2c^3(4y-c^2y)\right]}{4(1+2\check{q}+2\check{q}^2+\check{q}^3)(1+\check{q}+\check{q}^2+\check{q}^3)} \\ &- \frac{1\left[c^6\right]}{4(1+\check{q}+\check{q}^2)^3} - \frac{1\left[c^6+4(4-c^2)y^2+4c^3(4-c^2)y\right]}{16(1+\check{q}+\check{q}^2+\check{q}^3)^2} \\ &\leq -\frac{4}{(1+\check{q}+\check{q}^2+\check{q}^3)^2}, \text{ for } (c,y)\in\{(0,1)\}. \end{aligned} \tag{4.33}$$

(iv) When $x=1$.

$$\begin{aligned}\phi(c,1,y) &= \frac{c^6}{(1+\check{q})^6} - \frac{3\left[c^6+c^4(4-c^2)\right]}{2(1+2\check{q}+\check{q}^2)^2(1+\check{q}+\check{q}^2)} \\ &+ \frac{3\left[3c^6+c^2(4-c^2)+2c^4(4-c^2)\right]}{4(1+\check{q}+\check{q}^2)^2(1+2\check{q}+\check{q})} \\ &+ \frac{1}{16(1+\check{q}+\check{q}^2)(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4)}\Big[c^6+c^2(4-c^2)\left(c^2+4\right) \\ &+ c^4(4-c^2)y+(4-c^2)^2c^2(4)y^2-(1-y^2)\Big] \\ &- \frac{1\left[c^6+c^2(4-c^2)c^2+4+y^2-(1-y)^2\right]}{8(1+2\check{q}+\check{q}^2)(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4)} \\ &+ \frac{1}{4(1+2\check{q}+2\check{q}^2+\check{q}^3)(1+\check{q}+\check{q}^2+\check{q}^3)}\Big[c^6+3c^4(4-c^2)+2c^2(4-c^2)^2 \\ &- c^4(4-c^2)+2c^2(4-c^2)^2(1-x^2)y-c^2(4-c^2)^2\Big] \\ &- \frac{1\left[c^6+3c^4(4-c^2)+3c^2(4-c^2)^2+(4-c^2)^3\right]}{4(1+\check{q}+\check{q}^2)^3} \\ &- \frac{1}{(16)(1+\check{q}+\check{q}^2+\check{q}^3)^2}\Big[c^6+4c^2(4-c^2)^2+c^2(4-c^2)^2 \\ &+ c^4(16-4c^2)-c^4(8-2c^2)-4c^2(4-c^2)^2\Big] \\ &\leq \frac{4}{(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4(1+\check{q}+\check{q}^2} - \frac{16}{(1+\check{q}+\check{q}^2)^3}, \text{ for } c\in(0,2). \end{aligned} \tag{4.34}$$

(v) At $y=0$.

$$\phi(c,x,0)=\frac{c^6}{(1+\check{q})^6}-\frac{3\left[c^6+c^4(4x-c^2x)\right]}{2(1+2\check{q}+\check{q}^2)^2(1+\check{q}+\check{q}^2)}$$
$$+\frac{3\left[c^6+c^2(4-c^2)x^2+2c^4(4-c^2)x\right]}{4(1+2\check{q}+\check{q}^2)(1+\check{q}+\check{q}^2)^2}$$
$$+\frac{1}{16(1+\check{q}+\check{q}^2)(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^3)}\Big[c^6+c^2(4x-c^2x)\left(c^2(x^2-3x+3)+4x\right)$$
$$+(4-c^2)x^2(c^2(x^2-3x+3)+4x)+4(4-c^2)^2x(1-x^2)\Big]$$
$$-\frac{1}{8(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4)(1+2\check{q}+\check{q}^2)}\Big[c^6+c^2(4x-c^2x)(c^2(x^2-3x+3)$$
$$+4x)+c^2(4-c^2)(1-x^2)\Big]$$
$$+\frac{1}{4(1+2\check{q}+2\check{q}^2+\check{q}^3)(1+\check{q}+\check{q}^2+\check{q}^3}\Big[c^6+3c^4(4x-c^2x)+2c^2(4-c^2)^2x^2$$
$$-c^4(4x^2-c^2x^2)-c^2(4-c^2)^2x^3\Big]$$
$$-\frac{\left[c^6+3c^4(4x-c^2x)+3c^2(4-c^2)^2x^2+(4-c^2)^3x^3\right]}{4(1+\check{q}+\check{q}^2)^3}$$
$$-\frac{1}{(16)(1+\check{q}+\check{q}^2+\check{q}^3)^2}\Big[c^6+4c^2(4-c^2)^2x^2+c^2(4-c^2)^2x^4$$
$$+4c^4(4-c^2)^2x-2c^4(4-c^2)x^2-4c^2(4-c^2)^2x^3\Big]$$
$$\leq\frac{4}{(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4)(1+\check{q}+\check{q}^2)}-\frac{16}{(1+\check{q}+\check{q}^2)^3}$$
$$\text{for } c,x\in(0,2)\times(0,1). \tag{4.35}$$

(vi) When $y=1$ and $c=(0,2),x=(0,1)$, we have

$$\phi(c,x,1)=\frac{c^6}{(1+\check{q})^6}-\frac{3\left[c^6+c^4(4-c^2)x\right]}{2(1+2\check{q}+\check{q}^2)^2(1+\check{q}+\check{q}^2)}$$
$$+\frac{3\left[c^6+c^2(4-c^2)x^2+2c^4(4-c^2)x\right]}{4(1+2\check{q}+\check{q}^2)(1+\check{q}+\check{q}^2)^2}$$
$$+\frac{1}{16(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4)(1+\check{q}+\check{q}^2)}\Big[c^6+c^2(4x-c^2x)(c^2(x^2-3x+3)+4x)$$
$$-4c^2(4-c^2)(c(x-1)+x^2-(1-x^2))(1-x^2)+c^4(4-c^2)$$
$$+(4-c^2)^2x^2(c^2(x^2-3x+3)+4x)-4(4-c^2)^2x(1-x^2)(c(x-1)+x^2)\Big]$$
$$-\frac{\left[c^6+c^2(4-c^2)(c^2(x^2-3x+3)+4x)x-c^2(4-c^2)(1-x^2)(c(x-1)+x^2)\right]}{8(1+2\check{q}+\check{q}^2)(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4)}$$

$$+\frac{1}{4(1+2\check{q}+2\check{q}^2+\check{q}^3)(1+\check{q}+\check{q}^2+\check{q}^3)}\Big[c^6+3c^4(4x-c^2x)+2c^2(4-c^2)^2x^2$$
$$-c^4(4-c^2)x^2-c^2(4-c^2)^2x^3+2c^2(4-c^2)^2x(1-x^2)+2c^3(4-c^2)(1-x^2)\Big]$$
$$-\frac{\left[c^6+3c^4(4x-c^2x)+3c^2(4-c^2)^2x^2+(4-c^2)x^3\right]}{4(1+\check{q}+\check{q}^2)^3}$$
$$-\frac{1}{16(1+\check{q}+\check{q}^2+\check{q}^3)^2}\Big[c^6+4c^4(4-c^2)^2x^2+c^2(4-c^2)^2x^4+4(4-c^2)^2$$
$$(1-x^2)^2+4c^4(4-c^2)x-2c^4(4-c^2)x^2+4c^3(4-c^2)(1-x^2)-4c^2(4-c^2)^2x^3$$
$$-4c^2(4-c^2)^2x^2(1-x^2)+8c(4-c^2)^2x(1-x^2)\Big]$$
$$=g(c,x)\text{ where }c\in(0,2),x\in(0,1). \tag{4.36}$$

$$\frac{\partial g}{\partial c}=\frac{6c^5}{(1+\check{q})^6}-\frac{3\left[6c^5+16c^x-6c^5x\right]}{2(1+2\check{q}+\check{q}^2)^2(1+\check{q}+\check{q}^2)}$$
$$+\frac{3\left[6c^5+31cx^2+6c^5x^2-32c^3+32c^3x-12c^5x\right]}{4(1+2\check{q}+\check{q}^2)(1+\check{q}+\check{q}^2)^2}$$
$$+\frac{1}{16(1+\check{q}+\check{q}^2)(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4)}\Big[16c^3x^3-48c^3x^2+48c^3x+32cx^2-6c^5x^3$$
$$+18c^5x^2-18c^x-16c^3x^2-36c^2x+48c^2x^3-20c^4x^3+36c^2-48c^2x^2+20c^4x^2$$
$$+16cx^2+64cx^4-32c^3x^4+24c-32cx^2+16c^3x^2+4c^3$$
$$+32cx^4+6c^5x^4-32c^3x-96cx^3-18c^5x^3+96c^3x^3+96cx^2$$
$$+18c^5x^2-96c^3x^2+16c^3x^2-64cx^2-64x^2+5c^4x^2-32c^2x-64x-5c^4x$$
$$+24c^2x-64x^4-5c^4x^4+24c^2x^4+64x^3+5c^4x^3-24c^2x^3\Big]$$
$$-\frac{1}{8(1+2\check{q}+\check{q}^2)(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4)}\Big[6c^5+16c^3x^3-6c^5x^2-48c^3x+18c^5x$$
$$+48c^3x-6c^5x+32cx-16c^3x^2-12c^2x+5c^4x+12c^2x^3$$
$$-5c^4x^3+12c^2-5c^4-12c^2x^2+5c^4x\Big]$$
$$+\frac{1}{4(1+2\check{q}+2\check{q}^2+\check{q}^3)(1+\check{q}+\check{q}^2+\check{q}^3)}\Big[6c^5+32c^3x-12c^5x+64cx^2+12c^5x^2-64c^3x^2$$
$$-16c^3x^2+6c^5x^2-32cx^3-6c^5x^3+32c^3x^3+24c^2+24c^2x^2-48c^2x-10c^4-10c^4x^2\Big]$$
$$-\frac{1}{4(1+\check{q}+\check{q}^2)^3}\Big[6c^5+48c^3x-18c^5x+64c^3x^2+6c^5x^2-48c^5x^2+6c^5x^3$$
$$-96cx^3-48c^3x^3\Big]$$

$$
-\frac{1}{16(1+\check{q}+\check{q}^2+\check{q}^3)^2}\Big[6c^5+128cx^2+24c^5x^2-128c^3x^2-32cx^4
$$
$$
-6c^5x^4+32c^3x^4\Big], \tag{4.37}
$$

and

$$
\begin{aligned}
\frac{\partial g}{\partial x} =& \frac{3\left[c^4(4-c^2)\right]}{2(1+2\check{q}+\check{q}^2)^2(1+\check{q}+\check{q}^2)} \\
&+\frac{3\left[2c^2x(4-c^2)^2+2c^4(4-c^2)\right]}{(1+2\check{q}+\check{q}^2)(1+\check{q}+\check{q}^2)^2} \\
&+\frac{1}{16(1+\check{q}+\check{q}^2)(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4)}\Big[12c^4x^2-24c^4x+12c^4+32c^2x-3c^6x^2 \\
&+6c^6x+3c^6-8c^4x+16c^4x-32c^4x^3+16c^2+4c^4+48c^2x^2-12c^4x^2-32c^3x \\
&+8c^5x-32c^2x+8c^4x+64c^2x^3-144c^2x^2+96c^2x+19x^2+4c^6x^2-9c^6x^3+6c^6x \\
&+12c^4x^2-16c^4x+72c^4x^2-48c^3x-96+128cx+8c^5x-64c^3x-64c+4c^5 \\
&+32c^3+192x^2+12c^4x^2-96c^2x^2-256cx^3-16c^5x^3+128c^3x^3+192cx^2\Big] \\
&-\frac{1}{8(1+\check{q})^2(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4)}\Big[12c^4x^2-24c^4x+32c^2x-3c^6x^2+c6-4c^4 \\
&-4c^3-8c^2x+c5+2c^4x+12c^5x^2-8c^3x+16c^2x^3-3c^5x^2+2c^5x-4c^5x^3\Big] \\
&+\frac{1}{4(1+\check{q}+\check{q}^2)(1+\check{q})(1+\check{q}+\check{q}^2+\check{q}^3)}\Big[3c^4(4-c^2)+4xc^2(4-c^2)^2-2c^4x(4-c^2) \\
&-3c^2x^2(4-c^2)^2+32c^2+2c^6-16c^5-96c^2x^2-6c^6x^2+48c^4x^2\Big] \\
&-\frac{1}{4(1+\check{q}+\check{q}^2)^3}\Big[3c^4(4-c^2)+6c^2x(4-c^2)^2+3x^2(4-c^2)^3\Big] \\
&-\frac{1}{16(1+\check{q}+\check{q}^2+\check{q}^3)^2}\Big[8c^2(4-c^2)^2x-+4c^2(4-c^2)^2x^3+128x+8c^4x \\
&-64c^2x-128+8c^4+64c^2-4c^4(4-c^2)-32c^3x+8c^5x-24c^2x(4-c^2)^2\Big].
\end{aligned} \tag{4.38}
$$

The system's solution $\frac{\partial g}{\partial c}=0$ and $\frac{\partial g}{\partial x}=0$ belongs to the region $(0,2)*(0,1)$, is $(c_1,x_1)\approx(0.248233,0.445259)$. Butt at (c_1,x_1) we observe that

$$
(\frac{\partial^2 g}{\partial c^2})(\frac{\partial^2 g}{\partial x^2})-(\frac{\partial^2 g}{\partial x\partial c})^2<0.
$$

As a result, $y=1$ has no critical point.

D. Now, we look at the interior of the parallelepiped: $(0,2)\times(0,1)\times(0,1)$.

We possess $\frac{\partial \phi}{\partial y}$ if and only if

$$\begin{aligned} y_0(c,x) = & \frac{1\Big[-4c^2(4-c^2)(x-1+2xy)-4(4-c^2)^2x(1-x^2)(c(x-1)+2xy+2y)\Big]}{16(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4)(1+\check{q}+\check{q}^2)} \\ & -\frac{1\Big[-4c^2(4-c^2)(1-x^2)(c(x-1)+2xy+2y)\Big]}{8(1+2\check{q}+\check{q}^2)(1+\check{q}+\check{q}^2+\check{q}^3)} \\ & +\frac{1\Big[3c^3(4-c^2)(1-x^2)+2c(4-c^2)(1-x^2)x\Big]}{4(1+\check{q}+\check{q}^2)^2} \\ & -\frac{1}{16(1+\check{q}+\check{q}^2+\check{q}^3)^2}\Big[8(4-c^2)^2(1-x^2)^2y \\ & +4c^3(4-c^2)(1-x^2)+4c(4-c^2)^2x^2(1-x^2)-4c(4-c^2)^2x^2(1-x^2)\Big]. \end{aligned} \tag{4.39}$$

Therefore $\phi(c,x,y)$ does not possess a critical point within the parallelepiped interior. Based on A,B,C and D, we find

$$\begin{aligned} \max\{\phi(c,x,y); c\in(0,2),\ x\in(0,1),\ y\in(0,1)\} = & \frac{4}{(1+\check{q}+\check{q}^2)(1+\mathrm{q}+\check{q}^2+\check{q}^3+\check{q}^4)} \\ & -\frac{16}{(1+\check{q}+\check{q}^2)^3}, \end{aligned} \tag{4.40}$$

hence from (4.20) and (4.40), we get

$$|H_{3,1}(h^{-1})| \leq \frac{4}{(1+\check{q}+\check{q}^2)(1+\check{q}+\check{q}^2+\check{q}^3+\check{q}^4)} - \frac{16}{(1+\check{q}+\check{q}^2)^3}. \tag{4.41}$$

This completes the proof. □

The qualitative results align with the classical results.

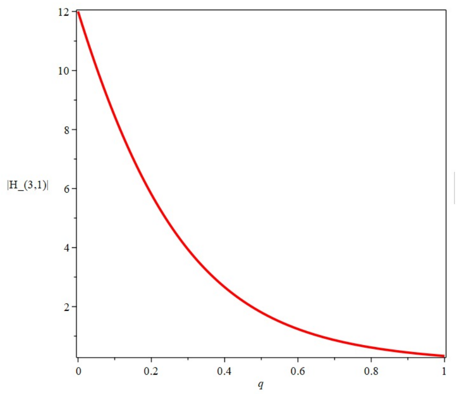

Figure 4.1: Variation of the Hankel determinant bounds $|H_{3,1}|$ with respect to parameter ǩ, for inverse ǩ bounded turning functions.

4.3 Remark

Taking $\check{q} \to 1^-$, then (4.41) will be reduced to results proved by Sanjay Kumar *et al.* [59].

4.4 Conclusion

We investigated the behavior of the Hankel determinant of order three $(H_{3,1})$, corresponding to the inverse function within the q̌-bounded turning function class, which is a q̌-analogue of classical bounded turning functions. Using the class $R_{\check{q}}$ we derived new estimates in the quantities of the magnitude of $\left|H_{3,1}(h^{-1})\right|$, where h is a member of the q̌-bounded turning class. These findings demonstrate that the theory of the classical geometrical functions can be continued by the use of the q̌-calculus and that the past results of classical functions of bounded turning can be generalized.

CHAPTER 5

CONCLUSION AND FUTURE DIRECTIONS

5.1 Conclusion

This thesis presents noval theoretical results and estimates for the Hankel determinant of inverse ǒ-bounded turning functions, contributing to mathematical aspects in the theory of inverse functions. While GFT addresses analytic and coefficient properties of univalent functions, this work extends the Hankel determinant of third order to inverse functions. The significance of q̌-bounded turning functions lies in their structural richness and their connection to convexity and starlikeness. The Hankel determinant $H_{3,1}(h)$ effectively captures nonlinear relations among Taylor coefficients and reveals key geometric features of analytic functions, yet its study for inverse functions within the class of q̌-bounded turning functions are unexplored.

In this research, we investigated coefficient bounds for inverse functions in the q̌-bounded turning class using inverse series and subordination, obtaining sharp upper bounds for the third-order Hankel determinant of inverse functions in class $R_{\check{q}}$ with $Re(D_{\check{q}}h(k)) > 0$. We analyzed that these bounds depend on q̌ and converge to the classical bounded-turning results as $\check{q} \to 1^-$. We also investigated the equality conditions for sharp bounds and examined that the extremal functions attain maximum values less than 1. Class $R_{\check{q}}$ was used to handle complex coefficient expressions to ensure accurate, sharp bounds. These results advance the theory of inverse-function coefficient problems for q̌-bounded classes and offer systematic analytical framework for analytic function classes.

5.2 Future Directions

Future research might focus on the higher-order Hankel determinants and inverse functions to relate the functions of q̌-convex, q̌-starlike, and bi-univalent functions. The Hankel determinants for the inverse of q̌-bounded turning functions may be extended to functions associated with various integral transforms and convolution operators. The theory can also be applied to the classical coefficient problems and the framework of the q̌-calculus and fractional q̌-operator. Additionally, this study can be connected with numerical and computational methods, which is future potential work.

5.3 Applications

The Hankel determinant and inverse functions are applied in control theory and signal processing, as Hankel matrices and their determinants helps in understanding the stability and identification in control systems. Together, they help understand how structure, stability, and connectivity behave in mathematical and physical systems, for detail see [60].

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