# On new classes of q-Janowski type functions of complex order

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Candidate of Master of Science in Mathematics at the National University of Modern Languages do hereby declare that the thesis On new classes of q-Janowski type functions of complex order submitted by me in partial fulfillment of MS/MATH degree, is my original work and has not been submitted or published earlier. I also solemnly declare that it shall not, in the future, be submitted by me for obtaining any other degree from this or any other university or institution. I also understand that if evidence of plagiarism is found in my thesis/dissertation at any stage, even after the award of a degree, the work may be canceled and the degree revoked.

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# ABSTRACT

**Title: On new classes of q-Janowski type functions of complex order**

This thesis introduces and investigates intriguing subclasses of q-starlike and q-convex functions related to Janowski-type functions of complex order. These classes are developed within the context of q-calculus, which extends traditional analytic function theory and provides a more complex structure for geometric function analysis. We examined these functions' essential properties, such as inclusion relations, distortion bounds, coefficient estimates, and the radius of convexity. Special emphasis is placed on their behavior under certain integral operators tailored to the q-calculus scenario. The given results not only expand conventional discoveries, but also provide larger generalizations and deeper insights into the geometric behavior of these q-analogues. Our findings give a more unified and comprehensive view than the previous research on Janowski-type analytic functions.

# TABLE OF CONTENTS

# LIST OF SYMBOLS

| | | |
|---|---|---|
| $\mathfrak{D}$ | - | Open unit disk |
| $\mathfrak{A}$ | - | Class of Analytical function |
| $\mathbb{S}$ | - | The Class of Univalent functions |
| $\mathfrak{S}$ | - | The Class of Starlike functions |
| $\mathsf{C}$ | - | The Class of Convex functions |
| $\mathfrak{C}$ | - | The Class of Quasi-Convex functions |
| $\mathsf{Q}$ | - | The Class of Close-to- Convex functions |
| $\mathfrak{P}$ | - | The Class of Caratheodory functions |
| $\prec$ | - | Subordination symbol |
| $\mathfrak{D}_{\mathrm{q}}$ | - | q-Derivative operator symbol |
| $\mathfrak{P}^{\alpha}[\xi,\varphi]$ | - | The Class of Janowski functions |
| $\tilde{C}^{\alpha}[\xi,\varphi]$ | - | The Class of Janowski Convex functions |
| $\mathfrak{S}^{\alpha}[\xi,\varphi]$ | - | The Class of Janowski Starlike functions |
| $\tilde{C}^{\alpha}_{\mathrm{q,c}}[\xi,\varphi]$ | - | The Class of q-Janowski Convex functions |
| $\mathfrak{S}^{\alpha}_{\mathrm{q,c}}[\xi,\varphi]$ | - | The Class of q-Janowski Starlike functions |

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May Allah accept our efforts and continue to guide us on the path of righteousness and wisdom.

# DEDICATION

*This thesis work is dedicated to my mother Memoona Tahira and my father Ashfaq Ahmed Gujjar for their unconditional love and support and to all my teachers throughout my education career whose good examples have taught me to work hard for the things that I aspire to achieve.*

# CHAPTER 1

# INTRODUCTION

## 1.1 Overview

Theory of Geometric Function is a field of mathematics that emphasizes the relevance of geometric concepts in Complex Function Theory, while same ideas may also be found in real analysis. Riemann, Cauchy, and Weierstrass all made significant advancements to this field of study. This theory focuses on functions inside the open unit disc $\mathfrak{D} = \{\mathfrak{s} \in \mathscr{C} : |\mathfrak{s}| < 1\}$, rather than arbitrary simply connected domains. Riemann's mapping theorem ensures that any such domain can be transferred conformally to the unit disc.

Koebe first proposed the concept of univalent functions in 1907.In 1916, Bieberbach proposed the "Bieberbach Conjecture", concerning the second coefficient of normalized univalent functions, a problem that was ultimately solved by de Branges in 1985. $\mathbb{S}$ denotes the collection of functions that are normalized, analytic, and univalent. Class of convex ($\mathbb{C}$) and starlike ($\mathfrak{S}$) univalent functions are the two notable subclasses of class $\mathbb{S}$. Kaplan proposed the $\mathbb{K}$ class, which comprises close-to-convex univalent functions, while Noor and Thomas created the $\mathfrak{C}$ subclass of quasi-convex functions. These classes are similar to class $\mathfrak{P}$, which includes functions having a positive real portion, often known as Carathéodory functions.

Geometric Function Theory relies mainly on subordination [1] and convolution [2] as tools for study. The subordination principle was defined by Lindelof [3] using the Schwarz function. Janowski [4] used subordination to investigate various interesting analytic function features.

Janowski type functions, developed by Wolfram Janowski in 1973, are an important expansion of geometric function theory. Janowski developed the idea of Janowski type functions. By investigating the univalence and geometric features of these functions, he established the framework for future research.

Classical Janowski type functions are analytic functions with specified growth requirements on their derivatives. The generic form is:

$$|\chi'(\mathfrak{s})| \leq \frac{\alpha}{(1-|\mathfrak{s}'|)^{\mathrm{r_u}}} \tag{1.1}$$

where $\chi \in \mathfrak{A}$ , $\alpha$ is constant and $\mathrm{r_u}$ is a real integer. These functions are being investigated for their distortion, growth, and mapping feature.

Quantum calculus, or $\mathfrak{q}$-calculus, is a mathematical technique comparable to classical calculus that focuses on finding $\mathfrak{q}$-analogous solutions without the usage of limits. Jackson proposed the systematic $\mathfrak{q}$-derivative in 1908. The development of $\mathfrak{q}$-calculus has connected physics with mathematics. Geometric Function Theory utilizes quantum calculus to build numerous subclasses of analytic functions.In 1990, Ismail [5] was the first to establish the class of $\mathfrak{q}$-starlike functions by applying the $\mathfrak{q}$-derivative

$$D_q\chi_1(\mathfrak{s}) = \frac{\chi_1(\mathfrak{q}\mathfrak{s}) - \chi_1(\mathfrak{s})}{(\mathfrak{q}-1)\mathfrak{s}}, (\mathfrak{s} \neq 0) \tag{1.2}$$

with the condition $D_{\mathfrak{q}}\chi_1(0) = \chi_1{}'(0)$, where $\mathfrak{q}$ is the interval (0, 1). For the specific case, If $\chi_1(\mathfrak{s}) = (\mathfrak{s})^{\underline{\mathfrak{n}}}$,where $\underline{\mathfrak{n}} = 1,2,3,....$ then $\mathfrak{q}$-difference operator yields:

$$D_{\mathfrak{q}}\chi_1(\mathfrak{s}) = [\underline{\mathfrak{n}}]_{\mathfrak{q}}(\mathfrak{s})^{\underline{\mathfrak{n}}-1} \tag{1.3}$$

where $[\underline{\mathfrak{n}}]_{\mathfrak{q}}$ can be expressed as follows:

$$[\underline{\mathfrak{n}}]_{\mathfrak{q}} = \frac{1-\mathfrak{q}^{\underline{\mathfrak{n}}}}{1-\mathfrak{q}} \tag{1.4}$$

Since $t \to 1^-$, $[\underline{\mathfrak{n}}]_{\mathfrak{q}} \to \underline{\mathfrak{n}}$, and $D_{\mathfrak{q}}(\mathfrak{s}) \to \chi'(\mathfrak{s})$. This is equivalent to:

$$D_{\mathfrak{q}}\chi(\mathfrak{s}) = 1 + \sum_{\underline{\mathfrak{n}}=2}^{\infty} a_{\underline{\mathfrak{n}}}\mathfrak{s}^{\underline{\mathfrak{n}}-1} \tag{1.5}$$

Jackson also presented the $\mathfrak{q}$-integral of a function $f_1$, which is as follows

$$\int_0^{\mathfrak{s}} \chi_1(\mathfrak{s})\, d_{\mathfrak{q}}\mathfrak{s} = \mathfrak{s}(1-\mathfrak{q}) \sum_{\underline{\mathfrak{n}}=0}^{\infty} \mathfrak{q}^{\underline{\mathfrak{n}}}\chi_1(\mathfrak{q}^{\underline{\mathfrak{n}}}\mathfrak{s}) \tag{1.6}$$

assuming the series converges.

## 1.2 Riemann Mapping Theorem

Bernard Riemann in 1851 established the Riemann Mapping Theorem (see [2]). Instead of using a complex random domain, this finding enables us to use an open unit disc, $\mathfrak{D} = \{\mathfrak{s} \in \mathscr{C} : |\mathfrak{s}| < 1\}$, as a domain. This theorem is necessary for the fundamental idea of geometric function theory. In the 19th century, Weierstrass, Riemann, and Cauchy contributed significantly to univalent function theory.

## 1.3 Analytic Function and Univalent Function

Univalent and analytic functions in $\mathfrak{D}$ were proposed by Koebe in 1907. In open unit disc $\mathfrak{D}$, he developed analytic, univalent, and normalized functions. Comprehensive research has been conducted on these functions (see references 1,2 and 3). Analytic functions were initially presented by Duren. He established class $\mathfrak{A}$, which comprises normalized and analytic functions with $\chi'(0) = 1$ and $\chi(0) = 0$ . It is possible to express analytical functions of class $\mathfrak{A}$ in series form as $\chi(\mathfrak{s}) = \mathfrak{s} + \sum_{k=2}^{\infty} c_k \mathfrak{s}^k$, $\mathfrak{s} \in \mathfrak{D}$, where $\mathfrak{D} = \{\mathfrak{s} \in \mathscr{C} : |\mathfrak{s}| < 1\}$. Analytic functions play a crucial role in Geometric Function Theory by providing a thorough framework for investigating complex function behavior. An analytic region is one where a function is differentiable at all points (see [6]). Analytic functions are categorized into classes and subclasses based on their image domain structure and geometry. Analytic function analysis relies heavily on the picture domain's geometric shape. New geometrical structures related to analytic functions have been introduced and studied by scholars. Robertson [7] introduces the notion of univalent functions in 1936. If a function is injective and accepts distinct values for various inputs within a region, it is said to be univalent in that area.

## 1.4 Subclasses of Analytic and Univalent Function

A key theory in complex analysis, univalent functions were first proposed by Koebe [6] in 1907. According to his proposal, functions that satisfy normalization conditions, are analytic, and are univalent in $\mathfrak{D}$ are categorized as $\mathbb{S}$ functions. The class $\mathbb{S}$ functions' subclasses will be the primary focus of this examination. In geometric function theory, normalized univalent functions, or class $\mathbb{S}$, are essential. The Class $\mathbb{S}$ is divided into four subclasses: the class of starlike functions $\mathfrak{S}$, quasi-convex functions $\mathfrak{C}$, close-to-convex functions $\mathsf{Q}$, and convex functions $\mathsf{C}$. The creation of this classification was an attempt to support the Bieberbach hypothesis [8] . The so-called Alexander relation, which links two distinct sets of convex and starlike functions, was first defined by Alexander [9] in 1915. In 1921, Nevanlinna [10] began to exhibit star-like characteristics. Convex and starlike functions of order $\alpha$ with negative coefficients were explored by Silverman [11] in 1975.

## 1.5 Coefficient Bounds

Geometric Function Theory focuses on identifying coefficient bounds and breaks down functions into subfamilies of class $\mathfrak{A}$. The Bierbach theorem, originally proposed by German mathematician Ludwig Bierberbach in 1916, is a foundational component of class $\mathbb{S}$. He calculated the class $\mathbb{S}$ univalent functions' second coefficient, $\acute{c}_2$. Bieberbach's conjecture, which resulted in significant progress in the field, was made possible by this theorem (see [12]). According to the coefficient conjecture, if $\chi$ is a member of class $\hat{\mathbb{S}}$, then the inequality $y \mid \acute{c}_k \mid \leq \check{k}$ for $\check{k} = \{2, 3, \dots\}$ holds for the kth coefficient of $\chi$, $\acute{c}_k$. From one of the Koebe function's rotations, sharp results are obtained. The extremal function corresponding to this inequality is the Koebe function, and the second coefficient $\acute{c}_2$ of $\chi$ has the constraint $|\acute{c}_2| \leq 2$, as demonstrated by Bieberbach [12] in 1916. Although mathematicians have made several attempts to prove this hypothesis, it is still a challenging task. A univalent function $\chi$ 's third coefficient, $\acute{c}_3$, is $|\acute{c}_3| \leq 3$, as demonstrated by Karl Loewner in 1923. The fourth coefficient problem which showed that $|\acute{c}_4| \leq 4$ was tackled in 1955 by Garabedian and Schiffer [13]. The generalized version of the Bieberbach conjecture asserts that $|a_k| \leq \check{k}$ for $\check{k} = \{2, 3, \dots\}$, was developed in 1985 by mathematician Louis de Branges [12]. For any $\check{k}$, the inequality is tight when $\omega$ is a rotation of a Koebe function (see [6]). For the q-starlike and q-convex function classes, Darus [14] discovered second and third coefficient estimates in 2016. Second and third coefficient for complex-order q-convex and q-starlike functions were,discussed by Seoudy and his colleagues in 2016 .

## 1.6 Janowski Type Function

Wolfram Janowski invented Janowski type functions in 1973, and they represent a significant generalization within theory of geometric function.The Janowski-type function class is determined by applying a subordination condition. $\frac{\mathfrak{s}k'(\mathfrak{s})}{k(\mathfrak{s})} \prec \frac{1+\xi\mathfrak{s}}{1+\varphi\mathfrak{s}}$ ,where $-1 \leq \varphi < \xi \leq 1$, where $\chi(\mathfrak{s}) \in \mathfrak{A}$. The parameters $\xi$ and $\varphi$ dictate the geometric behavior of the function.

The sign $\prec$ denotes subordination, which means that the image of $\frac{\mathfrak{s}k'(\mathfrak{s})}{k(\mathfrak{s})}$ is included within the image of the linear fractional transformation $\frac{1+\varphi\mathfrak{s}}{1+\xi\mathfrak{s}}$. This definition extends well-known univalent function subclasses, such as star-like and convex functions, by regulating their geometric features with parameters ($\xi$ and $\varphi$).

The Janowski class can be reduced to well-known function classes based on certain parameter values ($\xi$ and $\varphi$). For example: When $\xi = 1$ and $\varphi = 0$, the Janowski class corresponds to the class of star-like functions. When $\xi = 1$ and $\varphi = 1$, the class corresponds to convex functions.

When $\xi = 0$ and $\varphi = -1$, it represents Bazilevič functions. Because of its versatility in creating geometric classes of functions, Janowski type functions are an extremely useful tool in geometric function theory. Since the introduction of Janowski type functions of complex order, they have been the subject of extensive research. W. Janowski laid the platform for future investigation in the early 1980's by developing fundamental concepts with real order parameters. The 1990's saw the first attempts to extend these functions to complex orders, while research was mostly focused on real parameters. The early 2000's saw the start of increasingly significant generalizations, as academics began to study how complex factors impact function features. The 2010's saw deeper theoretical discoveries, with works looking at the behavior of Janowski type functions in complicated parameter spaces. Kumar and Singh made notable contributions to growth projections during this time period. The 2020's have been especially successful, with notable advances such as Arora and Sharma's investigation of border behavior applications and Miller and Zlotkowski's work on growth estimates and extreme points. Recent research by Hosseini and Moghimi, as well as Das and Gupta, has offered detailed analyses and generalizations of Janowski type functions, demonstrating the field's continual evolution and refinement.

## 1.7 Quantum Calculus

The idea of limits is not used in quantum calculus, which sets it apart from conventional calculus. In 1909 and 1910, Jackson introduced the $\mathfrak{q}$-integral and $\mathfrak{q}$-derivative methods. In 1740, Euler developed the concept of partitions, which led to the development of $\mathfrak{q}$-analysis. A key figure in the creation of quantum calculus was Gauß (1777–1855). Two important contributors to the creation of quantum calculus were Bernoulli and Euler. Because quantum calculus is utilized extensively in physics, mechanics, and mathematics, researchers are becoming more interested in it. In the study of classical mathematics, quantum calculus is a crucial field of study. The objective is to provide a theoretical synopsis of the differentiation and integration methods. One of the oldest and most extensive areas of mathematics is quantum calculus. The intricate calculations and computations required make it more difficult than other math subjects. The $\mathfrak{q}$-derivative is used in this study to assess the geometric characteristics of analytic functions. In 1990, Ismail et al. [5] proposed $\mathfrak{q}$-starlike functions with regard to the $\mathfrak{q}$-derivative. This is the first instance of Geometric Function Theory using $\mathfrak{q}$-calculus. He used the difference operator to do this. He reinstated his class the "class of $\mathfrak{q}$-starlike functions" when he initially presented it. Srivastava [15], in 2011, contributed significantly to the incorporation of $\mathfrak{q}$-calculus into geometric function theory by investigating the generalizations and $\mathfrak{q}$-extensions of classical polynomials such as Bernoulli, Euler, and Genocchi. For the open unit disc, Purohit [16] suggested a new class of multivalently analytic functions. Regarding this function class, he looked at distortion theorems and coefficient inequalities. He was the first to employ a unique $\mathfrak{q}$-derivative operator in a study. He contributed significantly to the theory of analytic functions by offering $\mathfrak{q}$-extensions for a number of results. Aldweby and Darus [14] used a generalized operator and a basic hypergeometric function to investigate complex valued harmonic univalent functions in 2013. For functions that belong to their class, they also include the coefficient requirements. In 2016, Darus [14] initiated the study of the $\mathfrak{q}$-derivative operator in combination with $\mathfrak{q}$-starlike and $\mathfrak{q}$-convex function classes. Seoudy and others [17] later that year presented generalized subclasses of these complex-order functions by including the $\mathfrak{q}$-derivative operator. $\mathfrak{q}$-calculus has been used by numerous researchers to significantly enhance geometric function theory.

## 1.8 Preface

A brief introduction to each chapter of the thesis is provided below:

In **Chapter 2**, several fundamentally relevant definitions and findings are explored. The chapter provides information about analytic and univalent functions, their subclasses, and Caratheodory function class P. Convolution and subordination methods are briefly discussed here. The study will include the analytic and univalent functions, classes of these functions, Caratheodory function, Janowski functions along with its subclasses. This chapter provides a comprehensive overview of $\mathfrak{q}$-calculus and the most current $\mathfrak{q}$-function classes. All of the contents of this chapter are precisely referred.

In **Chapter 3**, new subclasses of Janowski function of complex order will be examined. Inclusion characteristics, coefficient bounds, distortion theorems, and radius of convexity for these classes will be examined. Furthermore, analytic features of these classes under specific conditions will be studied.

In **Chapter 4**, new classes of Janowski-type functions of complex order will be developed on the basis of $\mathfrak{q}$-calculus. In the open unit disc, there are two new subclasses of $\mathfrak{q}$-Janowski type functions $\mathfrak{S}^{\alpha}_{\mathfrak{q},\mathfrak{c}}[\xi,\varphi]$ for $\mathfrak{q}$-starlike and $\tilde{C}^{\alpha}_{\mathfrak{q},\mathfrak{c}}[\xi,\varphi]$ for $\mathfrak{q}$-convex function.

It must be mentioned that new operators and new classes are extension of the existing ones.

In **Chapter 5**, the previous investigation's results will be discussed.

# CHAPTER 2

# PRELIMINARY CONCEPTS

## 2.1 Introduction

In this chapter some useful terms and conventional results that will pave the way for future study will be proposed. A comprehensive overview of the Caratheodory functions and normalized analytic univalent functions will be given. Well-known linear operators and certain specific functions and initial lemmas will be taken into account. A quick review of $\mathfrak{q}$-calculus's foundations is given. Several recent classes of $\mathfrak{q}$-analytic functions will also be discussed. Standard books [13, 18, 19, 20] have been used for the information in this chapter.

## 2.2 Analytic Functions and the Class A

In this section, definitions of class $\mathfrak{A}$ and class $\mathbb{S}$ along with their associated results are presented (see [13, 19, 21])

**Definition 2.2.1.** [22] The class $A$ is defined by analytic functions $\chi$ in the unit disk $\mathfrak{D}$ which are normalized such that:

$$\chi(\mathfrak{s}) = \mathfrak{s} + \sum_{\underline{\mathfrak{n}}=2}^{\infty} a_{\underline{\mathfrak{n}}}(\mathfrak{s})^{\underline{\mathfrak{n}}}, \quad \mathfrak{s} \in \mathfrak{D} \tag{2.1}$$

This normalization ensures that $\chi(0) = 0$ and $\chi'(0) = 1$.

**Definition 2.2.2.** [21] A function $\chi(\mathfrak{s}) \in \mathbb{S}$ is considered univalent in $\mathfrak{D}$ when it is both analytic and one-to-one,

$$\chi(0) = 0, \quad \chi'(0) = 1$$

**Definition 2.2.3.** [19] A functions $\chi(\mathfrak{s}) \in \mathbb{S}$ is called starlike if it is starlike with respect to the origin and satisfies the following condition:

$$\text{Re}\left(\frac{\mathfrak{s}\chi'(\mathfrak{s})}{\chi(\mathfrak{s})}\right) > 0, \quad \mathfrak{s} \in \mathfrak{D}. \tag{2.2}$$

**Definition 2.2.4.** [19] Let $\chi(\mathfrak{s}) \in \mathbb{S}$ is said to be be convex such that $\chi(\mathfrak{s})$ if:

$$\text{Re}\left(1 + \frac{\mathfrak{s}\chi''(\mathfrak{s})}{k'(\mathfrak{s})}\right) > 0, \quad \mathfrak{s} \in \mathfrak{D}. \tag{2.3}$$

**Definition 2.2.5.** [19] Let $\chi(\mathfrak{s}) \in \mathbb{S}$ are said to be close to convex if they are not necessarily starlike or convex but satisfy the following condition:

$$\text{Re}\left(e^{i\theta}\frac{\mathfrak{s}\chi'(\mathfrak{s})}{l(\mathfrak{s})}\right) > 0, \quad \mathfrak{s} \in \mathfrak{D}, \tag{2.4}$$

where $l(\mathfrak{s})$ is a starlike function and $\theta$ is a real parameter.

**Definition 2.2.6.** [13] Assume $\chi \in \mathfrak{A}$. Then $\chi(\mathfrak{s}) \in \mathfrak{S}(\gamma)$, if and only if

$$\text{Re}\left(\frac{\mathfrak{s}\chi'(\mathfrak{s})}{\chi(\mathfrak{s})}\right) > \gamma, \quad \mathfrak{s} \in \mathfrak{D}. \tag{2.5}$$

where $\gamma \in (0,1]$ is the order of starlikeness.

**Lemma 2.2.1.** [19] If $\chi \in \mathbb{S}$, the mapping of the unit disk under $\chi$ includes a disk of radius $1/4$ centered at the origin:

$$\chi(\mathfrak{D}) \supseteq \{\omega \in \mathscr{C} : |\omega| < 1/4\}. \tag{2.6}$$

If $\chi \in \mathfrak{S}$ and $\chi(\mathfrak{s}) = \mathfrak{s} + \sum_{\underline{\mathfrak{n}}=0}^{\infty} a_{\underline{\mathfrak{n}}}(\mathfrak{s})^{\underline{\mathfrak{n}}}$, then:

$$|a_{\underline{\mathfrak{n}}}| \leq \underline{\mathfrak{n}}, \quad \text{for all } \underline{\mathfrak{n}} \geq 2. \tag{2.7}$$

Equality holds for the Koebe function:

$$k(\mathfrak{s}) = \frac{\mathfrak{s}}{(1-\mathfrak{s})^2}. \tag{2.8}$$

**Lemma 2.2.2.** Let $\chi(\mathfrak{s}) \in \mathfrak{A}$, with $\chi(0) = 0$ and $|\chi(\mathfrak{s})| \leq 1$ for $\mathfrak{s} \in \mathfrak{D}$. Then:

$$|\chi(\mathfrak{s})| \leq |\mathfrak{s}|, \quad \text{and} \quad |k'(0)| \leq 1. \tag{2.9}$$

Equality holds if and only if $\chi(\mathfrak{s}) = e^{i\theta}\mathfrak{s}$, for some $\theta \in \mathbb{R}$.

**Lemma 2.2.3.** [13] For $\chi \in \mathfrak{S}$, if $|\mathfrak{s}| = \mathrm{r_u} < 1$, the following inequalities hold:

$$\frac{\mathrm{r_u}}{(1+\mathrm{r_u})^2} \leq |\chi(\mathfrak{s})| \leq \frac{\mathrm{r_u}}{(1-\mathrm{r_u})^2}, \tag{2.10}$$

and for derivative:

$$\frac{1-\mathrm{r_u}}{(1+\mathrm{r_u})^3} \leq |\chi'(\mathfrak{s})| \leq \frac{1+\mathrm{r_u}}{(1-\mathrm{r_u})^3}. \tag{2.11}$$

## 2.3 Carathéodory Class $\mathfrak{P}$

**Definition 2.3.1.** [13, 19] A function $\phi(\mathfrak{s})$ belongs to the Carathéodory class, denoted by $\mathfrak{P}$, if it satisfies the following conditions:

$\phi(\mathfrak{s})$ is analytic in $\mathfrak{D}$, $\mathrm{Re}(\phi(\mathfrak{s})) > 0$ for all $\mathfrak{s} \in \mathfrak{D}$,

$Re(\phi(\mathfrak{s})) > 0$ for all $\mathfrak{s} \in \mathfrak{D}$,

$\phi(\mathfrak{s})$ is normalized such that $\phi(0) = 1$.

A function in a Carathéodory class is represented as:

$$\phi(\mathfrak{s}) = 1 + \sum_{\underline{\mathfrak{n}}=1}^{\infty} b_{\underline{\mathfrak{n}}}(\mathfrak{s})^{\underline{\mathfrak{n}}}, \tag{2.12}$$

where $b_{\underline{\mathfrak{n}}} \in \mathscr{C}$ and $|b_{\underline{\mathfrak{n}}}|$ are constrained by the properties of $\phi(\mathfrak{s})$.

**Definition 2.3.2.** [23] By Herglotz's theorem, every $\phi(\mathfrak{s}) \in \mathfrak{P}$ can be expressed as:

$$\phi(\mathfrak{s}) = \int_{-\pi}^{\pi} \frac{1+e^{i\theta}\mathfrak{s}}{1-e^{i\theta}\mathfrak{s}}\, d\mu(\theta), \tag{2.13}$$

where $\mu$ is a probability measure on $[-\pi,\pi]$. This integral representation highlights that $\phi(\mathfrak{s})$ is a convex combination of the functions $\frac{1+e^{i\theta}\mathfrak{s}}{1-e^{i\theta}\mathfrak{s}}$, which are known as Herglotz functions.

**Lemma 2.3.1.** [21] For $\phi(\mathfrak{s}) = 1 + \sum_{\underline{\mathfrak{n}}=1}^{\infty} b_{\underline{\mathfrak{n}}}(\mathfrak{s})^{\underline{\mathfrak{n}}}$, the coefficients satisfy:

$$|b_{\underline{\mathfrak{n}}}| \leq 2, \quad \text{for all } \underline{\mathfrak{n}} \geq 1. \tag{2.14}$$

Equality is achieved for $\phi(\mathfrak{s}) = \frac{1+\mathfrak{s}}{1-\mathfrak{s}}$.

**Lemma 2.3.2.** If $\phi(\mathfrak{s}) \in \mathfrak{P}$, then the coefficients of its series expansion satisfy:

$$|b_{\underline{\mathfrak{n}}}| \leq 2, \quad \text{for all } \underline{\mathfrak{n}} \geq 1. \tag{2.15}$$

Equality holds for $\phi(\mathfrak{s}) = \frac{1+e^{i\theta}\mathfrak{s}}{1-e^{i\theta}\mathfrak{s}}$, $\theta \in \mathbb{R}$.

**Lemma 2.3.3.** If $\phi(\mathfrak{s}) \in \mathfrak{P}$, the following inequalities hold for $|\mathfrak{s}| = \mathrm{r_u} < 1$:

$$\frac{1-\mathrm{r_u}}{1+\mathrm{r_u}} \leq |\phi(\mathfrak{s})| \leq \frac{1+\mathrm{r_u}}{1-\mathrm{r_u}}. \tag{2.16}$$

In 1935, sufficient condition for univalency was derived by Noshiro [24] and Warschawski [25] as follows:

**Theorem 2.3.4.** (Noshiro-Warschawski Theorem) Assume that for each z be in convex domain $\mathbb{D}$, and $\xi \in \mathbb{R}$, $(e^{(i\theta)}\chi' \in \mathfrak{P}$. That is

$$Re(e^{(i\theta)}\chi'(\mathfrak{s})) \geq 0$$

Then $\chi$ is univalent in $\mathbb{D}$.

## 2.4 Janowski function

**Definition 2.4.1.** [4] A function $\chi(\mathfrak{s})$ falls within the Janowski class if it meets the subordination requirement stated as follows:

$$\frac{\mathfrak{s}k'(\mathfrak{s})}{\chi(\mathfrak{s})} \prec \frac{1+\xi\mathfrak{s}}{1+\varphi\mathfrak{s}}, \quad \mathfrak{s} \in \mathfrak{D},$$

where:

- $\chi(\mathfrak{s})$ is analytic in the unit disk $\mathfrak{D}$,
- $\xi$ and $\varphi$ are real constants with $-1 \leq \varphi < \xi \leq 1$,
- $\prec$ denotes the subordination relationship, meaning that $\frac{\mathfrak{s}\chi'(\mathfrak{s})}{\chi(\mathfrak{s})}$ is dominated by $\frac{1+\xi\mathfrak{s}}{1+\varphi\mathfrak{s}}$ in $\mathfrak{D}$.

The function $\frac{1+\xi\mathfrak{s}}{1+\varphi\mathfrak{s}}$ is a generalized Herglotz function, mapping $\mathfrak{D}$ onto a half-plane or disk depending on the values of $\xi$ and $\varphi$.

**Special Cases**

If $\xi = 1$ and $\varphi = -1$, the Janowski class will be reduced to the class of starlike functions:

$$\frac{\mathfrak{s}\chi'(\mathfrak{s})}{\chi(\mathfrak{s})} \prec \frac{1+\mathfrak{s}}{1-\mathfrak{s}}. \tag{2.17}$$

If $\xi = \varphi = 1$, the Janowski class reduces to the class of convex functions:

$$\frac{\mathfrak{s}\chi'(\mathfrak{s})}{\chi(\mathfrak{s})} \prec 1+\mathfrak{s}$$

For $-1 < \varphi < \xi \leq 1$, the Janowski class corresponds to functions starlike with respect to a boundary point or other general starlikeness conditions.

Alternatively it is defined as:

**Definition 2.4.2.** [4] If $\phi \in \mathfrak{P}$, then

$$\phi \in \mathfrak{P}[\xi,\varphi] \Leftrightarrow \phi(\mathfrak{s}) \prec \frac{1+\xi(\mathfrak{s})}{1+\varphi(\mathfrak{s})}, -1 \leq \varphi < \xi \leq 1, \mathfrak{s} \in \mathfrak{D}$$

The following relation shows the connection between the classes $\mathfrak{P}[\xi,\varphi]$ and $\mathfrak{P}$,

$$\mathfrak{h}^* \in \mathfrak{P} \Leftrightarrow \frac{(\xi+1)\mathfrak{h}^* - (\xi-1)}{(\varphi+1)\mathfrak{h}^* - (\varphi-1)} \in \mathfrak{P}[\xi,\varphi]$$

It is observed in [26] that $\mathfrak{P}[-1,1] = \mathfrak{P}$ and $\mathfrak{P}[\xi,\varphi]$ is a convex set.

**Definition 2.4.3.** [4] A function $\chi(\mathfrak{s})$ in the Janowski class can also be expressed through a subordinated Herglotz representation:

$$\frac{\chi'(\mathfrak{s})}{\chi(\mathfrak{s})} = \int_{-\pi}^{\pi} \frac{1+e^{i\phi}\mathfrak{s}}{1+e^{i\phi}B\mathfrak{s}}\, d\mu(\phi),$$

where $\mu$ is a probability measure.

**Lemma 2.4.1.** [4] For $\chi(\mathfrak{s}) \in$ Janowski class, the coefficients $a_2, a_3, \ldots$ satisfy:

$$|a_2| \leq \frac{2(\xi-\varphi)}{(1-\varphi)(1+\varphi)}.$$

Higher-order coefficients can be estimated using recursive relations or extremal function techniques.

**Lemma 2.4.2.** [4] Extreme points of the Janowski class can be expressed as functions of the form:

$$\chi(\mathfrak{s}) = \mathfrak{s}\exp\left(\int_{-\pi}^{\pi} \log\left(\frac{1+e^{i\phi}\mathfrak{s}}{1+e^{i\phi}B\mathfrak{s}}\right) d\mu(\phi)\right),$$

where $\mu$ is a probability measure.

**Definition 2.4.4.** The clasesses $\mathfrak{S}[\xi,\varphi]$ and $\tilde{C}[\xi,\varphi]$ of Janowski starlike and convex functions were initially defined by Janowski [27] as:

$$\mathfrak{S}[\xi,\varphi] = \left\{\chi \in \mathfrak{A} : \left(\frac{\mathfrak{s}\chi'}{\chi}\right) \in [\xi,\varphi], -1 \leq \varphi < \xi \leq 1, \mathfrak{s} \in \mathfrak{D}\right\},$$

$$\tilde{C}[\xi,\varphi] = \left\{\chi \in \mathfrak{A} : \left(\frac{(\mathfrak{s}\chi')'}{\chi'}\right) \in [\xi,\varphi], -1 \leq \varphi < \xi \leq 1, \mathfrak{s} \in \mathfrak{D}\right\}.$$

Relation between these classes is as follows:

$$\chi \in \tilde{C}[\xi,\varphi] \Leftrightarrow \mathfrak{s}\chi' \in \mathfrak{S}[\xi,\varphi].$$

Applications of these classes can be seen in [26, 28, 29]

## 2.5 Quantum Calculus

In Geometric Function Theory, quantum calculus is applied to develop subclasses of analytic functions. The class of $q$-starlike functions was initially introduced by Ismail [5] in 1990 using the $q$-derivative. The $q$-analogue of close-to-convex functions was examined in[16] .Raghavendar and Swaminathan [30] studied some basic properties of these function.

Some basic concepts of $\mathfrak{q}$-calculus are reviewed here. We will assume $0 < \mathfrak{q} < 1$, throughout this thesis.

**Definition 2.5.1.** [5] Let $\mathfrak{B} \subset \mathbb{C}$ is called geometric set, if $\mathfrak{q}\mathfrak{s} \in \mathfrak{B}$ whenever $\mathfrak{s} \in \mathfrak{B}, \mathfrak{q} \in (0,1)$,then all geometric sequences $\{\mathfrak{s}\mathfrak{q}^{\underline{\mathfrak{n}}}\}_0^\infty, \mathfrak{q}\mathfrak{s} \in \mathfrak{B}$ are contained in such a sets.

The concepts of the $q$-derivative and the $q$-integral were introduced by Jackson in 1908 [31, 32]

**Definition 2.5.2.** The $q$-difference operator or $q$-derivative of a normalized analytic function $\chi$ is defined in the context of $q$-calculus as follows:

$$\mathfrak{D}_{\mathfrak{q}}\chi(\mathfrak{s}) = \frac{\chi(\mathfrak{q}\mathfrak{s}) - \chi(\mathfrak{s})}{(\mathfrak{q}-1)\mathfrak{s}}, \quad (\mathfrak{s} \neq 0),$$

with the condition $\mathfrak{D}_{\mathfrak{q}}\chi(0) = \chi'(0)$, where $\mathfrak{q}$ is in the interval $(0,1)$.

For the specific case of $\chi(\mathfrak{s}) = \mathfrak{s}^{\underline{\mathfrak{n}}}$, the $q$-difference operator yields:

$$\mathfrak{D}_{\mathfrak{q}}\chi(\mathfrak{s}) = [\underline{\mathfrak{n}}]_{\mathfrak{q}}\mathfrak{s}^{\underline{\mathfrak{n}}-1},$$

where $[\underline{\mathfrak{n}}]_{\mathfrak{q}}$ is defined as:

$$[\underline{\mathfrak{n}}]_{\mathfrak{q}} = \frac{1-\mathfrak{q}^{\underline{\mathfrak{n}}}}{1-\mathfrak{q}}. \tag{2.18}$$

Taking the limit as $\mathfrak{q} \to 1^-$, we have $[\underline{\mathfrak{n}}]_{\mathfrak{q}} \to \underline{\mathfrak{n}}$, and the $q$-difference operator $\mathfrak{D}_{\mathfrak{q}}$ approaches the ordinary derivative $f'(\mathfrak{s})$. This corresponds to the classical derivative. The $q$-difference operator for a normalized analytic function $\chi(\mathfrak{s})$ can be expressed as:

$$\mathfrak{D}_{\mathfrak{q}}\chi(\mathfrak{s}) = 1 + \sum_{\underline{\mathfrak{n}}=2}^{\infty}[\underline{\mathfrak{n}}]_{\mathfrak{q}}a_{\underline{\mathfrak{n}}}\mathfrak{s}^{\underline{\mathfrak{n}}-1}.$$

**Remark.** [5] Following are some properties of $\mathfrak{q}$-derivative

I. Let $h_0(\mathfrak{s}) = \mathfrak{s}^{\underline{\mathfrak{n}}}$ be a function. Then $\mathfrak{q}$-derivative will be

$$\mathfrak{D}_{\mathfrak{q}} h_0(\mathfrak{s}) = [\underline{\mathfrak{n}}]_{\mathfrak{q}} \mathfrak{s}^{\underline{\mathfrak{n}}-1},$$

where $[\underline{\mathfrak{n}}]_q$ is given by (2.18)

II. The $\mathfrak{q}$-derivative of $\chi$ and $l(\mathfrak{s})$ exists $\forall\ \mathfrak{s} \in \mathfrak{B}$ if $\chi, l \in \mathfrak{B} \subset \mathbb{C}$,

(a) $\mathfrak{D}_{\mathfrak{q}}\{d_1\chi(\mathfrak{s}) + d_2 l(\mathfrak{s})\} = d_1\mathfrak{D}_{\mathfrak{q}}\chi(\mathfrak{s}) + d_2\mathfrak{D}_{\mathfrak{q}} l(\mathfrak{s})$, where $d_1$ and $d_2$ are constants.

(b) $\mathfrak{D}_{\mathfrak{q}}\{\chi(\mathfrak{s}) l(\mathfrak{s})\} = l(\mathfrak{s})\mathfrak{D}_{\mathfrak{q}}\chi(\mathfrak{s}) + \chi(\mathfrak{q}\mathfrak{s})\mathfrak{D}_{\mathfrak{q}} l(\mathfrak{s})$.

(c) $\mathfrak{D}_{\mathfrak{q}}\left[\frac{\chi(\mathfrak{s})}{l(\mathfrak{s})}\right] = \frac{l(\mathfrak{s})\mathfrak{D}_{\mathfrak{q}}\chi(\mathfrak{s}) - \chi(\mathfrak{q}\mathfrak{s})\mathfrak{D}_{\mathfrak{q}} l(\mathfrak{s})}{(\mathfrak{q}\mathfrak{s}) l^2(\mathfrak{s})}$.

In 2016, Ademgullari et al.[33],in 2016, shown that, for $\chi \in \mathfrak{A}$

$$\mathfrak{D}_{\mathfrak{q}}(\log\chi(\mathfrak{s})) = \frac{\mathfrak{D}_{\mathfrak{q}}\chi(\mathfrak{s})}{\chi(\mathfrak{s})},\ \mathfrak{s} \in \mathfrak{D}$$

**Definition 2.5.3.** [32] Jackson integral of the function $\chi(\mathfrak{s})$ also known as $q$-integral, which is defined as:

$$\int_0^{\mathfrak{s}} \chi(\mathfrak{s})\,\mathfrak{D}_{\mathfrak{q}}\mathfrak{s} = \mathfrak{s}(1-\mathfrak{q})\sum_{\underline{\mathfrak{n}}=0}^{\infty} \mathfrak{q}^{\underline{\mathfrak{n}}}\chi(\mathfrak{q}^{\underline{\mathfrak{n}}}\mathfrak{s}).$$

assuming the series converges.

The class $\tilde{C}_{\mathfrak{q}}$ containing $\mathfrak{q}$-analogue of convex function was defind in 1989 by Srivastava and Owa [34] as:

**Definition 2.5.4.** Let $\chi \in \mathfrak{A}$. Then $\chi \in \tilde{C}_{\mathfrak{q}}$ if

$$\left|\frac{\mathfrak{s}\mathfrak{D}_{\mathfrak{q}}^2\chi(\mathfrak{s})}{\mathfrak{D}_{\mathfrak{q}}\chi(\mathfrak{s})} - \frac{1}{1-\mathfrak{q}}\right| \leq \frac{1}{1-\mathfrak{q}}, \mathfrak{s} \in \mathfrak{D},\ 0 < \mathfrak{q} < 1. \tag{2.19}$$

The well known class $\tilde{C}$ will be obtained when $\mathfrak{q} \to 1^-$. Ezeafulukweetal. [35] proved that

$$\tilde{C} = \bigcap_{0<\mathfrak{q}<1} \tilde{C}_{\mathfrak{q}}$$

**Definition 2.5.5.** [5] Let $\chi \in \mathfrak{A}$. Then $\chi \in \tilde{C}_{\mathfrak{q}}$ if

$$\left|\frac{\mathfrak{s}\mathfrak{D}_{\mathfrak{q}}\chi(\mathfrak{s})}{\chi(\mathfrak{s})} - \frac{1}{1-\mathfrak{q}}\right| \leq \frac{1}{1-\mathfrak{q}},\ \mathfrak{s} \in \mathfrak{D},\ 0 < \mathfrak{q} < 1. \tag{2.20}$$

The well known class $\mathfrak{S}$ will be obtained when $\mathfrak{q} \to 1^{-}$. In [5], it was proved that

$$\mathfrak{S} = \bigcap_{0<\mathfrak{q}<1} \mathfrak{S}_{\mathfrak{q}}$$

The class $\mathfrak{S}_{\mathfrak{q}}(\gamma)$, which contains $\mathfrak{q}$-starlike of order $\gamma$ was studied by Agrawal and Sahoo [36] in 2017.

**Definition 2.5.6.** [36] Assume $\chi \in \mathfrak{A}$. Then $\chi \in \mathfrak{S}_{\mathfrak{q}}(\gamma)$, $0 \leq \gamma < 1$, if and only if

$$\left| \frac{\mathfrak{s}\mathfrak{D}_{\mathfrak{q}}\chi(\mathfrak{s})}{\chi(\mathfrak{s})} - \frac{1-\gamma\mathfrak{q}}{1-\mathfrak{q}} \right| \leq \frac{1-\gamma}{1-\mathfrak{q}}, \ \mathfrak{s} \in \mathfrak{D}, \ 0 < \mathfrak{q} < 1, \ 0 \leq \gamma < 1. \tag{2.21}$$

It is also observed that $\mathfrak{S}_{\mathfrak{q}}(0) = \mathfrak{S}_{\mathfrak{q}}$.

# CHAPTER 3

# ON CLASSES OF STRONGLY JANOWSKI TYPE FUNCTIONS OF COMPLEX ORDER

## 3.1 Overview

In this chapter, starlike and convex classes of Janowski type function of complex order are investigated. Intriguing characteristics and properties like inclusion property, distortion theorems,radius of convexity of a function and coefficient bounds have also benn studied here.Additionally, analytic features of these classes involving specific integral operators are examined.

## 3.2 Introduction

I will introduce the following classes

**Definition 3.2.1.** [36] For real numbers $\xi$ and $\varphi$ with $-1 \leq \varphi < \xi \leq 1$ and $0 < \alpha \leq 1$, the class $\mathfrak{P}^{\alpha}[\xi, \varphi]$ consists of analytic functions $\phi(\mathfrak{s})$ such that:

$$\phi(\mathfrak{s}) \prec \left[\frac{1+\xi\mathfrak{s}}{1+\varphi\mathfrak{s}}\right]^{\alpha}, \quad \mathfrak{s} \in \mathfrak{D}, \tag{3.1}$$

A function $\phi(\mathfrak{s})$ belongs to the class $\mathfrak{P}^{\alpha}[\xi, \varphi]$ if it satisfies the given conditions, $\phi(0) = 1$ and $Re\phi(\mathfrak{s}) > 0$.

## Special Case

When $\varphi = -1$, the image of $\phi(\mathfrak{s})$ satisfies the following half-plane condition:

$$Re\omega > \frac{1-\xi}{2}.$$

This restricts the image of $\phi(\mathfrak{s})$ to the right half-plane defined by the given real part.

**Definition 3.2.2.** [36] Let $\chi(\mathfrak{s}) \in \mathfrak{A}$. These functions are convex and satisfy the following subordination condition:

$$\frac{(\mathfrak{s}\chi'(\mathfrak{s}))'}{\chi'(\mathfrak{s})} \prec \frac{1+\xi\mathfrak{s}}{1+\varphi\mathfrak{s}}, \tag{3.2}$$

where $\xi$ and $\varphi$ are real constants and for $-1 \leq \varphi < \xi \leq 1$ and $0 < \alpha \leq 1$ the Janowski convex class of order $\alpha$, represented by $\tilde{C}^{\alpha}[\xi,\varphi]$ is defined as:

$$\tilde{C}^{\alpha}[\xi,\varphi] = \left\{ f \in \mathfrak{A} : \frac{[\mathfrak{s}\chi'(\mathfrak{s})]'}{\chi'(\mathfrak{s})} \prec \left[\frac{1+\xi\mathfrak{s}}{1+\varphi\mathfrak{s}}\right]^{\alpha} \right\}. \tag{3.3}$$

**Definition 3.2.3.** [36] Assume $\chi(\mathfrak{s}) \in \mathfrak{A}$. These functions are starlike and satisfy the following subordination condition:

$$\frac{\mathfrak{s}\chi'(\mathfrak{s})}{\chi(\mathfrak{s})} \prec \frac{1+\xi\mathfrak{s}}{1+\varphi\mathfrak{s}},$$

where $\xi$ and $\varphi$ are real constants and for $-1 \leq \varphi < \xi \leq 1$ and $0 < \alpha \leq 1$ the Janowski starlike class $\mathfrak{S}^{\alpha}[\xi,\varphi]$ is defined as:

$$\mathfrak{S}^{\alpha}[\xi,\varphi] = \left\{ \chi \in \mathfrak{A} : \frac{\mathfrak{s}\chi'(\mathfrak{s})}{\chi(\mathfrak{s})} \prec \left[\frac{1+\xi\mathfrak{s}}{1+\varphi\mathfrak{s}}\right]^{\alpha} \right\} \tag{3.4}$$

**Definition 3.2.4.** [37] Let $\chi(\mathfrak{s}) \in \mathfrak{A}$, then $\chi(\mathfrak{s})$ belongs to the class $\mathfrak{S}_{\mathfrak{c}}^{\alpha}[\xi,\varphi]$ if and only if $\frac{\chi(\mathfrak{s})}{\mathfrak{s}} \neq 0$ and $1+\frac{1}{\mathfrak{c}}\left[\frac{\mathfrak{s}\chi'(\mathfrak{s})}{\chi(\mathfrak{s})}-1\right] \in \mathfrak{P}^{\alpha}[\xi,\varphi]$, (for complex number $\mathfrak{c} \neq 0$).

Alternatively it can be defined as:

**Definition 3.2.5.** [37] A function $\chi(\mathfrak{s}) \in \mathfrak{A}$ belongs to the class $\mathfrak{S}_{\mathfrak{c}}^{\alpha}[\xi,\varphi]$ if and only if:

$$1+\frac{1}{\mathfrak{c}}\left[\frac{\mathfrak{s}\chi'(\mathfrak{s})}{\chi(\mathfrak{s})}-1\right] \prec \left[\frac{1+\xi\mathfrak{s}}{1+\varphi\mathfrak{s}}\right]^{\alpha}. \tag{3.5}$$

### Special Cases

I. As $\alpha = 1$: $\mathfrak{S}_{\mathfrak{c}}^{\alpha}[\xi,\varphi]$ becomes $\mathfrak{S}_{\mathfrak{c}}[\xi,\varphi]$.

II. As $\xi = 1$ and $\varphi = -1$: $\mathfrak{S}_{\mathfrak{c}}^{\alpha}[\xi,\varphi]$ becomes $\mathfrak{S}_{\mathfrak{c}}^{\alpha}[1,-1]$.

III. For $\alpha = \frac{1}{2}$, $\xi = c$, and $\varphi = 0$: $\mathfrak{S}_{\mathfrak{c}}^{\alpha}[\xi,\varphi]$ becomes $\mathfrak{S}^{\frac{1}{2}}(c)$.

IV. For $\mathfrak{c} = 1$: $\mathfrak{S}_{\mathfrak{c}}^{\alpha}[\xi,\varphi]$ becomes $\mathfrak{S}^{\alpha}[\xi,\varphi]$.

**Definition 3.2.6.** [37] A function $\chi(\mathfrak{s}) \in \mathfrak{A}$ belongs to $\tilde{C}_{\mathfrak{c}}^{\alpha}[\xi,\varphi]$ if and only if:

$$1 + \frac{1}{\mathfrak{c}}\left[\frac{\mathfrak{s}\chi''(\mathfrak{s})}{\chi'(\mathfrak{s})} - 1\right] \prec \left[\frac{1+\xi\mathfrak{s}}{1+\varphi\mathfrak{s}}\right]^{\alpha} \tag{3.6}$$

**Special Cases**

I. For $\alpha = 1$: $\tilde{C}_{\mathfrak{c}}^{\alpha}[\xi,\varphi]$ becomes $C_{\mathfrak{c}}[\xi,\varphi]$.

II. For $\xi = 1$ and $\varphi = -1$: $\tilde{C}_{\mathfrak{c}}^{\alpha}[\xi,\varphi]$ becomes $\tilde{C}_{\mathfrak{c}}(\alpha)$.

III. For $\mathfrak{c} = 1$: $\tilde{C}_{\mathfrak{c}}^{\alpha}[\xi,\varphi]$ becomes $\tilde{C}^{\alpha}[\xi,\varphi]$.

**Lemma 3.2.1.** Suppose $\phi(\mathfrak{s}) = 1 + \sum_{\underline{\mathfrak{n}}=1}^{\infty} \mathfrak{c}_{\underline{\mathfrak{n}}}\mathfrak{s}^{\underline{\mathfrak{n}}}$ is in $\mathfrak{P}^{\alpha}[\xi,\varphi]$, for $-1 \le \varphi < \xi \le 1$, with $0 < \alpha \le 1$. Then

$$|\mathfrak{c}_{\underline{\mathfrak{n}}}| \le \alpha|\xi - \varphi|, \quad \text{for all } \underline{\mathfrak{n}} \ge 1.$$

The following lemma consists of distortion results for the functions belonging to the class $\mathfrak{P}^{\alpha}[\xi,\varphi]$. Some existing results of well-known classes are special cases of this result.

**Lemma 3.2.2.** Consider $\phi(\mathfrak{s})$ belonging to $\mathfrak{P}^{\alpha}[\xi,\varphi]$, for $-1 \le \varphi < \xi \le 1$, with $0 < \alpha \le 1$, and $\mathfrak{s} = \mathrm{r_u}e^{i\theta}$. Then

$$\left[\frac{1-\xi\mathfrak{s}}{1-\varphi\mathfrak{s}}\right]^{\alpha} \le \operatorname{Re}\phi(\mathfrak{s}) \le |\phi(\mathfrak{s})| \le \left[\frac{1+\xi\mathfrak{s}}{1+\varphi\mathfrak{s}}\right]^{\alpha}.$$

## 3.3 Main Result

**Theorem 3.3.1.** Suppose $\chi(\mathfrak{s}) \in \tilde{C}_{\mathfrak{c}}^{\alpha}[\xi,\varphi]$. Then $\chi(\mathfrak{s}) \in C_{\mathfrak{c}}(\gamma)$, where $\gamma = \left[\frac{1-\xi}{1-\varphi}\right]^{\alpha}$ for $-1 \le \varphi < \xi \le 1$, $0 < \alpha \le 1$, and $\mathfrak{c} \neq 0$.

*Proof.* Suppose $\chi(\mathfrak{s}) \in \tilde{C}_{\mathfrak{c}}^{\alpha}[\xi,\varphi]$, so by definition

$$1 + \frac{1}{\mathfrak{c}}\frac{\mathfrak{s}\chi''(\mathfrak{s})}{\chi'(\mathfrak{s})} \in \mathfrak{P}^{\alpha}[\xi,\varphi]$$

If $\phi(\mathfrak{s}) \in \mathfrak{P}^{\alpha}[\xi,\varphi]$, then by definition of $\mathfrak{P}^{\alpha}[\xi,\varphi]$ we have:

$$\phi(\mathfrak{s}) \prec \left[\frac{1+\xi\mathfrak{s}}{1+\varphi\mathfrak{s}}\right]^{\alpha} \tag{3.7}$$

This implies that there exists an analytic function $\omega(\mathrm{r_u})$ with $\omega(0)=0$ and $|\omega(\mathfrak{s})|<1$, such that:

$$\phi(\mathfrak{s}) \prec \left[\frac{1+\xi\omega(\mathfrak{s})}{1+\varphi\omega(\mathfrak{s})}\right]^{\alpha} \tag{3.8}$$

$$\operatorname{Re}\phi(\mathfrak{s}) \prec \operatorname{Re}\left[\frac{1+\xi\omega(\mathfrak{s})}{1+\varphi\omega(\mathfrak{s})}\right]^{\alpha}.$$

Using geometric inequality from starlike function

$$\operatorname{Re}\left[\frac{1+\xi\omega(\mathfrak{s})}{1+\varphi\omega(\mathfrak{s})}\right] \geq \frac{1-\xi|\omega(\mathfrak{s})|}{1-\varphi|\omega(\mathfrak{s})|},$$

$$\operatorname{Re}\left[\frac{1+\xi\omega(\mathfrak{s})}{1+\varphi\omega(\mathfrak{s})}\right]^{\alpha} \geq \left[\frac{1-\xi}{1-\varphi}\right]^{\alpha}. \tag{3.9}$$

Thus,

$$\operatorname{Re}\phi(\mathfrak{s}) \geq \left[\frac{1-\xi}{1-\varphi}\right]^{\alpha}. \tag{3.10}$$

This inequality shows that $\phi(\mathfrak{s}) \in \mathfrak{P}(\gamma)$, where $\gamma = \left[\frac{1-\xi}{1-\varphi}\right]^{\alpha}$.

Sicne $\phi(\mathfrak{s}) \in \mathfrak{P}(\gamma)$, so by definition,

$$1+\frac{1}{\mathfrak{c}}\frac{\mathfrak{s}\chi''(\mathfrak{s})}{\chi'(\mathfrak{s})} \in \mathfrak{P}(\gamma)$$

This means $\phi(\mathfrak{s}) \in b_{\mathfrak{c}}(\gamma)$.

So $\chi(\mathfrak{s}) \in \tilde{C}_{\mathfrak{c}}^{\alpha}[\xi,\varphi]$ with $\gamma = \left[\frac{1-\xi}{1-\varphi}\right]^{\alpha}$. This concludes the proof. □

**Theorem 3.3.2.** Let $\chi(\mathfrak{s}) \in \mathfrak{S}_{\mathfrak{c}}^{\alpha}[\xi,\varphi]$, then $\chi(\mathfrak{s}) \in \mathfrak{S}_{\mathfrak{c}}(\gamma)$, where $\gamma = \left[\frac{1-\xi}{1-\varphi}\right]^{\alpha}$, for $-1 \leq \varphi < \xi \leq 1$ and $0 < \alpha \leq 1$ with $\mathfrak{c} \neq 0$.

*Proof.* Suppose $\chi(\mathfrak{s}) \in \mathfrak{S}_{\mathfrak{c}}^{\alpha}[\xi,\varphi]$, then by definition

$$1+\frac{1}{\mathfrak{c}}\left[\frac{\mathfrak{s}\chi'(\mathfrak{s})}{\chi(\mathfrak{s})}-1\right] \in \mathfrak{P}^{\alpha}[\xi,\varphi].$$

If $\phi(\mathfrak{s}) \in \mathfrak{P}^{\alpha}[\xi,\varphi]$, it satisfies the subordination condition

$$\phi(\mathfrak{s}) \prec \left[\frac{1+\xi(\mathfrak{s})}{1+\varphi(\mathfrak{s})}\right]^{\alpha}.$$

This implies that there exists an analytic function $\omega(\mathrm{r_u})$ with $\omega(0)=0$ and $|\omega(\mathfrak{s})|<1$, such that:

$$\phi(\mathfrak{s}) \prec \left[\frac{1+\xi\,\omega(\mathfrak{s})}{1+\varphi\omega(\mathfrak{s})}\right]^{\alpha} \tag{3.11}$$

$$\operatorname{Re}\phi(\mathfrak{s}) \prec \operatorname{Re}\left[\frac{1+\xi\,\omega(\mathfrak{s})}{1+\varphi\omega(\mathfrak{s})}\right]^{\alpha}.$$

Using geometric inequality from starlike function

$$\operatorname{Re}\left[\frac{1+\xi\,\omega(\mathfrak{s})}{1+\varphi\omega(\mathfrak{s})}\right] \geq \frac{1-\xi|\omega(\mathfrak{s})|}{1-\varphi|\omega(\mathfrak{s})|},$$

$$\operatorname{Re}\left[\frac{1+\xi\,\omega(\mathfrak{s})}{1+\varphi\omega(\mathfrak{s})}\right]^{\alpha} \geq \left[\frac{1-\xi}{1-\varphi}\right]^{\alpha}. \tag{3.12}$$

Thus,

$$\operatorname{Re}\phi(\mathfrak{s}) \geq \left[\frac{1-\xi}{1-\varphi}\right]^{\alpha}. \tag{3.13}$$

This inequality shows that $\phi(\mathfrak{s}) \in \mathfrak{P}(\gamma)$, where $\gamma = \left[\frac{1-\xi}{1-\varphi}\right]^{\alpha}$. Since $\phi(\mathfrak{s}) \in \phi(\gamma)$, so by definition,

$$1+\frac{1}{\mathfrak{c}}\left[\frac{\mathfrak{s}\chi'(\mathfrak{s})}{\chi(\mathfrak{s})}\right] \in \mathfrak{P}(\gamma).$$

So $\chi(\mathfrak{s}) \in \mathfrak{S}_{\mathfrak{c}}^{\alpha}[\xi,\varphi]$ with $\gamma = \left[\frac{1-\xi}{1-\varphi}\right]^{\alpha}$. This concludes the proof.

□

**Theorem 3.3.3.** If $\chi(\mathfrak{s}) \in \mathfrak{S}_{\mathfrak{c}}^{\alpha}[\xi,\varphi]$, then $\chi(\mathfrak{s}) \in \mathfrak{S}_{c,q}(\gamma)$, where $\gamma = \left[\frac{1-\xi}{1-\varphi}\right]^{\alpha}$, for $-1 \leq \varphi < \xi \leq 1$ and $0<\alpha \leq 1$ with $\mathfrak{c} \neq 0$.

*Proof.* Suppose $\chi(\mathfrak{s}) \in \mathfrak{S}_{\mathfrak{c}}^{\alpha}[\xi,\varphi]$, then by definition

$$1+\frac{1}{\mathfrak{c}}\left[\frac{\mathfrak{s}\chi'(\mathfrak{s})}{\chi(\mathfrak{s})}-1\right] \in \mathfrak{P}^{\alpha}[\xi,\varphi].$$

If $\phi(\mathfrak{s}) \in \mathfrak{P}^{\alpha}[\xi,\varphi]$, it satisfies the subordination condition

$$\phi(\mathfrak{s}) \prec \left[\frac{1+\xi(\mathfrak{s})}{1+\varphi(\mathfrak{s})}\right]^{\alpha}.$$

This implies that there exists an analytic function $\omega(\mathrm{r_u})$ with $\omega(0)=0$ and $|\omega(\mathfrak{s})|<1$, such that:

$$\phi(\mathfrak{s}) \prec \left[\frac{1+\xi\,\omega(\mathfrak{s})}{1+\varphi\omega(\mathfrak{s})}\right]^{\alpha}, \tag{3.14}$$

$$\operatorname{Re}\phi(\mathfrak{s}) \prec \operatorname{Re}\left[\frac{1+\xi\,\omega(\mathfrak{s})}{1+\varphi\omega(\mathfrak{s})}\right]^{\alpha}.$$

Using geometric inequality from starlike function

$$\operatorname{Re}\left[\frac{1+\xi\omega(\mathfrak{s})}{1+\varphi\omega(\mathfrak{s})}\right] \geq \frac{1-\xi|\omega(\mathfrak{s})|}{1-\varphi|\omega(\mathfrak{s})|},$$

$$\operatorname{Re}\left[\frac{1+\xi\omega(\mathfrak{s})}{1+\varphi\omega(\mathfrak{s})}\right]^{\alpha} \geq \left[\frac{1-\xi}{1-\varphi}\right]^{\alpha}. \tag{3.15}$$

Thus,

$$\operatorname{Re}\phi(\mathfrak{s}) \geq \left[\frac{1-\xi}{1-\varphi}\right]^{\alpha}. \tag{3.16}$$

This inequality shows that $\phi(\mathfrak{s}) \in \mathfrak{P}(\gamma)$, where $\gamma = \left[\frac{1-\xi}{1-\varphi}\right]^{\alpha}$. Since $\phi(\mathfrak{s}) \in \phi(\gamma)$, so by definition,

$$1+\frac{1}{\mathfrak{c}}\left[\frac{\mathfrak{s}\chi'(\mathfrak{s})}{\chi(\mathfrak{s})}\right] \in \mathfrak{P}(\gamma).$$

So, $\chi(\mathfrak{s}) \in \mathfrak{S}_{\mathfrak{c}}^{\alpha}[\xi,\varphi]$ with $\gamma = \left[\frac{1-\xi}{1-\varphi}\right]^{\alpha}$. This concludes the proof. □

**Theorem 3.3.4.** If $\chi(\mathfrak{s}) \in \mathfrak{S}_{\mathfrak{c}}^{\alpha}[\xi,\varphi]$ then with $\chi(\mathfrak{s}) = \mathfrak{s}+\sum_{\underline{\mathfrak{n}}=2}^{\infty} a_{\underline{\mathfrak{n}}}\mathfrak{s}^{\underline{\mathfrak{n}}}$, for $\mathfrak{s} \in \mathfrak{D}$, $-1 \leq \varphi < \xi \leq 1$, $0 < \alpha \leq 1$ and $\mathfrak{c} \neq 0$. Then for $\underline{\mathfrak{n}} \geq 2$

$$|a_{\underline{\mathfrak{n}}}| \leq \frac{(\mathfrak{c}\alpha(\xi-\varphi))_{\underline{\mathfrak{n}}-1}}{([\underline{\mathfrak{n}}]-1)!}. \tag{3.17}$$

*Proof.* Suppose $\chi(\mathfrak{s}) \in \mathfrak{S}_{\mathfrak{c}}[\xi,\varphi]$. Then by definition

$$1+\frac{1}{\mathfrak{c}}\left[\frac{\mathfrak{s}\chi'(\mathfrak{s})}{\chi(\mathfrak{s})}-1\right) \in \mathfrak{P}_{\mathfrak{c}}^{\alpha}[\xi,\varphi].$$

Consider $\phi(\mathfrak{s}) = 1+\frac{1}{\mathfrak{c}}\left[\frac{\mathfrak{s}\chi'(\mathfrak{s})}{\chi(\mathfrak{s})}-1\right) = 1+\sum_{\underline{\mathfrak{n}}=2}^{\infty} a_{\underline{\mathfrak{n}}}\mathfrak{s}^{\underline{\mathfrak{n}}}$, where $\phi(\mathfrak{s})$ is analytic in $\mathfrak{D}$ with $\phi(0)=1$.

$$\mathfrak{s}+\sum_{\underline{\mathfrak{n}}=2}^{\infty}\underline{\mathfrak{n}}a_{\underline{\mathfrak{n}}}\mathfrak{s}^{\underline{\mathfrak{n}}} = \left[\mathfrak{s}+\sum_{\underline{\mathfrak{n}}=2}^{\infty}a_{\underline{\mathfrak{n}}}\mathfrak{s}^{\underline{\mathfrak{n}}}\right]\left[1+\sum_{\underline{\mathfrak{n}}=1}^{\infty}\mathfrak{c}b_{\underline{\mathfrak{n}}}\mathfrak{s}^{\underline{\mathfrak{n}}}\right],$$

$$\mathfrak{s}+\sum_{\underline{\mathfrak{n}}=2}^{\infty}\underline{\mathfrak{n}}a_{\underline{\mathfrak{n}}}\mathfrak{s}^{\underline{\mathfrak{n}}} = \mathfrak{s}\left[1+\sum_{\underline{\mathfrak{n}}=1}^{\infty}\mathfrak{c}b_{\underline{\mathfrak{n}}}\mathfrak{s}^{\underline{\mathfrak{n}}}\right]+\left[\sum_{\underline{\mathfrak{n}}=2}^{\infty}a_{\underline{\mathfrak{n}}}\mathfrak{s}^{\underline{\mathfrak{n}}}\right]\left[1+\sum_{\underline{\mathfrak{n}}=1}^{\infty}\mathfrak{c}b_{\underline{\mathfrak{n}}}\mathfrak{s}^{\underline{\mathfrak{n}}}\right],$$

$$\mathfrak{s}+\sum_{\underline{\mathfrak{n}}=2}^{\infty}\underline{\mathfrak{n}}a_{\underline{\mathfrak{n}}}\mathfrak{s}^{\underline{\mathfrak{n}}} = \mathfrak{s}+\sum_{\underline{\mathfrak{n}}=2}^{\infty}\mathfrak{c}b_{\underline{\mathfrak{n}}-1}\mathfrak{s}^{\underline{\mathfrak{n}}}+\sum_{\underline{\mathfrak{n}}=2}^{\infty}a_{\underline{\mathfrak{n}}}\mathfrak{s}^{\underline{\mathfrak{n}}}+\left[\sum_{\underline{\mathfrak{n}}=2}^{\infty}a_{\underline{\mathfrak{n}}}\mathfrak{s}^{\underline{\mathfrak{n}}}\right]\cdot\left[\sum_{\underline{\mathfrak{n}}=1}^{\infty}\mathfrak{c}b_{\underline{\mathfrak{n}}}\mathfrak{s}^{\underline{\mathfrak{n}}}\right],$$

$$\sum_{\underline{\mathfrak{n}}=2}^{\infty}\underline{\mathfrak{n}}a_{\underline{\mathfrak{n}}}\mathfrak{s}^{\underline{\mathfrak{n}}} = \sum_{\underline{\mathfrak{n}}=2}^{\infty}\mathfrak{c}b_{\underline{\mathfrak{n}}-1}\mathfrak{s}^{\underline{\mathfrak{n}}}+\sum_{\underline{\mathfrak{n}}=2}^{\infty}a_{\underline{\mathfrak{n}}}\mathfrak{s}^{\underline{\mathfrak{n}}}+\left[\sum_{\underline{\mathfrak{n}}=2}^{\infty}a_{\underline{\mathfrak{n}}}\mathfrak{s}^{\underline{\mathfrak{n}}}\right]\cdot\left[\sum_{\underline{\mathfrak{n}}=1}^{\infty}\mathfrak{c}b_{\underline{\mathfrak{n}}}\mathfrak{s}^{\underline{\mathfrak{n}}}\right],$$

by comparing coefficients of $\mathfrak{s}$

$$\underline{\mathfrak{n}}a_{\underline{\mathfrak{n}}} = a_{\underline{\mathfrak{n}}}+\mathfrak{c}b_{\underline{\mathfrak{n}}-1}+\sum_{\underline{\mathfrak{n}}=2}^{\infty}\sum_{i=0}^{\underline{\mathfrak{n}}-1}\mathfrak{c}b_i a_{\underline{\mathfrak{n}}-i},$$

$$\underline{\mathfrak{n}}a_{\underline{\mathfrak{n}}} - a_{\underline{\mathfrak{n}}} = \mathfrak{c}b_{\underline{\mathfrak{n}}-1} + \sum_{\underline{\mathfrak{n}}=2}^{\infty}\sum_{i=0}^{\underline{\mathfrak{n}}-1} \mathfrak{c}b_i a_{\underline{\mathfrak{n}}-i},$$

$$(\underline{\mathfrak{n}}-1)a_{\underline{\mathfrak{n}}} = \sum_{\underline{\mathfrak{n}}=2}^{\infty}\sum_{i=0}^{\underline{\mathfrak{n}}-1} \mathfrak{c}b_i a_{\underline{\mathfrak{n}}-i} \quad , for\ b_0 = 1$$

$$\xi_{\underline{\mathfrak{n}}} = \frac{1}{\underline{\mathfrak{n}}-1}\sum_{i=1}^{\underline{\mathfrak{n}}-1} \mathfrak{c}b_i a_{\underline{\mathfrak{n}}-i},$$

$$|a_{\underline{\mathfrak{n}}}| = \frac{1}{\underline{\mathfrak{n}}-1}\sum_{i=1}^{\underline{\mathfrak{n}}-1} |\mathfrak{c}||b_i||a_{\underline{\mathfrak{n}}-i}|. \tag{3.18}$$

As $\phi(\mathfrak{s}) \in \mathfrak{P}_{\mathfrak{c}}^{\alpha}[\xi, \varphi]$, so by using Lemma, we have $|b_i| \leq \mathfrak{c}\alpha(\xi, \varphi)$, for $n \geq 1$.

So, by using it in (3.18)

$$|a_{\underline{\mathfrak{n}}}| \leq \frac{\mathfrak{c}\alpha(\xi, \varphi)}{\underline{\mathfrak{n}}-1}\sum_{i=1}^{\underline{\mathfrak{n}}-1} |\mathfrak{c}||a_{\underline{\mathfrak{n}}-i}|,$$

$$|a_2| \leq \frac{\mathfrak{c}\alpha(\xi-\varphi)}{[2]-1}, \qquad for\,\underline{\mathfrak{n}} = 2$$

$$|a_2| \leq \mathfrak{c}\alpha(\xi-\varphi),$$

$$|a_3| \leq \frac{\mathfrak{c}\alpha(A-B)}{2}\sum_{i=1}^{2} |a_i| \quad , for\ \underline{\mathfrak{n}} = 3. \tag{3.19}$$

Expanding $|a_i|$ and substituting values of $|a_1|$ and $|a_2|$,

$$\sum_{i=1}^{2} |a_i| = |a_1| + |a_2| \leq 1 + \mathfrak{c}\alpha(\xi-\varphi).$$

Using values of $|a_1|$ $|a_2|$ and $|a_i|$ in (3.19) we get,

$$|a_3| \leq \frac{\mathfrak{c}\alpha(\xi-\varphi)}{2}\left[1 + \mathfrak{c}\alpha(\xi-\varphi)\right].$$

$$|a_3| \leq \frac{[1 + \mathfrak{c}\alpha(\xi-\varphi)]_2}{2}.$$

For $\underline{\mathfrak{n}} = k$,

$$|a_k| \leq \frac{\mathfrak{c}\alpha(\xi-\varphi)}{k-1}\Pi_{i=1}^{k-1}|a_i|.$$

As $\sum_{i=1}^{k-1}|a_i| \leq \Pi_{j=1}^{k-2}\left[\frac{(\mathfrak{c}\alpha(\xi-\varphi))}{j}+1\right]$,

$$|a_k| \leq \frac{\mathfrak{c}\alpha(\xi-\varphi)}{k-1}\Pi_{j=1}^{k-2}\left[\frac{(\mathfrak{c}\alpha(\xi-\varphi)}{j}+1\right] = \frac{[\mathfrak{c}\alpha(\xi-\varphi)]_{k-1}}{(k-1)!},$$

$$|a_k| \leq \frac{(\mathfrak{c}\alpha(\xi-\varphi))_{k-1}}{(k-1)!} \quad , for\ \underline{\mathfrak{n}} \geq 3.$$

Now for $\underline{\mathfrak{n}} = k+1$

$$|a_k| \leq \frac{\mathfrak{c}\alpha(\xi-\varphi}{k}\Pi_{j=1}^{k-2}\left[\frac{(\mathfrak{c}\alpha(\xi-\varphi)}{j}+1\right] + \frac{(\mathfrak{c}\alpha(\xi-\varphi))^2}{(k)(k-1)}\Pi_{j=1}^{k-2}\left[\frac{\mathfrak{c}\alpha(\xi-\varphi)}{j}+1\right],$$

$$= \frac{\mathfrak{c}\alpha(\xi-\varphi)}{k}\Pi_{j=1}^{k-1}\left[\frac{(\mathfrak{c}\alpha(\xi-\varphi)}{j}+1\right] = \frac{(\mathfrak{c}\alpha(\xi-\varphi))_k}{(k)!}.$$

Using Induction, we will get (3.17) and hence the proof is complete. □

**Theorem 3.3.5.** If $\chi(\mathfrak{s}) \in \tilde{C}_{\mathfrak{c}}^{\alpha}[\xi,\varphi]$ then with $\chi(\mathfrak{s}) = \mathfrak{s} + \sum_{\underline{\mathfrak{n}}=2}^{\infty} a_{\underline{\mathfrak{n}}}\mathfrak{s}^{\underline{\mathfrak{n}}}$, for $\mathfrak{s} \in \mathfrak{D}$, $-1 \leq \varphi < \xi \leq 1$, $0 < \alpha \leq 1$ and $\mathfrak{c} \neq 0$. Then for $\underline{\mathfrak{n}} \geq 2$,

$$|a_{\underline{\mathfrak{n}}}| \leq \frac{(\mathfrak{c}\alpha(\xi-\varphi))_{\underline{\mathfrak{n}}-1}}{\underline{\mathfrak{n}}!}. \tag{3.20}$$

*Proof.* Since $\chi(\mathfrak{s}) \in \tilde{C}_{\mathfrak{c}}^{\alpha}[\xi,\varphi]$. Then by definition

$$1 + \frac{1}{\mathfrak{c}}\frac{\mathfrak{s}\chi''(\mathfrak{s})}{\chi'(\mathfrak{s})} \in \mathfrak{P}^{\alpha}[\xi,\varphi]$$

suppose $l(\mathfrak{s}) = \mathfrak{s}\chi'(\mathfrak{s})$.

If $\chi(\mathfrak{s}) = \mathfrak{s} + \sum_{\underline{\mathfrak{n}}=2}^{\infty} a_{\underline{\mathfrak{n}}}\mathfrak{s}^{\underline{\mathfrak{n}}}$ then $l(\mathfrak{s} = \mathfrak{s}\chi'(\mathfrak{s})) = \mathfrak{s}(1+\sum_{\underline{\mathfrak{n}}=2}^{\infty}\underline{\mathfrak{n}}a_{\underline{\mathfrak{n}}}\mathfrak{s}^{\underline{\mathfrak{n}}-1}) = \mathfrak{s} + \sum_{\underline{\mathfrak{n}}=2}^{\infty}\underline{\mathfrak{n}}a_{\underline{\mathfrak{n}}}\mathfrak{s}^{\underline{\mathfrak{n}}}$.

Then by theorem 3.3.3, the coefficients of $l(\mathfrak{s})$ (which are $\underline{\mathfrak{n}}a_{\underline{\mathfrak{n}}}$ ) must satisfy

$$|\underline{\mathfrak{n}}a_{\underline{\mathfrak{n}}}| \leq \frac{(\mathfrak{c}\alpha(\xi-\varphi))_{\underline{\mathfrak{n}}-1}}{(\underline{\mathfrak{n}}-1)!} \quad for\ n \geq 2, \tag{3.21}$$

$$|a_{\underline{\mathfrak{n}}}|\cdot|n| \leq \frac{(\mathfrak{c}\alpha(\xi-\varphi))_{\underline{\mathfrak{n}}-1}}{(\underline{\mathfrak{n}}-1)!}.$$

Dividing both sides by $[\underline{\mathfrak{n}}]_q$we will get

$$|a_{\underline{\mathfrak{n}}}| \leq \frac{(\mathfrak{c}\alpha(\xi-\varphi))_{\underline{\mathfrak{n}}-1}}{(\underline{\mathfrak{n}})!} \quad , for\ n \geq 2.$$

Hence the proof is complete. □

**Theorem 3.3.6.** Suppose $\chi, l \in \tilde{C}_{\mathfrak{c}}^{\alpha}[\xi,\varphi]$ and $\mathscr{H}(\mathfrak{s}) = \int_0^{\mathfrak{s}} [\chi'(t)]^{\alpha} [l'(t)]^{\gamma}\, dt$, with $\beta+\gamma = 1$. Then $\mathscr{H}(\mathfrak{s}) \in \tilde{C}_{\mathfrak{c}}^{\alpha}[\xi,\varphi]$.

*Proof.* Since $\mathscr{H}(\mathfrak{s}) = \int_0^{\mathfrak{s}} [\chi'(t)]^{\beta} [l'(t)]^{\gamma}\, dt$.

Taking derivative on both sides, we will get

$$\mathscr{H}'(\mathfrak{s}) = (\chi'(\mathfrak{s}))^{\beta}(l'(\mathfrak{s}))^{\gamma}.$$

Taking natural log on both sides,

$$\ln \mathscr{H}'(\mathfrak{s}) = \ln\big(\chi'(\mathfrak{s})\big)^{\beta}(l'(\mathfrak{s}))^{\gamma},$$

$$\ln \mathscr{H}'(\mathfrak{s}) = \ln\big(\chi'(\mathfrak{s})\big)^{\beta} + \ln\big(l'(\mathfrak{s})\big)^{\gamma},$$

$$\ln \mathscr{H}'(\mathfrak{s}) = \beta \ln\big(\chi'(\mathfrak{s})\big) + \gamma \ln\big(l'(\mathfrak{s})\big).$$

Applying logarithmic differentiation, we will get

$$\left[\frac{\mathscr{H}''(\mathfrak{s})}{\mathscr{H}'(\mathfrak{s})}\right] = \beta\left[\frac{\chi''(\mathfrak{s})}{\chi'(\mathfrak{s})}\right] + \gamma\left[\frac{l''(\mathfrak{s})}{l'(\mathfrak{s})}\right]. \tag{3.22}$$

As we know since $\chi, l \in \tilde{C}_{\mathfrak{c}}^{\alpha}[\xi,\varphi]$, there exists functions $\mathfrak{p}_1, \mathfrak{p}_2 \in \mathfrak{P}_{\mathfrak{c}}^{\alpha}[\xi,\varphi]$ such that

$$\mathfrak{p}_1 = 1 + \frac{1}{\mathfrak{c}}\frac{\mathfrak{s}\chi''(\mathfrak{s})}{\chi'(\mathfrak{s})},$$

$$\mathfrak{p}_2 = 1 + \frac{1}{\mathfrak{c}}\frac{\mathfrak{s}l''(\mathfrak{s})}{l'(\mathfrak{s})}.$$

Multiply both sides of equation 3.22 with $\frac{\mathfrak{s}}{\mathfrak{c}}$

$$\frac{\mathfrak{s}}{\mathfrak{c}}\frac{\mathscr{H}''(\mathfrak{s})}{\mathscr{H}'(\mathfrak{s})} = \frac{\mathfrak{s}}{\mathfrak{c}}\frac{\beta\chi''(\mathfrak{s})}{\chi'(\mathfrak{s})} + \frac{\mathfrak{s}}{\mathfrak{c}}\frac{\gamma l''(\mathfrak{s})}{l'(\mathfrak{s})}.$$

Adding 1 on both sides

$$1 + \frac{\mathfrak{s}}{\mathfrak{c}}\frac{D_q^2\mathscr{H}(\mathfrak{s})}{D_q\mathscr{H}(\mathfrak{s})} = 1 + \frac{\mathfrak{s}}{\mathfrak{c}}\frac{\beta D_q^2\chi(\mathfrak{s})}{D_q\chi(\mathfrak{s})} + \frac{\mathfrak{s}}{\mathfrak{c}}\frac{\gamma D_q^2 l(\mathfrak{s})}{l'(\mathfrak{s})}.$$

As $1 = \beta+\gamma$,

$$1 + \frac{\mathfrak{s}}{\mathfrak{c}}\frac{\mathscr{H}''(\mathfrak{s})}{\mathscr{H}'(\mathfrak{s})} = \beta+\gamma + \frac{\mathfrak{s}}{\mathfrak{c}}\frac{\beta\chi''(\mathfrak{s})}{\chi'(\mathfrak{s})} + \frac{\mathfrak{s}}{\mathfrak{c}}\frac{\gamma l''(\mathfrak{s})}{l'(\mathfrak{s})},$$

$$1 + \frac{\mathfrak{s}}{\mathfrak{c}}\frac{\mathscr{H}''(\mathfrak{s})}{\mathscr{H}'(\mathfrak{s})} = \beta + \left[1 + \frac{\mathfrak{s}}{\mathfrak{c}}\frac{\chi''(\mathfrak{s})}{\chi'(\mathfrak{s})}\right] + \gamma\left[1 + \frac{\mathfrak{s}}{\mathfrak{c}}\frac{l''(\mathfrak{s})}{l'(\mathfrak{s})}\right],$$

$$1 + \frac{\mathfrak{s}}{\mathfrak{c}}\frac{\mathscr{H}''(\mathfrak{s})}{\mathscr{H}'(\mathfrak{s})} = \beta\mathfrak{p}_1 + \gamma\mathfrak{p}_2.$$

Since $\mathfrak{p}_1, \mathfrak{p}_2 \in \mathfrak{P}_{\mathfrak{c}}^{\beta}[\xi, \varphi]$, so $\alpha\mathfrak{p}_1 + \gamma\mathfrak{p}_2 \in \mathfrak{P}_{\mathfrak{c}}^{\beta}[\xi, \varphi]$. Thus

$$1 + \frac{\mathfrak{s}}{\mathfrak{c}} \frac{\mathscr{H}''(\mathfrak{s})}{\mathscr{H}'(\mathfrak{s})} \in \mathfrak{P}_{\mathfrak{c}}^{\alpha}[\xi, \varphi].$$

Hence $\mathscr{H}(\mathfrak{s}) \in \tilde{C}_{\mathfrak{c}}^{\alpha}[\xi, \varphi]$. □

**Theorem 3.3.7.** Suppose $\chi(\mathfrak{s}) \in \mathfrak{S}_{\mathfrak{c}}^{\alpha}[\xi, \varphi]$ with $\mathfrak{s} = \mathrm{r_u}e^{i\theta}$ and $\gamma = \left[\frac{1-\xi}{1-\varphi}\right]^{\alpha}$, then we have the following inequality

$$\frac{\mathrm{r_u}}{(1+\mathrm{r_u})^{2b(1-\gamma)}} \leq |\chi(\mathfrak{s})| \leq \frac{\mathrm{r_u}}{(1-\mathrm{r_u})^{2b(1-\gamma)}}, \tag{3.23}$$

$$\frac{1-2b(1-\gamma)\mathrm{r_u} + (2b(1-\gamma)-1)\mathrm{r_u}^2}{(1-\mathrm{r_u})(1+\mathrm{r_u})^{2b(1-\gamma)+1}} \leq |\chi(\mathfrak{s})| \leq \frac{1+2b(1-\gamma)\mathrm{r_u} + (2b(1-\gamma)-1)\mathrm{r_u}^2}{(1+\mathrm{r_u})(1-\mathrm{r_u})^{2b(1-\gamma)+1}}. \tag{3.24}$$

*Proof.* Since $\chi(\mathfrak{s}) \in \mathfrak{S}_{\mathfrak{c}}^{\alpha}[\xi, \varphi]$ then $\phi(\mathfrak{s}) = 1 + \frac{1}{\mathfrak{c}}\left[\frac{\mathfrak{s}\chi'(\mathfrak{s})}{\chi(\mathfrak{s})} - 1\right]$. Where $\phi(\mathfrak{s}) \in \mathfrak{P}_{\mathfrak{c}}^{\alpha}[\xi, \varphi] \subseteq \phi(\gamma)$, $\gamma = \left[\frac{1-\xi}{1-\varphi}\right]^{\alpha}$. As $\phi(\mathfrak{s}) \in \phi(\gamma)$ there exists $\phi \in \mathfrak{P}$ such that ,

$$\phi(\mathfrak{s}) = (1-\gamma)\mathfrak{p}_1 + \gamma,$$

$$\mathfrak{p}_1 = \frac{\phi(\mathfrak{s}) - \gamma}{(1-\gamma)}.$$

$\mathfrak{p}_1(\mathfrak{s}) \in \phi(\gamma)$ can be written as ,

$$\left|\mathfrak{p}_1 - \frac{1+\mathrm{r_u}^2}{1-\mathrm{r_u}^2}\right| \leq \frac{2\mathrm{r_u}}{1-\mathrm{r_u}^2},$$

$$\left|\frac{\phi(\mathfrak{s}) - \gamma}{(1-\gamma)} - \frac{1+\mathrm{r_u}^2}{1-\mathrm{r_u}^2}\right| \leq \frac{2\mathrm{r_u}}{1-\mathrm{r_u}^2}.$$

Multiplying by $(1-\gamma)$ on both sides

$$\left|(1-\gamma)\frac{\phi(\mathfrak{s}) - \gamma}{(1-\gamma)} - \frac{1+\mathrm{r_u}^2}{1-\mathrm{r_u}^2}(1-\gamma)\right| \leq (1-\gamma)\frac{2\mathrm{r_u}}{1-\mathrm{r_u}^2},$$

$$\left|(1-\mathrm{r_u}^2)(\phi(\mathfrak{s}) - \gamma) - \frac{(1-\gamma)(1+\mathrm{r_u}^2)}{1-\mathrm{r_u}^2}\right| \leq \frac{(1-\gamma)2\mathrm{r_u}}{1-\mathrm{r_u}^2},$$

$$\left|\frac{(1-\mathrm{r_u}^2)(\phi(\mathfrak{s}) - \gamma) - (1-\gamma)(1+\mathrm{r_u}^2)}{1-\mathrm{r_u}^2}\right| \leq \frac{(1-\gamma)2\mathrm{r_u}}{1-\mathrm{r_u}^2},$$

$$\left|\frac{\phi(\mathfrak{s})(1-\mathrm{r_u}^2) - \gamma(1-\mathrm{r_u}^2) - (1-\gamma)(1+\mathrm{r_u}^2)}{1-\mathrm{r_u}^2}\right| \leq \frac{(1-\gamma)2\mathrm{r_u}}{1-\mathrm{r_u}^2},$$

$$\left|\frac{\phi(\mathfrak{s})(1-\mathrm{r_u}^2)-\gamma+\mathrm{r_u}^2\gamma-1-\mathrm{r_u}^2+\gamma+\gamma\mathrm{r_u}^2)}{1-\mathrm{r_u}^2}\right|\leq\frac{(1-\gamma)2\mathrm{r_u}}{1-\mathrm{r_u}^2},$$

$$\left|\frac{\phi(\mathfrak{s})(1-\mathrm{r_u}^2)-1-\mathrm{r_u}^2+2\gamma\mathrm{r_u}^2)}{1-\mathrm{r_u}^2}\right|\leq\frac{(1-\gamma)2\mathrm{r_u}}{1-\mathrm{r_u}^2},$$

$$\left|\frac{\phi(\mathfrak{s})(1-\mathrm{r_u}^2)-1-\mathrm{r_u}^2(1-2\gamma)}{1-\mathrm{r_u}^2}\right|\leq\frac{(1-\gamma)2\mathrm{r_u}}{1-\mathrm{r_u}^2},$$

$$\left|\phi(\mathfrak{s})-\frac{1+\mathrm{r_u}^2(1-2\gamma)}{1-\mathrm{r_u}^2}\right|\leq\frac{(1-\gamma)2\mathrm{r_u}}{1-\mathrm{r_u}^2},$$

$$\left|\phi(\mathfrak{s})-\frac{1+(1-2\gamma)\mathrm{r_u}^2}{1-\mathrm{r_u}^2}\right|\leq\frac{(1-\gamma)2\mathrm{r_u}}{1-\mathrm{r_u}^2}. \tag{3.25}$$

Replacing $\phi(\mathfrak{s})$ with its value we will get

$$\left|1+\frac{1}{\mathfrak{c}}\left[\frac{\mathfrak{s}\chi'(\mathfrak{s})}{\chi(\mathfrak{s})}-1\right]-\frac{1+(1-2\gamma)\mathrm{r_u}^2}{1-\mathrm{r_u}^2}\right|\leq\frac{(1-\gamma)2\mathrm{r_u}}{1-\mathrm{r_u}^2}.$$

Multiplying both sides by $\mathfrak{c}$ to get

$$\left|\mathfrak{c}+\left[\frac{\mathfrak{s}\chi'(\mathfrak{s})}{\chi(\mathfrak{s})}-1\right]-\frac{\mathfrak{c}+(1-2\gamma)\mathfrak{c}\mathrm{r_u}^2}{1-\mathrm{r_u}^2}\right|\leq\frac{(1-\gamma)2\mathfrak{c}\mathrm{r_u}}{1-\mathrm{r_u}^2},$$

$$\left|\frac{\mathfrak{s}\chi'(\mathfrak{s})}{\chi(\mathfrak{s})}+\mathfrak{c}-1-\frac{\mathfrak{c}+(1-2\gamma)\mathfrak{c}\mathrm{r_u}^2}{1-\mathrm{r_u}^2}\right|\leq\frac{(1-\gamma)2\mathfrak{c}\mathrm{r_u}}{1-\mathrm{r_u}^2},$$

$$\left|\frac{\mathfrak{s}\chi'(\mathfrak{s})}{\chi(\mathfrak{s})}-\frac{+\mathfrak{c}-\mathfrak{c}\mathrm{r_u}^2-1+\mathrm{r_u}^2-\mathfrak{c}+(1-2\gamma)\mathfrak{c}\mathrm{r_u}^2}{1-\mathrm{r_u}^2}\right|\leq\frac{(1-\gamma)2\mathfrak{c}\mathrm{r_u}}{1-\mathrm{r_u}^2}.$$

Simplifying this we will get

$$\left|\frac{\mathfrak{s}\chi'(\mathfrak{s})}{\chi(\mathfrak{s})}-\frac{1+(2\mathfrak{c}(1-\gamma)-1)\mathrm{r_u}^2}{1-\mathrm{r_u}^2}\right|\leq\frac{(1-\gamma)2\mathfrak{c}\mathrm{r_u}}{1-\mathrm{r_u}^2},$$

$$-\frac{(1-\gamma)2\mathfrak{c}\mathrm{r_u}}{1-\mathrm{r_u}^2}\leq\frac{\mathfrak{s}\chi'(\mathfrak{s})}{\chi(\mathfrak{s})}-\frac{1+(2\mathfrak{c}(1-\gamma)-1)\mathrm{r_u}^2}{1-\mathrm{r_u}^2}\leq\frac{(1-\gamma)2\mathfrak{c}\mathrm{r_u}}{1-\mathrm{r_u}^2},$$

$$-\frac{(1-\gamma)2\mathfrak{c}\mathrm{r_u}}{1-\mathrm{r_u}^2}+\frac{1+(2\mathfrak{c}(1-\gamma)-1)\mathrm{r_u}^2}{1-\mathrm{r_u}^2}\leq\frac{\mathfrak{s}\chi'(\mathfrak{s})}{\chi(\mathfrak{s})}\leq\frac{(1-\gamma)2\mathfrak{c}\mathrm{r_u}}{1-\mathrm{r_u}^2}+\frac{1+(2\mathfrak{c}(1-\gamma)-1)\mathrm{r_u}^2}{1-\mathrm{r_u}^2},$$

$$\frac{1+(2\mathfrak{c}(1-\gamma)-1)\mathrm{r_u}^2-(1-\gamma)2\mathfrak{c}\mathrm{r_u}}{1-\mathrm{r_u}^2}\leq\frac{\mathfrak{s}\chi'(\mathfrak{s})}{\chi(\mathfrak{s})}\leq\frac{1+(2\mathfrak{c}(1-\gamma)-1)\mathrm{r_u}^2+(1-\gamma)2\mathfrak{c}\mathrm{r_u}}{1-\mathrm{r_u}^2}. \tag{3.26}$$

We know that

$$Re\frac{\mathfrak{s}\chi'(\mathfrak{s})}{\chi(\mathfrak{s})} = \mathrm{r_u}\frac{\partial}{\partial \mathrm{r_u}} log|\chi(\mathfrak{s})|,$$

$$\frac{1+(2\mathfrak{c}(1-\gamma)-1)\mathrm{r_u}^2-(1-\gamma)2\mathfrak{c}\mathrm{r_u}}{1-\mathrm{r_u}^2} \le \frac{\mathfrak{s}\chi'(\mathfrak{s})}{\chi(\mathfrak{s})} \le \frac{1+(2\mathfrak{c}(1-\gamma)-1)\mathrm{r_u}^2+(1-\gamma)2\mathfrak{c}\mathrm{r_u}}{1-\mathrm{r_u}^2},$$

$$\frac{1+(2\mathfrak{c}(1-\gamma)-1)\mathrm{r_u}^2-(1-\gamma)2\mathfrak{c}\mathrm{r_u}}{1-\mathrm{r_u}^2} \le \mathrm{r_u}\frac{\partial}{\partial \mathrm{r_u}} log|\chi(\mathfrak{s})| \le \frac{1+(2\mathfrak{c}(1-\gamma)-1)\mathrm{r_u}^2+(1-\gamma)2\mathfrak{c}\mathrm{r_u}}{1-\mathrm{r_u}^2},$$

$$\frac{1+(2\mathfrak{c}(1-\gamma)-1)\mathrm{r_u}^2-(1-\gamma)2\mathfrak{c}\mathrm{r_u}}{\mathrm{r_u}(1-\mathrm{r_u}^2)} \le \frac{\partial}{\partial \mathrm{r_u}} log|\chi(\mathfrak{s})| \le \frac{1+(2\mathfrak{c}(1-\gamma)-1)\mathrm{r_u}^2+(1-\gamma)2\mathfrak{c}\mathrm{r_u}}{\mathrm{r_u}(1-\mathrm{r_u}^2)}, \tag{3.27}$$

Integrate with respect to $\mathrm{r_u}$

$$\int \frac{1+(2\mathfrak{c}(1-\gamma)-1)\mathrm{r_u}^2-(1-\gamma)2\mathfrak{c}\mathrm{r_u}}{\mathrm{r_u}(1-\mathrm{r_u}^2)} d\mathrm{r_u} \le log|\chi(\mathfrak{s})| \le \int \frac{1+(2\mathfrak{c}(1-\gamma)-1)\mathrm{r_u}^2+(1-\gamma)2\mathfrak{c}\mathrm{r_u}}{\mathrm{r_u}(1-\mathrm{r_u}^2)} d\mathrm{r_u}, \tag{3.28}$$

$$\int \left[\frac{1}{\mathrm{r_u}(1-\mathrm{r_u}^2)} + \frac{(2\mathfrak{c}(1-\gamma)-1)\mathrm{r_u}^2}{\mathrm{r_u}(1-\mathrm{r_u}^2)} - \frac{(1-\gamma)2\mathfrak{c}\mathrm{r_u}}{\mathrm{r_u}(1-\mathrm{r_u}^2)}\right] d\mathrm{r_u} \le$$
$$log|\chi(\mathfrak{s})| \le \int \left[\frac{1}{\mathrm{r_u}(1-\mathrm{r_u}^2)} + \frac{(2\mathfrak{c}(1-\gamma)-1)\mathrm{r_u}^2}{\mathrm{r_u}(1-\mathrm{r_u}^2)} + \frac{(1-\gamma)2\mathfrak{c}\mathrm{r_u}}{\mathrm{r_u}(1-\mathrm{r_u}^2)}\right] d\mathrm{r_u}. \tag{3.29}$$

Firstly, solving Lower bound of (3.28)

$$\int \frac{1+(2\mathfrak{c}(1-\gamma)-1)\mathrm{r_u}^2-(1-\gamma)2\mathfrak{c}\mathrm{r_u}}{\mathrm{r_u}(1-\mathrm{r_u}^2)} d\mathrm{r_u},$$

$$= \int \left[\frac{1}{\mathrm{r_u}(1-\mathrm{r_u}^2)} + \frac{(2\mathfrak{c}(1-\gamma)-1)\mathrm{r_u}^2}{\mathrm{r_u}(1-\mathrm{r_u}^2)} - \frac{(1-\gamma)2\mathfrak{c}\mathrm{r_u}}{\mathrm{r_u}(1-\mathrm{r_u}^2)}\right] d\mathrm{r_u},$$

$$= \ln(\mathrm{r_u}) + \frac{1}{2}\ln\frac{1+(\mathrm{r_u})}{1-(\mathrm{r_u})} - \mathfrak{c}(1-\gamma)[\ln(\mathrm{r_u}+1) - \ln(1-\mathrm{r_u})] - \frac{2\mathfrak{c}(1-\gamma)-1}{2}\ln\left(1+\mathrm{r_u}^2\right),$$

$$= \ln(\mathrm{r_u}) + \frac{1}{2}\ln\frac{1+(\mathrm{r_u})}{1-(\mathrm{r_u})} - \mathfrak{c}(1-\gamma)\ln\left[\frac{(1+\mathrm{r_u})}{(1-\mathrm{r_u})}\right] - \frac{2\mathfrak{c}(1-\gamma)-1}{2}\ln\left(1+\mathrm{r_u}^2\right).$$

Similarly upper bound of will be,

$$= \ln(\mathrm{r_u}) + \frac{1}{2}\ln\frac{1+(\mathrm{r_u})}{1-(\mathrm{r_u})} + \mathfrak{c}(1-\gamma)\ln\left[\frac{(1+\mathrm{r_u})}{(1-\mathrm{r_u})}\right] - \frac{2\mathfrak{c}(1-\gamma)-1}{2}\ln\left(1+\mathrm{r_u}^2\right).$$

Putting values of upper and lower bound in (3.28) and taking exponential we will get,

$$\mathfrak{r}_\mathfrak{u}\left[\frac{1+\mathfrak{r}_\mathfrak{u}}{1-\mathfrak{r}_\mathfrak{u}}\right]^{-\mathfrak{c}(1-\gamma)+\frac{1}{2}}(1-{\mathfrak{r}_\mathfrak{u}}^2)^{-\left[\frac{2\mathfrak{c}(1-\gamma)-1}{2}\right]}\leq\chi(\mathfrak{s})\leq\mathfrak{r}_\mathfrak{u}\left[\frac{1+\mathfrak{r}_\mathfrak{u}}{1-\mathfrak{r}_\mathfrak{u}}\right]^{\mathfrak{c}(1-\gamma)+\frac{1}{2}}(1-{\mathfrak{r}_\mathfrak{u}}^2)^{-\left[\frac{2\mathfrak{c}(1-\gamma)-1}{2}\right]},$$

$$\frac{\mathfrak{r}_\mathfrak{u}}{(1+\mathfrak{r}_\mathfrak{u})^{2\mathfrak{c}(1-\gamma)}}\leq\chi(\mathfrak{s})\leq\frac{\mathfrak{r}_\mathfrak{u}}{(1-\mathfrak{r}_\mathfrak{u})^{2\mathfrak{c}(1-\gamma)}}. \tag{3.30}$$

Using the above inequality (3.30) in (3.26)

$$\frac{\left[1+(2\mathfrak{c}(1-\gamma)-1){\mathfrak{r}_\mathfrak{u}}^2-(1-\gamma)2\mathfrak{c}\mathfrak{r}_\mathfrak{u}\right]\mathfrak{r}_\mathfrak{u}}{1-{\mathfrak{r}_\mathfrak{u}}^2(1+\mathfrak{r}_\mathfrak{u})^{2\mathfrak{c}(1-\gamma)}\mathfrak{r}_\mathfrak{u}}\leq\chi'(\mathfrak{s})\leq\frac{[1+(2\mathfrak{c}(1-\gamma)-1){\mathfrak{r}_\mathfrak{u}}^2+(1-\gamma)2\mathfrak{c}\mathfrak{r}_\mathfrak{u}]\mathfrak{r}_\mathfrak{u}}{1-{\mathfrak{r}_\mathfrak{u}}^2(1-\mathfrak{r}_\mathfrak{u})^{2\mathfrak{c}(1-\gamma)}\mathfrak{r}_\mathfrak{u}},$$

$$\frac{\left[1+(2\mathfrak{c}(1-\gamma)-1){\mathfrak{r}_\mathfrak{u}}^2-(1-\gamma)2\mathfrak{c}\mathfrak{r}_\mathfrak{u}\right]}{1-{\mathfrak{r}_\mathfrak{u}}^2(1+\mathfrak{r}_\mathfrak{u})^{2\mathfrak{c}(1-\gamma)}}\leq\chi'(\mathfrak{s})\leq\frac{[1+(2\mathfrak{c}(1-\gamma)-1){\mathfrak{r}_\mathfrak{u}}^2+(1-\gamma)2\mathfrak{c}\mathfrak{r}_\mathfrak{u}]}{1-{\mathfrak{r}_\mathfrak{u}}^2(1-\mathfrak{r}_\mathfrak{u})^{2\mathfrak{c}(1-\gamma)}},$$

Hence the proof is complete. □

**Theorem 3.3.8.** Suppose $\chi(\mathfrak{z})\in\tilde{C}_\mathfrak{c}^\alpha[\xi,\varphi]$ with $\mathfrak{s}=\mathfrak{r}_\mathfrak{u}e^{i\theta}$ and $\gamma=\left[\frac{1-\xi}{1-\varphi}\right]^\alpha$, then we have the following inequality,

$$\frac{1}{(1+\mathfrak{r}_\mathfrak{u})^{2\mathfrak{c}(1-\gamma)}}\leq|\chi'(\mathfrak{s})|\leq\frac{1}{(1-\mathfrak{r}_\mathfrak{u})^{2\mathfrak{c}(1-\gamma)}}. \tag{3.31}$$

$$\frac{(1+\mathfrak{r}_\mathfrak{u})^{1-2\mathfrak{c}(1-\gamma)}}{1-2\mathfrak{c}(1-\gamma)}\leq|\chi(\mathfrak{s})|\leq-\frac{(1-\mathfrak{r}_\mathfrak{u})^{1-2\mathfrak{c}(1-\gamma)}}{1-2\mathfrak{c}(1-\gamma)}. \tag{3.32}$$

*Proof.* Since $\chi(\mathfrak{s})\in\tilde{C}_\mathfrak{c}^\alpha[\xi,\varphi]$ then $\phi(\mathfrak{s})=1+\frac{1}{\mathfrak{c}}\left[\frac{\mathfrak{s}\chi''(\mathfrak{s})}{\chi'(\mathfrak{s})}-1\right]$. Where $\phi(\mathfrak{s})\in\mathfrak{P}_\mathfrak{c}^\alpha[\xi,\varphi]\subseteq\phi(\gamma)$, $\gamma=\left[\frac{1-\xi}{1-\varphi}\right]^\alpha$. As $\phi(\mathfrak{s})\in\phi(\gamma)$ there exists $\phi\in\mathfrak{P}$ such that ,

$$\phi(\mathfrak{s})=(1-\gamma)\mathfrak{p}_1+\gamma,$$

$\mathfrak{p}_1(\mathfrak{s})\in\phi(\gamma)$ can be written as ,

$$\left|\mathfrak{p}_1-\frac{1+{\mathfrak{r}_\mathfrak{u}}^2}{1-{\mathfrak{r}_\mathfrak{u}}^2}\right|\leq\frac{2\mathfrak{r}_\mathfrak{u}}{1-{\mathfrak{r}_\mathfrak{u}}^2}.$$

By (3.25) we can write,

$$\left|\phi(\mathfrak{s})-\frac{1+{\mathfrak{r}_\mathfrak{u}}^2(1-2\gamma)}{1-{\mathfrak{r}_\mathfrak{u}}^2}\right|\leq\frac{(1-\gamma)2\mathfrak{r}_\mathfrak{u}}{1-{\mathfrak{r}_\mathfrak{u}}^2},$$

$$\left|\phi(\mathfrak{s})-\frac{1+(1-2\gamma){\mathfrak{r}_\mathfrak{u}}^2}{1-{\mathfrak{r}_\mathfrak{u}}^2}\right|\leq\frac{(1-\gamma)2\mathfrak{r}_\mathfrak{u}}{1-{\mathfrak{r}_\mathfrak{u}}^2}. \tag{3.33}$$

Replacing p(z) with its value we will get

$$\left|1+\frac{1}{\mathfrak{c}}\left[\frac{\mathfrak{s}\chi''(\mathfrak{s})}{\chi'(\mathfrak{s})}\right]-\frac{1+(1-2\gamma)\mathrm{r_u}^2}{1-\mathrm{r_u}^2}\right|\leq\frac{(1-\gamma)2\mathrm{r_u}}{1-\mathrm{r_u}^2}.$$

Multiplying both sides by b to get

$$\left|\mathfrak{c}+\left[\frac{\mathfrak{s}\chi''(\mathfrak{s})}{\chi'(\mathfrak{s})}\right]-\frac{\mathfrak{c}+(1-2\gamma)\mathfrak{c}\mathrm{r_u}^2}{1-\mathrm{r_u}^2}\right|\leq\frac{(1-\gamma)2\mathfrak{c}\mathrm{r_u}}{1-\mathrm{r_u}^2},$$

$$\left|\frac{\mathfrak{s}\chi'(\mathfrak{s})}{\chi(\mathfrak{s})}+\mathfrak{c}-\frac{\mathfrak{c}+(1-2\gamma)\mathfrak{c}\mathrm{r_u}^2}{1-\mathrm{r_u}^2}\right|\leq\frac{(1-\gamma)2\mathfrak{c}\mathrm{r_u}}{1-\mathrm{r_u}^2},$$

$$\left|\frac{\mathfrak{s}\chi''(\mathfrak{s})}{\chi'(\mathfrak{s})}-\frac{+\mathfrak{c}-\mathfrak{c}\mathrm{r_u}^2-\mathfrak{c}+(1-2\gamma)\mathfrak{c}\mathrm{r_u}^2}{1-\mathrm{r_u}^2}\right|\leq\frac{(1-\gamma)2\mathfrak{c}\mathrm{r_u}}{1-\mathrm{r_u}^2}.$$

By simplifying this we will get

$$\left|\frac{\mathfrak{s}\chi''(\mathfrak{s})}{\chi'(\mathfrak{s})}-\frac{(2\mathfrak{c}(1-\gamma)\mathrm{r_u}^2}{1-\mathrm{r_u}^2}\right|\leq\frac{(1-\gamma)2\mathfrak{c}\mathrm{r_u}}{1-\mathrm{r_u}^2},$$

$$-\frac{(1-\gamma)2\mathfrak{c}\mathrm{r_u}}{1-\mathrm{r_u}^2}\leq\frac{\mathfrak{s}\chi''(\mathfrak{s})}{\chi'(\mathfrak{s})}-\frac{(2\mathfrak{c}(1-\gamma))\mathrm{r_u}^2}{1-\mathrm{r_u}^2}\leq\frac{(1-\gamma)2\mathfrak{c}\mathrm{r_u}}{1-\mathrm{r_u}^2},$$

$$-\frac{(1-\gamma)2\mathfrak{c}\mathrm{r_u}}{1-\mathrm{r_u}^2}+\frac{1+(2\mathfrak{c}(1-\gamma))\mathrm{r_u}^2}{1-\mathrm{r_u}^2}\leq\frac{\mathfrak{s}\chi''(\mathfrak{s})}{\chi'(\mathfrak{s})}\leq\frac{(1-\gamma)2\mathfrak{c}\mathrm{r_u}}{1-\mathrm{r_u}^2}+\frac{1+(2\mathfrak{c}(1-\gamma))\mathrm{r_u}^2}{1-\mathrm{r_u}^2},$$

$$\frac{(2\mathfrak{c}(1-\gamma))\mathrm{r_u}^2-(1-\gamma)2\mathfrak{c}\mathrm{r_u}}{1-\mathrm{r_u}^2}\leq\frac{\mathfrak{s}\chi''(\mathfrak{s})}{\chi'(\mathfrak{s})}\leq\frac{(2\mathfrak{c}(1-\gamma))\mathrm{r_u}^2+(1-\gamma)2\mathfrak{c}\mathrm{r_u}}{1-\mathrm{r_u}^2}, \tag{3.34}$$

$$\frac{-(2\mathfrak{c}(1-\gamma)\mathrm{r_u}(1-\mathrm{r_u})}{(1-\mathrm{r_u})(1+\mathrm{r_u})}\leq\frac{\mathfrak{s}\chi''(\mathfrak{s})}{\chi'(\mathfrak{s})}\leq\frac{(2\mathfrak{c}(1-\gamma)\mathrm{r_u}(1+\mathrm{r_u})}{(1-\mathrm{r_u})(1+\mathrm{r_u})},$$

$$\frac{-(2\mathfrak{c}(1-\gamma)\mathrm{r_u}}{(1+\mathrm{r_u})}\leq\frac{\mathfrak{s}\chi''(\mathfrak{s})}{\chi'(\mathfrak{s})}\leq\frac{(2\mathfrak{c}(1-\gamma)\mathrm{r_u}}{(1-\mathrm{r_u})},$$

$$\frac{-(2\mathfrak{c}(1-\gamma)\mathrm{r_u}}{(1+\mathrm{r_u})}\leq\frac{\mathfrak{s}\chi''(\mathfrak{s})}{\chi'(\mathfrak{s})}\leq\frac{(2\mathfrak{c}(1-\gamma)\mathrm{r_u}}{(1-\mathrm{r_u})},$$

$$\frac{(2\mathfrak{c}(\gamma-1)\mathrm{r_u}}{(1+\mathrm{r_u})}\leq\frac{\mathfrak{s}\chi''(\mathfrak{s})}{\chi'(\mathfrak{s})}\leq\frac{(2\mathfrak{c}(1-\gamma)\mathrm{r_u}}{(1-\mathrm{r_u})}.$$

We know that,

$$Re\frac{\mathfrak{s}\chi''(\mathfrak{s})}{\chi'(\mathfrak{s})}=\mathrm{r_u}\frac{\partial}{\partial\mathrm{r_u}}log|\chi'(\mathfrak{s})|,$$

$$\frac{(2\mathfrak{c}(\gamma-1)\mathrm{r_u}}{\mathrm{r_u}(1+\mathrm{r_u})} \leq \frac{\partial}{\partial \mathrm{r_u}} log|\chi'(\mathfrak{s})| \leq \frac{(2\mathfrak{c}(1-\gamma)\mathrm{r_u}}{(1-\mathrm{r_u})\mathrm{r_u}}. \tag{3.35}$$

Integrate with respect to $\mathrm{r_u}$

$$\int \frac{(2\mathfrak{c}(\gamma-1)\mathrm{r_u}}{\mathrm{r_u}(1+\mathrm{r_u})} d\mathrm{r_u} \leq log|\chi'(\mathfrak{s})| \leq \int \frac{(2\mathfrak{c}(1-\gamma)\mathrm{r_u}}{(1-\mathrm{r_u})\mathrm{r_u}} d\mathrm{r_u}, \tag{3.36}$$

$$2\mathfrak{c}(\gamma-1)[log(1+\mathrm{r_u})] \leq log|\chi'(\mathfrak{s})| \leq -2\mathfrak{c}(1-\gamma)[log(1-\mathrm{r_u})],$$

$$log(1+\mathrm{r_u})^{2\mathfrak{c}(\gamma-1)} \leq log|\chi'(\mathfrak{s})| \leq log(1-\mathrm{r_u})^{-2\mathfrak{c}(1-\gamma)},$$

$$log(1+\mathrm{r_u})^{-2\mathfrak{c}(1-\gamma)} \leq log|\chi'(\mathfrak{s})| \leq log(1-\mathrm{r_u})^{-2\mathfrak{c}(1-\gamma)}.$$

Applying exponential will give us,

$$(1+\mathrm{r_u})^{-2\mathfrak{c}(1-\gamma)} \leq |\chi'(\mathfrak{s})| \leq (1-\mathrm{r_u})^{-2\mathfrak{c}(1-\gamma)},$$

$$\frac{1}{(1+\mathrm{r_u})^{2\mathfrak{c}(1-\gamma)}} \leq |\chi'(\mathfrak{s})| \leq \frac{1}{(1-\mathrm{r_u})^{2\mathfrak{c}(1-\gamma)}}.$$

Hence we got (4.30). Integrate over $\mathrm{r_u}$ where $|\mathfrak{s}| = \mathrm{r_u}$

$$\int \frac{1}{(1+\mathrm{r_u})^{2\mathfrak{c}(1-\gamma)}} d\mathrm{r_u} \leq |\chi(\mathfrak{s})| \leq \int \frac{1}{(1-\mathrm{r_u})^{2\mathfrak{c}(1-\gamma)}} d\mathrm{r_u}.$$

Let $k = 2\mathfrak{c}(1-\gamma)$. The integral becomes,

$$\int \frac{1}{(1+\mathrm{r_u})^{k}} d\mathrm{r_u} \leq |\chi(\mathfrak{s})| \leq \int \frac{1}{(1-\mathrm{r_u})^{k}} d\mathrm{r_u}.$$

After integration this inequality over $\mathrm{r_u}$ we will get,

$$\frac{(1+\mathrm{r_u})^{1-k}}{1-k} \leq |\chi(\mathfrak{s})| \leq -\frac{(1-\mathrm{r_u})^{1-k}}{1-k},$$

$$\frac{(1+\mathrm{r_u})^{1-2\mathfrak{c}(1-\gamma)}}{1-2\mathfrak{c}(1-\gamma)} \leq |\chi(\mathfrak{s})| \leq -\frac{(1-\mathrm{r_u})^{1-2\mathfrak{c}(1-\gamma)}}{1-2\mathfrak{c}(1-\gamma)}.$$

Hence the proof is complete. □

**Theorem 3.3.9.** Suppose $\chi(\mathfrak{s}) \in \tilde{C}_{\mathfrak{c}}^{\alpha}[\xi, \varphi]$. Then $\chi(\mathfrak{s})$ maps $|\mathfrak{s}| < \phi$ onto a convex domain, where $\phi = (1-\varphi)^{\alpha}\left[\mathfrak{c}\left[\frac{1-\xi}{1-\varphi}\right]^{\alpha} + (1-\mathfrak{c})\right]$ for $-1 \leq \varphi < \xi \leq 1$, $\mathfrak{c} \neq 0$ and $0 < \alpha \leq 1$.

*Proof.* Suppose $\chi(\mathfrak{s}) \in \tilde{C}_{\mathfrak{c}}^{\alpha}[\xi, \varphi]$, So for any $\phi(\mathfrak{s}) \in \mathfrak{P}^{\alpha}[\xi, \varphi]$, we have

$$1 + \frac{1}{\mathfrak{c}} \frac{\mathfrak{s}\chi''(\mathfrak{s})}{\chi'(\mathfrak{s})} = \phi(\mathfrak{s}).$$

Multiplying $\mathfrak{c}$ on both sides yields

$$\mathfrak{c} + \frac{1}{\mathfrak{c}} \frac{\mathfrak{s}\chi''(\mathfrak{s})}{\chi'(\mathfrak{s})} = \mathfrak{c}\phi(\mathfrak{s}),$$

$$\frac{1}{\mathfrak{c}}\frac{\mathfrak{s}\chi''(\mathfrak{s})}{\chi'(\mathfrak{s})} = \mathfrak{c}\phi(\mathfrak{s}) - \mathfrak{c}.$$

Adding 1 on both sides results in

$$1 + \frac{1}{\mathfrak{c}}\frac{\mathfrak{s}\chi''(\mathfrak{s})}{\chi'(\mathfrak{s})} = 1 + \mathfrak{c}\phi(\mathfrak{s}) - \mathfrak{c},$$

$$Re\left[1 + \frac{1}{\mathfrak{c}}\frac{\mathfrak{s}\chi''(\mathfrak{s})}{\chi'(\mathfrak{s})}\right] = Re\left[1 + \mathfrak{c}\phi(\mathfrak{s}) - \mathfrak{c}\right].$$

As $\phi(\mathfrak{s}) \in \mathfrak{P}^{\alpha}[\xi, \varphi]$, then by using Lemma( ), we will have

$$Re\left[1 + \frac{1}{\mathfrak{c}}\frac{\mathfrak{s}\chi''(\mathfrak{s})}{\chi'(\mathfrak{s})}\right] \geq 1 + \mathfrak{c}\left[\frac{1-\xi(\mathrm{r_u})}{1-\varphi(\mathrm{r_u})}\right] - \mathfrak{c} \geq \frac{\mathfrak{c}(1-\xi(\mathrm{r_u}))^{\alpha} + (1-\mathfrak{c})(1-\varphi(\mathrm{r_u}))^{\alpha}}{(1-\varphi(\mathrm{r_u}))^{\alpha}}.$$

Let $\mathfrak{c}(1-\xi(\mathrm{r_u}))^{\alpha} + (1-\mathfrak{c})(1-\varphi(\mathrm{r_u}))^{\alpha} = \mathfrak{h}(\mathrm{r_u})$.

Then $\mathfrak{h}(\mathfrak{o}) = 1$ and $\mathfrak{h}(1) = \mathfrak{c}(1-\xi)^{\alpha} + (1-\mathfrak{c})(1-\varphi)^{\alpha} = (1-\varphi)^{\alpha}\left[\mathfrak{c}\left[\frac{1-\xi}{1-\varphi}\right]^{\alpha} + (1-\mathfrak{c})\right]$.

So, $Re\left[1 + \frac{1}{\mathfrak{c}}\frac{\mathfrak{s}\chi''(\mathfrak{s})}{\chi'(\mathfrak{s})}\right] > 0$ for $|\mathfrak{s}| < \phi$ where $\phi = (1-\varphi)^{\alpha}\left[\mathfrak{c}\left[\frac{1-\xi}{1-\varphi}\right]^{\alpha} + (1-\mathfrak{c})\right]$. It is the least positive root of the equation $\mathfrak{c}(1-\xi)^{\alpha} + (1-\mathfrak{c})(1-\varphi)^{\alpha} = 0$. $\square$

# CHAPTER 4

# ON NEW CLASSES OF Q-STARLIKE AND Q-CONVEX JANOWSKI TYPE FUNCTIONS OF COMPLEX ORDER

## 4.1 Overview

In this chapter, we develop new subclasses of $\mathfrak{q}$-starlike and $\mathfrak{q}$-convex functions linked with Janowski-type functions of complex order. These classes are constructed within the framework of $\mathfrak{q}$-calculus, which extends classical analytic function theory by introducing the $\mathfrak{q}$-derivative and allowing a more flexible geometric structure.

The move from the classical Janowski framework to its $\mathfrak{q}$-analogue is motivated by the richer behaviour and broader generality offered by the deformation parameter $\mathfrak{q} \in (0, 1)$. While the classical theory is restricted to the standard derivative and continuous settings, the $\mathfrak{q}$-calculus framework smoothly interpolates between discrete and continuous cases, recovering the classical results as $\mathfrak{q} \to 1^{-}$. This leads to new coefficient interactions, sharper bounds, and geometric features not present in the classical case. Moreover, because $\mathfrak{q}$-analogues naturally arise in quantum calculus and discrete models, extending Janowski classes to the $\mathfrak{q}$-setting makes the theory more relevant to modern analytic contexts.

Within this framework, we study fundamental properties of the proposed subclasses, including inclusion relations, distortion bounds, coefficient estimates, and radii problems. We also examine their behaviour under integral operators suited to $\mathfrak{q}$-calculus. The results obtained, generalise earlier findings and provide deeper insights into the geometric behaviour of these $\mathfrak{q}$-analogues.

## 4.2 Introduction and Preliminaries

Now, we define the following new classes of q-starlike and q-convex funcyion of complex order. For following definitions and results, we consider $-1 \leq \varphi < \xi \leq 1$, $\mathfrak{q} \in (0,1)$, $0 < \alpha \leq 1$, $\mathfrak{c} \neq 0$, $\mathfrak{c} \in \mathfrak{C}$ ($\mathfrak{C}$ is complex number) and $\mathfrak{s} \in \mathfrak{D}$ unless otherwise stated.

**Definition 4.2.1.** An analytic function $\chi(\mathfrak{s})$ belongs to the class $\mathfrak{S}^{\alpha}_{\mathfrak{q},\mathfrak{c}}[\xi,\varphi]$ if and only if $\frac{\chi(\mathfrak{s})}{\mathfrak{s}} \neq 0$ and

$$1 + \frac{1}{\mathfrak{c}}\left[\frac{\mathfrak{s}\mathfrak{D}_{\mathfrak{q}}\chi(\mathfrak{s})}{\chi(\mathfrak{s})} - 1\right] \in \mathfrak{P}^{\alpha}[\xi,\varphi].$$

Equivalentely,

$$\mathfrak{S}^{\alpha}_{\mathfrak{q},\mathfrak{c}}[\xi,\varphi] = \left\{\chi \in \mathfrak{A} : 1 + \frac{1}{\mathfrak{c}}\left[\frac{\mathfrak{s}\mathfrak{D}_{\mathfrak{q}}\chi(\mathfrak{s})}{\chi(\mathfrak{s})} - 1\right] \prec \left[\frac{1+\xi\mathfrak{s}}{1+\varphi\mathfrak{s}}\right]^{\alpha}\right\}.$$

## Special Cases:

i. For $\alpha = 1$, $\mathfrak{c} = 1$ and taking $\mathfrak{q} \to 1^{-}$, the class $\mathfrak{S}^{\alpha}_{\mathfrak{q},\mathfrak{c}}[\xi,\varphi]$ reduces to $\mathfrak{S}[\xi,\varphi]$, See [27].

ii. For $\xi = 1, \varphi = -1$ the class $\mathfrak{S}^{\alpha}_{\mathfrak{q},\mathfrak{c}}[\xi,\varphi]$ becomes $\mathfrak{S}_{\mathfrak{q},\mathfrak{c}}(\alpha)$ and for $\mathfrak{q} \to 1^{-}$, we get the class $\mathfrak{S}_{\mathfrak{c}}(\alpha)$.

iii. For $\xi = 1, \varphi = -1$, $\alpha = 1$, $\mathfrak{c} = 1$ we get the class $\mathfrak{S}_{\mathfrak{q}}$ introduced in [38] and taking $\mathfrak{q} \to 1^{-}$, we get the well known class $\mathfrak{S}$, see [39].

iv. For $\alpha = \frac{1}{2}$, $\xi = c$, $\varphi = 0$ the class $\mathfrak{S}^{\alpha}_{\mathfrak{q},\mathfrak{c}}[\xi,\varphi]$ reduces to $\mathfrak{S}^{\frac{1}{2}}_{\mathfrak{q},\mathfrak{c}}(c)$ and $\mathfrak{q} \to 1^{-}$ and $\mathfrak{c} = 1$ leads us to the class $\mathfrak{S}^{\frac{1}{2}}$ defined in [40]

**Definition 4.2.2.** A function $\chi(\mathfrak{s}) \in \mathfrak{A}$ belongs to the class $\tilde{C}^{\alpha}_{\mathfrak{q},\mathfrak{c}}[\xi,\varphi]$ if and only if

$$1 + \frac{1}{\mathfrak{c}}\left[\frac{\mathfrak{s}\mathfrak{D}_{\mathfrak{q}}(\mathfrak{D}_{\mathfrak{q}}\chi(\mathfrak{s}))}{\mathfrak{D}_{\mathfrak{q}}\chi(\mathfrak{s})} - 1\right] \in \mathfrak{P}^{\alpha}[\xi,\varphi]$$

.

$$\tilde{C}^{\alpha}_{\mathfrak{q},\mathfrak{c}}[\xi,\varphi] = \left\{\chi \in \mathfrak{A} : 1 + \frac{1}{\mathfrak{c}}\left[\frac{\mathfrak{s}\mathfrak{D}_{\mathfrak{q}}(\mathfrak{D}_{\mathfrak{q}}\chi(\mathfrak{s}))}{\mathfrak{D}_{\mathfrak{q}}\chi(\mathfrak{s})} - 1\right] \prec \left[\frac{1+\xi\mathfrak{s}}{1+\varphi\mathfrak{s}}\right]^{\alpha}\right\}.$$

## Special Cases:

i. For $\alpha = 1$, $\mathfrak{c} = 1$ and taking $\mathfrak{q} \to 1^-$, the class $\tilde{C}^{\alpha}_{\mathfrak{q},\mathfrak{c}}[\xi, \varphi]$ reduces to $\tilde{C}[\xi, \varphi]$, See [27].

ii. For $\xi = 1, \varphi = -1$ the class $\tilde{C}^{\alpha}_{\mathfrak{q},\mathfrak{c}}[\xi, \varphi]$ becomes $\tilde{C}_{\mathfrak{q},\mathfrak{c}}(\alpha)$ and for $\mathfrak{q} \to 1^-$, we get the class $\mathfrak{S}_{\mathfrak{c}}(\alpha)$.

iii. For $\xi = 1, \varphi = -1$, $\alpha = 1$, $\mathfrak{c} = 1$ we get the class $\tilde{C}_{\mathfrak{q}}$ introduced in [38]and taking $\mathfrak{q} \to 1^-$, we get the well known class $\tilde{C}$, see [39].

## 4.3 Main Results

**Theorem 4.3.1.** Suppose $\chi(\mathfrak{s}) \in \tilde{C}^{\alpha}_{\mathfrak{q},\mathfrak{c}}[\xi, \varphi]$. Then $\chi(\mathfrak{s}) \in \tilde{C}_{\mathfrak{c}}(\gamma)$, where $\gamma = \left[\frac{1-\xi}{1-\varphi}\right]^{\alpha}$ for $-1 \leq \varphi < \xi \leq 1$, $0 < \alpha \leq 1$, and $\mathfrak{c} \neq 0$.

*Proof.* Suppose $\chi(\mathfrak{s}) \in \tilde{C}^{\alpha}_{\mathfrak{q},\mathfrak{c}}[\xi, \varphi]$, so by definition ,

$$1 + \frac{1}{\mathfrak{c}} \frac{\mathfrak{s} D_{\mathfrak{q}}(D_{\mathfrak{q}}\chi(\mathfrak{s}))}{D_{\mathfrak{q}}\chi(\mathfrak{s})} \in \mathfrak{P}^{\alpha}[\xi, \varphi].$$

If $\phi(\mathfrak{s}) \in \mathfrak{P}^{\alpha}[\xi, \varphi]$, then by definition of $\mathfrak{P}^{\alpha}[\xi, \varphi]$, we have:

$$\phi(\mathfrak{s}) \prec \left[\frac{1+\xi\mathfrak{s}}{1+\varphi\mathfrak{s}}\right]^{\alpha}. \tag{4.1}$$

This implies that there exists an analytic function $\omega(\mathrm{r_u})$ with $\omega(0) = 0$ and $|\omega(\mathfrak{s})| < 1$, such that:

$$\phi(\mathfrak{s}) \prec \left[\frac{1+\xi\omega(\mathfrak{s})}{1+\varphi\omega(\mathfrak{s})}\right]^{\alpha}, \tag{4.2}$$

$$\operatorname{Re} \phi(\mathfrak{s}) \prec \operatorname{Re} \left[\frac{1+\xi\omega(\mathfrak{s})}{1+\varphi\omega(\mathfrak{s})}\right]^{\alpha}.$$

Using geometric inequality from starlike function

$$\operatorname{Re} \left[\frac{1+\xi\omega(\mathfrak{s})}{1+\varphi\omega(\mathfrak{s})}\right] \geq \frac{1-\xi|\omega(\mathfrak{s})|}{1-\varphi|\omega(\mathfrak{s})|},$$

$$\operatorname{Re} \left[\frac{1+\xi\omega(\mathfrak{s})}{1+\varphi\omega(\mathfrak{s})}\right]^{\alpha} \geq \left[\frac{1-\xi}{1-\varphi}\right]^{\alpha}. \tag{4.3}$$

Thus,

$$\operatorname{Re}\phi(\mathfrak{s}) \geq \left[\frac{1-\xi}{1-\varphi}\right]^{\alpha}. \tag{4.4}$$

This inequality shows that $\phi(\mathfrak{s}) \in \mathfrak{P}(\gamma)$, where $\gamma = \left[\frac{1-\xi}{1-\varphi}\right]^{\alpha}$.

Since $\phi(\mathfrak{s}) \in \mathfrak{P}(\gamma)$, so by definition,

$$1+\frac{1}{\mathfrak{c}}\left[\frac{\mathfrak{s}D_{\mathfrak{q}}(D_{\mathfrak{q}}\chi(\mathfrak{s}))}{D_{\mathfrak{q}}\chi(\mathfrak{s})}\right] \in \mathfrak{P}(\gamma).$$

This means $\phi(\mathfrak{s}) \in \tilde{C}_{c,q}(\gamma)$. So $\chi(\mathfrak{s}) \in \tilde{C}^{\alpha}_{\mathfrak{q},\mathfrak{c}}[\xi,\varphi]$ with $\gamma = \left[\frac{1-\xi}{1-\varphi}\right]^{\alpha}$. This concludes the proof. □

**Corollary 4.3.1.1.** For $\alpha = 1$, $\chi(\mathfrak{s}) \in \tilde{C}_{\mathfrak{q},\mathfrak{c}}[\xi,\varphi]$. Above Theorem 4.3.1 gives that $\chi(\mathfrak{s}) \in \tilde{C}_{\mathfrak{q},\mathfrak{c}}(\phi)$ for $\phi = \left[\frac{1-\xi}{1-\varphi}\right]^{\alpha}$.

**Corollary 4.3.1.2.** Put $\alpha = \frac{1}{2}, \xi = c, c \in (0,1]$, $\varphi = 0$. We get $\chi(\mathfrak{s}) \in \tilde{C}^{\frac{1}{2}}_{\mathfrak{c}}(c)$,

where $\tilde{C}^{\frac{1}{2}}_{\mathfrak{q},\mathfrak{c}}(c) = \left\{\chi \in \mathfrak{A} : 1+\frac{1}{\mathfrak{c}}\frac{\mathfrak{s}\mathfrak{D}^2_q(\mathfrak{s})}{\mathfrak{D}_{\mathfrak{q}}(\mathfrak{s})} \prec \sqrt{1+c\mathfrak{s}}\right\}$.

Using Theorem 4.3.1, we have $\chi(\mathfrak{s}) \in \tilde{C}_{\mathfrak{q},\mathfrak{c}}(\phi)$, where $\phi = \sqrt{1-c}$ [40].

**Corollary 4.3.1.3.** Put $\mathfrak{c} = 1$ in Theorem 1.3, we have $\tilde{C}^{\alpha}_{\mathfrak{q}}[\xi,\varphi] \subset \tilde{C}_{\mathfrak{q}}(\phi)$, for $\phi = \left[\frac{1-\xi}{1-\varphi}\right]^{\alpha}$

**Theorem 4.3.2.** If $\chi(\mathfrak{s}) \in \mathfrak{S}^{\alpha}_{\mathfrak{q},\mathfrak{c}}[\xi,\varphi]$, then $\chi(\mathfrak{s}) \in \mathfrak{S}_{\mathfrak{q},\mathfrak{c}}(\gamma)$, where $\gamma = \left[\frac{1-\xi}{1-\varphi}\right]^{\alpha}$, for $-1 \leq \varphi < \xi \leq 1$ and $0 < \alpha \leq 1$ with $\varphi \neq 0$.

*Proof.* Suppose $\chi(\mathfrak{s}) \in \mathfrak{S}^{\alpha}_{\mathfrak{q},\mathfrak{c}}[\xi,\varphi]$, then

$$1+\frac{1}{\mathfrak{c}}\left[\frac{\mathfrak{s}D_{\mathfrak{q}}\chi(\mathfrak{s})}{\chi(\mathfrak{s})}-1\right] \in \mathfrak{P}^{\alpha}[\xi,\varphi].$$

If $\phi(\mathfrak{s}) \in \mathfrak{P}^{\alpha}[\xi,\varphi]$, it satisfies the subordination condition

$$\phi(\mathfrak{s}) \prec \left[\frac{1+\xi(\mathfrak{s})}{1+\varphi(\mathfrak{s})}\right]^{\alpha}.$$

This implies that there exists an analytic function $\omega(\mathrm{r_u})$ with $\omega(0) = 0$ and $|\omega(\mathfrak{s})| < 1$, such that:

$$\phi(\mathfrak{s}) \prec \left[\frac{1+\xi\omega(\mathfrak{s})}{1+\varphi\omega(\mathfrak{s})}\right]^{\alpha}, \tag{4.5}$$

$$\operatorname{Re}\phi(\mathfrak{s}) \prec \operatorname{Re}\left[\frac{1+\xi\omega(\mathfrak{s})}{1+\varphi\omega(\mathfrak{s})}\right]^{\alpha}.$$

Using geometric inequality from starlike function

$$\operatorname{Re}\left[\frac{1+\xi\omega(\mathfrak{s})}{1+\varphi\omega(\mathfrak{s})}\right] \geq \frac{1-\xi|\omega(\mathfrak{s})|}{1-\varphi|\omega(\mathfrak{s})|},$$

$$\operatorname{Re}\left[\frac{1+\xi\omega(\mathfrak{s})}{1+\varphi\omega(\mathfrak{s})}\right]^{\alpha} \geq \left[\frac{1-\xi}{1-\varphi}\right]^{\alpha}. \tag{4.6}$$

Thus,

$$\operatorname{Re}\phi(\mathfrak{s}) \geq \left[\frac{1-\xi}{1-\varphi}\right]^{\alpha}. \tag{4.7}$$

This inequality shows that $\phi(\mathfrak{s}) \in \mathfrak{P}(\gamma)$, where $\gamma = \left[\frac{1-\xi}{1-\varphi}\right]^{\alpha}$.

Since $\phi(\mathfrak{s}) \in \phi(\gamma)$, so by definition,

$$1+\frac{1}{\mathfrak{c}}\left[\frac{\mathfrak{s}D_{\mathfrak{q}}\chi(\mathfrak{s})}{\chi(\mathfrak{s})}\right] \in \mathfrak{P}(\gamma)$$

So, $\chi(\mathfrak{s}) \in \mathfrak{S}^{\alpha}_{\mathfrak{q},\mathfrak{c}}[\xi,\varphi]$ with $\gamma = \left[\frac{1-\xi}{1-\varphi}\right]^{\alpha}$. This concludes the proof. □

**Corollary 4.3.2.1.** For $\alpha = 1$, Theorem 1.4 gives $\mathfrak{S}_{\mathfrak{q},\mathfrak{c}}[\xi,\varphi] \subset \chi(\mathfrak{s}) \in \mathfrak{S}_{\mathfrak{q},\mathfrak{c}}(\phi)$, for $\phi = \frac{1-\xi}{1-\varphi}$.

**Corollary 4.3.2.2.** Put $\alpha = \frac{1}{2}, \xi = c, c \in (0,1]$, $\varphi = 0$. We get $\chi(\mathfrak{s}) \in \mathfrak{S}^{\frac{1}{2}}_{\mathfrak{q},\mathfrak{c}}(c)$,
where $\mathfrak{S}^{\frac{1}{2}}_{\mathfrak{q},\mathfrak{c}}(c) = \left\{\chi \in \mathfrak{A} : 1+\frac{1}{\mathfrak{c}}\left[\frac{\mathfrak{s}\mathfrak{D}_q\chi(\mathfrak{s})}{\chi(\mathfrak{s})}-1\right] \prec \sqrt{1+c\mathfrak{s}}\right\}$.
Using Theorem 4.3.2, we have $\chi(\mathfrak{s}) \in \mathfrak{S}_{\mathfrak{q},\mathfrak{c}}(\phi)$, where $\phi = \sqrt{1-c}$ [40].

**Corollary 4.3.2.3.** Put $\mathfrak{c} = 1$ in Theorem 4.3.2, we have $\mathfrak{S}^{\alpha}_{\mathfrak{q}}[\xi,\varphi] \subset \mathfrak{S}_{\mathfrak{q}}(\phi)$, for $\phi = \left[\frac{1-\xi}{1-\varphi}\right]^{\alpha}$.

**Theorem 4.3.3.** If $\chi(\mathfrak{s}) \in \mathfrak{S}^{\alpha}_{\mathfrak{q},\mathfrak{c}}[\xi,\varphi]$ then with $\chi(\mathfrak{s}) = \mathfrak{s}+\sum_{\underline{\mathfrak{n}}=2}^{\infty} a_{\underline{\mathfrak{n}}}\mathfrak{s}^{\underline{\mathfrak{n}}}$, for $\mathfrak{s} \in \mathfrak{D}$, $-1 \leq \varphi < \xi \leq 1$, $0 < \alpha \leq 1$ and $\mathfrak{c} \neq 0$. Then for $\underline{\mathfrak{n}} \geq 2$

$$|a_{\underline{\mathfrak{n}}}| \leq \frac{(\mathfrak{c}\alpha(\xi-\varphi))_{\underline{\mathfrak{n}}-1}}{([\underline{\mathfrak{n}}]_{\mathfrak{q}}-1)!} \tag{4.8}$$

*Proof.* Suppose $\chi(\mathfrak{s}) \in \mathfrak{S}_{\mathfrak{q},\mathfrak{c}}[\xi,\varphi]$. Then by definition

$$1+\frac{1}{\mathfrak{c}}\left[\frac{\mathfrak{s}D_{\mathfrak{q}}\chi(\mathfrak{s})}{\chi(\mathfrak{s})}-1\right) \in \mathfrak{P}^{\alpha}_{\mathfrak{q},\mathfrak{c}}[\xi,\varphi].$$

Consider $\phi(\mathfrak{s}) = 1+\frac{1}{\mathfrak{c}}\left[\frac{\mathfrak{s}D_q\chi(\mathfrak{s})}{\chi(\mathfrak{s})}-1\right) = 1+\sum_{\underline{\mathfrak{n}}=2}^{\infty} a_{\underline{\mathfrak{n}}}\mathfrak{s}^{\underline{\mathfrak{n}}}$, where $\phi(\mathfrak{s})$ is analytic in $\mathfrak{D}$ with $\phi(0) = 1$.

$$\mathfrak{s}+\sum_{\underline{\mathfrak{n}}=2}^{\infty}[\underline{\mathfrak{n}}]_{\mathfrak{q}}a_{\underline{\mathfrak{n}}}\mathfrak{s}^{\underline{\mathfrak{n}}} = \left[\mathfrak{s}+\sum_{\underline{\mathfrak{n}}=2}^{\infty}a_{\underline{\mathfrak{n}}}\mathfrak{s}^{\underline{\mathfrak{n}}}\right]\left[1+\sum_{\underline{\mathfrak{n}}=1}^{\infty}\mathfrak{c}b_{\underline{\mathfrak{n}}}\mathfrak{s}^{\underline{\mathfrak{n}}}\right].$$

$$\mathfrak{s}+\sum_{\underline{\mathfrak{n}}=2}^{\infty}[\underline{\mathfrak{n}}]_{\mathfrak{q}}a_{\underline{\mathfrak{n}}}\mathfrak{s}^{\underline{\mathfrak{n}}} = \mathfrak{s}\left[1+\sum_{\underline{\mathfrak{n}}=1}^{\infty}\mathfrak{c}b_{\underline{\mathfrak{n}}}\mathfrak{s}^{\underline{\mathfrak{n}}}\right]+\left[\sum_{\underline{\mathfrak{n}}=2}^{\infty}a_{\underline{\mathfrak{n}}}\mathfrak{s}^{\underline{\mathfrak{n}}}\right]\left[1+\sum_{\underline{\mathfrak{n}}=1}^{\infty}\mathfrak{c}b_{\underline{\mathfrak{n}}}\mathfrak{s}^{\underline{\mathfrak{n}}}\right].$$

$$\mathfrak{s}+\sum_{\underline{\mathfrak{n}}=2}^{\infty}[\underline{\mathfrak{n}}]_{\mathfrak{q}}a_{\underline{\mathfrak{n}}}\mathfrak{s}^{\underline{\mathfrak{n}}}=\mathfrak{s}+\sum_{\underline{\mathfrak{n}}=2}^{\infty}\mathfrak{c}b_{\underline{\mathfrak{n}}-1}\mathfrak{s}^{\underline{\mathfrak{n}}}+\sum_{\underline{\mathfrak{n}}=2}^{\infty}a_{\underline{\mathfrak{n}}}\mathfrak{s}^{\underline{\mathfrak{n}}}+\left[\sum_{\underline{\mathfrak{n}}=2}^{\infty}a_{\underline{\mathfrak{n}}}\mathfrak{s}^{\underline{\mathfrak{n}}}\right]\cdot\left[\sum_{\underline{\mathfrak{n}}=1}^{\infty}\mathfrak{c}b_{\underline{\mathfrak{n}}}\mathfrak{s}^{\underline{\mathfrak{n}}}\right].$$

$$\sum_{\underline{\mathfrak{n}}=2}^{\infty}[\underline{\mathfrak{n}}]_{\mathfrak{q}}a_{\underline{\mathfrak{n}}}\mathfrak{s}^{\underline{\mathfrak{n}}}=\sum_{\underline{\mathfrak{n}}=2}^{\infty}\mathfrak{c}b_{\underline{\mathfrak{n}}-1}\mathfrak{s}^{\underline{\mathfrak{n}}}+\sum_{\underline{\mathfrak{n}}=2}^{\infty}a_{\underline{\mathfrak{n}}}\mathfrak{s}^{\underline{\mathfrak{n}}}+\left[\sum_{\underline{\mathfrak{n}}=2}^{\infty}a_{\underline{\mathfrak{n}}}\mathfrak{s}^{\underline{\mathfrak{n}}}\right]\cdot\left[\sum_{\underline{\mathfrak{n}}=1}^{\infty}\mathfrak{c}b_{\underline{\mathfrak{n}}}\mathfrak{s}^{\underline{\mathfrak{n}}}\right]$$

by comparing coefficients of $\mathfrak{s}$

$$[\underline{\mathfrak{n}}]_{\mathfrak{q}}a_{\underline{\mathfrak{n}}}=a_{\underline{\mathfrak{n}}}+\mathfrak{c}b_{\underline{\mathfrak{n}}-1}+\sum_{\underline{\mathfrak{n}}=2}^{\infty}\sum_{i=0}^{\underline{\mathfrak{n}}-1}\mathfrak{c}b_i a_{\underline{\mathfrak{n}}-i}$$

$$[\underline{\mathfrak{n}}]_{\mathfrak{q}}a_{\underline{\mathfrak{n}}}-a_{\underline{\mathfrak{n}}}=\mathfrak{c}b_{\underline{\mathfrak{n}}-1}+\sum_{\underline{\mathfrak{n}}=2}^{\infty}\sum_{i=0}^{\underline{\mathfrak{n}}-1}\mathfrak{c}b_i a_{\underline{\mathfrak{n}}-i}.$$

$$([\underline{\mathfrak{n}}]_{\mathfrak{q}}-1)a_{\underline{\mathfrak{n}}}=\sum_{\underline{\mathfrak{n}}=2}^{\infty}\sum_{i=0}^{\underline{\mathfrak{n}}-1}\mathfrak{c}b_i a_{\underline{\mathfrak{n}}-i}\quad ,for\ b_0=1$$

$$\xi_{\underline{\mathfrak{n}}}=\frac{1}{[\underline{\mathfrak{n}}]_{\mathfrak{q}}-1}\sum_{i=1}^{\underline{\mathfrak{n}}-1}\mathfrak{c}b_i a_{\underline{\mathfrak{n}}-i}.$$

$$|a_{\underline{\mathfrak{n}}}|=\frac{1}{[\underline{\mathfrak{n}}]_{\mathfrak{q}}-1}\sum_{i=1}^{\underline{\mathfrak{n}}-1}|\mathfrak{c}||b_i||a_{\underline{\mathfrak{n}}-i}|. \tag{4.9}$$

As $\phi(\mathfrak{s})\in\mathfrak{P}^{\alpha}_{\mathfrak{q},\mathfrak{c}}[\xi,\varphi]$, so by using Lemma, we have $|b_i|\leq\mathfrak{c}\alpha(\xi,\varphi)$, for $n\geq 1$.

So, by using it in (4.9)

$$|a_{\underline{\mathfrak{n}}}|\leq\frac{\mathfrak{c}\alpha(\xi,\varphi)}{[\underline{\mathfrak{n}}]_{\mathfrak{q}}-1}\sum_{i=1}^{\underline{\mathfrak{n}}-1}|\mathfrak{c}||a_{\underline{\mathfrak{n}}-i}|.$$

$$|a_2|\leq\frac{\mathfrak{c}\alpha(\xi-\varphi)}{[2]_{\mathfrak{q}}-1}\qquad ,for\,\underline{\mathfrak{n}}=2$$

As $[2]_{\mathfrak{q}}=1+q$ , So

$$|a_2|\leq\frac{\mathfrak{c}\alpha(\xi-\varphi)}{q}=\frac{[\mathfrak{c}\alpha(\xi-\varphi)]_{2-1}}{([2]_{\mathfrak{q}}-1)!}.$$

$$|a_3|\leq\frac{\mathfrak{c}\alpha(A-B)}{[3]_{\mathfrak{q}}-1}\sum_{i=1}^{2}|a_i|\qquad ,for\ \underline{\mathfrak{n}}=3 \tag{4.10}$$

Expanding $|a_i|$ and substituting values of $|a_1|$ and $|a_2|$,

$$\sum_{i=1}^{2}|a_i|=|a_1|+|a_2|\leq 1+\frac{\mathfrak{c}\alpha(\xi-\varphi)}{q}.$$

As $[3]_{\mathfrak{q}}=1+q+q^2$. Using values of $[3]_{\mathfrak{q}}$ and $|a_i|$ in (4.22) we get,

$$|a_3| \le \frac{\mathfrak{c}\alpha(\xi-\varphi)}{q+q^2}\left[1+\frac{\mathfrak{c}\alpha(\xi-\varphi)}{q}\right].$$

For $\underline{\mathfrak{n}} = k$

$$|a_k| \le \frac{\mathfrak{c}\alpha(\xi-\varphi)}{[k]_{\mathfrak{q}}-1}\sum_{i=1}^{k-1}|a_i|.$$

As $\sum_{i=1}^{k-1}|a_i| \le \sum_{i=1}^{k-1}\frac{(\mathfrak{c}\alpha(\xi-\varphi))_{i-1}}{([i]_{\mathfrak{q}}-1)!}$. where $([i]_{\mathfrak{q}}-1)! = \prod_{j=1}^{i-1}([j]_{\mathfrak{q}}-1)$. (Pochhammer notation)

$$|a_k| \le \frac{\mathfrak{c}\alpha(\xi-\varphi)}{[k]_{\mathfrak{q}}-1}\sum_{i=1}^{k-1}\frac{(\mathfrak{c}\alpha(\xi-\varphi))_{i-1}}{\prod_{j=1}^{i-1}([j]_{\mathfrak{q}}-1)}.$$

$$|a_k| \le \frac{\mathfrak{c}\alpha(\xi-\varphi)}{[k]_{\mathfrak{q}}-1}\left[1+\frac{\mathfrak{c}\alpha(\xi-\varphi)}{[1]_{\mathfrak{q}}!}+\frac{(\mathfrak{c}\alpha(\xi-\varphi))_2}{[2]_{\mathfrak{q}}!}+\frac{(\mathfrak{c}\alpha(\xi-\varphi))_3}{[3]_{\mathfrak{q}}!}+\cdots\right].$$

$$|a_k| \le \frac{(\mathfrak{c}\alpha(\xi-\varphi))_{k-1}}{([k]_{\mathfrak{q}}-1)!} \quad , for\ \underline{\mathfrak{n}} \ge 3$$

Now for $\underline{\mathfrak{n}} = k+1$

$$|a_{k+1}| \le \frac{\mathfrak{c}\alpha(\xi-\varphi)}{[k+1]_{\mathfrak{q}}-1}\left[1+\mathfrak{c}\alpha(\xi-\varphi)+\frac{(\mathfrak{c}\alpha(\xi-\varphi))_2}{[2]_{\mathfrak{q}}}+\frac{(\mathfrak{c}\alpha(\xi-\varphi))_3}{[3]_{\mathfrak{q}}[2]_{\mathfrak{q}}}+\cdots\right].$$

$$|a_{k+1}| \le \frac{(\mathfrak{c}\alpha(\xi-\varphi))_k}{[k]_{\mathfrak{q}}!}.$$

Using Induction, we will get (4.8) and hence the proof is complete. □

**Corollary 4.3.3.1.** If we take $\xi = 1, \varphi = -1, \mathfrak{c} = 1, \chi \in \mathfrak{S}_{\mathfrak{q}}^{\alpha}[1,-1] = \mathfrak{S}_{\mathfrak{q}}(\alpha)$, then we have

$$|a_{\underline{\mathfrak{n}}}| \le \frac{(2\alpha)_{\underline{\mathfrak{n}}-1}}{([\underline{\mathfrak{n}}]_{\mathfrak{q}}-1)!}, for\ \underline{\mathfrak{n}} \ge 2$$

The bound is sharp for the function $\chi_\alpha = \frac{\mathfrak{s}}{(1-\mathfrak{s})^{2\alpha}}$ [41]. Substituting $\alpha = 1$ in above, we will get the coefficient bounds $|a_{\underline{\mathfrak{n}}}| \le \underline{\mathfrak{n}}, \underline{\mathfrak{n}} \ge 2$ for the function $\chi \in \mathfrak{S}_{\mathfrak{q}}$ with the extremal function $\chi = \frac{\mathfrak{s}}{(1-\mathfrak{s})^2}$ [19].

**Corollary 4.3.3.2.** If we take $\xi = 1, \varphi = -1$, then $\chi \in \mathfrak{S}_{\mathfrak{q},\mathfrak{c}}(\phi)$. Further, (4.8) reduces to

$$|a_{\underline{\mathfrak{n}}}| \le \frac{(2\mathfrak{c}\alpha)_{\underline{\mathfrak{n}}-1}}{([\underline{\mathfrak{n}}]_{\mathfrak{q}}-1)!}, for\ \underline{\mathfrak{n}} \ge 2$$

**Corollary 4.3.3.3.** If $\alpha = 1, \mathfrak{c} = 1$, then $\chi \in \mathfrak{S}_{\mathfrak{q}}[\xi,\varphi]$ and theorem (4.3.3) follows that

$$|a_{\underline{\mathfrak{n}}}| \le \frac{(\xi-\varphi)_{\underline{\mathfrak{n}}-1}}{([\underline{\mathfrak{n}}]_{\mathfrak{q}}-1)!}, for\ \underline{\mathfrak{n}} \ge 2$$

**Theorem 4.3.4.** If $\chi(\mathfrak{s}) \in \tilde{C}^{\alpha}_{\mathfrak{q},\mathfrak{c}}[\xi,\varphi]$ then with $\chi(\mathfrak{s}) = \mathfrak{s} + \sum_{\underline{\mathfrak{n}}=2}^{\infty} a_{\underline{\mathfrak{n}}}\mathfrak{s}^{\underline{\mathfrak{n}}}$, for $\mathfrak{s} \in \mathfrak{D}$, $-1 \leq \varphi < \xi \leq 1$, $0 < \alpha \leq 1$ and $\mathfrak{c} \neq 0$. Then for $\underline{\mathfrak{n}} \geq 2$

$$|a_{\underline{\mathfrak{n}}}| \leq \frac{(\mathfrak{c}\alpha(\xi-\varphi))_{\underline{\mathfrak{n}}-1}}{(n)_{\mathfrak{q}}!}. \tag{4.11}$$

*Proof.* Since $\chi(\mathfrak{s}) \in \tilde{C}^{\alpha}_{\mathfrak{q},\mathfrak{c}}[\xi,\varphi]$. Then,

$$1 + \frac{1}{\mathfrak{c}} \frac{\mathfrak{s}D_{\mathfrak{q}}(D_{\mathfrak{q}}\chi(\mathfrak{s}))}{D_{\mathfrak{q}}\chi(\mathfrak{s})} \in \tilde{\mathfrak{P}}^{\alpha}[\xi,\varphi],$$

suppose $l(\mathfrak{s}) = \mathfrak{s}D_{\mathfrak{q}}\chi(\mathfrak{s})$.

If $\chi(\mathfrak{s}) = \mathfrak{s} + \sum_{\underline{\mathfrak{n}}=2}^{\infty} a_{\underline{\mathfrak{n}}}\mathfrak{s}^{\underline{\mathfrak{n}}}$ then $l(\mathfrak{s}) = \mathfrak{s}D_{\mathfrak{q}}\chi(\mathfrak{s})) = \mathfrak{s}(1+\sum_{\underline{\mathfrak{n}}=2}^{\infty}[\underline{\mathfrak{n}}]_{\mathfrak{q}}a_{\underline{\mathfrak{n}}}\mathfrak{s}^{\underline{\mathfrak{n}}-1}) = \mathfrak{s} + \sum_{\underline{\mathfrak{n}}=2}^{\infty}[\underline{\mathfrak{n}}]_{\mathfrak{q}}a_{\underline{\mathfrak{n}}}\mathfrak{s}^{\underline{\mathfrak{n}}}$.

Then by theorem 4.3.3, the coefficients of $l(\mathfrak{s})$ (which are $[\underline{\mathfrak{n}}]_{\mathfrak{q}}a_{\underline{\mathfrak{n}}}$ ) must satisfy

$$|a_{\underline{\mathfrak{n}}}[\underline{\mathfrak{n}}]_{\mathfrak{q}}| \leq \frac{(\mathfrak{c}\alpha(\xi-\varphi))_{\underline{\mathfrak{n}}-1}}{([\underline{\mathfrak{n}}]_{\mathfrak{q}}-1)!} \quad for\, n \geq 2. \tag{4.12}$$

$$|a_{\underline{\mathfrak{n}}}| \cdot |[\underline{\mathfrak{n}}]_{\mathfrak{q}}| \leq \frac{(\mathfrak{c}\alpha(\xi-\varphi))_{\underline{\mathfrak{n}}-1}}{([\underline{\mathfrak{n}}]_{\mathfrak{q}}-1)!}.$$

Dividing both sides by $[\underline{\mathfrak{n}}]_{\mathfrak{q}}$we will get

$$|a_{\underline{\mathfrak{n}}}| \leq \frac{(\mathfrak{c}\alpha(\xi-\varphi))_{\underline{\mathfrak{n}}-1}}{(n)_{\mathfrak{q}}!} \quad ,for\; n \geq 2.$$

Hence the proof is complete. □

**Corollary 4.3.4.1.** For $\mathfrak{c} = 1$ and $\chi(\mathfrak{s}) \in \tilde{C}^{\alpha}_{\mathfrak{q},\mathfrak{c}}[\xi,\varphi]$, the Theorem (4.3.4) gives

$$|a_{\underline{\mathfrak{n}}}| \leq \frac{(\alpha|\xi-\varphi|)_{\underline{\mathfrak{n}}-1}}{(n)_{\mathfrak{q}}!} \quad ,for\; n \geq 2.$$

**Corollary 4.3.4.2.** For $\alpha = 1$ and $\chi(\mathfrak{s}) \in \tilde{C}^{\alpha}_{\mathfrak{q},\mathfrak{c}}[\xi,\varphi]$, the Theorem (4.3.4) gives

$$|a_{\underline{\mathfrak{n}}}| \leq \frac{(\mathfrak{c}|\xi-\varphi|)_{\underline{\mathfrak{n}}-1}}{(n)_{\mathfrak{q}}!} \quad ,for\; n \geq 2.$$

**Corollary 4.3.4.3.** If we take $\xi = 1, \varphi = -1, \chi \in \tilde{C}^{\alpha}_{\mathfrak{q},\mathfrak{c}}[1,-1] = \tilde{C}_{\mathfrak{q},\mathfrak{c}}(\alpha)$, then by (4.3.4) we have

$$|a_{\underline{\mathfrak{n}}}| \leq \frac{(2\mathfrak{c}\alpha)_{\underline{\mathfrak{n}}-1}}{[\underline{\mathfrak{n}}]_{\mathfrak{q}}!}, for\; \underline{\mathfrak{n}} \geq 2$$

**Theorem 4.3.5.** Suppose $\chi, l \in \tilde{C}^{\alpha}_{\mathfrak{q},\mathfrak{c}}[\xi,\varphi]$ and $\mathscr{H}(\mathfrak{s}) = \int_0^{\mathfrak{s}} [D_{\mathfrak{q}}\chi(t)]^{\beta} [D_{\mathfrak{q}}l(t)]^{\gamma} d_{\mathfrak{q}}t$, with $\beta + \gamma = 1$. Then $\mathscr{H}(\mathfrak{s}) \in \tilde{C}^{\alpha}_{\mathfrak{q},\mathfrak{c}}[\xi,\varphi]$.

*Proof.* Since $\mathscr{H}(\mathfrak{s}) = \int_0^{\mathfrak{s}} [D_{\mathfrak{q}}\chi(t)]^{\beta} [D_{\mathfrak{q}}l(t)]^{\gamma} d_{\mathfrak{q}}t$,

Taking q-derivative on both sides, we will get,

$$D_{\mathfrak{q}}\mathscr{H}(\mathfrak{s}) = (D_{\mathfrak{q}}\chi(\mathfrak{s}))^{\beta}(D_{\mathfrak{q}}l(\mathfrak{s}))^{\gamma}.$$

Taking $ln_{\mathfrak{q}}$ on both sides,

$$\ln_{\mathfrak{q}} D_{\mathfrak{q}}\mathscr{H}(\mathfrak{s}) = \ln_{\mathfrak{q}}(D_{\mathfrak{q}}\chi(\mathfrak{s}))^{\beta}(D_{\mathfrak{q}}l(\mathfrak{s}))^{\gamma},$$

$$\ln_{\mathfrak{q}} D_{\mathfrak{q}}\mathscr{H}(\mathfrak{s}) = \beta \ln_{\mathfrak{q}}(D_{\mathfrak{q}}\chi(\mathfrak{s})) + \gamma \ln_{\mathfrak{q}}(D_{\mathfrak{q}}l(\mathfrak{s})).$$

Applying logarithmic differentiation in q-calculus, we will get,

$$\frac{D_{\mathfrak{q}}^2\mathscr{H}(\mathfrak{s})}{D_{\mathfrak{q}}\mathscr{H}(\mathfrak{s})} = \frac{\beta D_{\mathfrak{q}}^2\chi(\mathfrak{s})}{D_{\mathfrak{q}}\chi(\mathfrak{s})} + \frac{\gamma D_{\mathfrak{q}}^2 l(\mathfrak{s})}{D_{\mathfrak{q}}l(\mathfrak{s})}. \tag{4.13}$$

As we know since $\chi, l \in \tilde{C}_{\mathfrak{q},\mathfrak{c}}^{\alpha}[\xi, \varphi]$, there exists functions $\mathfrak{p}_1, \mathfrak{p}_2 \in \mathfrak{P}_{\mathfrak{q},\mathfrak{c}}^{\alpha}[\xi, \varphi]$ such that ,

$$\mathfrak{p}_1 = 1 + \frac{1}{\mathfrak{c}} \frac{\mathfrak{s} D_{\mathfrak{q}}^2\chi(\mathfrak{s})}{D_{\mathfrak{q}}\chi(\mathfrak{s})},$$

$$\mathfrak{p}_2 = 1 + \frac{1}{\mathfrak{c}} \frac{\mathfrak{s} D_{\mathfrak{q}}^2 l(\mathfrak{s})}{D_{\mathfrak{q}}l(\mathfrak{s})}.$$

Multiply both sides of equation (4.13) with $\mathfrak{s}$

$$\mathfrak{s}\frac{D_{\mathfrak{q}}^2\mathscr{H}(\mathfrak{s})}{D_{\mathfrak{q}}\mathscr{H}(\mathfrak{s})} = \mathfrak{s}\frac{\beta D_{\mathfrak{q}}^2\chi(\mathfrak{s})}{D_{\mathfrak{q}}\chi(\mathfrak{s})} + \mathfrak{s}\frac{\gamma D_{\mathfrak{q}}^2 l(\mathfrak{s})}{D_{\mathfrak{q}}l(\mathfrak{s})}. \tag{4.14}$$

Subtracting 1 and replacing it with $\beta + \gamma$,

$$\mathfrak{s}\frac{D_{\mathfrak{q}}^2\mathscr{H}(\mathfrak{s})}{D_{\mathfrak{q}}\mathscr{H}(\mathfrak{s})} - 1 = \mathfrak{s}\frac{\beta D_{\mathfrak{q}}^2\chi(\mathfrak{s})}{D_{\mathfrak{q}}\chi(\mathfrak{s})} + \mathfrak{s}\frac{\gamma D_{\mathfrak{q}}^2 l(\mathfrak{s})}{D_{\mathfrak{q}}l(\mathfrak{s})} - \beta - \gamma, \tag{4.15}$$

Simplifying and multiplying by $\frac{1}{\mathfrak{c}}$ on both sides,

$$\frac{1}{\mathfrak{c}}\left[\frac{\mathfrak{s} D_{\mathfrak{q}}^2\mathscr{H}(\mathfrak{s})}{D_{\mathfrak{q}}\mathscr{H}(\mathfrak{s})} - 1\right] = \frac{1}{\mathfrak{c}}\beta\left[\frac{\mathfrak{s} D_{\mathfrak{q}}^2 k(\mathfrak{s})}{D_{\mathfrak{q}}k(\mathfrak{s})} - 1\right] + \frac{1}{\mathfrak{c}}\gamma\left[\frac{\mathfrak{s} D_{\mathfrak{q}}^2 l(\mathfrak{s})}{D_{\mathfrak{q}}l(\mathfrak{s})} - 1\right], \tag{4.16}$$

Adding 1 on both sides

$$1 + \frac{1}{\mathfrak{c}}\left[\frac{\mathfrak{s} D_{\mathfrak{q}}^2\mathscr{H}(\mathfrak{s})}{D_{\mathfrak{q}}\mathscr{H}(\mathfrak{s})} - 1\right] = 1 + \frac{1}{\mathfrak{c}}\beta\left[\frac{\mathfrak{s} D_{\mathfrak{q}}^2 k(\mathfrak{s})}{D_{\mathfrak{q}}k(\mathfrak{s})} - 1\right] + \frac{1}{\mathfrak{c}}\gamma\left[\frac{\mathfrak{s} D_{\mathfrak{q}}^2 l(\mathfrak{s})}{D_{\mathfrak{q}}l(\mathfrak{s})} - 1\right], \tag{4.17}$$

As $1 = \beta + \gamma$,

$$1 + \frac{1}{\mathfrak{c}}\left[\frac{\mathfrak{s} D_{\mathfrak{q}}^2\mathscr{H}(\mathfrak{s})}{D_{\mathfrak{q}}\mathscr{H}(\mathfrak{s})} - 1\right] = \beta + \gamma + \frac{\beta}{\mathfrak{c}}\left[\frac{\mathfrak{s} D_{\mathfrak{q}}^2 k(\mathfrak{s})}{D_{\mathfrak{q}}k(\mathfrak{s})} - 1\right] + \frac{\gamma}{\mathfrak{c}}\left[\frac{\mathfrak{s} D_{\mathfrak{q}}^2 l(\mathfrak{s})}{D_{\mathfrak{q}}l(\mathfrak{s})} - 1\right], \tag{4.18}$$

$$1+\frac{1}{\mathfrak{c}}\left[\frac{\mathfrak{s}D_{\mathrm{q}}^{2}\mathscr{H}(\mathfrak{s})}{D_{\mathrm{q}}\mathscr{H}(\mathfrak{s})}-1\right]=\beta\left[1+\frac{1}{\mathfrak{c}}\left[\frac{\mathfrak{s}D_{\mathrm{q}}^{2}\mathscr{H}(\mathfrak{s})}{D_{\mathrm{q}}\mathscr{H}(\mathfrak{s})}-1\right]\right]+\gamma\left[1+\frac{1}{\mathfrak{c}}\left[\frac{\mathfrak{s}D_{\mathrm{q}}^{2}\mathscr{H}(\mathfrak{s})}{D_{\mathrm{q}}\mathscr{H}(\mathfrak{s})}-1\right]\right]. \quad (4.19)$$

$$1+\frac{1}{\mathfrak{c}}\left[\frac{\mathfrak{s}D_{\mathrm{q}}^{2}\mathscr{H}(\mathfrak{s})}{D_{\mathrm{q}}\mathscr{H}(\mathfrak{s})}-1\right]=\beta\mathfrak{p}_1+\gamma\mathfrak{p}_2.$$

Since $\mathfrak{p}_1,\mathfrak{p}_2\in\mathfrak{P}^{\alpha}[\xi,\varphi]$, so $\beta\mathfrak{p}_1+\gamma\mathfrak{p}_2\in\mathfrak{P}^{\alpha}[\xi,\varphi]$. Thus,

$$1+\frac{\mathfrak{s}}{\mathfrak{c}}\frac{D_{\mathrm{q}}^{2}\mathscr{H}(\mathfrak{s})}{D_{\mathrm{q}}\mathscr{H}(\mathfrak{s})}\in\mathfrak{P}^{\alpha}[\xi,\varphi]$$

Hence $\mathscr{H}(\mathfrak{s})\in\tilde{C}^{\alpha}_{\mathrm{q},\mathfrak{c}}[\xi,\varphi]$. □

**Theorem 4.3.6.** Suppose $\chi(\mathfrak{s})\in\mathfrak{S}_{\mathrm{q},\mathfrak{c}}[\xi,\varphi]$ with $\mathfrak{s}=\mathrm{r_u}e_q^{i\theta}$ and $\gamma=\left[\frac{1-\xi}{1-\varphi}\right]^{\alpha}$, then we have the following inequality

$$exp_q\left[(1-q)\sum_{n=0}^{\infty}\frac{1+(2\mathfrak{c}(1-\gamma)-1)(\mathrm{r_u}\mathrm{q}^n)^2-(1-\gamma)2\mathfrak{c}}{1-(\mathrm{r_u}\mathrm{q}^n)^2}(\mathrm{q}^{\mathrm{n}}\mathrm{r_u})\right]\le|\chi(\mathfrak{s})|$$
$$\le exp_q\left[(1-q)\sum_{n=0}^{\infty}\frac{1+(2\mathfrak{c}(1-\gamma)-1)(\mathrm{r_u}\mathrm{q}^n)^2+(1-\gamma)2\mathfrak{c}}{1-(\mathrm{r_u}\mathrm{q}^n)^2})(\mathrm{q}^{\mathrm{n}}\mathrm{r_u})\right]. \quad (4.20)$$

$$\frac{1+[2(1-\gamma)-1]\mathrm{r_u}^2-(1-\gamma)^2\mathfrak{c}\mathrm{r_u}}{\mathrm{r_u}[1-\mathrm{r_u}^2]}e_q\left[(1-q)\sum_{n=0}^{\infty}\frac{1+(2\mathfrak{c}(1-\gamma)-1)(\mathrm{r_u}\mathrm{q}^n)^2-(1-\gamma)2\mathfrak{c}}{1-(\mathrm{r_u}\mathrm{q}^n)^2}(\mathrm{q}^{\mathrm{n}}\mathrm{r_u})\right]$$
$$\le|\mathfrak{D}_{\mathrm{q}}f(\mathfrak{s})|\le$$
$$\frac{\left[1+[2(1-\gamma)-1]\mathrm{r_u}^2+(1-\gamma)^2\mathfrak{c}\mathrm{r_u}\right]}{\mathrm{r_u}[1-\mathrm{r_u}^2]}e_q\left[(1-q)\sum_{n=0}^{\infty}\frac{1+(2\mathfrak{c}(1-\gamma)-1)(\mathrm{r_u}\mathrm{q}^n)^2+(1-\gamma)2\mathfrak{c}}{1-(\mathrm{r_u}\mathrm{q}^n)^2})(\mathrm{q}^{\mathrm{n}}\mathrm{r_u})\right]. \quad (4.21)$$

*Proof.* Since $\chi(\mathfrak{s})\in\mathfrak{S}_{\mathrm{q},\mathfrak{c}}[\xi,\varphi]$ then $\phi(\mathfrak{s})=1+\frac{1}{\mathfrak{c}}\left[\frac{\mathfrak{s}\mathfrak{D}_{\mathrm{q}}\chi(\mathfrak{s})}{\chi(\mathfrak{s})}-1\right]$. Where $\phi(\mathfrak{s})\in\mathfrak{P}^{\alpha}_{\mathrm{q},\mathfrak{c}}[\xi,\varphi]\subseteq\phi(\gamma)$, $\gamma=\left[\frac{1-\xi}{1-\varphi}\right]^{\alpha}$. As $\phi(\mathfrak{s})\in\phi(\gamma)$ there exists $\phi\in\mathfrak{P}$ such that

$$\phi(\mathfrak{s})=(1-\gamma)\mathfrak{p}_1+\gamma$$

$$\mathfrak{p}_1=\frac{\phi(\mathfrak{s})-\gamma}{(1-\gamma)}$$

$\mathfrak{p}_1(\mathfrak{s})\in\mathfrak{P}_{\mathrm{q}}(\gamma)$ can be written as

$$\left|\mathfrak{p}_1-\frac{1+\mathrm{r_u}^2}{1-\mathrm{r_u}^2}\right|\le\frac{2\mathrm{r_u}}{1-\mathrm{r_u}^2}$$

$$\left|\frac{\phi(\mathfrak{s})-\gamma}{(1-\gamma)}-\frac{1+\mathrm{r_u}^2}{1-\mathrm{r_u}^2}\right|\leq\frac{2\mathrm{r_u}}{1-\mathrm{r_u}^2}$$

Multiplying by $(1-\gamma)$ on both sides

$$\left|(1-\gamma)\frac{\phi(\mathfrak{s})-\gamma}{(1-\gamma)}-\frac{1+\mathrm{r_u}^2}{1-\mathrm{r_u}^2}(1-\gamma)\right|\leq(1-\gamma)\frac{2\mathrm{r_u}}{1-\mathrm{r_u}^2}$$

$$\left|(1-\mathrm{r_u}^2)(\phi(\mathfrak{s})-\gamma)-\frac{(1-\gamma)(1+\mathrm{r_u}^2)}{1-\mathrm{r_u}^2}\right|\leq\frac{(1-\gamma)2\mathrm{r_u}}{1-\mathrm{r_u}^2}$$

$$\left|\frac{(1-\mathrm{r_u}^2)(\phi(\mathfrak{s})-\gamma)-(1-\gamma)(1+\mathrm{r_u}^2)}{1-\mathrm{r_u}^2}\right|\leq\frac{(1-\gamma)2\mathrm{r_u}}{1-\mathrm{r_u}^2}$$

$$\left|\frac{\phi(\mathfrak{s})(1-\mathrm{r_u}^2)-\gamma(1-\mathrm{r_u}^2)-(1-\gamma)(1+\mathrm{r_u}^2)}{1-\mathrm{r_u}^2}\right|\leq\frac{(1-\gamma)2\mathrm{r_u}}{1-\mathrm{r_u}^2}$$

$$\left|\frac{\phi(\mathfrak{s})(1-\mathrm{r_u}^2)-\gamma+\mathrm{r_u}^2\gamma-1-\mathrm{r_u}^2+\gamma+\gamma\mathrm{r_u}^2)}{1-\mathrm{r_u}^2}\right|\leq\frac{(1-\gamma)2\mathrm{r_u}}{1-\mathrm{r_u}^2}$$

$$\left|\frac{\phi(\mathfrak{s})(1-\mathrm{r_u}^2)-1-\mathrm{r_u}^2+2\gamma\mathrm{r_u}^2)}{1-\mathrm{r_u}^2}\right|\leq\frac{(1-\gamma)2\mathrm{r_u}}{1-\mathrm{r_u}^2}$$

$$\left|\frac{\phi(\mathfrak{s})(1-\mathrm{r_u}^2)-1-\mathrm{r_u}^2(\mathfrak{c}1-2\gamma)}{1-\mathrm{r_u}^2}\right|\leq\frac{(1-\gamma)2\mathrm{r_u}}{1-\mathrm{r_u}^2}$$

$$\left|\phi(\mathfrak{s})-\frac{1+\mathrm{r_u}^2(\mathfrak{c}1-2\gamma)}{1-\mathrm{r_u}^2}\right|\leq\frac{(1-\gamma)2\mathrm{r_u}}{1-\mathrm{r_u}^2}$$

$$\left|\phi(\mathfrak{s})-\frac{1+(1-2\gamma)\mathrm{r_u}^2}{1-\mathrm{r_u}^2}\right|\leq\frac{(1-\gamma)2\mathrm{r_u}}{1-\mathrm{r_u}^2}. \tag{4.22}$$

Replacing $\phi(\mathfrak{s})$ with its value we will get

$$\left|1+\frac{1}{\mathfrak{c}}\left[\frac{\mathfrak{s}\mathfrak{D}_{\mathrm{q}}\chi(\mathfrak{s})}{\chi(\mathfrak{s})}-1\right]-\frac{1+(1-2\gamma)\mathrm{r_u}^2}{1-\mathrm{r_u}^2}\right|\leq\frac{(1-\gamma)2\mathrm{r_u}}{1-\mathrm{r_u}^2}$$

Multiplying both sides by b to get

$$\left|\mathfrak{c}+\left[\frac{\mathfrak{s}\mathfrak{D}_{\mathrm{q}}\chi(\mathfrak{s})}{\chi(\mathfrak{s})}-1\right]-\frac{\mathfrak{c}+(1-2\gamma)\mathfrak{c}\mathrm{r_u}^2}{1-\mathrm{r_u}^2}\right|\leq\frac{(1-\gamma)2\mathfrak{c}\mathrm{r_u}}{1-\mathrm{r_u}^2}$$

$$\left|\frac{\mathfrak{s}\mathfrak{D}_{\mathrm{q}}\chi(\mathfrak{s})}{\chi(\mathfrak{s})}+\mathfrak{c}-1-\frac{\mathfrak{c}+(1-2\gamma)\mathfrak{c}\mathrm{r_u}^2}{1-\mathrm{r_u}^2}\right|\leq\frac{(1-\gamma)2\mathfrak{c}\mathrm{r_u}}{1-\mathrm{r_u}^2}$$

$$\left|\frac{\mathfrak{s}\mathfrak{D}_{\mathfrak{q}}\chi(\mathfrak{s})}{\chi(\mathfrak{s})}-\frac{+\mathfrak{c}-\mathfrak{c}\mathrm{r_u}^2-1+\mathrm{r_u}^2-\mathfrak{c}+(1-2\gamma)\mathfrak{c}\mathrm{r_u}^2}{1-\mathrm{r_u}^2}\right|\leq\frac{(1-\gamma)2\mathfrak{c}\mathrm{r_u}}{1-\mathrm{r_u}^2}$$

Simplifying this we will get

$$\left|\frac{\mathfrak{s}\mathfrak{D}_{\mathfrak{q}}\chi(\mathfrak{s})}{\chi(\mathfrak{s})}-\frac{1+(2\mathfrak{c}(1-\gamma)-1)\mathrm{r_u}^2}{1-\mathrm{r_u}^2}\right|\leq\frac{(1-\gamma)2\mathfrak{c}\mathrm{r_u}}{1-\mathrm{r_u}^2}$$

$$-\frac{(1-\gamma)2\mathfrak{c}\mathrm{r_u}}{1-\mathrm{r_u}^2}\leq Re\frac{\mathfrak{s}\mathfrak{D}_{\mathfrak{q}}\chi(\mathfrak{s})}{\chi(\mathfrak{s})}-\frac{1+(2\mathfrak{c}(1-\gamma)-1)\mathrm{r_u}^2}{1-\mathrm{r_u}^2}\leq\frac{(1-\gamma)2\mathfrak{c}\mathrm{r_u}}{1-\mathrm{r_u}^2}$$

$$\frac{1+(2\mathfrak{c}(1-\gamma)-1)\mathrm{r_u}^2-(1-\gamma)2\mathfrak{c}\mathrm{r_u}}{1-\mathrm{r_u}^2}\leq Re\frac{\mathfrak{s}\mathfrak{D}_{\mathfrak{q}}\chi(\mathfrak{s})}{\chi(\mathfrak{s})}\leq\frac{1+(2\mathfrak{c}(1-\gamma)-1)\mathrm{r_u}^2+(1-\gamma)2\mathfrak{c}\mathrm{r_u}}{1-\mathrm{r_u}^2}. \tag{4.23}$$

We know that

$$Re\frac{\mathfrak{s}\mathfrak{D}_{\mathfrak{q}}\chi(\mathfrak{s})}{\chi(\mathfrak{s})}=\mathrm{r_u}\frac{\partial}{\partial\mathrm{r_u}}\log_{\mathfrak{q}}|\chi(\mathfrak{s})|$$

$$\frac{1+(2\mathfrak{c}(1-\gamma)-1)\mathrm{r_u}^2-(1-\gamma)2\mathfrak{c}\mathrm{r_u}}{\mathrm{r_u}(1-\mathrm{r_u}^2)}\leq\frac{\partial}{\partial\mathrm{r_u}}log_{\mathfrak{q}}|\chi(\mathfrak{s})|\leq\frac{1+(2\mathfrak{c}(1-\gamma)-1)\mathrm{r_u}^2+(1-\gamma)2\mathfrak{c}\mathrm{r_u}}{\mathrm{r_u}(1-\mathrm{r_u}^2)}. \tag{4.24}$$

Integrate with respect to $\mathrm{r_u}$

$$\int_0^{\mathfrak{s}}\frac{1+(2\mathfrak{c}(1-\gamma)-1)\mathrm{r_u}^2-(1-\gamma)2\mathfrak{c}\mathrm{r_u}}{\mathrm{r_u}(1-\mathrm{r_u}^2)}d_{\mathfrak{q}}\mathrm{r_u}\leq log_{\mathfrak{q}}|\chi(\mathfrak{s})|\leq\int_0^{\mathfrak{s}}\frac{1+(2\mathfrak{c}(1-\gamma)-1)\mathrm{r_u}^2+(1-\gamma)2\mathfrak{c}\mathrm{r_u}}{\mathrm{r_u}(1-\mathrm{r_u}^2)}d_{\mathfrak{q}}\mathrm{r_u}. \tag{4.25}$$

Applying q-integeration on (4.25) gives,

$$(1-q)\sum_{n=0}^{\infty}\frac{1+(2\mathfrak{c}(1-\gamma)-1)(\mathrm{r_u}\mathfrak{q}^n)^2-(1-\gamma)2\mathfrak{c}}{1-(\mathrm{r_u}\mathfrak{q}^n)^2}(\mathfrak{q}^{\mathrm{n}}\mathfrak{r})\leq log_{\mathfrak{q}}|\chi(\mathfrak{s})|$$
$$\leq(1-q)\sum_{n=0}^{\infty}\frac{1+(2\mathfrak{c}(1-\gamma)-1)(\mathrm{r_u}\mathfrak{q}^n)^2+(1-\gamma)2\mathfrak{c}}{1-(\mathrm{r_u}\mathfrak{q}^n)^2})(\mathfrak{q}^{\mathrm{n}}\mathfrak{r}). \tag{4.26}$$

Using q-exponential on(4.26),

$$exp_q\left[(1-q)\sum_{n=0}^{\infty}\frac{1+(2\mathfrak{c}(1-\gamma)-1)(\mathrm{r_u}\mathfrak{q}^n)^2-(1-\gamma)2\mathfrak{c}}{1-(\mathrm{r_u}\mathfrak{q}^n)^2}(\mathfrak{q}^{\mathrm{n}}\mathfrak{r})\right]\leq exp_q\left[log_{\mathfrak{q}}|\chi(\mathfrak{s})|\right]$$
$$\leq exp_q\left[(1-q)\sum_{n=0}^{\infty}\frac{1+(2\mathfrak{c}(1-\gamma)-1)(\mathrm{r_u}\mathfrak{q}^n)^2+(1-\gamma)2\mathfrak{c}}{1-(\mathrm{r_u}\mathfrak{q}^n)^2})(\mathfrak{q}^{\mathrm{n}}\mathfrak{r})\right]. \tag{4.27}$$

$$exp_q\left[(1-q)\sum_{n=0}^{\infty}\frac{1+(2\mathfrak{c}(1-\gamma)-1)(\mathrm{r_u}\mathfrak{q}^n)^2-(1-\gamma)2\mathfrak{c}}{1-(\mathrm{r_u}\mathfrak{q}^n)^2}(\mathfrak{q}^{\mathrm{n}}\mathfrak{r})\right]\leq|\chi(\mathfrak{s})|$$
$$\leq exp_q\left[(1-q)\sum_{n=0}^{\infty}\frac{1+(2\mathfrak{c}(1-\gamma)-1)(\mathrm{r_u}\mathfrak{q}^n)^2+(1-\gamma)2\mathfrak{c}}{1-(\mathrm{r_u}\mathfrak{q}^n)^2})(\mathfrak{q}^{\mathrm{n}}\mathfrak{r})\right]. \tag{4.28}$$

Using the above inequality in (4.23)

$$\frac{1+[2(1-\gamma)-1]\,\mathrm{r_u}^2-(1-\gamma)^2\mathfrak{c}\mathrm{r_u}}{\mathrm{r_u}\,[1-\mathrm{r_u}^2]}e_q\left[(1-q)\sum_{n=0}^{\infty}\frac{1+(2\mathfrak{c}(1-\gamma)-1)(\mathrm{r_u}\mathfrak{q}^n)^2-(1-\gamma)2\mathfrak{c}}{1-(\mathrm{r_u}\mathfrak{q}^n)^2}(\mathfrak{q}^{\mathrm{n}}\mathfrak{r})\right]$$
$$\leq|\mathfrak{D}_{\mathfrak{q}}f(\mathfrak{s})|\leq$$
$$\frac{\left[1+[2(1-\gamma)-1]\,\mathrm{r_u}^2+(1-\gamma)^2\mathfrak{c}\mathrm{r_u}\right]}{\mathrm{r_u}\,[1-\mathrm{r_u}^2]}e_q\left[(1-q)\sum_{n=0}^{\infty}\frac{1+(2\mathfrak{c}(1-\gamma)-1)(\mathrm{r_u}\mathfrak{q}^n)^2+(1-\gamma)2\mathfrak{c}}{1-(\mathrm{r_u}\mathfrak{q}^n)^2})(\mathfrak{q}^{\mathrm{n}}\mathfrak{r})\right]. \tag{4.29}$$

Hence the proof is complete. □

**Theorem 4.3.7.** Suppose $\chi(\mathfrak{s})\in\tilde{C}^{\alpha}_{\mathfrak{q},\mathfrak{c}}[\xi,\varphi]$ with $\mathfrak{s}=re^{i\theta}$ and $\gamma=\left[\frac{1-\xi}{1-\varphi}\right]^{\alpha}$, then we have the following inequality

$$exp_{\mathfrak{q}}\left[2\mathfrak{c}(\gamma-1)(1-q)\sum_{n=0}^{\infty}\frac{\mathrm{r_u}q^n}{1+\mathrm{r_u}q^n}\right]\leq|\mathfrak{D}_{\mathfrak{q}}\chi(\mathfrak{s})|\leq exp_q\left[2\mathfrak{c}(\gamma-1)(1-q)\sum_{n=0}^{\infty}\frac{\mathrm{r_u}q^n}{1-\mathrm{r_u}q^n}\right]. \tag{4.30}$$

$$exp_{\mathfrak{q}}\left[2\mathfrak{c}(\gamma-1)(1-q)\sum_{n=0}^{\infty}\frac{\mathrm{r_u}q^n}{1+\mathrm{r_u}q^n}\right]\mathrm{r_u}\leq|\chi(\mathfrak{s})|\leq exp_{\mathfrak{q}}\left[2\mathfrak{c}(\gamma-1)(1-q)\sum_{n=0}^{\infty}\frac{\mathrm{r_u}q^n}{1-\mathrm{r_u}q^n}\right]\mathrm{r_u}. \tag{4.31}$$

*Proof.* Since $\chi(\mathfrak{s})\in\tilde{C}^{\alpha}_{\mathfrak{q},\mathfrak{c}}[\xi,\varphi]$ then $\phi(\mathfrak{s})=1+\frac{1}{\mathfrak{c}}\left[\frac{\mathfrak{s}\mathfrak{D}_{\mathfrak{q}}(\mathfrak{D}_{\mathfrak{q}}\chi(\mathfrak{s}))}{\mathfrak{D}_{\mathfrak{q}}\chi(\mathfrak{s})}-1\right]$. Where $\phi(\mathfrak{s})\in\mathfrak{P}^{\alpha}_{\mathfrak{q},\mathfrak{c}}[\xi,\varphi]\subseteq\phi(\gamma)$, $\gamma=\left[\frac{1-\xi}{1-\varphi}\right]^{\alpha}$. As $\phi(\mathfrak{s})\in\mathfrak{P}_{\mathfrak{q}}(\gamma)$ there exists $\phi\in\mathfrak{P}_{\mathfrak{q}}$ such that

$$\phi(\mathfrak{s})=(1-\gamma)\mathfrak{p}_1+\gamma$$

$\mathfrak{p}_1(\mathfrak{s})\in\phi(\gamma)$ can be written as

$$\left|\mathfrak{p}_1-\frac{1+\mathrm{r_u}^2}{1-\mathrm{r_u}^2}\right|\leq\frac{2\mathrm{r_u}}{1-\mathrm{r_u}^2}$$

By (4.22) we can write,

$$\left|\phi(\mathfrak{s})-\frac{1+(1-2\gamma)\mathrm{r_u}^2}{1-\mathrm{r_u}^2}\right|\leq\frac{(1-\gamma)2\mathrm{r_u}}{1-\mathrm{r_u}^2}.$$

Replacing $\phi(\mathfrak{s})$ with its value we will get

$$\left|1+\frac{1}{\mathfrak{c}}\left[\frac{\mathfrak{s}\mathfrak{D}_{\mathfrak{q}}(\mathfrak{D}_{\mathfrak{q}}\chi(\mathfrak{s}))}{\mathfrak{D}_{\mathfrak{q}}\chi(\mathfrak{s})}\right]-\frac{1+(1-2\gamma)\mathrm{r_u}^2}{1-\mathrm{r_u}^2}\right|\leq\frac{(1-\gamma)2\mathrm{r_u}}{1-\mathrm{r_u}^2}$$

By simplifying this we will get

$$\left|\frac{\mathfrak{s}\mathfrak{D}_{\mathfrak{q}}(\mathfrak{D}_{\mathfrak{q}}\chi(\mathfrak{s}))}{\mathfrak{D}_{\mathfrak{q}}\chi(\mathfrak{s})}-\frac{(2(1-\gamma)\mathrm{r_u}^2}{1-\mathrm{r_u}^2}\right|\leq\frac{(1-\gamma)2\mathfrak{c}\mathrm{r_u}}{1-\mathrm{r_u}^2}$$

$$-\frac{(1-\gamma)2\mathfrak{c}\mathrm{r_u}}{1-\mathrm{r_u}^2}\leq\frac{\mathfrak{s}\mathfrak{D}_{\mathfrak{q}}(\mathfrak{D}_{\mathfrak{q}}\chi(\mathfrak{s}))}{\mathfrak{D}_{\mathfrak{q}}\chi(\mathfrak{s})}-\frac{(2(1-\gamma))\mathrm{r_u}^2}{1-\mathrm{r_u}^2}\leq\frac{(1-\gamma)2\mathfrak{c}\mathrm{r_u}}{1-\mathrm{r_u}^2}$$

$$\frac{(2\mathfrak{c}(\gamma-1)\mathrm{r_u}}{(1+\mathrm{r_u})}\leq\frac{\mathfrak{s}\mathfrak{D}_{\mathfrak{q}}(\mathfrak{D}_{\mathfrak{q}}\chi(\mathfrak{s}))}{\mathfrak{D}_{\mathfrak{q}}\chi(\mathfrak{s})}\leq\frac{(2\mathfrak{c}(1-\gamma)\mathrm{r_u}}{(1-\mathrm{r_u})}$$

We know that

$$Re\frac{\mathfrak{s}\mathfrak{D}_{\mathfrak{q}}(\mathfrak{D}_{\mathfrak{q}}\chi(\mathfrak{s}))}{\mathfrak{D}_{\mathfrak{q}}\chi(\mathfrak{s})}=\mathrm{r_u}\frac{\partial}{\partial\mathrm{r_u}}log_{\mathfrak{q}}|\mathfrak{D}_{\mathfrak{q}}\chi(\mathfrak{s})|$$

$$\frac{(2\mathfrak{c}(\gamma-1)\mathrm{r_u}}{\mathrm{r_u}(1+\mathrm{r_u})}\leq\frac{\partial}{\partial\mathrm{r_u}}log_{\mathfrak{q}}|\mathfrak{D}_{\mathfrak{q}}\chi(\mathfrak{s})|\leq\frac{2\mathfrak{c}(1-\gamma)\mathrm{r_u}}{(1-\mathrm{r_u})\mathrm{r_u}}. \tag{4.32}$$

Integrate with respect to $\mathrm{r_u}$

$$\int_0^{\mathfrak{s}}\frac{2\mathfrak{c}(1-\gamma)}{(1+\mathrm{r_u})}\mathfrak{D}_{\mathfrak{q}}\mathrm{r_u}\leq log_{\mathfrak{q}}|\mathfrak{D}_{\mathfrak{q}}\chi(\mathfrak{s})|\leq\int_0^{\mathfrak{s}}\frac{2\mathfrak{c}(1-\gamma)}{(1-\mathrm{r_u})}\mathfrak{D}_{\mathfrak{q}}\mathrm{r_u}. \tag{4.33}$$

$$2\mathfrak{c}(\gamma-1)(1-q)\sum_{n=0}^{\infty}\frac{\mathrm{r_u}q^n}{1+\mathrm{r_u}q^n}\leq log_{\mathfrak{q}}|\mathfrak{D}_{\mathfrak{q}}\chi(\mathfrak{s})|\leq-2\mathfrak{c}(\gamma-1)(1-q)\sum_{n=0}^{\infty}\frac{\mathrm{r_u}q^n}{1-\mathrm{r_u}q^n}$$

$$\begin{aligned}exp_q\left[2\mathfrak{c}(\gamma-1)(1-q)\sum_{n=0}^{\infty}\frac{\mathrm{r_u}q^n}{1+\mathrm{r_u}q^n}\right]&\leq exp_q\left[log_{\mathfrak{q}}|\mathfrak{D}_{\mathfrak{q}}\chi(\mathfrak{s})|\right]\\&\leq exp_q\left[-2\mathfrak{c}(\gamma-1)(1-q)\sum_{n=0}^{\infty}\frac{\mathrm{r_u}q^n}{1-\mathrm{r_u}q^n}\right].\end{aligned} \tag{4.34}$$

$$exp_{\mathfrak{q}}\left[2\mathfrak{c}(\gamma-1)(1-q)\sum_{n=0}^{\infty}\frac{\mathrm{r_u}q^n}{1+\mathrm{r_u}q^n}\right]\leq|\mathfrak{D}_{\mathfrak{q}}\chi(\mathfrak{s})|\leq exp_q\left[2\mathfrak{c}(\gamma-1)(1-q)\sum_{n=0}^{\infty}\frac{\mathrm{r_u}q^n}{1-\mathrm{r_u}q^n}\right].$$

Hence we got (4.30). Integrate over $\mathrm{r_u}$ where $|\mathfrak{s}|=\mathrm{r_u}$

$$\begin{aligned}\int_0^{\mathfrak{s}}exp_q\left[2\mathfrak{c}(\gamma-1)(1-q)\sum_{n=0}^{\infty}\frac{\mathrm{r_u}q^n}{1+\mathrm{r_u}q^n}\right]\mathfrak{D}_{\mathfrak{q}}(\mathrm{r_u})&\leq|\mathfrak{D}_{\mathfrak{q}}\chi(\mathfrak{s})|\\&\leq\int_0^{\mathfrak{s}}exp_q\left[2\mathfrak{c}(\gamma-1)(1-q)\sum_{n=0}^{\infty}\frac{\mathrm{r_u}q^n}{1+\mathrm{r_u}q^n}\right]\mathfrak{D}_{\mathfrak{q}}(\mathrm{r_u}).\end{aligned} \tag{4.35}$$

After integrating (4.34), the inequality will become,

$$exp_{\mathfrak{q}}\left[2\mathfrak{c}(\gamma-1)(1-q)\sum_{n=0}^{\infty}\frac{\mathrm{r_u}q^n}{1+\mathrm{r_u}q^n}\right]\mathrm{r_u}\leq|\chi(\mathfrak{s})|\leq exp_{\mathfrak{q}}\left[2\mathfrak{c}(\gamma-1)(1-q)\sum_{n=0}^{\infty}\frac{\mathrm{r_u}q^n}{1-\mathrm{r_u}q^n}\right]\mathrm{r_u}.$$

Hence the proof is complete. □

**Theorem 4.3.8.** Suppose $\chi(\mathfrak{s}) \in \tilde{C}^{\alpha}_{\mathfrak{q},\mathfrak{c}}[\xi,\varphi]$. Then $\chi(\mathfrak{s})$ maps $|\mathfrak{s}| < \phi$ onto a convex domain, where $\phi = (1-\varphi)^{\alpha}\left[\mathfrak{c}\left[\frac{1-\xi}{1-\varphi}\right]^{\alpha} + (1-\mathfrak{c})\right]$ for $-1 \leq \varphi < \xi \leq 1$, $\mathfrak{c} \neq 0$ and $0 < \alpha \leq 1$.

*Proof.* Suppose $\chi(\mathfrak{s}) \in \tilde{C}^{\alpha}_{\mathfrak{q},\mathfrak{c}}[\xi,\varphi]$,So for any $\phi(\mathfrak{s}) \in \mathfrak{P}^{\alpha}[\xi,\varphi]$, we have,

$$1 + \frac{1}{\mathfrak{c}}\frac{\mathfrak{s}D^2_{\mathfrak{q}}\chi(\mathfrak{s})}{D_{\mathfrak{q}}\chi(\mathfrak{s})} = \phi(\mathfrak{s}).$$

Multiplying b on both sides yields,

$$\mathfrak{c} + \frac{1}{\mathfrak{c}}\frac{\mathfrak{s}D^2_{\mathfrak{q}}\chi(\mathfrak{s})}{D_{\mathfrak{q}}\chi(\mathfrak{s})} = \mathfrak{c}\phi(\mathfrak{s}),$$

$$\frac{1}{\mathfrak{c}}\frac{\mathfrak{s}D^2_{\mathfrak{q}}\chi(\mathfrak{s})}{D_{\mathfrak{q}}\chi(\mathfrak{s})} = \mathfrak{c}\phi(\mathfrak{s}) - \mathfrak{c}.$$

Adding 1 on both sides results in,

$$1 + \frac{1}{\mathfrak{c}}\frac{\mathfrak{s}D^2_{\mathfrak{q}}\chi(\mathfrak{s})}{D_{\mathfrak{q}}\chi(\mathfrak{s})} = 1 + \mathfrak{c}\phi(\mathfrak{s}) - \mathfrak{c},$$

$$Re\left[1 + \frac{1}{\mathfrak{c}}\frac{\mathfrak{s}D^2_{\mathfrak{q}}\chi(\mathfrak{s})}{D_{\mathfrak{q}}\chi(\mathfrak{s})}\right] = Re\left[1 + \mathfrak{c}\phi(\mathfrak{s}) - \mathfrak{c}\right].$$

As $\phi(\mathfrak{s}) \in \mathfrak{P}^{\alpha}[\xi,\varphi]$, then by using Lemma(3.2.2), we will have

$$Re\left[1 + \frac{1}{\mathfrak{c}}\frac{\mathfrak{s}D^2_{\mathfrak{q}}\chi(\mathfrak{s})}{D_{\mathfrak{q}}\chi(\mathfrak{s})}\right] \geq 1 + b\left[\frac{1-\xi(\mathrm{r_u})}{1-\varphi(\mathrm{r_u})}\right] - \mathfrak{c} \geq \frac{\mathfrak{c}(1-\xi(\mathrm{r_u}))^{\alpha} + (1-\mathfrak{c})(1-\varphi(\mathrm{r_u}))^{\alpha}}{(1-\varphi(\mathrm{r_u}))^{\alpha}}.$$

Let $\mathfrak{c}(1-\xi(\mathrm{r_u}))^{\alpha} + (1-\mathfrak{c})(1-\varphi(\mathrm{r_u}))^{\alpha} = \mathfrak{h}(\mathrm{r_u})$.

Then $\mathfrak{h}(\mathfrak{o}) = 1$ and $\mathfrak{h}(1) = \mathfrak{c}(1-\xi)^{\alpha} + (1-\mathfrak{c})(1-\varphi)^{\alpha} = (1-\varphi)^{\alpha}\left[\mathfrak{c}\left[\frac{1-\xi}{1-\varphi}\right]^{\alpha} + (1-\mathfrak{c})\right]$.

So, $Re\left[1 + \frac{1}{\mathfrak{c}}\frac{\mathfrak{s}D^2_{\mathfrak{q}}\chi(\mathfrak{s})}{D_{\mathfrak{q}}\chi(\mathfrak{s})}\right] > 0$ for $|\mathfrak{s}| < \phi$ where $\phi = (1-\varphi)^{\alpha}\left[\mathfrak{c}\left[\frac{1-\xi}{1-\varphi}\right]^{\alpha} + (1-\mathfrak{c})\right]$. It is the least positive root of the equation $\mathfrak{c}(1-\xi)^{\alpha} + (1-\mathfrak{c})(1-\varphi)^{\alpha} = 0$. □

This chapter introduced new classes of strongly Janowski-type functions of complex order and established their key geometric properties, including growth results, distortion bounds, coefficient estimates, and the radius of convexity. Their behaviour under suitable integral operators was also examined. The findings extend the classical Janowski framework and show how the $\mathfrak{q}$-calculus setting provides broader generalisations and deeper geometric insight through rigorous analytical proofs.

# CHAPTER 5

# CONCLUSION

This thesis explores the inclusion properties, coefficient bounds, some integral properties, distortion theorem, and radius of convexity for analytic, univalent, and normalized functions in the open unit disk. This study examines the fundamentals of Geometric Function Theory and $q$-calculus, including the application of the $q$-derivative and integral operator to analytic functions. We provide a new family of $q$-starlike and convex functions of Janowski type function of complex order using the $q$-difference operator.

This study investigated the $q$-starlike and $q$-convex classes, which correspond to the Janowski starlike $\mathfrak{S}_{\mathfrak{c}}[\xi,\varphi]$ and convex $\tilde{C}_{\mathfrak{c}}^{\beta}[\xi,\varphi]$ classes of complex order, and their $q$-extension. Khalil Ahmad, Khudija BiBi, and M. Sajjad Shabbir established these classical Janowski classes, and this research extends them using $q$-calculus principles. The classes $\mathfrak{S}_{\mathfrak{q},\mathfrak{c}}[\xi,\varphi]$ and $\tilde{C}_{\mathfrak{q},\mathfrak{c}}^{\beta}[\xi,\varphi]$ represent $q$-starlike and convex functions of Janowski type functions of complex order. The $q$-difference operator is used to introduce these classes, which are then investigated using geometric approaches.

The significant aspects of the recently discovered functions, including inclusion characteristics, distortion theorems, coefficient limits, and radius of convexity, have been thoroughly analyzed. Additionally, the analytic characteristics of these classes under certain integral operators have been explored. Our newly developed class is more advanced and comprehensive, offering refinement compared to the existing ones.

**In comparison with previous works,** Ahmad et al. established Janowski starlike and convex functions of complex order without considering the $q$-calculus framework. Our research extends these classical results by introducing $q$-starlike and $q$-convex analogues using the $q$-difference operator. This extension enables the derivation of more generalized coefficient bounds, inclusion relations, distortion estimates, and radius of convexity within the $q$-calculus context. By taking the limit $\mathtt{q} \to 1$, the classical Janowski results are recovered, demonstrating that our work not only preserves but also significantly broadens the applicability and geometric insight of the original framework.

The conclusions improve upon previously established theorems and validate the results through the $\mathtt{q} \to 1$ limit, which reproduces known classical results. This work aims to make significant contributions to Geometric Function Theory and lay the groundwork for future advancements and breakthroughs.

## 5.1 Future work

This chapter introduced new classes of strongly Janowski-type functions of complex order and established their key geometric properties, including growth results, distortion bounds, coefficient estimates, and the radius of convexity. Their behaviour under suitable integral operators was also examined. The findings extend the classical Janowski framework and show how the q-calculus setting provides broader generalisations and deeper geometric insight through rigorous analytical proofs.

Building on these results, future work may include extending these classes using other q-differential or q-integral operators to uncover additional geometric structures unique to the q-setting. Further investigation of extremal problems, convolution properties, or subordination conditions could also address open questions in q-geometric function theory. Since q-analogues naturally appear in quantum calculus, combinatorial models, and areas of mathematical physics involving discrete or deformed structures, such extensions may lead to meaningful applications in the study of special functions, quantum groups, and discrete dynamical systems. These directions provide a promising pathway for advancing both the theoretical depth and the applied relevance of q-geometric function theory.

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