TELEPARALLEL KILLING VECTOR FIELDS OF STATIC SPHERICALLY SYMMETRIC SPACE-TIMES IN *f(T)* GRAVITY

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Abstract

Title: Teleparallel Killing Vector Fields of Static Spherically Symmetric Space-Times in $f(T)$ gravity.

Symmetries principle play a crucial role in solving various problems,as they underpin conservation laws and fundamental interactions. The connection between conservation principle and symmetry in physics is both fundamentally important and highly useful. Curved space-time symmetries are produced by Killing vectors, also known as isometries. These symmetries aid in the classification and solution solving in connection with Einstein field equations (EFEs). As a result, symmetries are important for explaining space-time geometry. A fascinating theory that has gained traction in recent decades is teleparallel gravity (TG) in which torsion takes the place of curvature. It accomplishes by this replacing the Levi-Civita connection, which is built on curvature, with a teleparallel connection, which is based on torsion. This thesis provides a comprehensive analysis of Teleparallel Killing vector fields (TKVFs) of static spherically symmetric space-times within the framework of $f(T)$ gravity, an extended theory of gravity based on torsion rather than curvature. Static spherically symmetric solutions to Einstein field equations in $f(T)$ gravity have already been existed in the literature.The classification of those solutions via TKVFs have been done. In this study, 20 distinct solutions have been explored. Ten coupled partial differential equations were obtained for each solution. To determine the TKVFs these equations were solved using the direct integration technique. Teleparallel Killing vector fields were found in all those 20 cases. Every scenario is thoroughly examined to investigate the ways in which the changed gravity framework affects the presence and characteristics of teleparallel Killing vector fields. The study concludes with a thorough analysis of the find-

ings that emphasizes the main distinctions from general relativity.

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Dedication

This thesis is dedicated to my parents, and my supervisor who always supported and taught me to work hard for the things that I aspire to achieve. All of them have been a source of motivation and strength during moments of despair and discouragement

Chapter 1

The Theory of General Relativity

1.1 Introduction

In the interplay of gravity and geometry, mathematics helps us understand the beautiful order of the universe — this thesis begins that exploration. In modern theoretical physics, one of the most essential areas of study has been gravitational theory. Science has always relied heavily on experimentation in physics. When a new phenomena in nature is found that cannot be explained, there are two possible explanations: either there is something new and unexplored in physics, or our comprehension of the established theory is lacking. It is noteworthy that the most mysterious interaction is still gravity, despite the fact that it is the easiest to explain without specialist knowledge and is highly connected to everyday happenings. Actually, since it was so simple to construct an appropriate experimental setup, gravitational interaction was the first to be investigated through the prism of experimental research [1].

1.2 Newton Theory of Gravity

Newton introduced the idea of a hard body, or particle, in his Principia. He clarified that mass is the result of volume times density. Since density was not adequately defined when it was rejected by many critics as a fake definition, this definition may seem tautological. This method, while not illogical, is unable to restructure Newton's theory of the structure of matter, starting with the idea of atoms. In Newton's view, matter consists of extremely tiny particles, or atoms. The advancement of atomic physics in the nineteenth century effectively validated this idea. Newton was the first to understand the significance of a product's speed, which we refer to as its momentum. This quantity remains constant in the absence of outside influences. It was initially noted by Galileo that the Aristotelian principle needed to be revised. Galileo conducted numerous experiments to determine that a body that is not affected by an outside force can yet retain its motion. It never stops, but it keeps moving forever [2]. Galileo Galilei used pendulums and inclined planes to perform ground-breaking research on terrestrial gravity in the late 16th century [1]. The movements of all the bodies in a particular space move in the same way, whether the space is moving in a straight line without any circular motion or is at rest [3].

1.3 Special Relativity

Two physical theories are included in the term relativity. Established in 1905, the more traditional theory known as special relativity describes electromagnetic and mechanical events occurring in a reference system that moves rapidly in relation to an observer but is unaffected by gravitational forces. As a closed theory, it is regarded. In 1915, General Relativity was published. Under the effect of gravity, it explained the core principles governing time alongside space, as well as mechanical and electromagnetic processes [4]. A thorough analysis of Einstein's special relativity theory and his writings from modern scientific and philosophical viewpoints revealed that, at the turn of the 20th century, these viewpoints were not fully developed to comprehend the issues that physicists were facing and that Einstein attempted to address with his theory. No matter how great a scientist he was, Einstein was constrained in his quest by false philosophical beliefs that were common at the time. The principles of Einstein's special relativity theory, such as the invariance of the velocity of light, the relativity of simultaneity, and the principle of relativity, can no longer be defended. Consequently, Einstein's endeavor to integrate light and electromagnetic with mechanics, his notion of light, space, and time, as well as the entirety of relativity and its ramifications, are unable to accurately depict the actualities of the physical universe [5]. As the father of relativity, Einstein is consistently portrayed. Both special and general relativity are viable theories. Electron microscopy (EM) is described by a set of formulas called Maxwell's equations (1873). These computations show that our common concept of speed—that the speed of light is constant—is flawed. It is resolved by special relativity. In 1892, the Lorentz transformation was introduced as the first step towards special relativity (Lorentz, 1892). Notably, Einstein said that Lorentz's basic research was necessary for him to discover special relativity. When a clock approaches the speed of light, it appears to run more slowly than when it is at rest due to the Lorentz transformation, which alters the flow of time. We term this dilation of time. Lorentz presented his work in 1892. Poincaré (1900) then worked on the synchronization of clocks with the notion that time-flow is changing and came to the conclusion that simultaneity is lost if time-flow varies. Einstein's initial works on special relativity, which defended the Lorentz transformation, were published

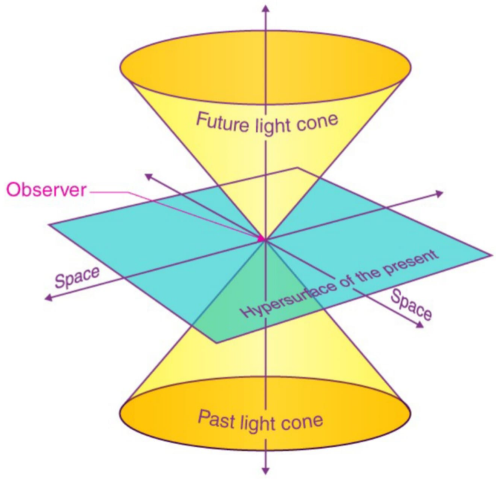

Figure 1: Light Cone

in 1905 (Einstein, 1905) [6]. The relationship between space and time is stated by the physics theory known as special relativity or the special theory of relativity. This is commonly referred to as STR hypothesis. The special theory of relativity is based on two basic postulates.

(1) The laws of physics are invariant.

(2) In a vacuum, light travels at the same speed in every other space, independent of the light source.

Entitled "On the Electrodynamics of Moving Bodies," Albert Einstein first put out this idea in 1905. Special relativity entails the following concepts: universal speed limit, relativity of simultaneity, mass-energy equivalency, and length contraction. The conventional notion of an absolute universal time is replaced by a time that depends on the physical context and reference frame.

The special theory of relativity's definitions are based on the idea that light moves at a constant speed in a vacuum. The shape of causality and the four-dimensional continuum that surrounds tiny phenomena are determined by this principle. These concepts from relativity theory are carried over to a distinct universe in contemporary theory, which is fundamentally different from the world of elementary particles. However, the history of science also teaches us that qualitative changes in the physical laws typically coincide with a shift in scale. As a result, it stands to reason that there would be an elementary length that acts as a scale for the region of space-time where the structure of space-time may, in theory, differ from that which is known to us in the macroworld [7]. The special theory of relativity, formulated by Einstein, governs motion at relative velocities below the speed of light. Above all, the new transformations give well-defined formulas without introducing complicated physics or fictitious masses at all. They behave singularly when the relative velocity becomes close to the speed of light and function within the same mathematical framework as the Lorentz transformation. [8]. The Special Theory of Relativity is generalized into the General Theory of Relativity, as the name suggests. Einstein created it, and it is unquestionably one of the most amazing scientific discoveries to date. The scientific work of Einstein was situated within this theoretical framework of physics at the start of the twentieth century:

a. The connection imposed by Michelson-Morley's experiment, which demonstrated the absence of interference and, consequently, the absence of ether.

b. The widely held belief among physicists that Newton's absolute reference frame, from which Fitzgerald-Poincarè-Lorentz's scientific publications were formed, is necessary to preserve the ether and, by extension, the absolute reference frame.

c. Maxwell's finished electromagnetic theory created doubts about the constancy of the speeds of electromagnetic waves and light because it appeared to go against the Relativity principle [9].

1.4 General Relativity

In 1915, Einstein essentially finished the theory of General Relativity (GR), which encompassed gravity and any accelerating frame. According to his theory, gravity is a geometric phenomena because matter and energy shape space-time. The idea that one particle is drawn to another is produced by the way that energy sources curve space-time. A malleable space-time is proposed in place of Newtonian gravity, which is based on the gravitational force. A number of assumptions are necessary for it to work, including the equivalency principle, which holds that gravitational fields and accelerated frames are equivalent [10]. GR has shown to be a very fruitful theory. It has undergone testing at Solar System scales with good precision. Additionally, additional phenomena that are absent from normal Newtonian gravity are described by GR. Black holes, gravitational waves, gravitational lensing, and redshift are a few of them. A few of these impacts have been measured in various contexts. Moreover, the discovery of gravitational waves in 2016 provided the final unchallenged confirmation of general relativity (GR) and opened up a new window for astronomical observations [11, 12]. General relativity built upon special relativity as a more comprehensive theory. Its validity was surprisingly corroborated by experimental evidence such as the Lense-Thirring effect, the orbital precession of Mercury, and the gravitomagnetic wave precession (1918) .[13–15]. Albert Einstein made a connection between space-time and gravity through the theory of equivalency. This signified the introduction of a new paradigm in understanding gravity, meant to confront many lingering mysteries—though not all of them [16]. GR has proven to be a highly effective hypothesis. The discovery of gravitational waves in 2016 also provided us with the final unproven confirmation of general relativity (GR) and a new window for astronomical observations [11].

1.5 Teleparallel Killing Vector Field

Symmetry principles play a crucial role in solving various physics problem. The connection between conservation principles and symmetry in physics is both fundamentally important and highly useful. Killing vectors, also referred to as isometries, are the source of curved space-time symmetries. Other space-time symmetries include matter collineations (MCs), curvature collineations (CCs), and Ricci collineations (RCs). These symmetries aid in the classification and solution solving in connection with Einstein field equations (EFEs). As a result, symmetries are important for explaining space-time geometry [17]. A fascinating theory that has gained traction in recent decades, teleparallel gravity (TG) has emerged as a prominent topic in the literature, in which torsion takes the place of curvature as the method via which geometric distortion results in a gravitational field. It accomplishes by this replacing the Levi-Civita connection, which is built on curvature, with a teleparallel connection, which is based on torsion. There are actually thousands of papers on the topic in the literature at current time. Torsion-based gravity has given rise to a number of theories, including the teleparallel equivalent of general relativity (TEGR), which is dynamically similar to GR but indiscernible by conventional testing[18]. General relativity's teleparallel formulation, which we'll just call teleparallel gravity from here on, allows for an alternative scientific explanation of the gravitational interaction in terms of torsion rather than curvature. Its ability to understand general relativity as a gauge theory has drawn attention in the past. In an attempt to combine gravity and electromagnetic that failed, Einstein proposed the teleparallel theory of gravity (TPG). This is seen as an alternate interpretation of gravity that relates to a Weitzenb*ö*ck geometry-based gauge theory for the translation group. In this case, the curvature tensor vanishes in the same way that when the torsion is not zero assumes the function of force. This method eliminates geodesics and uses force equations to characterize the gravitational interaction that resemble the Lorentz force in electrodynamics explained mathematically. As a result, we are able to state that gravitational pull might be expressed either in the context of torsion, as in TPG, in the context of curvature, as in GR [17]. It wasn't long after Einstein's general theory of relativity was developed that modifications were investigated. Constructing a cogent geometric framework that could incorporate the two then-known forces of nature, gravity and electromagnetic, was the main objective of these early studies. Late in the 1920s, Einstein [19] tried to unite electromagnetic and gravity through the use of the teleparallelism (sometimes called absolute parallelism) mathematical framework. Teleparallalism's fundamental ability is in order to determine the angle between two distant vectors. In particular, Einstein suggested the creation of the tetrad field, a field of Tangent space vectors with orthonormal bases that is produced at every point on the structure of space-time in four dimensions. According to Einstein, there should be a correlation between the six components of the electromagnetic field and six extra field properties, as the tetrad comprises 16 linearly independent components, whereas the metric consists of only 10. It was ultimately discovered that the additional components are connected to the theory's Lorentz invariance, but regrettably, this attempt likewise failed. Despite the first unification attempt's failure, teleparallel gravity—a novel description of gravity—was developed.

1.6 Generalization of Teleparallel Gravity

Recent efforts have focused on extending teleparallel gravity frameworks,particularly through $f(T)$ gravity to generalize Einstein theory of general relativity. The expansion of the theories' Lagrangian was expressed as $f(T)$, where T is the Lagrangian of teleparallel gravity and f is a suitably differentiable function. The claim that these theories' dynamics deviate from general relativity's while maintaining second-order derivative in their equations piqued interest because it meant that these theories might be able to explain the universe's accelerated expansion while avoiding pathologies [20]. It has been common practice to match two distinct space-times in order to explain certain intriguing physical phenomena or to build new models that shed light on theoretical issues that the original divided models were unable to address. Analyzing the relationship between two different and well-known space-times allows one to investigate models of cosmic inhomogeneities, gravitational wave interactions, collapsing or expanding stars, and other related phenomena.

1.7 Literature Review

The study of teleparallel gravity has gained significant attention as an alternative to general relativity, particularly in the context of modified gravity theories. In order to provide insight into significant advancements, commonly used approaches, and the urgent need for more research within $f(T)$ gravity, this review of the literature attempts to summarize previous studies on (TKVFs) in static spherically symmetric space-times. Here are the work of some authors. Fayos gave his general results on matching spherically symmetric space-times through a time like hypersurface are presented in this study, exposing straightforward requirements for viability. It reveals all feasible models by applying these techniques to general flat Robertson-Walker space-time matching and Vaidya's radiating metric. These models, which were not previously taken into consideration, explain intriguing physical circumstances [21]. In [22] the author discussed the stationary solutions in massive gravity that have spherically symmetry and are produced by "stars" that have regular interiors. It refutes the idea that discontinuity close to the source can be healed by nonlinear effects. Numerical investigation shows that for tiny m, solutions lead to singularities. On the other hand, a unique class of solutions that resemble general relativistic solutions and have spontaneous symmetry breaking characteristics are discovered. In [23] the author investigated spherically symmetric space-time properties, with particular attention to a practical metric in polar-areal coordinates. It draws attention to the Hawking mass's monotonicity, its consistency with Newtonian mechanics, and its use in resolving the spherically symmetric Einstein-Klein-Gordon equations. Wagh S, M discussed in [24] that the idea of self-similarity in spherically symmetric space-times results in a freely distributed radial matter profile, a separate metric with non-vanishing energy flux and shear, and the separability of the space-time metric using co-moving coordinates. Complete analytical solutions in terms of Kleinian sigma functions are offered for the geodesic equation of heavy test particles in higher dimensions. The particle's energy, mass, angular momentum, gravitational source charge, and cosmological constant all affect its orbits. For orbits up to 11 dimensions, when the escape and bound orbits cross separate universes, the solution is explicit [25]. For four-dimensional $f(T)$ gravitational theories, Schwarzschild geometry continues to be a vacuum solution, functioning as ultraviolet deformations of general relativity. The vacuum solutions of infrared-deformed $f(T)$ gravities are circularly symmetric, and their effective cosmological constant varies with infrared scale [26]. According to Wei, S. W. [27] particle orbits with distinct characteristics can yield distinct traces of gravitational observable occurrences, which can be helpful in evaluating compact astrophysical objects in general relativity or altered theories of gravity. Furthermore, it demonstrates that in an asymptotically flat space-time, stable and unstable static spheres are always found in pairs. This paper focuses on teleparallel conformal Killing vector fields (CKVFs) in non-static plane-symmetric spacetimes. The ten derived linear CKVF equations are given a generic solution under integrability requirements. Four examples give accurate CKVFs, but three show that CKVFs drop to teleparallel homothetic or Killing vector fields.[28].

In [29] the author compares teleparallel conformal Killing vector fields (CKVFs) with general relativity in LRS Bianchi type V space-times. Killing's equations have a generic solution, and in one instance the LRS Bianchi type V space-times admit proper CKVF, whereas in the other cases KVFs are found. Ganiou, M. G. [30] uses flat Friedmann-Robertson-Walker equations to examine autonomous dynamical systems within the context of gravity. The analysis of these systems in vacuum and non-vacuum gravities is the main emphasis of the work. In the quasi-de Sitter inflationary age, the study finds stable de Sitter attractors and unstable fixed points; unstable fixed points indicate eras dominated by matter, whereas stable attractor fixed points describe eras dominated by dark energy. In [17] Teleparallel theory of gravitation has been used to assess Killing vectors of spherically space-times. We also look into the Friedmann metrics' Killing vectors. It is discovered that there are seven Killing vectors for static spherically space-times and six teleparallel Killing vectors for Friedmann models. Next, a comparison is made between the outcomes and General Relativity. In [31] Direct integration and algebraic methods are used to derive and solve teleparallel Killing equations. The space-time admits two, three, four, five, or six teleparallel Killing vectors, according to this analysis. The results are described in comparison with general relativity, in which two, three, four, or six Killing vectors are admitted in the same space-time. M-Sharif in [32] discussed the Lie derivative of a generic tensor of rank $p+q$, as well as the second-order tensor in the context of teleparallel gravity, have also been investigated. This definition is then applied to find the Killing vectors of the Einstein world. It is discovered that the Killing vectors in Einstein's cosmos are identical in both teleparallel theory and general relativity. Suhail khan [33] Killing and proper Homothetic vector fields were examined for a non-diagonal tetrad of Kantowski-Sachs space-time in the context of teleparallel gravity. The direct integration approach has been applied for the goal. It comes out that there are either four or seven Killing vector fields. Three generators are lost and the remaining seven teleparallel Killing vector fields generators, which are in charge of spin angular momentum, linear momentum, and energy conservation, are regained. Gulam Shabir [34] Teleparallel conformal vector fields in non-static plane-symmetric space-times were found by researchers using diagonal tetrads and direct integration. The study also investigated static planar symmetric space-times. Non-static plane-symmetric space-times in teleparallel theory have zero curvature, whereas additional symmetries are made possible via torsion. In [35] the Einstein field equations (EFEs) for static spherically symmetric (SS) ideal fluid space-times in the context of $f(T,B)$ gravity are developed algebraically. The retrieved solutions are then used to obtain the conformal vector fields (CVFs). It is crucial to remember that our methods make it easier to modify previously created $f(T,B)$ gravity models to produce a variety of space-times structures.

Chapter 2

Basic Concepts and Definitions

Basic concepts and guidance are given in this chapter to assist readers in understanding analysis. Complex ideas are made simpler for easy understanding. The strategy makes sure that readers can understand the content efficiently and clearly.

2.1 Space-Time

Einstein's physical intuition inspired the formulation of special relativity, while Hermann Minkowski's mathematical formulation was required for the development of general relativity. Minkowski's inclusion of time to the three spatial dimensions created a four-dimensional manifold for representing space-time. A manifold is a topological space with locally flat points, as represented by Euclidean geometry. This implies that a flat neighbourhood exists around each point. A manifold is a surface with an infinite number of flat space-times that overlap smoothly and continuously. Our planet, Earth, exemplifies this principle clearly. The Earth seems flat to the human eye, despite its spherical shape. Special relativity theory states that the space-time manifold is flat, not only in the near region but also elsewhere. However, when discussing general relativity, this becomes less applicable. We define events in space-time as points on a manifold. Given the four-dimensional nature of space-time, each point requires four coordinates to be uniquely characterized. Coordinates are typically interpreted as having three spatial coordinates and one time coordinate. The conventional notation is x^0, x^1, x^2, x^3, with x^0 representing the time coordinate. However, we have the flexibility to use any option as needed [36].

2.2 Metric Tensor

Defining the metric tensor is important for calculating angles and distances between locations in space-time. This tensor, $g_{\alpha\beta}$, is a rank-2 symmetric tensor defined on a smooth manifold. The metric requires a Lorentzian signature to accurately characterize space-time. Depending on tradition, the signature could be $(+---)$or $(-+++)$. The non-degenerate metric has three eigenvalues with the same sign and one with the opposite sign. Given a metric and coordinates x^α, the local line element in space-time can be defined as follows:

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta. \tag{2.1}$$

This study follows Einstein's summation convention, summing repeated indices across their whole range. The line element calculated the distance between two infinitesimal points located at x^α and $x^\alpha + dx^\alpha$. The metric's determinant, denoted by g, is neither zero or disappear since it does not degenerate. In non-degenerated space-times, the inverse metric $g^{\alpha\beta}$ is always well-defined. This meets the condition: $g_{\alpha\beta} g^{\beta\lambda} = \delta_\alpha^\lambda$. The metric and its inverse allow us to map both contravariant and covariant vectors. Additionally, we can map both covariant and contravariant vectors thanks to the metric and its inverse. If one were to take any random covariant vector v^{α} and contract it using the metric $v^{\alpha} = g^{\alpha\beta} v_{\beta}$, one would receive the corresponding contravariant vector v_{α}. For every covariant vector,this is possible [37, 38].

2.3 The Connection

A vector U^{α} can be translated in parallel from a point X^{α} to a point $X^{\alpha} + dx^{\alpha}$, which transforms the vector specified by [37].

$$dU^{\alpha} = -\Gamma^{\alpha}_{\beta\gamma} U^{\beta} dx^{\gamma}, \tag{2.2}$$

where $\Gamma^{\alpha}_{\beta\gamma}$ is the connection on manifold. There are $4^3 = 64$ distinct component of this connection. Specifically, this link offers a tensorial generalization of partial derivatives on manifolds, which is referred to as the covariant derivative. The covariant derivative $\delta_{\alpha} U^{\beta}$ of a vector U^{β} defined as

$$\nabla_{\alpha} U^{\beta} = \partial_{\alpha} U^{\beta} + \Gamma^{\beta}_{\alpha\gamma} U^{\gamma} \tag{2.3}$$

. And higher rank tensors can be easily generalized to using this equation (2.3). This relationship physically identifies the observers who are inertial to their own local coordinate system in addition to specifying our space-time's geodesic structure. Specifically, connection does not form a tensor.

2.4 Curvature Tensor

The Riemann curvature for a particular connection is defined as follows [39]:

$$R^{b}_{adc} = \Gamma^{b}_{ad,c} - \Gamma^{b}_{cd,a} + \Gamma^{\nu}_{ad} \Gamma^{b}_{c\nu} - \Gamma^{\nu}_{cd} \Gamma^{b}_{a\nu}, \tag{2.4}$$

where comma ”, ”depicts the partial derivative or ordinary derivative operator. The Riemann curvature tensor is a tool used to quantify a manifold’s curvature with regard to its surrounding space.The Riemann tensor measures the displacement of a vector from its initial location in the tangent space following its movement along a closed loop on a manifold. By showing how parallel transport around the loop alters the vector’s orientation, it calculates the space’s curvature. Intrinsic curvature is revealed by this discrepancy. It is crucial to remember that the curvature tensor’s final two indices are anti symmetric, with

$$R^b_{acd} = -R^b_{adc}. \tag{2.5}$$

2.5 Einstein Field Equations

Newton’s theory of gravity was derived from his universe model, which was characterized by a great deal of simplicity. Newton’s model states that particles move and interact within of space, which is always equal to itself. A more precise version of the field equations was needed to remedy this issue. After much trial and error, Einstein came to the conclusion that the Riemann geometry, which Gauss had first postulated and Riemann had later extended to any dimension, could account for the curvature of space caused by the different distributions of matter. In 1915, Einstein developed the field Equations [40]

$$R_{ab} - \frac{1}{2}Rg_{ab} = \kappa^2 U_{ab}. \tag{2.6}$$

Einstein proposed this famous equation almost immediately, and in 1917 he proposed the first variation. In order to reach his objective of a static universe, Einstein calculated that equation

(2.6) needed to have one more term added. According to his logic, the cosmos had to be static and it had to accept the principle of Mach. Furthermore, he thought that this extra component was necessary for his equation to work and stop the gravitational collapse. The factor that he discovered can be used to modify his field equations in order to also satisfy the requirement of a static Universe is referred to as the cosmological constant. This modification is given explicitly by and is referred to as the Einstein field equations with cosmological constant [41]

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = \kappa^2 U_{ab}, \tag{2.7}$$

where the cosmological constant is Λ, and $\kappa^2 = \frac{8\pi G}{c^4}$ and a, b from 0,1,2,3.

2.6 Lie Derivative

The Lie derivative makes it easier to calculate the changes in a tensor field as it travels down the manifold M from one point to another. The congruence of curves that have been built so that there is precisely one curve that passes through each point in the manifold will be looked at first. Afterwards, we can utilize any curve that constitutes the congruence,$X^a = x^\alpha(\nu)$, to find the tangent vector field $\frac{dx^a}{dv}$ along the curve. Upon repeating this procedure for every curve included in the congruence, we will ultimately reach a vector field X^α that is defined throughout the entire manifold [42]. The Lie derivative of the tensor, say W,of type (u,v) is given by,

$$L_x W^{a_1 \ldots a_u}{}_{b_1 \ldots b_v} = W^{a_1 \ldots a_u}{}_{b_1 \ldots b_v};_c X^c - \sum_{i=1}^{u} W^{a_1 .. c .. a_u}{}_{b_1 \ldots b_v} X^{a_i}, c + \sum_{j=1}^{v} W^{a_1 \ldots a_u}{}_{b_1 .. c .. b_v} X^c, b_j. \tag{2.8}$$

2.7 Symmetries in General Relativity

The idea of "symmetry" is important in physics since it can help to simplify a wide range of issues. The theory of classical general relativity gives us a comprehensive understanding of space-time symmetries by using vector fields on space-time. Because the Einstein field equations form a complex system of nonlinear partial differential equations, solving them is extremely challenging. From a number of angles, space-time symmetries' implementation is crucial. The main benefit of utilizing symmetries is that they provide restrictions that make PDEs easier to manage by converting them into ODEs. Secondly, symmetries have a tendency to group and arrange the precise answers to EFEs. It is helpful to remember that the covariant derivative of ξ may always be split into its symmetric and skew-symmetric parts (hereξ is a smooth vector field defined on manifold M) such that [38]

$$\xi_{j;h} = \frac{1}{2} h_{jh} + F_{jh} \tag{2.9}$$

Where $h_{jh} = 2\xi_{(j;h)} = L_{jh}$ and $F_{jh} = 2\xi_{[j;h]}$. The conformal vector field (CVF) denoted by ξ and equipped with a metric g_{jh} can be represented mathematically as follows:

$$L_{\xi} g_{jh} = g_{jh,b} \xi^{b} + g_{jb} \xi^{b}{}_{,h} + g_{hb} \xi^{b}{}_{,j} = 2\psi g_{jh} \tag{2.10}$$

where ψ (ψ:$M \to R$) is a real valued function on M.

$$\xi = \begin{cases} KVFs, & if\ \psi = 0, \\ HVFs, & if\ \psi = constant, \\ Proper\ CVFs, & Otherwise. \end{cases} \tag{2.11}$$

2.8 Weitzenb$\ddot{o}$ck Connection

The metric theory of gravity is called general relativity that functions in Riemannian space, where the Einstein field equations determine the Levi-Civita and the metric link defines the Riemann curvature tensor. By modeling gravity through torsion in a flat environment, teleparallel theory differs from general relativity's curvature-based methodology. Although it is not an easy task, a universally flat space can be obtained by selecting a suitable connection as indicated by the Weitzenb$\ddot{o}$ck connection. [43]. A smooth tetrad field serves as the basis for inducing the Weitzenb$\ddot{o}$ck connection on the manifold.

$$\Gamma^{\alpha}{}_{\beta\gamma} = e_b{}^{\alpha}\, \partial_\gamma\, e^b{}_\beta = -e^b{}_\beta\, \partial_\gamma\, e_b{}^{\alpha}. \tag{2.12}$$

The lower two indices of the Weitzenb$\ddot{o}$ck link exhibit non-symmetry. Regardless of the line connecting two tangent spaces, two vectors are parallel if their projections on the tetrad are proportionate. Absolute parallelism is present on a manifold if and only if there is no curvature [44].

2.9 Tetrad Field

The dynamical object that TEGR is interested in is the tetrad. At every point q on the manifold M, there are four orthonormal vectors that make up the tetrad field, represented by the notation $e_a(x)$. The tangent space represented by the symbol is T_qM based on these vectors. The co-tangent space, represented by T_q^*M, is based on the dual co-frame, indicated by the notation $e_a(x)$ [45].

$$e^{\alpha} = e^{\alpha}{}_{\mu}\, dx^{\mu} \qquad \text{and} \qquad e_{\alpha} = e_{\alpha}{}^{\mu}\, \partial_{\mu}, \tag{2.13}$$

where $e^{\alpha}{}_{\mu}$ and $e_{\alpha}{}^{\mu}$ are the respective components that satisfy

$$e^{\alpha}{}_{\mu}\, e_{\beta}{}^{\mu} = \delta^{\alpha}_{\beta} \qquad and \qquad e^{\alpha}{}_{\mu}\, e_{\alpha}{}^{\nu} = \delta^{\nu}_{\mu}. \tag{2.14}$$

A tetrad field $e_\alpha = e_\alpha{}^{\mu}\, \partial_\mu$, also known as vierbein meaning "four legs" connects g to the metric on tangent-space $\eta = \eta_{pq} dx^p dx^q$ by the relation [46]

$$\eta_{mn} = g_{\mu\nu}\, e_m{}^{\mu}\, e_n{}^{\nu}, \tag{2.15}$$

The Minkowski metric is represented here by $\eta_{mn} = diag(1,1,1,-1)$. The metric may be extracted from the tetrad thanks to the orthonormality criterion. In fact, equation (2.14) can be used to reverse this relation and derive the metric Obtained from the tetrad as follows:

$$g_{\mu\nu} = \eta_{mn}\, e^{m}{}_{\mu}\, e^{n}{}_{\nu} \text{ or } \quad g^{\mu\nu} = \eta^{mn}\, e_{m}{}^{\mu}\, e_{n}{}^{\nu}. \tag{2.16}$$

Furthermore, $e = det(e^{\alpha}{}_{\mu}) = \sqrt{-g}$ connects the metric's determinant to the tetrad's determinant.

2.10 Modified Theory of Gravity

Various conceptual and experimental issues in astrophysics, cosmology, and high-energy physics, such as those pertaining to inflation, dark energy, dark matter, large-scale structure, and quantum gravity, are intended to be resolved by extended gravity theories. Two theoretical challenges confronting contemporary astrophysical and cosmological models are dark energy and dark matter issues. A number of candidates, such as modified gravity and dark energy models, have been put up to explain the Universe's accelerated expansion. Vacuum energy, dynamical fields, and Einstein's General Relativity are some examples of dark energy. Galactic dynamics can be explained by modified gravity in the absence of dark matter, and dark matter may exist at both galactic and extra-galactic scales [47]. Further insights into general relativity (GR) can be gained by studying modified gravity theories (TEGR). As a generalization of GR, TEGR makes it possible to find new properties that are difficult to find through direct study of GR. GR can be adjusted in a number of ways, including by creating modified theories with screening processes, adding new terms, and forming an Einstein-Hilbert action. There isn't yet a comprehensively updated theory that can address every query, though.

2.11 Teleparallel Theory

A very successful theory that shows excellent agreement with observations is general relativity. Nonetheless, the theory encounters certain difficulties, which are commonly encapsulated as the dark energy and dark matter issues. One way that the dark matter issue shows up in the universe is through flattened galaxy rotation curves. About 27% of the matter in the universe dark matter, a major component of the structure of the cosmos, has a significant impact on its dynamics. The remaining matter is made up of dark energy, which makes up roughly 68% of the matter in the universe. However, the observed faster expansion of the Universe can be attributed to dark energy. The cosmological constant λ could be accepted in theory as an extra component of physics, but this presents significant issues when the cosmological term is understood as a vacuum expectation value. Considerations for modifications to general relativity (GR) began nearly immediately after the theory was developed. A wide range of models were motivated by developments in other areas of theoretical physics, and many of those early research focused on integrating electromagnetism into the new geometrical framework. One method based on this it is constructed using a geometric framework. finding that dates back to Weitzenb$\ddot{o}$ck Connection, who noted that a specific connection may always be defined so that the space is universally flat. The geometrical framework consists of a manifold that has the so-called Weitzenb$\ddot{o}$ck Connection and curves and torsion. This serves as the foundation for what is currently known as general relativity's teleparallel equivalent. Since the torsion scalar T is also not invariant under local Lorentz transformations—there is a total derivative term between the torsion Ricci scalar and the scalar—it is widely known that $f(T)$ gravity is not invariant under these transformations. This method simplifies things compared to the fourth-order derivatives of $f(R)$ gravity, the resulting $f(T)$ gravity theory is a second-order theory.Through a new analysis of these models, we determine the teleparallel counterpart of $f(R)$ gravity as a specific subset of models that rely on a boundary term and the torsion scalar. We prove that this theory is the only one of its kind that remains invariant under the local Lorentz transformation. Additionally,we are able to demonstrate that $f(T)$ gravity is the only theory that admits field equations of second order [48].

2.12 $f(R)$ Gravity

Even if one determines that altering gravity is the best course of action, this is a difficult undertaking. First off, there are lots of ways to stray from GR. Setting aside scalar-tensor theory, the most well-known alternative to general relativity, and the early attempts to generalize Einstein's theory Will, 1981, most of which have been shown to be impractical. There are still many arguments for modified gravity in modern literature, including those made by Brans and Dicke (1961), Dicke (1962), Bergmann (1968) and others. This review focuses on a distinct category of ideas known as f(R) theories of gravity.A simple generalization of the Lagrangian in the Einstein-Hilbert action leads to these theories.

$$S_{f(R)} = \frac{1}{2\kappa^2}\int \sqrt{-g}f(R)d^4x. \tag{2.17}$$

.

2.13 $f(T)$ Gravity

The $f(T)$ gravity is one of the finest modified theory introduced by Bengochea where T represent torsion scalar. It makes sense to consider the $f(R)$ gravity when discussing the metric formulation of gravity. It is no longer possible to examine the feasibility of carrying out work using a function of the Ricci scalar. a plausible notion if we were to suppose that the TEGR was the underlying theory instead. By changing the teleparallel Lagrangian, the theory of $f(T)$ gravity is created, which enables T to be represented as a generic function $f(T)$[49].

$$S_{f(T)} = \frac{1}{2\kappa^2}\int e\, f(T) d^4x. \tag{2.18}$$

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2.14 Symmetries in Teleparallel Theory of Gravity

In covariant teleparallel gravity theories, the function of an isometry is less evident than in metric-based theories. The tetrad [or (co)frame] and spin-connection take the place of the metric as the main subject of investigation in a teleparallel geometry since they are utilized in the computation of the field equations and the torsion tensor. Because of this, we can think of the metric as a derived tensor made up of the symmetric products of the frame elements. Asking if the collection of isometries and the symmetries of a specific teleparallel geometry coincide is worthwhile [50].

$$L_K^T\, g_{\lambda\rho} = g_{\lambda\rho},_j\, K^j, + g_{j\rho}\, K^j_\lambda \;+\; g_{\lambda j}\, K^j,_\rho + K^j\,(g_{\sigma\rho}\, T^\sigma_{\lambda\;j} \;+\; g_{\lambda\sigma}\, T^\sigma_{\rho\;j}) \;=\; 2\eta\; g_{\lambda\rho} \tag{2.19}$$

$$\xi = \begin{cases} TKVF, & if\ \eta = 0, \\ THVF, & if\ \eta = constant, \\ Proper\ TCVF, & Otherwise. \end{cases} \tag{2.20}$$

Chapter 3

Killing Symmetries of Static Spherically Symmetric Space-Times within General Relativity

3.1 Introduction

The work done by A. Qadir and A. H. Bokhari and [51] is thoroughly examined in this chapter. This study attempts to explore and comprehend the symmetries of static, spherically symmetric space-times in order to establish the relationship between symmetries and conserved quantities as specified by Noether's theorem. The motion of particles in gravitational fields are represented by geometries such as Kerr-Newman, Schwarzschild, and Reissner-Nordström space-times has been studied using this concept. These symmetries are expressed as Killing vector fields, which correspond to conserved properties of space-time in the context of general relativity. The complexity and structure of the symmetries in space-time are reflected in the number of independent Killing vectors. Instead of investigating these symmetries through group theory, this method uses a methodical elimination process to find every potential Killing vector field inside a spherically symmetric, unchanging space-time framework.

3.2 Mathematical Formulation

The study found that when space-time symmetry is reduced to minimal static spherical symmetry, the number of Killing vector fields decreases from 10 to 4. Regardless of Einstein's field equations, maximal symmetry is a characteristic unique to the de Sitter, anti-de Sitter, and Minkowski metrics. The metric tensor g must have a Lie derivative equal to zero for J to be considered a Killing vector fields that is, $L_J g = 0$. In (2.10) if ψ =0 we get Killing vector field in Torsion-free space. The most thorough static spherically symmetric line element taken into account in this investigation is provided by.

$$ds^2 = -e^{\lambda(r)}dr^2 + e^{\nu(r)}dt^2 - r^2 d\theta^2 - r^2 \sin^2 d\phi^2. \tag{3.1}$$

We will consider the Killing equations for all possibilities.

3.3 Procedure Adopted

To produce an entire collection of connected partial differential equations of the first order, first substitute Equation (3.1) into Equation (2.9), from which the Killing vectors (KVF's) for the metric specified by Equation (3.1) has been obtained. We obtain identities between equation pairs by differentiating these equations, leading to decoupled partial differential equations of either the first or second order. The separation of variables method is then used to solve these decoupled equations. After that, the integration and separation constants are allowed to have any value, including zero, negative, and positive. Nevertheless, in some situations, the selection of these integration constants is limited by the need for the positivity of particular functions, such as $e^{\lambda(r)}$ and $e^{\nu(r)}$.

3.4 Main Results

By expanding (2.10) by taking ψ =0 the following set of equations has been obtained:

$$\nu^{*}(r)R^{1}+2\,R^{0}{}_{,0}=0, \tag{3.2}$$

$$e^{\nu(r)}R^{0}{}_{,1}-e^{\lambda(r)}R^{1}{}_{,0}=0, \tag{3.3}$$

$$e^{\nu(r)}R^{0}{}_{,2}-r^{2}R^{2}{}_{,0}=0, \tag{3.4}$$

$$e^{\nu(r)}R^{0}{}_{,3}-r^{2}\sin^{2}\theta R^{3}{}_{,0}=0, \tag{3.5}$$

$$\lambda^{*(r)}R^{*}+2R^{1}{}_{,1}=0, \tag{3.6}$$

$$e^{\lambda(r)}R^{1}{}_{,2}+r^{2}R^{2}{}_{,1}=0, \tag{3.7}$$

$$e^{\lambda(r)}R^{1}{}_{,3}+r^{2}\sin^{2}\theta R^{3}{}_{,1}=0, \tag{3.8}$$

$$R^{1}+rR^{2}{}_{,2}=0, \tag{3.9}$$

$$R^{2}{}_{,3}+\sin^{2}\theta R^{3}{}_{,2}=0, \tag{3.10}$$

$$R^{1}+r\cot\theta R^{2}+rR^{3}{}_{,3}=0, \tag{3.11}$$

Here, the asterik ($*$) represents differentiation with respect to r. The differential equation in this equation is dependent on R and its derivative with respect to r (3.6). Consequently, it might be combined with regard to r to produce

$$R^{1}=B(t,\theta,\phi)e^{-\lambda(r)/2}, \tag{3.12}$$

The function $B(t,\theta,\phi)$ acts as the "constant" of integration in this context. Two different possibilities arise:

(I) B $\neq$ 0, and (II) B = 0. We examine case (I) first. Equations (3.9) and (3.7) can be compared by taking the derivatives of Equation (3.9) with respect to r and θ, respectively. (as B $\neq$0)

$$B(\theta,\phi,t)_{\theta\theta}/B(\phi,t,\theta) = -(1+r\lambda^{*}(r)/2)e^{-\lambda(r)} = -\alpha, \tag{3.13}$$

where $B_{\theta\theta} = \partial^2 B/\partial\theta^2$. Given that the right-hand side of Equation (3.13) is a function of r while the left-hand side is not, α is a separation constant. Now there are three possibilities; (1) $\alpha < 0$, (2) $\alpha > 0$, or (3) $\alpha = 0$.

Consider first **Case 1**. Here Equation (3.13) can be resolved rapidly to produce

$$B(t,\theta,\phi) = \cos\sqrt{\alpha\theta}B_1(t,\phi) + \sin\sqrt{\alpha\theta}, B_2(t,\phi), \tag{3.14}$$

$$e^{-\lambda(r)} = (\alpha + \beta^{r2}), \tag{3.15}$$

where $B_2(t,\theta)$, $B_1(t,\theta)$, and β are constants of integration. Once more, there are three possible situations;(a) β>0,(b) β<0, or (c) β=0. We will begin with **Case 1**.

Differentiating Equation (3.2) with respect to r and Equation (3.3) with respect to t and then comparing the results with Equations (3.2), (3.14), and (3.15), we draw the following conclusion:

$$\frac{B_1(t,\phi)_{tt}\cos\sqrt{\alpha\theta} + B_2(t,\phi)_{tt}\sin\sqrt{\alpha\theta}}{B_1(t,\phi)\cos\sqrt{\alpha\theta} + B_2(t,\phi)\sin\sqrt{\alpha\theta}} = \nu^{**}(\alpha+\beta r^2)\frac{1}{2} + \beta\nu^{*}r = \gamma, \tag{3.16}$$

where γ is the separation constant.Once more there are three options: (a1) γ> 0, (b2) $\gamma < 0$, or (c3) $\gamma = 0$. We consider **case (a1)**. Equation (3.16) after some algebric maniplution give the following results,

$$B_1(t,\phi) = B_{11}(\phi)\cosh\sqrt{\gamma t} + B_{12}(\phi)\sinh\sqrt{\gamma t}, \tag{3.17}$$

$$B_2(t,\phi) = B_{21}(\phi)\cosh\sqrt{\gamma t} + B_{22}(\phi)\sinh\sqrt{\gamma t}, \tag{3.18}$$

$$e^{\nu(r)} = -(\gamma/\alpha\beta)(\alpha+\beta r^2) = -(\gamma/\alpha\beta)e^{-\lambda(r)}. \tag{3.19}$$

From equation (3.19) it can be seen that for $e^{\nu(r)}$ to be positive, α must be nonzero and γ and β must have opposing signs, Using the value of R^1 in equation (3.9) and after some simflication we get

$$\begin{aligned} R^2 = &-[(\alpha+\beta r^2)/\sqrt{\alpha r}][B_{11}(\phi)\cosh\sqrt{\gamma t} - B_{21}(\phi)\cosh\sqrt{\gamma t} + A_1(t,r,\phi) \\ &+ B_{12}(\phi)\sinh\sqrt{\gamma t}\sin\sqrt{\alpha\theta} + B_{22}(\phi)\sinh\sqrt{\gamma t}\cos\sqrt{\alpha\theta}]. \end{aligned} \tag{3.20}$$

Integrating Equation (3.4) w.r.t θ and by using equation (3.20), we obtain

$$\begin{aligned} R^0 = &[\beta r/(\gamma(\alpha+\beta r^2))]^{\frac{1}{2}}[B_{11}(\phi)\sinh\sqrt{\gamma t} + B_{12}(\phi)\cosh\sqrt{\gamma t}\cos\sqrt{\alpha\theta} + A_2(t,r,\pi) \\ &+ B_{22}(\phi)\cosh\sqrt{\gamma t}] - \alpha\beta\theta A_1(t,\phi)_t/\gamma(\alpha+\beta r^2) + B_{21}(\phi)\sinh\sqrt{\gamma t}. \end{aligned} \tag{3.21}$$

Differentiating equation (3.21) with respect to r and compared with Equation (3.3),it has been observe that A_1 is a function of ϕ only and A_2 is a function of t and ϕ. Integratiting Equation (3.8) with respect to r and Using R^1 one obtains:

$$\begin{aligned} R^3 = &[(\alpha+\beta r^2)^{\frac{1}{2}}/\alpha r\sin^2\theta][B_{11}(\phi)_\phi\cosh\sqrt{\gamma t} + B_{12}(\phi)_\phi\sinh\sqrt{\gamma t}\cos\sqrt{\alpha\theta} \\ &+ (B_{21}(\phi)_\phi\cosh\sqrt{\gamma t} + B_{22}(\phi)_\phi\sinh\sqrt{\gamma t})\sin\sqrt{\alpha\theta}] + A_3(t,\theta,\phi). \end{aligned} \tag{3.22}$$

Using equations (3.21) and (3.22) A_3 is a function of t and ϕ, but A_2 is a function of t alone. Utilize Equations (3.20) and (3.22) in Equation (3.10) to verify consistency, which suggests that it is satisfied only if

$$-2[(\alpha+\beta r^2)/\alpha r]^{\frac{1}{2}}[(B_{11}(\phi)_\phi \cosh\sqrt{\gamma t}+B_{12}(\phi)_p hi \sinh\sqrt{\gamma t})(\sqrt{\alpha}\sin\theta\sin\sqrt{\alpha\theta}+\cos\theta\cos\sqrt{\alpha\theta})$$

$$-(B_{12}(\phi)_\phi \cosh\sqrt{\gamma t}+B_{22}(\phi)_\phi \sinh\sqrt{\gamma t}\times(\sqrt{\alpha}\sin\theta\cos\sqrt{\alpha}\theta-\cos\theta-\cos\theta\sin\sqrt{\alpha}\theta]$$

$$[A_3(\phi,\theta)\sin^2\theta+A_1(\phi)_\phi]\sin\theta=0.$$

(3.23)

The above equation is satisfied when each of the coefficients of sin 0 and r is zero. Thus

$$\begin{aligned}&(B_{11}(\phi)_\phi \cosh\sqrt{\gamma t}+B_{12}(\phi)_\phi \sinh\sqrt{\gamma t})(\sqrt{\alpha}\sin\theta\sin\sqrt{\theta}+\cos\theta\cos\sqrt{\alpha\theta}\\&-(B_{21}(\phi)_\phi \cosh\sqrt{\gamma t}+B_{22}(\phi)_\phi \sinh\sqrt{\gamma t})(\alpha\sin\theta\cos\sqrt{\alpha\theta}-\cos\theta\sin\sqrt{\alpha\theta})=0,\end{aligned} \tag{3.24}$$

$$A_3(\theta,\phi)=A_4(\phi)+\cot\theta A_1(\phi)_\phi. \tag{3.25}$$

Equation (3.24), may hold in two different situations. The first is ($\otimes$) α= 1 and the second is ($\perp$) $\alpha \neq$1.For the first case, we obtain the de Sitter metric with $\beta=-\frac{1}{R^2}$.. From Equation (3.24) we see that B_{11} and B_{12} are constants, say c_1 and c_2,respectively. Differentiating and using Equation (3.22) and (3.11), we obtain (3.22) we obtain

$$\begin{cases}B_{21}=c_3\cos\phi+c_4\sin\phi,\\ B_{22}=c_5\cos\phi+c_6\sin\phi,\\ A_1=c_7\cos\phi+c_8\sin\phi,\\ A_4=c_9.\end{cases} \tag{3.26}$$

Using Equation (3.3) Verifying that A_2 is an integration constant is now easy let's say c_{10}. As a result, we have ten KVs for the de Sitter metric:

$$\begin{cases} R^0 = [r/T^0/(\gamma(1-r^2/T^2))^{\frac{1}{2}}][(c_1 \sinh\sqrt{rt} + c_2 \cosh\sqrt{\gamma t}) * \cos\theta + \\ (c_3\cos\phi + c_4 \sin\phi)\sinh\sqrt{\gamma t} + (c_6 \sin\phi + c_5\cos\phi)\cosh\sqrt{rt}\sin\theta] + c_7, \\ R^1 = (1-r^2/T^2)^{\frac{1}{2}}[\cos\theta(c_1\cosh\sqrt{\gamma t} + c_2\sinh\sqrt{\gamma t}) + \\ (c_3\cos\phi + c_4\sin\phi)\cosh\sqrt{\gamma t} + (c_5\cos\phi + c_6\sin\phi)\sinh\sqrt{\gamma t}\sin\theta], \\ R^2 = -[(1-r^2/T^2)^{\frac{1}{2}}/r][(c_1\cosh\sqrt{\gamma t} + c_2\sinh\sqrt{\gamma t}) * \sin\theta - \\ (c_3\cos\phi c_4\sin\phi)\cosh\sqrt{\gamma t} + c_5\cos\phi + c_6\sin\phi)\sinh\sqrt{\gamma t}\cos\theta] + (c_8\cos\phi + c_9\sin\phi), \\ R^3 = [(1-r^2/T^2)^{\frac{1}{2}}/r\sin\theta][(-c_3\sin\phi + c_4\cos\phi) * \cosh\sqrt{\gamma t} + \\ (-c_5\sin\phi + c_6\cos\phi)\sinh\sqrt{\gamma t}] + \cos\theta(-c_8\sin\phi + c_9\cos\phi) + c_{10}. \end{cases} \tag{3.27}$$

Anti-de Sitter metric: In this case, there is an additional possibility (1.b.b2.$\otimes$). Using the same process as in the initial instance (replacing γ by -γ and β by $\frac{1}{R^2}$). The anti-de Sitter metric has unique and accessible Killing vector fields. Once more, the fields in the Killing vector are 10 with $\sinh\sqrt{\gamma t}(\cosh\sqrt{\gamma t})$ replaced by $\sin\sqrt{\gamma}t(\cos\sqrt{\gamma}t$ in Equations (3.27).

The **Minkowski metric:** Consider the case now. (1.c3.c.$\otimes$). Equation (3.16) give

$$e^{\nu(r)} = e^{br+a}. \tag{3.28}$$

Two choices are now available:(Θ) b=0,(ω) b=0. The Minkowski metric is defined in the first situation (assuming that the zero-zero coefficient is unity, modulo a constant). We have again 10 KVFs:

$$R^0 = r[c_2\cos(\phi + c_3)\sin\theta + c_1\cos\theta +] + c_4, \tag{3.29}$$

$$R^1 = t[c_1 \cos\theta - c_2 \cos(\phi + c_3)\sin\theta] + c_5 \cos\theta + c_6 \cos(\phi + c_7)\sin\theta, \tag{3.30}$$

$$\begin{aligned} R^2 = &-(t/r)[c_1 \sinh\theta - c_2 cos(\phi + c_3)\cos\theta] - (1/r)[c_5 \sin\theta \\ &- c_6 \cos(\phi + c_7)\cos\theta] + c_8 \cos(\phi + c_9), \end{aligned} \tag{3.31}$$

$$R^3 = -(1/r \sin\theta[tc_2 \sin(\phi + c_3) + c_6 \sin(\phi + c_7) - c_8 \sin(\phi + c_9)\cot\theta] + c_{10}). \tag{3.32}$$

Now considering the instance ($\perp$) where the reduction is readily apparent and KVs from 4 to 10 only. In case ($\perp$) Equation (3.24) is satisfied if B_{11}, B_{12}, B_{21}, and B_{22} are all constants. Using equations, we can verify consistency (3.12), (3.14), (3.15), and (3.19) in Equation (3.11). As it happens, every one of the aforementioned constants is exactly zero. In this instance A_1 and A_4 are provided by Equations (3.26). The KVs are

$$\begin{cases} R^0 = c_1, \\ R^1 = 0, \\ R^2 = \sin\phi c_3 + \cos\phi c_2, \\ R^3 = \cot\theta(c_3 \cos\phi - \sin\phi c_2) + c_4. \end{cases} \tag{3.33}$$

The Schwarzschild metric's Killing vectors are normal, it should be mentioned. The one-one and zero-zero components of the metric tensor in this instance are given by Equations (3.15) and (3.18). According to the positivity of the following, the other subcases in case I are acceptable $e^{\lambda(r)}$ and $e^{\nu(r)}$ requirements, we now write $e^{\nu(r)}$:

Cases: $e^{\nu(r)}$

(1.a.c3) $a + (b/\sqrt{\beta})\sinh^{-1}\sqrt{\beta/\alpha r},$

(1.b.c3) $a + (b/\sqrt{\beta})\sinh^{-1}\sqrt{\beta/\alpha r},$

(1.c.a1) $-2\gamma/\alpha\nu^{**},$

(1.c.c3 $\neq$) $a+br$,

(2.b.b2) $-(\gamma/\alpha\beta)(\alpha+\beta r^2)$,

(2.b.c3) $a+(b/\sqrt{\beta})\sinh^{-1}\sqrt{\beta/\alpha r}$,

(3.b.b2) $-2\gamma/\beta(\nu^* r)^*$,

(3.b.c3) ar.

The identical process yields only four KVs in each of the aforementioned scenarios. These vector fields can be seen in set of Equations (3.33).

Case 2

Now discuss case (II) $B=0$. In this case Equation (3.12) gives $R^1=0$

Using the value of R in Equations (3.2) and (3.3) It is evident that R^0 is a function of θ and ϕ only. Also Equations (3.7) and (3.9) suggest that R^2 can only rely on t and ϕ. Differentiating Equations (3.5) with respect to t we get

$$R^3 = A_1(\phi,\theta)+A_2(\phi,\theta)t. \tag{3.34}$$

Equation (3.4) is now being differentiated with regard to θ, and following a process we obtain

$$R^0 = A_3(\phi)+A_6(\phi)\theta. \tag{3.35}$$

Additionally, in order to get, differentiate Equation (3.4) with respect to t first, and then integrate with respect to t.

$$R^2 = A_6(\phi)t+A_5(\phi). \tag{3.36}$$

Now we can solve Equations (3.10), by using Equation (3.34) and (3.36) we get

$$\begin{cases} A_1 = \cot\theta A_5(\phi)_\phi + A_7(\phi), \\ A_2 = \cot\theta A_6(\phi)_\phi + A_8(\phi). \end{cases} \tag{3.37}$$

For uniformity using values of R^0 and R^2 in Equation (3.4) it happens that,

$$A_4(\phi)e^{\nu(r)} = r^2 A_6(\phi). \tag{3.38}$$

Equation (3.38) may be divided in r and ϕ with the separation constant γ and solved to give

$$e^{\nu(r)} = \gamma r^2, A_6(\phi) = \gamma A_4(\phi). \tag{3.39}$$

Observe that in this case, the separation constant may exceed 0 [because $\gamma \leq 0$ in Equation (3.36) is not allowed]. Applying the aforementioned findings to Equation (3.11), we obtain

$$\begin{cases} A_4 = \cos\phi c_3 + \sin_4, \\ A_5 = \cos_1 + \sin_2, \\ A_7 = c_5, \\ A_8 = c_6. \end{cases} \tag{3.40}$$

It is clear from Equation (3.14) that A_3 is a constant and that c_3, c_4,, and c_6 are all zero. Using the values from Equations (3.37) and (3.40) into Equations (3.34)-(3.36) we get the same four KVs given by Equation (3.33). Equation (3.38), in **case II**, where $A_4 = 0$, suggests that A must be zero. Thus $e^{\nu(r)}$ or $e^{\lambda(r)}$ in case (II) have no constraints. For arbitrary $\nu^{(r)}$ and $\lambda^{(r)}$ this results in the form of KVF's being provided by Equation (3.33) while preserving spherical symmetry and staticity.

3.5 Summary and Conclusion

The number of Killing vectors (KVF's) in a static, spherically symmetric space-time can be either 4 or 10. The maximum degree of symmetry, represented by $\alpha = 1$, permits 10KVF's in space-times like de Sitter, anti-de Sitter, and Minkowski. In the case where $\alpha \neq 1$ and some terms (such as B_11 to B_22) are zero, the symmetry decreases to 4KVF's. Further requirements must be enforced by the Einstein field equations in this instance since the metric components $e^{\nu(r)}$ and$e^{\lambda(r)}$ are not limited. Depending on the field equations and separation constants, several metrics with 4 KVF's emerge, illustrating the connection between space-time symmetry and its geometric characteristics.

Chapter 4

Investigating Static Spherically Symmetric Space-times through Teleparallel Killing Symmetrizes in $f(T)$ Gravity

The primary focus of this chapter is to classify static spherically symmetric space-times via Teleparallel Killing vector fields in $f(T)$ gravity. It expended the work work done by A. Qadir and A. H. Bokhari and [51]. System of highly non-linear partial differential equations have been obtained that are solved using direct integration technique. Ten cases are discussed in detail whlie the rest are tabulated in the form of table.

4.1 Static Spherically Symmetric Solutions of the Einstein Field Equations and their Teleparallel Killing Vector Fields in $f(T)$ Gravity

The line element to generate a static spherically symmetric space-time in the typical coordinate (t, r, θ, ϕ) with the labels (y^0, y^1, y^2, y^3) is respectively given as [52]

$$ds^2 = -e^{i(r)}dt^2 + e^{j(r)}dr^2 + Q^2(r)[d\theta^2 + \sin^2\theta d\phi^2], \tag{4.1}$$

where the radial coordinate r has unknown functions $i = i(r)$, $j = j(r)$, and $Q = Q(r)$. The minimum Killing vector fields that the aforementioned space-times allow are [53]: $Y_1 = \partial_t, Y_2 = \cos\phi\cot\theta\partial\phi + \sin\phi\partial_\theta, Y_3 = -\sin\phi\cot\theta\partial\phi + \cos\phi\partial\theta$ and $Y_4 = \partial_\phi$. The solutions of static spherically symmetric space-times in $f(T)$ gravity is given in [53]. The following situations arise from this [53].

1. $i =$ constant, $j = j(r), e^{-j}j'r + 2e^{-j} - 2 = 0$ implies $j = \ln(\frac{1}{1+E_1r^2}), T = (\frac{2}{r^2} + 2E_1)$and $Q = r,$where $E_1 \in \Re/\{0\}$.

2. $i = i(r),\ j = j(r), i = j^{-1},\ e^i(\frac{i''}{2} + \frac{i'^2}{2} - \frac{1}{r^2}) + \frac{1}{r^2} = 0 = i = \ln(1 - \frac{2M}{r}),\ j = \ln(1 - \frac{2M}{r})^{-1}$ and $T = \frac{2}{r^2}$,and $Q = r$ where the Arnowitt-Deser-Misner mass is denoted by M.

3. $i = i(r),\ j = j(r),\ i = j^{-1},\ r^2(i'' + i'^2) - 2(1 - e^{-i})$ implies $i = \ln(1 - \frac{\Lambda r^2}{3},\ j = \ln(1 - \frac{\Lambda r^2}{3})^{-1}$, $T = (\frac{2}{r^2} - 2\Lambda)$ and $Q = r,$where the cosmological constant is Λ.

4. $i =$ constant $= E_1 \neq 0$, $j =$ constant $= E_2 \neq 0,\ \ j = \ln(E_2), i = \ln(E_1)\ QQ'' - Q'^2 + E_2 = 0$ implies $T = \frac{2}{E_2r^2}$ and $Q = r\sqrt{E_2}$,where $E_1, E_2 \in \Re/\{0\}$ with $E_1 \neq E_2$.

5. $j == E_1 \neq 0$ constant $i = i(r)\ \ ,r^2i'' - ri' - 2 = 0 \rightarrow i = (\frac{E_2r^2}{2} + E_3 - \ln r),\ \ T = \frac{2E_2}{e^{E_1}},$ and $Q = r$ where $E_1,\ E_2,\ E_3 \in \Re\ (E_1, E_2 \neq 0)$.

6. $j = j(r), i = i(r),\ (\frac{i''}{2} + \frac{i'^2}{4}) = 0 \rightarrow i = \ln(\frac{E_1 r + E_2}{2})^2,\ -E_1 j' e^{-j} + 2(E_1 r + E_2) = 0$, which implies $j = \ln\left(\frac{E_1}{E_1 E_3 - E_1 r^2 - 2k_2 r}\right)$, $T = 0$, and $Q = 1$ where $E_1,\ E_2,\ E_3 \in \Re/\{0\}$.

7. $i = i(r),\ j = j(r),\ ri'' - i' = 0$ implies $i = (\frac{E_1 r^2}{2} + E_2),\ rj'(ri' + 1) + 2 = 0 \rightarrow\ j = \ln\left(\frac{E_3\sqrt{E_1 r^2 + 1}}{r}\right)^2$, $T = \frac{2}{E_3^2,}$ and $Q = r$ where $E_1,\quad E_2,\ E_3 \in \Re(E_1,\ E_3 \neq 0)$.

8. $i = i(r),\ j =$ constant $2ri'' + ri'^2 - 2i' = 0\ \rightarrow i = \ln r^4,\ e^j = 1, T = 10r^{-2}$, and $Q = r$.

9. $i = i(r),\ j = j(r),\ 1 + rj' = 0$ implies $j = \ln(\frac{E_1}{r}),\ r^2 i'' - rj' - 2 = 0\ \rightarrow\ a = \ln(\frac{E_3 e^{E_2 r}}{r}$, $T = \frac{2k_2}{k_3}$,and $Q = r$ where $E_1,\ E_2,\ E_3\ \in \Re(E_1, E_3 \neq 0)$.

10. $i = i(r),\ j = j(r),\ i = j^{-1},\ r^2(j'' - j'^2) + 2(1 - e^j) = 0$ implies $j = \ln(1 - \frac{E_1}{r} + \frac{E_2 r^2}{3})^{-1},\ i = \ln(1 - \frac{E_1}{r} + \frac{E_2 r^2}{3}), T = (\frac{2}{r^2} + 2k_2)$ and $Q = r$ where $E_1,\ E_2\ \in\ \Re/\{0\}$.

11. $i = i(r),\ j = j(r),\ ri' + 1 = 0$ which gives $i = \ln(\frac{E_1}{r}),\ 4e^j - rj' + 1 = 0$ implies $j = \ln(\frac{r}{E_2 - 4r)}$ $T = 0$, and $Q = r$ where $E_1,\ E_2\ \in\ \Re/\{0\}$.

12. $i = i(r),\ j = j(r),\ ri' - 2 = 0$ implies $i = \ln(E_1 r^2),\ e^j - rj' - 2 = 0 \rightarrow\ j = \ln(\frac{2}{1 + 2E_2 r^2})$, $T = \frac{3(1 + 2E_2 r^2)}{r^2}$, and $Q = r$ where $E_1,\ E_2\ \in\ \Re/\{0\}$.

13. $i = i(r),\ j = j(r),\ i'' = 0$ implies $i(E_1 r + E_2),\ ri'(1 + rj') + rj' + 2 = 0\ \rightarrow j =\ \ln\left[\frac{E_3(E_1 r + 1)}{r^2}\right]$ and $T = \frac{2}{E_3,}$ where $E_1,\ E_2,\ E_3,\ \in\ \Re(E_1,\ E_3 \neq 0)$.

14. $i = i(r),\ j = j(r),\ i = j^{-1},\ i'' + i'^2 + 2e^{-i} = 0$ implies $i = \ln(E_2 - E_1 r - r^2),\ j = \ln(E_2 - E_1 r - r^2)^{-1}, T = 0$ and $Q = 1$ where $E_1,\ E_2\ \in\ \Re/\{0\}$.

15. $i = i(r),\ j = j(r),\ 2 + rj' = 0$ implies $j = \ln(\frac{E_1}{r^2}),\ ri'' - i'(1 + rj') = 0\ \rightarrow i = \ln(E_3 r^{E_2),}$ $T = \frac{2(E_2 + 1)}{E_1}$ and $Q = r$ where $E_1,\ E_2,\ E_3\ \in\ \Re\ (E_1,\ E_2,\ E_3 \neq 0)$.

16. $j = j(r), i = i(r),\ 2 + ri' = 0$ implies $i = \ln(\frac{E_1}{r^2}), ri'' - j'(1 + ri') = 0\ \rightarrow\ j = \ln\left(\frac{E_2}{r^2}\right)$, $T = \frac{-2}{E_2}$, and $Q = r$ where $E_1,\ E_2\ \in\ \Re(E_1,\ E_2 \neq 0)$.

17. $i = i(r),\ j = j(r), r^2 i'' - 2 = 0$ implies $i = \ln(\frac{e^{E_1^{rE_2}}}{r^2}),\ i' + j'(1 + ri') = 0\ \rightarrow\ j = \ln\left[\frac{E_3(E_1 r - 1)}{r^2}\right]$, $T = \frac{2}{k_3}$ and $Q = r$ where $E_1,\ E_2,\ E_3\ \in\ \Re(E_1,\ E_3 \neq 0)$.

18. $i=$ constant $=E_1\neq 0,\ j=$ constant $=E_2\neq 0, i=\ln(E_1),\ j=\ln(E_2),\ QQ''-Q'^2=0$ implies $Q=e^{E_3^r+E_4}$ and $T=\frac{2E_3^2}{e^{E_2}}$ where $E_1,\ E_2,\ E_3,\ E_4\ \in\ \Re(E_3\neq 0)$ with $E_1\neq E_2$.

19. $i=$ constant $=E_1\neq 0,\ j=j(r),\ Q''=0$ implies $Q=(E_2r+E_3),\ j'Q+2Q'=0\ \rightarrow\ j=\ln\left[\frac{E_4}{(E_2r+E_3)^2}\right]$ and $T=\frac{2E_2^2}{E_4}$, where $E_1,\ E_2,\ E_3,\ E_4\ \in\ \Re(E_1,\ E_2,\ E_4\ \neq 0)$.

20. $i=$ constant, $j=j(r),\ rj'+2=0\rightarrow\ j=\ln(\frac{E_1}{r^2}),\ e^i=1,$ and $T=\frac{2}{E_1}$,and $Q=r$ where $E_1\ \in\ \Re/\{0\}$.

In this section, we will look at ten situations in detail, while a table presents the key conclusions for the remaining cases.

4.2 Main Results

We will discuss in detail some of the cases rest will be present in the form of table.

Case 1:

$i=$ constant, $j=j(r), e^{-j}j'r+2e^{-j}-2=0$ implies $b=\ln(\frac{1}{1+E_1r^2}), T=(\frac{2}{r^2}+2E_1)$and $Q=r$, where $E_1\in\ \Re\ /\{0\}$.

The space-times (4.1) takes from

$$ds^2=-dt^2+\left[\frac{1}{1+k_1r^2}\right]dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2). \tag{4.2}$$

Expanding (2.19) and using (4.2) we get following set of equations

$$X^0{}_{,0}=0, \tag{4.3}$$

$$r^2X^2{}_{,0}-X^0{}_{,2}=0, \tag{4.4}$$

$$r^2\sin^2\theta X^3{}_{,0}-X^0{}_{,3}=0, \tag{4.5}$$

$$-X^0{}_{,1}+\frac{1}{1+K_1r^2}X^1{}_{,0}=0, \tag{4.6}$$

$$-\frac{K_1 r}{1+K_1 r^2}X^1 + X^1{}_{,1} = 0, \tag{4.7}$$

$$r^2 X^2{}_{,1} + \frac{1}{1+K_1 r^2}X^1{}_{,2} + rX^2 = 0, \tag{4.8}$$

$$r^2 \sin^2\theta X^3{}_{,1} + \frac{1}{1+K_1 r^2}X^1{}_{,3} + r\sin^2\theta X^3 = 0, \tag{4.9}$$

$$X^2{}_{,2} = 0, \tag{4.10}$$

$$\sin^2\theta X^3{}_{,2} + X^2{}_{,3} = 0, \tag{4.11}$$

$$\cot\theta X^2 + X^3{}_{,3} = 0. \tag{4.12}$$

From Eqation (4.3), Equation (4.10), Equation (4.11) and Equation (4.7) the following system of Equation is obtain.

$$\begin{cases} X^0 = K^1(r,\theta,\phi), \\ X^1 = \sqrt{1+k_1 r^2}\, K^4(t,\theta,\phi), \\ X^2 = K^2(t,r,\phi), \\ X^3 = K^2_\phi(\phi,r,t)\cot\theta + K^3(\phi,r,t), \end{cases} \tag{4.13}$$

where $K^1(r,\phi,\theta), K^3(t,r,\phi), K^2(r,\phi,t)$ and $K^4(\phi,\theta,t)$ are function of integration. Using the Equation (4.13) in Equation (4.4) we obtain

$$r^2 K^2_t(t,r,\phi) - K^1_\theta(r,\theta,\phi) = 0. \tag{4.14}$$

Differentiating w.r.t θ and after some simplification we get

$K^1(\phi,r,\theta) = \theta D^1(\phi,r) + D^2(\phi,r)$,

where $D^1(\phi,r)$ and $D^2(\phi,r)$ are constants of integration. Substituted all values in Equation (4.14) one gets

$$r^2 K^2_t(t,r,\phi) - D^1(\phi,r) = 0. \tag{4.15}$$

Taking the derivative of Equation (4.15) with respect to t we obtain $K^2(t,\phi,r) = tD^3(\phi,r) + D^4(\phi,r)$. So, the system of Equation (4.13) takes the form

$$\begin{cases} X^0 = \theta D^1(r,\phi) + D^2(r,\phi), \\ X^1 = \sqrt{1+k_1 r^2} K^4(t,\theta,\phi), \\ X^2 = D^4(r,\phi) + tD^3(r,\phi), \\ X^3 = K^2_\phi(t,\phi,r)\cot\theta + K^3(t,\phi,r). \end{cases} \tag{4.16}$$

Now we consider Equation (4.5) and using set of Equation (4.16) we get

$$r^2 \sin^2\theta [D^3_\phi(\phi,r)]\cot\theta + K^3_t(t,\phi,r) - D^1(\phi,r) = 0. \tag{4.17}$$

Differentiate with respect to t and after some algebraic manipulation we obtain $K^2_\phi(\phi,r,t) = tD^5(\phi,r) + D^6(\phi,r)$.

$$\begin{cases} X^0 = \theta D^1(\phi,r) + D^2(\phi,r), \\ X^1 = \sqrt{1+k_1 r^2}\, K^4(t,\phi,\theta), \\ X^2 = tD^3(\phi,r) + D^4(\phi,r), \\ X^3 = [tD^5(\phi,r) + D^6(\phi,r)]\cot\theta + K^3(t,r,\phi). \end{cases} \tag{4.18}$$

Using Equation (4.11) we obtain $D^5(r,\phi) = D^3_\phi(r,\phi)$ and in the view of (4.18) we get the following set of Equation

$$\begin{cases} X^0 = \theta D^1(r,\phi) + D^2(r,\phi), \\ X^1 = \sqrt{1+k_1 r^2}\, K^4(t,\phi,\theta), \\ X^2 = tD^3(\phi,r) + D^4(\phi,r), \\ X^3 = [tD^3_\phi(r,\phi) + D^6(r,\phi)]\cot\theta + K^3(t,\phi,r). \end{cases} \tag{4.19}$$

Equation (4.4) we obtain $D^1(r,\phi) = r^2 D^3(r,\phi)$ and using (4.5) we get $K^3(t,\phi,r) = tD^7(\phi,r) + D^8(\phi,r)$ →$\mathrm{D}^3(r,\phi) = 0$ putting all these values and in the light of (4.19) we get the following set of equation

$$\begin{cases} X^0 = \theta D^1(r,\phi) + D^2(r,\phi), \\ X^1 = \sqrt{1+k_1 r^2}\, K^4(t,\phi,\theta), \\ X^2 = D^4(\phi,r) + tS^1(r), \\ X^3 = D^6(\phi,r)\cot\theta + tD^7(\phi,r) + D^8(\phi,r). \end{cases} \tag{4.20}$$

Using Equation (4.4) and considering set of Equations (4.20) we get the system of Equation

$$\begin{cases} X^0 = \theta D^1(r,\phi) + D^2(r,\phi), \\ X^1 = \sqrt{1+k_1 r^2}\, K^4(t,\theta,\phi), \\ X^2 = D^4(r,\phi) + t\frac{S^2(r)}{r^2}, \\ X^3 = D^6(r,\phi)\cot\theta + tD^7(r,\phi) + D^8(r,\phi). \end{cases} \tag{4.21}$$

Now using Equation (4.8) and in the light of Equation (4.21) we get

$$-2tS_r^2(r) + tS_r^2(r) + r^2 D_r^4(r,\phi) + \frac{1}{\sqrt{1+k_1 r^2}} K_\theta^4(t,\theta,\phi) + t\frac{s^2}{r} + D^4(r,\phi) = 0. \tag{4.22}$$

Differentiating w.r.t. θ and after some calculations, we get the value $K^4(t,\theta,\phi) = \theta D^9(t,\phi) + D^{10}(t,\phi)$. Substituting into Equation (4.22) we obtain $D^9(t,\phi) = 0$. By using all these information in Equation (4.21) we get the system of Equation

$$\begin{cases} X^0 = \theta S^2(r) + D^2(r,\phi), \\ X^1 = \sqrt{1+k_1 r^2}\, D^{10}(t,\phi), \\ X^2 = D^4(r,\phi) + t\frac{S^2(r)}{r^2}, \\ X^3 = D^6(r,\phi)\cot\theta + tD^7(r,\phi) + D^8(r,\phi). \end{cases} \quad (4.23)$$

Now using Equation (4.23) in (4.6) we get

$$-2\theta S_r^2(r) + D_r^2(r,\theta) + \frac{2}{1+k_1 r^2}\sqrt{1+k_1 r^2}\, D_t^{10}(t,\phi) = 0. \quad (4.24)$$

Differentiating w.r.t t we obtain $D^{10}(t,\phi) = tS^3(\phi) + S^4(\phi)$. Using (4.24) we get $S^2(r) = c_1$. Implementing all these information in Equation (4.23) we get

$$\begin{cases} X^0 = \theta c_1 + D^2(r,\phi), \\ X^1 = \sqrt{1+k_1 r^2} t\, S^3(\phi) + S^4(\phi), \\ X^2 = D^4(r,\phi) + t\frac{S^2(r)}{r^2}, \\ X^3 = D^6(\phi,r)\cot\theta + tD^7(\phi,r) + D^8(\phi,r). \end{cases} \quad (4.25)$$

Now we can use Equation (4.5) we obtain $D^7(r,\phi) = 0$ and $D^2(r,\phi) = S^5(r)$, Equation (4.6) after some simplification and in the light of Equation (4.25) we get

$$\begin{cases} X^0 = \theta c_1 + S^5(r), \\ X^1 = \sqrt{1+k_1 r^2}\, tc_2 + S^4(\phi), \\ X^2 = D^4(r,\phi) + t\frac{c_1}{r^2}, \\ X^3 = D^6(r,\phi)\cot\theta + tD^7(\phi,r) + D^8(\phi,r). \end{cases} \quad (4.26)$$

Now using Equation (4.8) we obtain $c_1 = 0$, Equation (4.3), Equation (4.7) after some algebraic manipulation and considering Equation (4.26) we get

$$\begin{cases} X^0 = S^5(r), \\ X^1 = 0, \\ X^2 = \frac{S^5(\phi)}{r}, \\ X^3 = \frac{S^5(\phi)}{r^2} \cot\theta + D^8(r,\phi). \end{cases} \tag{4.27}$$

Now using Equation (4.6) we obtain $S^5(r) = c_4$. Now using Equation (4.12) we obtain $S^5(\phi) = c_5 \cos\phi + c_6 \sin\phi$ and in the light of Equation (4.27) we get so the system of Equation is

$$\begin{cases} X^0 = c_4, \\ X^1 = 0, \\ X^2 = \frac{1}{r}(c_5 \cos\phi + c_6 \sin\phi), \\ X^3 = \frac{1}{r}(-c_5 \sin\phi + c_6 \cos\phi) \cot\theta + D^8(r,\phi). \end{cases} \tag{4.28}$$

Now we can use Equation (4.12) we obtain $D^8(r,\phi) = S^6(r)$.Using Equation (4.11) we obtain $S^6(r) = 0$ and using Equation (4.9) after some algebraic manipulation we get the final result

$$\begin{cases} X^0 = c_4, \\ X^1 = 0, \\ X^2 = \frac{1}{r}(c_5 \cos\phi + c_6 \sin\phi), \\ X^3 = \frac{1}{r}(-\sin_5 + \cos\phi c_6) \cot\theta + c_{10} - e^r. \end{cases} \tag{4.29}$$

Thus the **case 1** we get the following TKVS: $\partial_t, \frac{1}{r}\cos\phi\,\partial_\theta - \frac{1}{r}\sin\phi\cot\theta\,\partial_\phi, \frac{1}{r}\sin\phi\,\partial_\theta + \frac{1}{r}\cos\phi\cot\theta\,\partial_\phi, e^r\partial_\phi$

Case 2

$i = i(r),\ j = j(r), i = j^{-1},\ e^i(\frac{i''}{2} + \frac{i'^2}{2} - \frac{1}{r^2}) + \frac{1}{r^2} = 0 = i = \ln(1 - \frac{2M}{r}),\ j = \ln(1 - \frac{2M}{r})^{-1}$ and $T = \frac{2}{r^2}$,and $Q = r$ where the Arnowitt-Deser-Misner mass is denoted by M. The space-time takes from.

$$ds^2 = 1 - \frac{2M}{r}dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + Q^2(r)[d\theta^2 + \sin^2 d\phi^2]. \tag{4.30}$$

Expanding Equation (2.19) and using Equation (4.30) we get

$$X^0{}_{,0} = 0, \tag{4.31}$$

$$r^2X^2{}_{,0} - \left(\frac{r-2M}{r}\right)X^0{}_{,2} = 0, \tag{4.32}$$

$$r^2\sin^2\theta X^3{}_{,0} - \left[\frac{r-2M}{r}\right]X^0{}_{,3} = 0, \tag{4.33}$$

$$\left[-\frac{r+2M}{r}\right]X^0{}_{,1} + \left[\frac{r}{r-2M}\right]X^1{}_{,0} - \left(\frac{M}{r^2}\right)X^0 = 0, \tag{4.34}$$

$$\frac{-2M}{r(r-2M)}X^1 + 2X^1{}_{,1} = 0, \tag{4.35}$$

$$r^2X^2{}_{,1} + \left[\frac{r}{r-2M}\right]X^1{}_{,2} + rX^2 = 0, \tag{4.36}$$

$$r^2\sin^2\theta X^3{}_{,1} + \left[\frac{r}{r-2M}\right]X^1{}_{,3} + r\sin^2\theta X^3 = 0, \tag{4.37}$$

$$X^2{}_{,2} = 0, \tag{4.38}$$

$$\sin^2\theta X^3{}_{,2} + X^2{}_{,3} = 0, \tag{4.39}$$

$$\cot X^2 + X^3{}_{,3} = 0. \tag{4.40}$$

From Equation (4.31), Equation (4.38), Equation (4.39) and Equation (4.35) the following system of equation is obtain.

$$\begin{cases} X^0 = K^1(r,\phi,\theta), \\ X^1 = \frac{\sqrt{-2M+r}}{r} \ K^4(t,\theta,\phi), \\ X^2 = K^2(\phi,t,r), \\ X^3 = K^2_{\phi}(\phi,r,t)\cot\theta + K^3(\phi,r,t), \end{cases} \tag{4.41}$$

where $K^1(\phi,\theta,r), K^2(\phi,t,r), K^3(r,\phi,t)$ and $K^4(t,\phi,\theta)$, are function of integration. We can use Equation (4.36) and after some simplification we get $K^4(t,,\phi,\theta) = \theta D^1(t,\phi) + D^2(t,\phi)$, where $D^1(\phi,t), D^2(\phi,t)$ are function of integration. Putting all these values in set of Equation (4.41) we obtain

$$\begin{cases} X^0 = K^1(r,\phi,\theta), \\ X^1 = \frac{\sqrt{r-2M}}{r} \ \theta D^1(t,\phi) + D^2(t,\phi), \\ X^2 = K^2(t,r,\phi), \\ X^3 = K^2_{\phi}(t,r,\phi)\cot\theta + K^3(t,r,\phi). \end{cases} \tag{4.42}$$

Using Equation (4.32) in Equation (4.42) we get.

$$r^2 \ K^2_t(t,r,\phi) - \left(\frac{r-2M}{r}\right) K^1_{\theta} \ (r,\theta,\phi) = 0 \tag{4.43}$$

Differentiating w.r.t t and after some algebraic manipulation we derive $K^2(\phi,t,r) = tD^3(r,\phi) + D^4(r,\phi)$. Differentiate w.r.t. ϕ we obtain $K^2_{\phi}(t,r,\phi) = tD^3_{\phi}(\phi,r) + D^4_{\phi}(\phi,r)$ substituting back in Equation (4.43) we derive $K^1(\theta,\phi,r) = D^5(\phi,r)\theta + D^6(\phi,r)$. In the light of Equation (4.42) and using all these information we get

$$\begin{cases} X^0 = \theta D^5(\phi,r) + D^6(\phi,r), \\ X^1 = \frac{\sqrt{r-2M}}{r} \ [\theta D^1(t,\phi) + D^2(t,\phi)], \\ X^2 = tD^3(\phi,r) + D^4(\phi,r), \\ X^3 = [tD^3_\phi(\phi,r) + D^4_\phi(\phi,r)]\cot\theta + K^3(t,\phi,r). \end{cases} \tag{4.44}$$

Considering Equation (4.34) and after some algebraic manipulation the above system of Equation takes the form

$$\begin{cases} X^0 = \theta D^5(\phi,r) + D^6(\phi,r), \\ X^1 = \frac{\sqrt{r-2M}}{r} \ [\theta\{tS^1(\phi) + S^2(\phi)\} + D^2(t,\phi)], \\ X^2 = tD^3(\phi,r) + D^4(\phi,r), \\ X^3 = [tD^3_\phi(r,\phi)]\cot\theta + K^3(t,r,\phi). \end{cases} \tag{4.45}$$

Next we use Equation (4.33) and Equation (4.37) and after some calculations we get

$$\begin{cases} X^0 = \theta S^4(r) + D^6(\phi,r), \\ X^1 = \frac{\sqrt{r-2M}}{r} \ [\theta\{tS^1(\phi) + S^2(\phi)\} + D^2(t,\phi)], \\ X^2 = D^4(r,\phi) + tS^3(r), \\ X^3 = D^4_\phi(\phi,r)\cot\theta + D^8(\phi,r). \end{cases} \tag{4.46}$$

Now from Equation (4.40) we get $S^3(r) = 0$. Similarly from Equation (4.36) we get $S^1(\phi) = c_1$ and after some simplifications we obtain $c_1 = 0$. From Equation (4.34) we get $D^2_t = (t,\phi) = S^5(\phi) \Rightarrow D^2_t = (t,\phi) = tS^5(\phi) + S^6(\phi_)$. Substituting all these information the above system of equation takes the form

$$\begin{cases} X^0 = \theta S^4(r) + D^6(\phi, r), \\ X^1 = \frac{\sqrt{r-2M}}{r} \; [\theta S^2(\phi) + tS^5(\phi) + S^6(\phi)], \\ X^2 = D^4(r, \phi), \\ X^3 = D^4_\phi(\phi, r)\cot\theta + D^8(\phi, r). \end{cases} \tag{4.47}$$

Using the above system of equation into Equation (4.30) we get $S^4(r) = 0$. Similarly from equation (4.31) and Equation (4.32) one gets $D^6(r, \phi) = S^7(r)$ and $S^5(r) = c_2$ respectively. By updating the above system we obtain

$$\begin{cases} X^0 = S^7(r), \\ X^1 = \frac{\sqrt{r-2M}}{r} \; [\theta S^2(\phi) + tc_2 + S^6(\phi)], \\ X^2 = D^4(r, \phi), \\ X^3 = D^4_\phi(r, \phi)\cot\theta + D^8(r, \phi). \end{cases} \tag{4.48}$$

Now substituting the values from system of equation (4.48) into Equation (4.40)and Equation (4.37) and after some calculations we get $D^8(r, \phi) = S^{10}(r)$ and $D^4 = \frac{c_3}{r}\cos\phi + \frac{c_4}{r}\sin\phi$ substituting back these values we get the following system of Equation

$$\begin{cases} X^0 = S^7(r), \\ X^1 = \frac{\sqrt{r-2M}}{r} \; [\theta S^2(\phi) + tc_2 + S^6(\phi)], \\ X^2 = \frac{c_4}{r}\sin\phi + \frac{c_3}{r}\cos\phi, \\ X^3 = -[\frac{c_3}{r}\cos\phi + \frac{c_4}{r}\sin\phi]\cot\phi + S^{10}(r). \end{cases} \tag{4.49}$$

Now we can use Equation (4.35), Equation (4.34) and Equation (4.37) after performing calculations. We get the final system of the Equation given below

$$\begin{cases} X^0 = [\frac{c_1(r-2M)}{r}]^{\frac{-1}{2}}, \\ X^1 = 0, \\ X^2 = \frac{c_3}{r}\cos\phi + \frac{c_4}{r}\sin\phi, \\ X^3 = [-\frac{c_3}{r}\cos\phi + \frac{c_4}{r}\sin\phi]\cot\phi + c_5 e^{-x}. \end{cases} \tag{4.50}$$

Thus for **Case 2** we get the following TKVF: $[\frac{r-2M}{r}]^{\frac{1}{2}}\partial_t,\ \frac{1}{r}\cos\phi\,\partial_\theta - \frac{1}{r}\cos\phi\cot\theta\,\partial_\phi,\ \frac{1}{r}\sin\phi\,\partial_\theta + \frac{1}{r}\sin\phi\cot\theta\,\partial_\phi,\ \ e^{-x}\partial_\phi.$

Case 3 $i = i(r),\ j = j(r),\ i = j^{-1},\ r^2(i'' + i'^2) - 2(1 - e^{-i})$ implies $i = \ln(1 - \frac{\Lambda r^2}{3}),\ j = \ln(1 - \frac{\Lambda r^2}{3})^{-1}$, $T = (\frac{2}{r^2} - 2\Lambda)$ and $Q = r$,where the cosmological constant is Λ.

The space-times takes from

$$ds^2 = -\left(\frac{3 - Ar^2}{3}\right)dt^2 + \left(\frac{3}{3 - Ar^2}\right)dr^2 + r^2[d\theta^2 + \sin^2\theta d\phi^2]. \tag{4.51}$$

Expanding (2.19) and using (4.51) we get

$$X^0{}_{,0} = 0, \tag{4.52}$$

$$r^2 X^2{}_{,0} - \left(\frac{3 - Ar^2}{3}\right)X^0{}_{,3} = 0, \tag{4.53}$$

$$r^2\sin^2\theta X^3{}_{,0} - \left(\frac{3 - Ar^2}{3}\right)X^0{}_{,3} = 0, \tag{4.54}$$

$$\left[\frac{Ar^2 - 3}{3}\right]X^0{}_{,1} + \left(\frac{3 - Ar^2}{3}\right)X^1{}_{,0} + \left(\frac{Ar}{3}\right)X^0 = 0, \tag{4.55}$$

$$\left(\frac{Ar}{3 - Ar^2}\right)X^1 + X^1{}_{,1} = 0, \tag{4.56}$$

$$r^2 X^2{}_{,1} + \frac{3}{3 - Ar^2}x^1{}_{,2} + rX^2 = 0, \tag{4.57}$$

$$r^2\sin^2\theta X^3{}_{,1} + \left(\frac{3}{3 - Ar^2}\right)X^1{}_{,3} + r\sin^2\theta X^3 = 0, \tag{4.58}$$

$$X^2{}_{,2} = 0, \tag{4.59}$$

$$\sin^2\theta X^3{}_{,2} + X^2{}_{,3} = 0, \tag{4.60}$$

$$\cot X^2 + X^3{}_{,3} = 0. \tag{4.61}$$

From Equation (4.52), Equation (4.59), Equation (4.60) and Equation (4.56) the following system of Equation is obtain.

$$\begin{cases} X^0 = K^1(r,\theta,\phi), \\ X^1 = \sqrt{\frac{3-Ar^2}{3}}\ K^4(t,\theta,\phi), \\ X^2 = K^2(\phi,t,r), \\ X^3 = \cot\theta K^2_\phi(\phi,t,r) + K^3(\phi,t,r), \end{cases} \tag{4.62}$$

where $K^1(r,\theta,\phi), K^2(\phi,t,r), K^3(t,\phi,r), K^4(t,\phi,\theta)$ are function of integration. We can use Equation (4.53) and after some simplification we get $K^1(r,\phi,\theta) = \theta D^1(\phi,r) + D^2(\phi,r)$. Using Equation (4.57) and after some calculation we get $K^4(t,\theta,\phi) = D^3(\phi,t)\theta + D^4(\phi,t)$. And in the light of (4.62) we get

$$\begin{cases} X^0 = \theta D^1(\phi,r) + D^2(\phi,r), \\ X^1 = \sqrt{\frac{3-Ar^2}{3}}\ [\theta D^3(\phi,t) + D^4(\phi,t)], \\ X^2 = K^2(t,r,\phi), \\ X^3 = K^2_\phi(r,\phi,t)\cot\theta + K^3(r,\phi,t),. \end{cases} \tag{4.63}$$

Equation (4.55), and after some calculations we get $D^3(t,\phi) = S^1(\phi)$, and $D^4(t,\phi) = tS^2(\phi) + S^3(\phi)$. Equation (4.53) and after some simplification we get $K^2(t,\phi,r) = tD^5(\phi,r) + D^6(\phi,r)$ and $K^2_\phi(t,r,\phi) = tD^5_\phi(r,\phi) + D^6_\phi(r,\phi)$ and Equation (4.54) and after some simplification we obtain $D^7(\phi,r)=0$ and $D^5(\phi,r)=S^4(r)$ after using all the values and in the light of (4.63) we get the following set of Equation

$$\begin{cases} X^0 = \theta D^1(r,\phi) + D^2(r,\phi), \\ X^1 = \sqrt{\frac{3-Ar^2}{3}}\ [\theta S^1(\phi) + tS^2(\phi) + S^3(\phi)], \\ X^2 = tS^4(\phi) + D^6(r,\phi), \\ X^3 = D^6_\phi(r,\phi)\cot\theta + D^8(r,\phi). \end{cases} \tag{4.64}$$

By using equation (4.54) we get $D^1(r,\phi)=S^5(r)$ and $D^2(r,\phi)=S^6(r)$. And using (4.55) and after some calculation we get $S^2(\phi)=c_1$. Using all these information and in the light of (4.64) we get

$$\begin{cases} X^0 = \theta S^5(r) + S^6(r), \\ X^1 = \sqrt{\frac{3-Ar^2}{3}}\ [\theta S^1(\phi) + tc_1 + S^3(\phi)], \\ X^2 = tS^4(\phi) + D^6(r,\phi), \\ X^3 = D^6_\phi(r,\phi)\cot\theta + D^8(r,\phi). \end{cases} \tag{4.65}$$

Using Equation (4.57) and after performing calculation $S^4(\phi)=0$ and using (4.53) we get $S^5(r)=0$ and in the light of (4.65) we get

$$\begin{cases} X^0 = S^6(r), \\ X^1 = \sqrt{\frac{3-Ar^2}{3}}\ [\theta S^1(\phi) + tc_1 + S^3(\phi)], \\ X^2 = D^6(r,\phi), \\ X^3 = D^6_\phi(r,\phi)\cot\theta + D^8(r,\phi). \end{cases} \tag{4.66}$$

Using (4.61) and after performing some calculations we get $D^6(\phi,r)=S^7(r)\cos\phi+S^8(r)\sin\phi$ and $D^6_\phi(\phi,r)=-\sin\phi S^7(r)+\cos\phi S^8(r)$ and $D^8(r,\phi)=S^9(r)$. Using (4.58) we get $D^6(r,\phi)=$

$\frac{c_2}{r}\cos\phi + \frac{c_3}{r}\sin\phi$ and $S^9(r) = \frac{c_4}{r}$ and in the light of (4.66) we get

$$\begin{cases} X^0 = s^6(r) \\ X^1 = \sqrt{\frac{3-Ar^2}{3}}\ [\theta S^1(\phi) + tc_1 + S^3(\phi)], \\ X^2 = \frac{c_3}{r}\cos\phi + \frac{c_4}{r}\sin\phi, \\ X^3 = [-\frac{c_3}{r}\cos\phi + \frac{c_4}{r}\sin\phi]\cot\phi + \frac{c_4}{r}. \end{cases} \tag{4.67}$$

Using Equation (4.56) and after some calculation we get $S^1(\phi) = 0$, $c_1 = 0$ and $S^3(\phi) = 0$. Using Equation (4.55) after some calculation and in the light of (4.67) we get the final system of Equation

$$\begin{cases} X^0 = \frac{C_5}{\sqrt{Ar^2-3}}, \\ X^1 = \sqrt{\frac{3-Ar^2}{3}}\ [\theta S^1(\phi) + tc_1 + S^3(\phi)], \\ X^2 = \frac{c_3}{r}\cos\phi + \frac{c_4}{r}\sin\phi, \\ X^3 = [-\frac{c_3}{r}\cos\phi + \frac{c_4}{r}\sin\phi]\cot\phi + \frac{c_4}{r}. \end{cases} \tag{4.68}$$

Thus for **case 3** we get the following TKVF: $\frac{1}{\sqrt{Ar^2=-3}}\partial_t, \sqrt{\frac{3-Ar^2}{3}}t\partial_r$ $\frac{1}{r}\cos\phi\partial_\theta - \frac{1}{r}\cos\phi\cot\theta\partial_\phi$, $\frac{1}{r}\sin\phi\partial_\theta + \frac{1}{r}\sin\phi\cot\phi\partial_\phi$, $\frac{1}{r}\partial_\phi$.

Case 4 $i = \text{constant} = E_1 \neq 0$, $j = \text{constant} = E_2 \neq 0$, $i = \ln(E_1)$, $j = \ln(E_2)$, $QQ'' - Q'^2 + E_2 = 0$ implies $T = \frac{2}{E_2 r^2}$ and $Q = r\sqrt{E_2}$, where $E_1, E_2 \in \Re/\{0\}$ with $E_1 \neq E_2$. The space-time takes from

$$ds^2 = -k_1 dt^2 + k_2 dr^2 + r^2 k_2[d\theta^2 + \sin^2\theta d\phi^2]. \tag{4.69}$$

Expanding (2.19) and using (4.69) we get

$$X^0{}_{,0} = 0, \tag{4.70}$$

$$r^2 K_2 X^2{}_{,0} - K_1 X^0{}_{,2} = 0, \tag{4.71}$$

$$r^2K_2\sin^2\theta X^3{}_{,0} - K_1X^0{}_{,3} = 0, \tag{4.72}$$

$$-2K_1X^0{}_{,1} + 2K_2X^1{}_{,0} = 0, \tag{4.73}$$

$$X^1{}_{,1} = 0, \tag{4.74}$$

$$r^2X^2{}_{,1} + X^1{}_{,2} + X^2 = 0, \tag{4.75}$$

$$r^2\sin^2\theta X^3{}_{,1} + X^1{}_{,3} + e\sin^2\theta X^3 = 0, \tag{4.76}$$

$$X^2{}_{,2} = 0, \tag{4.77}$$

$$\sin^2\theta X^3{}_{,2} + X^2{}_{,3} = 0, \tag{4.78}$$

$$\cot X^2 + X^3{}_{,3} = 0. \tag{4.79}$$

From Equation (4.70), Equation (4.77), Equation (4.78) and Equation (4.74) and after some calculations we get

$$\begin{cases} X^0 = K^1(r,\theta,\phi), \\ X^1 = \frac{1}{\sqrt{k^2}}\ K^4(t,\theta,\phi), \\ X^2 = K^2(t,r,\phi), \\ X^3 = K^2_{\phi}(\phi,t,r)\cot\theta + k^3(\phi,t,r). \end{cases} \tag{4.80}$$

$K^1(\phi,\theta,r), K^2(r,t,\phi), k^3(t,\phi,r)$ and $K^4(\phi,\theta,t)$ are function of integration. Now, we use Equation (4.71) and after simplification we get $K^2(r,t,\phi) = tD^3(\phi,r) + D^4(\phi,r)$ and derivative w.r.t ϕ we obtain $K^2_{\phi}(t,\phi,r) = tD^3_{\phi}(\phi,r) + D^4_{\phi}(\phi,r)$. Equation (4.72) and after performing calculations we obtain $K^3(\phi,t,r) = tD^5(\phi,r) + D^6(\phi,r)$ and $D^3(\phi,r) = S^1(\phi)$ and $D^5(\phi,r) = 0$ Equation (4.71) and some calculation we get$D^1(\phi,r) = S^2(r)$ and again using Equation (4.72) and after some calculations and considering (4.80) we get

$$\begin{cases} X^0 = \theta S^2(r) + S^3(r), \\ X^1 = \frac{1}{\sqrt{k^2}} \ K^4(\phi, t, \theta), \\ X^2 = tS^1(r) + D^4(r, \phi), \\ X^3 = D^4_\phi(r, \phi)\cot\theta + D^6(r, \phi). \end{cases} \tag{4.81}$$

Now using Equation (4.73) we gets $K^4(r,t,\phi) = tD^7(\phi,r) + D^8(\phi,r)$, $S^2(r) = c_1$ and $D^7(\phi,r) = S^4(r)$. Equation (4.74), Equation (4.76), one gets $S^4(r) = c_2 \rightarrow \mathrm{D}^8(r,\phi) = S^5(\phi)$ and others $c_2 = 0$.

$$\begin{cases} X^0 = S^3(r) + \theta c_1, \\ X^1 = \frac{1}{\sqrt{k^2}} \ S^5(\phi), \\ X^2 = tS^1(r) + D^4(\phi, r), \\ X^3 = \cot\theta D^4_\phi(r, \phi) + D^6(\phi, r). \end{cases} \tag{4.82}$$

Equation (4.79) and Equation (4.71) after some calculations and considering (4.82) we get the following set of Equation

$$\begin{cases} X^0 = c_2, \\ X^1 = \frac{1}{\sqrt{k^2}} \ S^5(\phi), \\ X^2 = S^6(r)\cos\phi + S^7(r)\sin\phi, \\ X^3 = (-S^6(r)\sin\phi + S^7(r)\cos\phi)\cot\theta - S^8(r). \end{cases} \tag{4.83}$$

Now using (4.76) and after some algebric maniplution we get $S^6(r) = \frac{c_3}{r}$, $S^7(r) = \frac{c_4}{r}$ and

$S^8(r) = \frac{c_5}{r}$ and $S^5(\phi) = c_6$ substituting back into (4.83) we get the final system of the equation

$$\begin{cases} X^0 = c_2, \\ X^1 = \frac{1}{\sqrt{k^2}} \ c_6, \\ X^2 = \cos\phi \frac{c_3}{r} + \frac{c_4}{r}\sin\phi, \\ X^3 = [-\sin\phi \frac{c_3}{r} + \cos\phi \frac{c_4}{r}]\cot\theta - \frac{c_5}{r}. \end{cases} \tag{4.84}$$

Thus for **case 4** we gets the following TKVF: ∂_t, $\frac{1}{\sqrt{k^2}}\partial_r$ $\frac{1}{r}\cos\phi\,\partial\theta - \frac{1}{r}\sin\phi\cot\theta\,\partial_\phi$, $\frac{1}{r}\sin\phi\,\partial_\theta + \frac{1}{r}\cos\phi\cot\theta, -\frac{1}{r}\partial_\phi$

Case 5

$j = E_1 \neq 0$ constant $i = i(r)$ $,r^2 i'' - ri' - 2 = 0 \rightarrow i = (\frac{E_2 r^2}{2} + E_3 - \ln r)$, $T = \frac{2E_2}{e^{E_1,}}$ and $Q = r$ where $E_1,\ E_2,\ E_3 \in \Re\ (E_1, E_2 \neq 0)$.

The space-times takes from

$$ds^2 = exp\left(\frac{k_2 r^2}{2} - \ln r + k_3\right) dt^2 + dr^2 + r^2\ [d\theta^2 + \sin^2\theta\ d\phi^2]. \tag{4.85}$$

Expanding Equation (2.19) and using Equation (4.85) we get

$$X^0{}_{,0} = 0, \tag{4.86}$$

$$r^2 X^2{}_{,0} - e^{(}\frac{k_2 r^2}{2} - \ln r + k_3) X^0{}_{,2} = 0, \tag{4.87}$$

$$r^2 k_2 \sin^2\theta X^3{}_{,0} - e^{(}\frac{k_2 r^2}{2} - \ln r + k_3)\ X^0{}_{,3} = 0, \tag{4.88}$$

$$-2exp^{\left(\frac{k_2 r^2}{2} - \ln r + k_3\right)} X^0{}_{,1} + 2X^1{}_{,0} - e^{\left(\frac{k_2 r^2}{2} - \ln r + k_3\right)(k_2 r - \frac{1}{r})}\ X^0 = 0, \tag{4.89}$$

$$2X^1{}_{,1} = 0, \tag{4.90}$$

$$r^2 X^2{}_{,1} + X^1{}_{,2} + rX^2 = 0, \tag{4.91}$$

$$r^2\sin^2\theta X^3{}_{,1} + X^1{}_{,3} + r\sin^2\theta X^3 = 0, \tag{4.92}$$

$$X^2{}_{,2} = 0, \tag{4.93}$$

$$\sin^2\theta X^3{}_{,2} + X^2{}_{,3} = 0, \tag{4.94}$$

$$\cot X^2 + X^3{}_{,3} = 0. \tag{4.95}$$

From Equation (4.86), Equation (4.93), Equation (4.94) and Equation (4.90) and after determining we get

$$\begin{cases} X^0 = K^1(r,\theta,\phi), \\ X^1 = e^{\frac{-k_1}{2}} \; K_4(\theta,t,\phi), \\ X^2 = K^2(r,\phi,t), \\ X^3 = K^2_\phi(\phi,t,r)\cot\theta + K^3(\phi,t,r), \end{cases} \tag{4.96}$$

where $K^1(r,\phi,\theta), K^2(r,\phi,t), K^3(t,\phi,r)$ and $K_4(t,\theta,\phi)$ are constant of integration. Now,we use Equation (4.87) and after some calculation we derive $K^2(r,\phi,t) = tD^3(\phi,r) + D^4(\phi,r)$ and derivative w.r.t ϕ we obtain $K^2_\phi(r,\phi,t) = tD^3_\phi(\phi,r) + D^4_\phi(\phi,r)$. Now using Equation (4.88) we get $K^3(\phi,t,r) = tD^5(\phi,r) + D^6(\phi,r)$ and $D^3(r,\phi) = S^1(r)$ and $D^5(r,\phi) = 0$ Equation (4.91) after deriving and in the view of Equation (4.96) we get

$$\begin{cases} X^0 = \theta D^1(r,\phi) + D^2(r,\phi), \\ X^1 = e^{\frac{-k_1}{2}}[c_2 + \theta(tc_1) + D^8(t,\phi)], \\ X^2 = tS^1(r) + D^4(r,\phi), \\ X^3 = D^4_\phi(r,\phi)\cot\theta + D^6(r,\phi). \end{cases} \tag{4.97}$$

Using Equation (4.95) upon solving and in the view of Equation (4.97) we get

$$\begin{cases} X^0 = \theta D^1(r,\phi) + D^2(r,\phi), \\ X^1 = e^{\frac{-k_1}{2}} \ [c_1\theta + S^6(\phi)t + c_3], \\ X^2 = \cos\phi S^3(r) + S^4(r)\sin\phi, \\ X^3 = (-\sin\phi S^3(r) + \cos\phi S^4(r))\cot\theta - S^5(r). \end{cases} \tag{4.98}$$

Now using Equation (4.87) we get $D^1(r,\phi) = 0$ Equation (4.89) we get $c_2 = 0$, and $D^8(t,\phi) = tS^6(\phi) + c_3$ Equation (4.88) $D^2(r,\phi) = S^7(r)$ and Equation (4.92) we get $S^3(r) = \frac{c_4}{r}, S^4(r) = \frac{c_5}{r}$ and $S^5(r) = \frac{c_6}{r}$ after determining results and in the view of Equation (4.98) we get

$$\begin{cases} X^0 = S^7(r), \\ X^1 = e^{\frac{-k_1}{2}}[c_1\theta + tS^6(\phi) + c_3], \\ X^2 = \cos\phi \frac{c_4}{r} + \frac{c_5}{r}\sin\phi, \\ X^3 = [-\sin\phi \frac{c_4}{r} + \cos\phi \frac{c_5}{r}]\cot\theta - \frac{c_6}{r}. \end{cases} \tag{4.99}$$

Using Equation (4.91) we get $c_1 = 0$, Equation (4.89) and after determining results and considering Equation (4.99) we get the final system of the Equation

$$\begin{cases} X^0 = c_7 . \int e^{\frac{1}{2r}} - \frac{k_2 r}{2} \, dr + c_8, \\ X^1 = e^{\frac{-k_1}{2}}[tc_7 + c_3], \\ X^2 = \frac{c_4}{r}\cos\phi + \frac{c_5}{r}\sin\phi, \\ X^3 = [-\frac{c_4}{r}\sin\phi + \frac{c_5}{r}\cos\phi]\cot\theta - \frac{c_6}{r}. \end{cases} \tag{4.100}$$

Thus for **case 5** we get the following TKVF: $\int e^{\frac{1}{2r}} - \frac{k_2 r}{2} \, dr\partial_t, e^{\frac{-k_1}{2}}\partial_r, \frac{1}{r}\cos\phi\partial_\theta - \frac{1}{r}\sin\phi\cot\theta\partial_\phi, \frac{1}{r}\sin\phi\partial_\theta + \frac{1}{r}\cos\phi\cot\theta\partial_\theta, \frac{1}{r}\partial_\phi$

Case 6 $j = j(r), i = i(r),\ (\frac{i''}{2} + \frac{i'^2}{4}) = 0 \rightarrow i = \ln(\frac{E_1 r + E_2}{2})^2,\ -E_1 j' e^{-j} + 2(E_1 r + E_2) = 0,$ which implies $j = \ln\left(\frac{E_1}{E_1 E_3 - E_1 r^2 - 2k_2 r}\right)$, $T = 0$, and $Q = 1$ where $E_1,\ E_2,\ E_3 \in \Re/\{0\}$. The space-times takes place

$$ds^2 = +[\frac{k_2}{r^2}]dr^2 - \frac{k_1}{r^2}dt^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{4.101}$$

Expanding (2.19) and using (4.101) We get

$$X^0{}_{,0} = 0, \tag{4.102}$$

$$r^2 X^2{}_{,0} - \frac{k_1}{r^2} X^0{}_{,2} = 0, \tag{4.103}$$

$$r^2 \sin^2\theta X^3{}_{,0} - \frac{k_1}{r^2} X^0{}_{,3} = 0, \tag{4.104}$$

$$-k_1 X^0{}_{,1} + k_2 X^1{}_{,0} + \frac{k_1}{r} X^0 = 0, \tag{4.105}$$

$$-\frac{-1}{r} X^1 + X^1{}_{,1} = 0, \tag{4.106}$$

$$r^2 X^2{}_{,1} + \frac{k_2}{r_2} X^1{}_{,2} + r X^2 = 0, \tag{4.107}$$

$$r^2 \sin^2\theta X^3{}_{,1} + \frac{K_2}{r^2} X^1{}_{,3} + r\sin^2\theta X^3 = 0, \tag{4.108}$$

$$X^2{}_{,2} = 0, \tag{4.109}$$

$$\sin^2\theta X^3{}_{,2} + X^2{}_{,3} = 0, \tag{4.110}$$

$$\cot\theta X^2 + X^3{}_{,3} = 0. \tag{4.111}$$

From Equation (4.102), Equation (4.109), Equation (4.110) and Equation (4.106) and after deriving results we get

$$\begin{cases} X^0 = K^1(r, \theta, \phi), \\ X^1 = \sqrt{\frac{r^2}{k_2}}\, K^4(t, \phi, \theta), \\ X^2 = K^2(t, \phi, r), \\ X^3 = K^2_\phi(t, \phi, r)\cot\theta + K^3(t, \phi, r), \end{cases} \tag{4.112}$$

where $K^1(r,\phi,\theta), K^2(r,\phi,t), K^3(t,\phi,r)$ and $K^4(\theta,\phi,t)$ are constant of integration. Now, we use Equation (4.103) we get $K^2(t,\phi,r) = tD^1(\phi,r) + D^2(\phi,r)$ and derivative w.r.t ϕ we get $K^2_\phi(r,\phi,t) = tD^1_\phi(\phi,r) + D^2\phi(\phi,r)$ and $K^1(\theta,\phi,r) = \theta D^3(\phi,r) + D^4(\phi,r)$, Equation (4.104) $K^3(t,r,\phi) = tD^5(\phi,r) + D^6(\phi,r)$, $D^1(\phi,r) = S^1(r)$ and $D^5(\phi,r) = 0$. Equation (4.105) we get $K^4(\theta,\phi,t) = tD^7(\phi,\theta) + D^8(\phi,\theta) \rightarrow \mathrm{D}^7(\theta,\phi) = \theta S^2(\phi) + S^3(\phi)$. Equation (4.108) and after determining results and in the light of Equation (4.112) we get

$$\begin{cases} X^0 = \theta D^3(r,\phi) + D^4(r,\phi), \\ X^1 = \sqrt{\frac{r^2}{k_2}}\ t[\theta c_1 + c_2] + D^8(\theta,\phi), \\ X^2 = tS^1(r) + D^2(r,\phi), \\ X^3 = D^2_\phi(r,\phi)\cot\theta + D^6(r,\phi). \end{cases} \tag{4.113}$$

Using Equation (4.103) upon solving we get

$$[D^3(r,\phi)] = S^4(r), \tag{4.114}$$

where $S^1(r), D^3(r,\phi)$ are function of integration. Now using Equation (4.104) we get $D^4(r,\phi) = S^5(r)$. Now using Equation (4.107) we get $D^8(\theta,\phi) = \theta S^6(\phi) + c_3$, Equation (4.111) we get $S^1(r) = 0$, and using Equation (4.103) we get$S^4(r) = 0$. Now the system of the Equation is

$$\begin{cases} X^0 = S^5(r), \\ X^1 = \sqrt{\frac{r^2}{k^2}} t[\theta c_1 + c_2] + \theta S^6(\phi) + c_3, \\ X^2 = D^2(r,\phi), \\ X^3 = D^2_\phi(r,\phi)\cot\theta + D^6(r,\phi), \end{cases} \tag{4.115}$$

where c_1, c_2, c_3 are constants. Now using Equation (4.105) we get $c_1 = 0$ Equation (4.106) and after some calculations and in the view of Equation (4.115) we get

$$\begin{cases} X^0 = S^5(r), \\ X^1 = 0, \\ X^2 = D^2(r,\phi), \\ X^3 = D^2_\phi(r,\phi)\cot\theta + D^6(r,\phi). \end{cases} \quad (4.116)$$

Using Equation (4.111) and after processing and in the view of Equation (4.116) we get

$$\begin{cases} X^0 = \frac{c_4}{r}, \\ X^1 = 0, \\ X^2 = S^7(r)\cos\phi + S^8(r)\sin\phi, \\ X^3 = [-S^7(r)\sin\phi + S^8(r)\cos\phi]\cot\theta + S^9(r). \end{cases} \quad (4.117)$$

Now using Equation (4.108) and after some simplification and in the view of (4.117) we obtain the final set of Equation

$$\begin{cases} X^0 = \frac{c_4}{r}, \\ X^1 = 0, \\ X^2 = \cos\phi\frac{c_5}{r} + \frac{c_6}{r}\sin\phi, \\ X^3 = [-\sin\phi\frac{c_5}{r} + \frac{c_6}{r}\cos\phi]\cot\theta + \frac{c_7}{r}. \end{cases} \quad (4.118)$$

Thus for **case 6** we get the following TKVF: $\frac{1}{r}\partial_t, \frac{1}{r}[\cos\phi\partial_\theta - \sin\phi\cot\theta\partial_\phi], \frac{1}{r}[\sin\phi\partial_\theta + \cos\phi\partial_\theta], \frac{1}{r}\partial_\phi$.

Case 7 $j = j(r),\ i = i(r),\ ri'' - i' = 0$ implies $i = (\frac{E_1 r^2}{2} + E_2),\ rj'(ri'+1) + 2 = 0 \rightarrow\ j = \ln\left(\frac{E_3\sqrt{E_1 r^2+1}}{r}\right)^2$, $T = \frac{2}{E_3^2,}$ and $Q = r$ where $E_1,\ E_2,\ E_3 \in \Re(E_1,\ E_3 \neq 0)$

The space-times takes from

$$ds^2 = -k_1 r\, exp\left(\frac{k_1 r^2}{2} +^{k_2}\right)\, dt^2 + \frac{k_3\sqrt{k_1 r^2+1}}{r}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (4.119)$$

Expanding equation (2.19) and Using (4.119) we get

$$X^0{}_{,0} = 0, \tag{4.120}$$

$$r^2 X^2{}_{,0} - \left(e^{\frac{k_1 r^2}{2}+k_2}\right) X^0{}_{,2} = 0, \tag{4.121}$$

$$r^2 \sin^2\theta X^3{}_{,0} - \left(e^{\frac{k_1 r^2}{2}+k_2}\right) X^0{}_{,3} = 0, \tag{4.122}$$

$$-2(e^{\frac{k_1 r^2}{2}+k2}) X^0_1 + 2\frac{k_3\sqrt{k_1 r^2+1}}{r} dX^1{}_{,0} - \left(e^{\frac{k_1 r^2}{2}+k_2}\right) X^0 = 0, \tag{4.123}$$

$$-\frac{1}{r(k_1 r^2+1)} X^1 + X^1{}_{,1} = 0, \tag{4.124}$$

$$r^2 X^2{}_{,1} + \left[\frac{k_3\sqrt{k_1 r^2+1}}{r}\right]^2 X^1{}_{,2} + r X^2 = 0, \tag{4.125}$$

$$r^2 \sin^2\theta X^3{}_{,1} + \left[\frac{k_3\sqrt{k_1 r^2+1}}{r}\right]^2 X^1{}_{,3} + r\sin^2\theta X^3 = 0, \tag{4.126}$$

$$X^2{}_{,2} = 0, \tag{4.127}$$

$$\sin^2\theta X^3{}_{,2} + X^2{}_{,3} = 0, \tag{4.128}$$

$$\cot\theta X^2 + X^3{}_{,3} = 0. \tag{4.129}$$

From Equation (4.120), Equation (4.127), Equation (4.128) and Equation (4.124) and after deriving results we get the following set of Equation

$$\begin{cases} X^0 = K^1(r,\theta,\phi), \\ X^1 = \frac{r}{k_3\sqrt{k_1 r^2+1}} K^4(t,\theta,\phi), \\ X^2 = K^2(r,t,\phi), \\ X^3 = K^2_\phi(r,t,\phi)\cot\theta + K^3(r,t,\phi), \end{cases} \tag{4.130}$$

where $K^1(\phi,\theta,r)$, $k^2(\phi,t,r)$, $k^3(\phi,t,r)$ and $K^4(\phi,t,\theta)$ are function of integration. Now, we use Equation (4.121) we get $K^1(r,\theta,\phi) = \theta D^1(\phi,r) + D^2(\phi,r)$ $\rightarrow$K$^2(r,t,\phi) = tD^3(\phi,r) +$

$D^4(\phi,r)$, derivative w.r.t ϕ we derive $K^2_\phi(r,t,\phi) = tD^3_\phi(\phi,r)+D^4_\phi(\phi,r)$. Equation (4.122) we get $K^3(r,t,\phi) = tD^5(\phi,r)+D^6(\phi,r) \rightarrow \mathrm{D}^3(\phi,r) = S^1(r)$, and $D^5(\phi,r) = 0$. Equation (4.121) we get $D^1(\phi,r) = S^2(r)$ and Equation (4.123) after deriving results and in the view of (4.130) we get

$$\begin{cases} X^0 = \theta S^2(r)+S^3(r), \\ X^1 = \frac{r}{k_3\sqrt{k_1r^2+1}}\,[t(c_1\theta+S^5(\phi))+D^8(\theta,\phi)], \\ X^2 = tS^1(r)+D^4(\phi,r), \\ X^3 = \cot\theta D^4_\phi(\phi,r)+D^6(\phi,r). \end{cases} \tag{4.131}$$

Using Equation (4.129) after some calculations and considering Equation (4.131) we get

$$\begin{cases} X^0 = \theta S^2(r)+S^3(r), \\ X^1 = \frac{r}{k_3\sqrt{k_1r^2+1}}\,[t(\theta c_1+S^5(\phi))+D^8(\theta,\phi)], \\ X^2 = \cos\phi S^6(r)+\sin\phi S^7(r), \\ X^3 = (-\sin\phi S^6(r)+\cos\phi S^7(r))\cot\theta - S^8(r). \end{cases} \tag{4.132}$$

Now using Equation (4.121) we get $S^2(r) = 0$, Equation (4.123) we get $S^5(\phi) = c_2$ and $c_1 = 0$, Equation (4.125) we obtain $D^8(\theta,\phi) = S^9(\phi)\theta+S^{10}(\phi)$,and Equation (4.126) we the values of $S^6(r) = \frac{c_3}{r}$ $S^7(r) = \frac{c_4}{r}$ and $S^8(r) = \frac{c_5}{r}$ and after deriving results and in the light of Equation (4.132) we get

$$\begin{cases} X^0 = S^3(r), \\ X^1 = \frac{r}{k_3\sqrt{k_1r^2+1}}\,[tc_2)+\theta S^9(\phi)+S^{10}(\phi)], \\ X^2 = \cos\phi\frac{c_3}{r}+\frac{c_4}{r}\sin\phi, \\ X^3 = \cot\theta[-\sin\phi\frac{c_3}{r}+\frac{c_4}{r}\cos\phi]-\frac{c_5}{r}. \end{cases} \tag{4.133}$$

Using Equation (4.125) we obtain $S^9(\phi)=0$, Equation (4.123) we obtain $S^3(r)=\frac{-1}{2}c_7e^{\frac{-1}{2}r}$, Equation (4.126) deriving results $S^{10}(\phi)=c_8$ where c_8 is constant and in the light of Equation (4.133) we get the following system

$$\begin{cases} X^0=\frac{-1}{2}c_7e^{\frac{-1}{2}r}, \\ X^1=\frac{r}{k_3\sqrt{k_1r^2+1}}[tc_2+c_8], \\ X^2=\frac{c_3}{r}\cos\phi+\sin\phi\frac{c_4}{r}, \\ X^3=[-\sin\phi\frac{c_3}{r}+\frac{c_4}{r}\cos\phi]\cot\theta-\frac{c_5}{r}. \end{cases} \tag{4.134}$$

Thus for **case 7** we get the following TKVF: $\frac{-1}{2}e^{\frac{-1}{2}r}\partial_t, \frac{r}{k_3\sqrt{k_1r^2+1}}\partial_r\cos\phi\frac{1}{r}+\sin\phi\frac{1}{r}, \frac{1}{r}\cos\phi\partial_\theta-\frac{1}{r}\cot\theta\cos\phi\partial_\phi, \frac{1}{r}\sin\phi\partial_\theta+\frac{1}{r}\cos\phi\cot\theta\partial_\theta, c_4\partial_\phi$.

Case 8

$i=i(r),\ j=$ constant $2ri''+ri'^2-2i'=0\ \rightarrow i=\ln r^4,\ e^j=1,\ T=10r^{-2}$, and $Q=r$. The space-times takes from

$$ds^2=dr^2-r^4dt^2+(d\theta^2+\sin^2\theta d\phi^2)r^2. \tag{4.135}$$

Expanding (2.19) and using (4.135) we get

$$X^0{}_{,0}=0, \tag{4.136}$$

$$X^2{}_{,0}-r^2X^0{}_{,2}=0, \tag{4.137}$$

$$\sin^2\theta X^3{}_{,0}-r^2X^0{}_{,3}=0, \tag{4.138}$$

$$-r^4X^0{}_{,1}+X^1{}_{,0}-2r^3X^0=0, \tag{4.139}$$

$$X^1{}_{,1}=0, \tag{4.140}$$

$$r^2X^2{}_{,1}+X^1{}_{,2}+rX^2=0, \tag{4.141}$$

$$r^2 \sin^2\theta X^3{}_{,1} + X^1{}_{,3} + r\sin^2\theta X^3 = 0, \tag{4.142}$$

$$X^2{}_{,2} = 0, \tag{4.143}$$

$$\sin^2\theta X^3{}_{,2} + X^2{}_{,3} = 0, \tag{4.144}$$

$$\cot\theta X^2 + X^3{}_{,3} = 0. \tag{4.145}$$

From Equation (4.136), Equation (4.143), Equation (4.144) and Equation (4.140) after deriving results we get

$$\begin{cases} X^0 = K^1(r,\theta,\phi), \\ X^1 = K^4(t,\phi,\theta), \\ X^2 = K^2(t,\phi,r), \\ X^3 = K^2_\phi(t,\phi,r)\cot\theta + K^3(t,\phi,r), \end{cases} \tag{4.146}$$

where $K^1(r,\phi,\theta), K^2(t,\phi,r), K^3(t,\phi,r)$ and $K^4(t,\theta,\phi)$ are the function of integration. We can use Equation (4.137) we obtain $K^1(r,\phi,\theta) = \theta D^1(\phi,r) + D^2(\phi,r) \rightarrow \mathrm{K}^2(t,\phi,r) = tD^3(\phi,r) + D^4(\phi,r)$ and derivative w.r.t ϕ we obtain $K^2_\phi(t,r,\phi) = tD^3_\phi(r,\phi) + D^4_\phi(r,\phi)$. Equation (4.141) we get $K^4(t,\phi,\theta) = \theta D^5(\phi,t) + D^6(\phi,t) \rightarrow \mathrm{D}^4(t,\phi) = tS^1(\phi) + S^2(\phi)$. Equation (4.139) we get $D^6(t,\phi) = tS^3(\phi) + S^4(\phi)$. Equation (4.138) we obtain $K^3(t,\phi,r) = tD^7(\phi,r) + D^8(\phi,r)$, and Equation (4.140) after determining we get

$$\begin{cases} X^0 = \theta D^1(r,\phi) + D^2(r,\phi), \\ X^1 = \theta S^2(\phi) + S^4(\phi), \\ X^2 = tD^2(\phi,r) + D^3(\phi,r), \\ X^3 = [tD^2_\phi(\phi,r) + D^3_\phi(\phi,r)]\cot\theta + tD^4(\phi,r) + D^5(\phi,r). \end{cases} \tag{4.147}$$

Using Equation (4.138) we obtain $D^2(r,\phi) = S^5(r) \rightarrow \mathrm{D}^4(r,\phi) = 0$. Equation (4.145) after determining results and considering (4.147) we get

$$\begin{cases} X^0 = \theta S^2(r) + S^3(r), \\ X^1 = \theta S^2(\phi) + S^4(\phi), \\ X^2 = \cos\phi S^6(r) + \sin\phi S^7(r), \\ X^3 = (-\sin\phi S^6(r) + \cos\phi S^7(r))\cot\theta - S^8(r). \end{cases} \tag{4.148}$$

Now using Equation (4.137) we obtain $D^1(r,\phi) = 0$, Equation (4.138) we get $D^2(r,\phi) = S^9(r)$, Equation (4.139) we obtain $S^9(r) = -\frac{c_1}{2r}$ and Equation (4.142) after solving and in the light of (4.148) we get

$$\begin{cases} X^0 = -\frac{c_1}{2r}, \\ X^1 = \theta S^2(\phi) + S^4(\phi), \\ X^2 = \frac{c_2}{r}\cos\phi + \frac{c_3}{r}\sin\phi, \\ X^3 = [-\frac{c_2}{r}\sin\phi + \frac{c_3}{r}\cos\phi]\cot\theta - \frac{c_4}{r}. \end{cases} \tag{4.149}$$

Using Equation (4.142) and in the view of (4.149) we obtain the final set of the Equation

$$\begin{cases} X^0 = -\frac{c_1}{2r}, \\ X^1 = \theta c_5, \\ X^2 = \frac{c_2}{r}\cos\phi + \frac{c_3}{r}\sin\phi, \\ X^3 = [-\frac{c_2}{r}\sin\phi + \frac{c_3}{r}\cos\phi]\cot\theta - \frac{c_4}{r}, \end{cases} \tag{4.150}$$

where, c_1, c_2, c_3, c_4 and c_5 are constants. Thus for **case 8** we get the following TKVS: $\frac{1}{2r}\partial_t, \theta\partial_r, \frac{1}{r}\cos\phi\partial_\theta - \frac{1}{r}\cos\phi\cot\theta\partial_\phi, \frac{1}{r}\sin\phi\partial_\theta + \frac{1}{r}\sin\phi\cot\theta\partial_\theta, \frac{1}{r}\partial_\phi$.

Case 9

$i=i(r),\ j=j(r),\ 1+rj'=0$ implies $j=\ln(\frac{E_1}{r}),\ r^2i''-rj'-2=0\ \rightarrow\ i=\ln(\frac{E_3e^{E_2r}}{r})$, $T=\frac{2k_2}{k_3}$,and $Q=r$ where $E_1,\ E_2,\ E_3\ \in\Re(E_1,E_3\neq 0)$. The space-times takes from

$$ds^2=-\frac{k_3e^{k_2r}}{r}dt^2+\left[\frac{k_1}{r}\right]dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2). \tag{4.151}$$

Expanding (2.19) and using (4.151) we get

$$X^0{}_{,0}=0, \tag{4.152}$$

$$r^2X^2{}_{,0}--\frac{K_3e^{k_2r}}{r}X^0{}_{,2}=0, \tag{4.153}$$

$$r^2\sin^2\theta X^3{}_{,0}--\frac{K_3e^{k_2r}}{r}X^0{}_{,3}=0, \tag{4.154}$$

$$-X^0{}_{,1}+\frac{1}{1+K_1r^2}X^1{}_{,0}-\frac{k_3e^{k_2r-1}}{r^2}X^0=0, \tag{4.155}$$

$$-\frac{-1}{r}X^1+2X^1{}_{,1}=0, \tag{4.156}$$

$$r^2X^2{}_{,1}+\frac{k_1}{r}X^1{}_{,2}+rX^2=0, \tag{4.157}$$

$$r^2\sin^2\theta X^3{}_{,1}+\frac{k_1}{r}X^1{}_{,3}+r\sin^2\theta X^3=0, \tag{4.158}$$

$$X^2{}_{,2}=0, \tag{4.159}$$

$$\sin^2\theta X^3{}_{,2}+X^2{}_{,3}=0, \tag{4.160}$$

$$\cot\theta X^2+X^3{}_{,3}=0. \tag{4.161}$$

From Equation (4.152), Equation (4.159), Equation (4.160) and Equation (4.156) and after deriving results we get

$$
\begin{cases}
X^0 = K^1(r,\theta,\phi), \\
X^1 = \sqrt{\frac{r}{k_1}}\, K^4(t,\theta,\phi), \\
X^2 = K^2(\phi,t,r), \\
X^3 = K^2_\phi(\phi,t,r)\cot\theta + K^3(t,\phi,r).
\end{cases}
\tag{4.162}
$$

where $K^1(\theta,\phi,r)$, $K^4(t,\theta,\phi)$, $K^2_\phi(\phi,t,r)$ and $K^3(r,t,\phi)$ are constant of integration. Now,we use Equation (4.153) we obtain $K^1(r,\theta,\phi) = \theta D^1(\phi,r) + D^2(\phi,r) \rightarrow \mathrm{K}^2(r,t,\phi) = tD^3(\phi,r) + D^4(\phi,r)$, derivative w.r.t ϕ $K^2_\phi(r,t,\phi) = tD^3_\phi(\phi,r) + D^4_\phi(\phi,r)$. Equation (4.154) $K^3(r,t,\phi) = tD^5(\phi,r) + D^6(\phi,r) \rightarrow \mathrm{D}^3(\phi,r) = S^1(r)$ and $D^5(\phi,r) = 0$. Equation (4.153) we obtain $D^1(\phi,r) = S^2(r)$, and Equation (4.155) deriving results and considering Equation (4.162) we get

$$
\begin{cases}
X^0 = \theta S^2(r) + D^1(r,\phi), \\
X^1 = \sqrt{\frac{r}{k_1}}\, [t(\theta C_1 + S^4(\phi)) + D^8(\theta,\phi)], \\
X^2 = tS^1(r) + D^4(\phi,r), \\
X^3 = D^4_\phi(\phi,r)\cot\theta + D^6(\phi,r).
\end{cases}
\tag{4.163}
$$

Using Equation (4.154) we obtain $D^2(\phi,r) = S^5(r)$ and Equation (4.161) we obtain $D^4(\phi,r) = S^6(r)\cos\phi + S^7(r)\sin\phi$ and $D^6(\phi,r) = S^8(r)$ and in the view of Equation (4.163) we get

$$
\begin{cases}
X^0 = \theta S^2(r) + S^3(r), \\
X^1 = \sqrt{\frac{r}{k_1}}\, [t(\theta C_1 + S^4(\phi)) + D^8(\theta,\phi)], \\
X^2 = \cos\phi S^6(r) + \sin\phi S^7(r), \\
X^3 = (-\sin\phi S^6(r) + \cos\phi S^7(r))\cot\theta - S^8(r).
\end{cases}
\tag{4.164}
$$

where $D^8(\theta,\phi), S^2(r), S^3(r), S^4(\phi), yS^6(r), S^7(r)$ and $S^8(r)$ are constant of integration. Now using Equation (4.153) we obtain $S^2(r) = 0$, Equation (4.155) we obtain $c_1 = 0$ and $S^4(\phi) = c_2$, Equation (4.157) we obtain $D^8(\phi,\theta) = S^9(\phi)\theta + S^{10}(\phi)$, Equation (4.156) we obtain $c_1 = 0, S^9(\phi) = 0$ and $S^{10}(\phi) = 0$ Equation (4.158) we obtain $S^6(r) = \frac{c_3}{r}, S^7(r) = \frac{c_4}{r}$ and $S^8(r) = \frac{c_5}{r}$ and in the light of Equation (4.164) we get

$$\begin{cases} X^0 = S^5(r), \\ X^1 = 0, \\ X^2 = \cos\phi\frac{c_3}{r} + \frac{c_4}{r}\sin\phi, \\ X^3 = -[\sin\phi\frac{c_3}{r} + \frac{c_4}{r}\cos\phi]\cot\theta - \frac{c_5}{r}. \end{cases} \tag{4.165}$$

Using Equation (4.158) after some simplification and in the view of (4.165) the set of Equation is

$$\begin{cases} X^0 = c_7 e^{k_2 r^2 - 2r\ln r + 2rc_6}, \\ X^1 = 0, \\ X^2 = \cos\phi\frac{c_3}{r} + \sin\phi\frac{c_4}{r}, \\ X^3 = [-\frac{c_3}{r}\sin\phi + \frac{c_4}{r}\cos\phi]\cot\theta - \frac{c_5}{r}. \end{cases} \tag{4.166}$$

where, c_3, c_4, c_5, c_6 and c_7 are constants. Thus for **case 9** we get the following are TKVS: $e^{k_2 r^2 - 2r\ln r + 2r}\ \partial_t, \frac{1}{r}\cos\phi\partial_\theta - \frac{1}{r}\sin\phi\cot\theta\partial_\phi,\ \frac{1}{r}\sin\phi\partial_{\theta+}\frac{1}{r}\cos\phi\cot\theta\partial_\theta, \frac{1}{r}\partial_\phi$.

Table 1: Killing Vector Fields of Static Spherically Symmetric Space-times in $f(T)$ gravity.

Case No	Metric Components	Killing Vector Fields	Killing Factor	Description
(10)	$i=i(r),\ \ j=j(r),\ \ i=\ln\left(1-\frac{E_1}{r}+\frac{E_2r^2}{3}\right),$ $j=\ln\left(1-\frac{E_1}{r}+\frac{E_2r^2}{3}\right)^{-}1,\ \ Q=r$.	$X^0=c_6\,e^{\int\frac{\frac{k_1}{r^2}+\frac{2k_2r}{3}}{-2+\frac{2k_1}{r}-\frac{2k_2r}{3}}}$ $X^1=0$ $X^2=\sin\phi\frac{c_2}{r}+\frac{c_1}{r}\cos\phi,$ $X^3=\cot\theta[\cos\phi\frac{c_2}{r}-\frac{c_1}{r}\sin\phi]+\frac{c_3}{r}$	$\eta=0\cdot$	TKVF
(11)	$i=i(r),\ \ j=j(r)\ \ Q=ri=\ln(\frac{E_1}{r}),$ $j=\ln\left(\frac{r}{E_2=4r}\right)\ ,$	$X^0=c_6r^{\frac{1}{2}},$ $X^1=0,$ $X^2=\frac{c_3}{r}\cos\phi+\sin\phi\frac{c_4}{r}$ $X^3=[-\sin\phi\frac{c_3}{r}+\cos\phi\frac{c_4}{r}]\cot\theta+\frac{c_5}{r}$	$\eta=0\cdot$	TKVF
(12)	$i=i(r),\ \ j=j(r),$ $i=\ln(E_1r^2),\ \ j=\ln(\frac{2}{1+2E_2r^2}),\ \ Q=r$	$X^0=c_6r^{\frac{1}{2}},$ $X^1=0,X^2=\frac{c_3}{r}\cos\phi+\frac{c_4}{r}\sin\phi$ $X^3=[\frac{c_4}{r}\cos\phi-\frac{c_3}{r}\sin\phi]\cot\theta+\frac{c_5}{r}$	$\eta=0.$	TKVF
(13)	$i=i(r),\ \ j=j(r),i=\ln\left(\frac{E_1r+E_2}{2}\right)^2$ $j=\ln\left(\frac{E_1}{E_1E_3-E_1r^2-2E_2r}\right),Q=r$	$X^0=\frac{k_1}{2}c_7e^{-r},X^1=0,$ $X^2=c_3\cos\phi+c_4\sin\phi$ $X^3=[-c_3\sin\phi+c_4$ $cos\phi]\cot\theta+c_5$	$\eta=0\cdot$	TKVF
(14)	$i=i(r),\ \ j=j(r),i=\ln(E_2-E_1r-r^2),$ $j=\ln(E_2-E_1r-r^2)^{-1},\ \ Q=1$	$X^0=c_8\sqrt{r^2-k_1r+K_2},$ $X^1=0,$ $X^2=(\sin\phi c_6+c_5\cos\phi),$ $X^3=[c_6\cos\phi-c^5\sin\phi]\cot\theta+c_7$	$\eta=0.$	TKLF
(15)	$i=i(r),\ \ j=j(r),\ \ i=\ln(E_3\ r^{E_2)},$ $j=\ln(\frac{E_1}{r^2}),\ \ Q=r$	$X^0=c_8\ e^{(\frac{-1}{2}-\frac{rk_2}{k_2})},$ $X^1=0,$ $X^2=-\frac{c_5}{r}\cos\phi+\frac{c_6}{r}\sin\phi,$ $X^3=\left[-\frac{c_5}{r}\sin\phi+\frac{c_6}{r}\cos\phi\right]\cot\theta+\frac{c_7}{r}.$	$\eta=0.$	TKVF

Table 2: Killing Vector of Static Spherically Symmetric space-times in $f(T)$ gravity.

Case No	Metric Components	Killing Vector Fields	Killing Factor	Description
(16)	$i=i(r),\ \ j=j(r)\ ,\ \ i=(E_1r+E_2),$ $j=\ln\left[\frac{E_3(E_1r+1}{r^2}\right]\ \ Q=r$	$X^0=C_8\ e^{(\frac{-1}{2}-\frac{rk_2}{k_2})},$ $X^1=0,$ $X^2=-\frac{c_5}{r}\cos\phi+\frac{c_6}{r}\sin\phi,$ $X^3=\left[-\frac{c_5}{r}\sin\phi+\frac{c_6}{r}\cos\phi\right]\cot\theta+\frac{c_7}{r}.$	$\eta=0.$	TKVF
(17)	$i=i(r),\ \ j=j(r),\ \ i=\ln\frac{e^{E^{1}r+\frac{E}{2}}}{r^2},$ $j=\ln\left[\frac{E_3(E_1r+1}{r^2}\right]r^2),$ $j=\ \ Q=r,$	$X^0=c_7.\frac{r}{e^{\frac{k_1}{2}}r},$ $X^1=0,$ $X^2=\frac{c_3}{r}\cos\phi+\frac{c_4}{r}\sin\phi$ $X^3=\left[-\frac{c_3}{r}\sin\phi+\frac{c_4}{r}\cos\phi\right]\cot\theta+\frac{c_5}{r}$	$\eta=0\cdot$	TKVF
(18)	$i=\text{constant},\ \ j=\text{constant},$ $i=\ln(E_1),\ \ j=\ln(E_2),\ \ Q=e^{E_3^{r+E}{}_{4}}$	$X^0=c_1$ $X^1=0,$ $X^2=c_2e^{rk_3\cos\phi+}$ $c_3e^{rk_3\sin\phi}$ $X^3=\Big[-c_2e^{rk_3\sin\phi+}$ $c_3e^{rk_3\cos\phi}$ $\cot\theta+(\frac{c_5}{r})$	$\eta=0.$	TKVF
(19)	$i=\text{constant},\ \ j=j(r),$ $j=\ln\left[\frac{E_4}{(E_2r-E_3)^2}\right],Q=r$	$X^0=\frac{E_1}{2}c_7e^{-r},X^1=0,$ $X^2=c_3\cos\phi+c_4\sin\phi$ $X^3=[-c_3\sin\phi+c_4$ $cos\phi]\cot\theta+c_5$	$\eta=0\cdot$	TKVF
(20)	$i=\text{constant},\ \ j=\text{j(r)},$ $j=\ln\frac{E_1}{r^2},\ \ Q=r$	$X^0=c_6$ $X^1=0,$ $X^2=-\frac{c_5}{r}\sin\phi+\frac{c_4}{r}\cos\phi$ $X^3=[\frac{c_5}{r}\cos\phi-\frac{c_4}{r}\sin\phi]$ $\cot\theta+c^7e^{-r}.$	$\eta=0.$	TKVF

Chapter 5

Conclusion

In this thesis, we investigated the Teleparallel Killing vector fields (TKVFs) of static spherically symmetric space-times in $f(T)$ gravity. Specifically, we classified 20 known solutions by employing the direct integration technique to determine the existence of TKVFs in each case. Our analysis revealed that Teleparallel Killing vector fields exist for all 20 solutions, indicating the presence of underlying symmetries in these space-times.

The existence of TKVFs in all cases suggests that teleparallel gravity, particularly in the context of $f(T)$ gravity, maintains certain structural symmetries similar to those found in General Relativity. This reinforces the role of Killing vector fields in characterizing the geometric and physical properties of modified gravitational theories. Additionally, our classification provides a systematic framework for identifying symmetries in teleparallel formulations of gravity, which may have implications for further theoretical developments.

5.1 Future work

The analysis of Teleparallel Killing vector Fields of static spherically symmetric space-times in $f(T)$ gravity can be extended to generalized $f(T)$ models with higher-order torsion or matter couplings, as well as more complex geometries like rotating or cylindrically symmetric space-times. Research opportunities are provided by applications in cosmology and astrophysics, such as early-universe scenarios, neutron stars, and black holes. Numerical simulations can be used to study dynamic or perturbative space-times, and the generated Killing Vector Fields can be used to compute conserved quantities such as energy and angular momentum. Future research could possibly extend the framework to higher-dimensional space-times or compare theoretical predictions with observational evidence, connecting $f(T)$ gravity with contemporary physical theories and observations.

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