TELEPARALLEL HOMOTHETIC VECTOR FIELDS OF STATIC SPHERICALLY SYMMETRIC SPACE-TIMES IN *f(T)* GRAVITY

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Candidate of Master of Science in Mathematics at the National University of Modern Languages do hereby declare that the thesis Teleparallel Homothetic Vector Fields of Static Spherically Symmetric Space-Times in $f(T)$ gravity submitted by me in partial fulfillment of MS degree, is my original work and has not been submitted or published earlier. I also solemnly declare that it shall not, in the future, be submitted by me for obtaining any other degree from this or any other university or institution. I also understand that if evidence of plagiarism is found in my thesis/dissertation at any stage, even after the award of a degree, the work may be canceled and the degree revoked.

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Abstract

Title: Teleparallel Homothetic Vector Fields of Static Spherically Symmetric Space-Times in $f(T)$ gravity

In general relativity, Einstein Field Equations (EFEs) are fundamental equations which are used to explain how geometry of a space-time is affected with the presence of massive objects. EFEs are the set of non-linear differential equations of second order that govern the behavior of metric tensor. Since the beginning of this theory, a wide range of physically interesting exact solutions to these equations have been discovered. Considering the non-linearity of EFEs, finding the exact solutions to them is a task. This task may be achieved by placing certain symmetry limitations on the metrics. The static spherically symmetric (SS) solutions to EFEs in $f(T)$ gravity have already been existed in the literature. These solutions are further classified which arose 20 cases. In this paper, we solved each case individually to see the existence of Teleparallel Homothetic Vector Fields (THVFs) of static SS space-times in $f(T)$ gravity. We find that no such case exists for which the space-time admit THVFs and for all the cases THVFs become TKVFs. To complete the study, the energy density and pressure of each model is computed. Additionally, the solutions are categorized based on energy conditions. Not to mention, the results of all the cases have been shown by designing certain tables.

Contents

List of Tables

List of Figures

List of Symbols

NIL

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Dedication

I dedicate this thesis to my parents, my siblings, my husband and to my mentor Dr. Shabeela Malik, whose guidance and wisdom, have illuminated my path and made this achievement possible.

Chapter 1

Introduction

1.1 Classical Gravity to Modified Theories: *a journey*

1.1.1 Gravitational Phenomena

Despite tremendous advances in science and technology, the universe is huge and full of mysteries that remain beyond our comprehension. From the star studies of ancient civilizations to the exploration of far-off galaxies and celestial events by contemporary telescopes and space missions, the understanding of humans for the cosmos has expanded enormously throughout the ages. Despite this in-depth understanding, humans cannot fully understand the mystery behind gravity which considered to be the most captivating topic for scholarly work. Common individuals view gravity as a force responsible for keeping them on the crust of the earth. Gravity's significance goes beyond mere attraction; it also plays a vital role in determining the orbital paths taken by planets as they revolve around the sun. Gravitational interaction, in fact, was the initial subject to be experimentally investigated because it was relatively simple to construct the necessary experimental equipment [1].

Among the philosophers of ancient times, Aristotle possessed a remarkable understanding of substantial bodies. He asserted that universe was made up of exactly five elements which include four earthly elements: earth, water, air, fire and one heavenly element which was 'ether.' According to his law of terrestrial movement, 'all terrestrial bodies are inclined to return to their natural state of rest.' So, anything taken from the earth will eventually fall back to it, as it's their natural state of rest. In the late 16th century, Galileo Galilei initiated the concept of terrestrial gravity by employing pendulums and inclined planes in his scientific investigation [2–5]. In one of the studies carried out by Galileo, he simultaneously dropped two unequally sized cannonballs from the top of the Leaning Tower of Pisa, and they appeared to fall freely to the Earth at the same time. Numerous comparable experiments have been conducted, all yielding comparable absolute results. Thus, it is well known that all bodies, regardless of mass, free fall (accelerate) equally under the force of gravity. The study of gravity profoundly impacted Galileo's views on the essentiality of experimentation in scientific investigation, which had a significant impact on the advancement of scientific thinking [6]. This basic yet remarkably persuasive empirical law convinced Issac Newton to incorporate it into his 'Law of Universal Gravitational Attraction.'

1.1.2 Law of Universal Gravitational Attraction: Issac Newton

Drawing from Galileo's law of equal gravitational acceleration, Newton inferred that all planets descend equally towards the sun regardless of their significantly different masses [7]. He even made efforts to explain Galileo's paradoxical law through his own observations and experiments involving pendulums [8]. However, in Galileo's work, terrestrial gravity was not directly linked to celestial gravity in a single theory until, in 1665, Isaac Newton gave his Law of Universal Gravitational Attraction. Newton asserted that gravity is an intrinsic quality possessed by every particle. All mass-containing bodies are subject to gravitational attraction from one another, even if this force might become infinitesimally tiny at larger separations between bodies. His law articulates the idea that "The gravitational attraction situated between two mass containing bodies (m_1) and (m_2) grows proportionally with their mass product $(m_1 m_2)$ and varies inversely with the the squared distance separating their centers(i.e. r^2)". This is also renowned as a 'Universal law,' also Terrestrial gravity and Celestial gravity were unified under a single theory [9, 10]

Considering that every consistent theory is evidently *'right'*, it will not be appropriate to declare if Newton's theory, or any other physical theory, is correct or incorrect. Instead, it would be more relevant to ask how well this theory aligns with the real world. The predictions made by Newton's theory proved to be accurate for various scenarios and for multiple sizes, such as planetary body motion and terrestrial tests. Newton's Gravitational Theory comprised of two fundamental concepts:

1. The concept of absolute space; assumes that space is a rigid environment in which physical processes take place, is actually fixed, unaffected structure.

2. The idea behind the Weak Equivalence Principle, that in terms of Newtonian Theory, proclaims that gravitational and inertial masses hold equal value.

During the first two decades after the inception of Newtonian gravity, it was obvious that it successfully explained every aspect of gravity which should be explained at that time. But sooner or later, all of the aforementioned concerns were raised. In the 19th century, there were several experimental findings which Newton's theory of gravity could not articulate. These include 1855's discovery by Urbain Le Verrier, in which he revealed an irregularity in Mercury's orbital path, with an excess precession of 35 arc second [11, 12]. Afterwards, more accurate measurement of a 43 arc-seconds excess precession made by Newcomb [13]. In addition, Dicke proposed that the gravitational constant should depend on the mass distribution [14] unlike Newton, who believed that the gravitational constant ought to be a universal constant. These events caused scientists of the era to question the underlying axioms of Newton's gravitational theory.

1.1.3 Einstein Theories

Newtonian theory faced a major challenge when Albert Einstein finished the Theory of Special Relativity in 1905. This theory is referred to as the "Special Theory of Relativity" as it applies only to bodies that adhere to the principles of inertial motion and no external forces acting upon them. Through a synthesis of experimental data and physical arguments made so far, Albert Einstein constructed a set of novel principles that provide an enriched knowledge of the intrinsic nature of space, time, matter and energy. In adherence to the Special theory of Relativity, observers in inertial reference frames perceive that the speed of light and the laws of physics as invariant. This theory relies on a flat space-time structure and managed to describe a number of Non-Gravitational events related to physics [15]. Time dilation and length contraction, two perplexing characteristics of time and length, led to the popularity of this theory, which was later shown to be accurate. By conducting the theoretical framework and experimental investigations into the propagation of light observed from the perspectives of observers in motion, the properties of time and length were demonstrated. The mass-energy equivalency relation, which is expressed by the well-known equation- $E = mc^2$, is another important aspect of Special Relativity Theory. This theory emphasizes the proportional relationship between mass and energy within the framework of physical system and the relationship between mass and energy equivalence is determined by the numerical constant representing the speed of light, abbreviated as "c." It appeared that non-inertial frames should be included in the topic of Special Relativity in a certain manner.

In 1907, Albert Einstein proposed the idea that gravity and inertia are equivalent, and he later used this theory to forecast an exact gravitational redshift. Following that, he rounded off his General Relativity (GR) theory in 1915. GR was an extended framework of Special Relativity that considers the effects of gravity with any frame experiencing acceleration. The fundamental strategy of this conceptual framework was to visualize space-time as a geometric structure, enabling us to understand how gravity operates between massive objects. At the heart of this theory is the proposition that "gravity leads to curvature." As an illustration, when a heavy ball is positioned at the center of a trampoline, gravity will cause the center of trampoline to sink, creating a deviation from a perfectly flat surface. The aforementioned example can be explained as "the mass of the ball is causing the trampoline to bend."

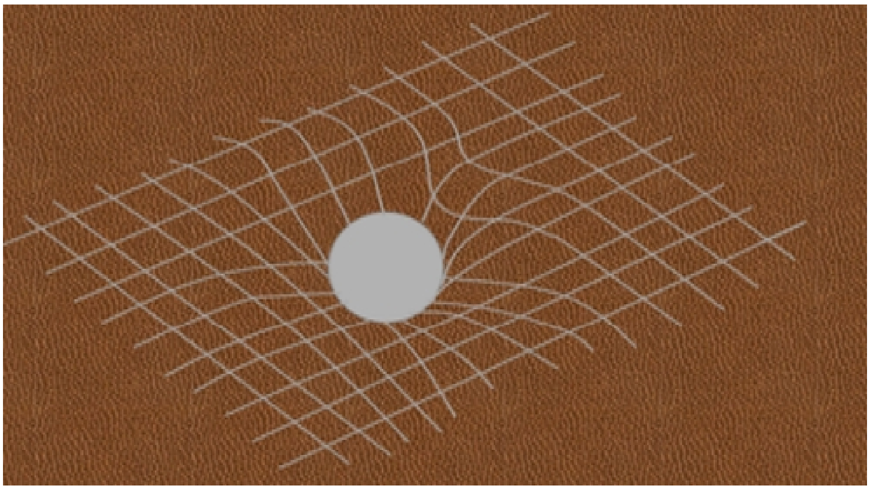

Figure 1: A distorted space-time due to the gravitational force of a mass.

Remarkably, the Lense-Thirring gravitomagnetic effect (1918) and other experimental results, including the Mercury's orbital precession matched perfectly with General Relativity theory [16–19]. General Relativity (GR) replaces Newtonian Gravity, commonly accepted theory of gravity today. Like many other scientific hypotheses, Newtonian gravity continues to be widely accepted by scientists. Although it has been acknowledged that its applicability is limited in comparison to General Relativity, yet it is enough for the majority of gravity-related applications. Under conditions of weak gravitational field strengths and low velocities, GR ultimately reduces to Newtonian gravity. Although generalizations have been made to Newton's gravity equations, some fundamental axioms (like the existence of an absolute frame) have been abandoned. However, several fundamental principles from his theory continue to shape the core of GR. Though it has been rephrased to be more appropriate for the situation. The Equivalence Principle is the most well-known illustration of this.

To understand our universe, having a clear understanding on the physical and geometric nature of our space-times is essential. GR appears to be the most relativistic theory in order to understand these geometrical features among all the theories that have been developed so far, at least on a classical level. In General Relativity, gravity is understood as the distortion of space-time, which occurs due to matter and energy. This distortion of space-time indicates how much the fabric of space-time bends away from being perfectly flat and is exclusively defined through its metric tensor. In GR, the metric of space-time does not have to be flat, unlike the scenario presented in Special Relativity theory. The structure and space-time dynamics are determined by "Einstein's Field Equations" or "EFEs" which exhibit a notable degree of non-linearity and play a vital role in explaining how mass and energy determine space-times geometry and how that geometry affects the motion of matter. Thus, in order to deal with EFEs effectively, certain symmetry restrictions must be imposed. That's the reason why certain symmetries are acknowledged in GR. Even though Einstein's General Relativity has been passed through numerous astrophysical trails and is a very successful theory for understanding the structure of the universe [20, 21], however, it is restricted by several prime limitations.

1.1.4 Existence of Invisible Matter Components

Theoretical models have yet to offer a straightforward approach to integrate quantum effects into the framework of General Relativity. At present, there is no unified quantum theory that clearly describes the behavior of gravity. Recent research highlights that the behavior of accelerated expansion of universe have become more significant yet interesting subject in today's cosmological and astrophysical fields. Current advancements in cosmological fields have revealed innovative ideas to understand the pivotal and observational advancements that explain the universe's rapid expansion and the mysterious existence of invisible matter components gravitating towards the clusters of celestial bodies. Different observations might offer an evidence supporting accelerated expansion as a result of high red-shift supernova experiments [22, 23]. The intriguing phenomenon of expansion is governed by Einstein's relativity theory and the classical dynamics of solar objects. An apparent explanation for these atypical phenomena could involve the occurrence of some non-standard matter components, including dark matter and energy [24]. Nevertheless, GR theory has certain limitations, as it fails to properly address some significant issues like dark energy (DE), dark matter, late-time cosmic acceleration, initial singularity and flatness problems. Such concerns stimulate the effort to explore the modified or extended theory of gravity that can address scenarios where GR yields inadequate results [25, 26].

1.1.5 Teleparallelism

Finding a stable theory that combines the essential concepts of Quantum Mechanics and the primary objective of theoretical physics is to develop the General Theory of Gravity. In 1920's, Einstein tried to develop such a theory. He combined electromagnetic and gravity by taking the help of the mathematical framework of Teleparallelism, which is also called "Absolute Parallelism." The fundamental property of Teleparallelism is the ability to calculate the angle between vectors that are far away. Einstein suggested the introduction of the tetrad field as an essential component, which represents a field of orthonormal basis existing within the tangent space generated for every coordinate within the four-dimensional space-time manifold. In total, there are 16 individual components within the tetrad framework and metric only has

10 [27]. Einstein postulated that there must be some sort of relationship between the six components of the electromagnetic field and these additional six aspects of the field. Unfortunately, this attempt was likewise unsuccessful, and it eventually came to light that the additional components pertain to Lorentz invariance of the theory. Despite the failure of the first attempt at unification, a new way of describing gravity called "Teleparallel Gravity" was born [28]. A different interpretation of gravity within the domain of GR is provided by the “Teleparallel Theory” (TT). In the framework of general relativity, space-time’s curvature defines its overall geometric layout, represented by the Riemann Curvature tensor. However, in Teleparallelism, space-time’s shape is flat, with no curvature, but it does possess a distinctive feature known as ’Torsion.’

We are basically going to find the Teleparallel Homothetic Vector Fields (THVFs) of Static Spherically Symmetric (SS) Space-Times in *f(T)* gravity. By looking at gravity through the lens of teleparallelism, vector fields exhibiting homothetic qualities are called Teleparallel Homothetic Vector Fields. These fields provide scaling transformations (where distances are uniformly scaled at each point in space by a constant factor). A static SS space-time is a solution to field equations of Einstein in GR where the gravitational force field strength is spherically symmetric and is independent to time. *f(T)* gravity is an alternative explanation for gravitational phenomenon that modifies Einstein’s general relativity by considering functions of the torsion scalar T, which emerges as a geometric parameter derived from the torsion tensor in teleparallel gravity.

1.2 Literature Review

Over the past few years, there has been a heightened enthusiasm among researchers for investigating the symmetries which also involves homothetics vector fields. McIntosh *et al.* [29] examined the characteristics of homothetic motion in GR by focusing on vacuum and perfect-fluid space-times. It was proved that vacuum space-times can only admit nontrivial homothetic motions if the HVF is non-null and not hypersurface orthogonal. In 2015 [30], Azeb Alghanemi *et al.* classified the Bianchi type I space-times based on their homothetic vectors in Lyra geometry. Non-linear coupled Lyra homothetic equations were formulated and solved for several cases. Homothetic and Killing vectors were obtained for Bianchi type I space-times in GR by considering the displacement vector in Lyra geometry to be zero. It was seen that some space-times admit proper Lyra homothetic vectors (LHVs) for specific metric functions, while others only admit Lyra Killing vectors (LKVs). Eardley *et al.* [31] presented some results which revealed that Einstein's equations do not have solutions with conformal or homothetic symmetry. It was shown that homothetic or conformal killing fields are Killing in spatially compact space-times. M. Jamil Khan *et al.* made an attempt to find the proper HVFs in plane symmetric perfect fluid static space-times in *f(R,T)* gravity theory by simple integration technique. The existence of six cases was seen in which proper Homothetic Vector Fields (HVFs) exists in four cases whereas in the other two situations HVFs become Killing Vector Fields (KVFs) [32]. In theoretical physics, the idea of teleparallelism provides a different way to express gravity. If we talk about Teleparallel Homothetic Vector Fields (THVFs), different researchers in relativity, work in these kinds of fields. In 2010, Shabbir *et al.* made used of the direct integration strategy so that they can examine THVFs associated with TT of Bianchi type

I space-times. Dimensions of THVFs were discovered which were four, five, seven, or eleven and were exactly identical in quantity as those observed in general relativity. For special choice of space-times, proper THVFs exist in dimensions four, five or seven. And eleven THVFs exhibit zero torsion components entirely. It was concluded that General Relativity's homothetic vector fields (HVFs) are retrieved, and space-time is converted to Minkowski space [33]. In another article, authors Ghulam Shabbir and Suhail Khan opted the direct integration approach for categorization of cylindrically symmetric static space-times based on their THVFs. It was examined that the dimensions of the THVFs are four, five, seven or eleven, which are exactly identical in numbers to the dimensions in GR. For special choice of space-times, Proper THVFs exist into the dimensions four, five or seven. In the case of dimension eleven of Teleparallel Homothetic Vector Fields, every component has zero torsion which results in the transition of space-times to Minkowski. In the end, it was concluded that THVFs in this particular scenario are exactly similar to vector fields in GR [34]. In 2011, Ghulam Shabbir and Suhail Khan used the direct integration to categorize plane symmetric non-static space-times based on THVFs. It was seen that the dimensions of the THVF's are six, seven, eight or eleven. For the dimensions six, seven or eight, the existence of Proper THVFs for special choice of the space-times is shown. When it comes to eleven, all components of torsion vanish and this lack of torsion components causes the space-time to transition into the Minkowski phase which completely agreed to GR. After a lot of discussion, it was made clear that for non-static plane symmetric space-times, the existence of torsion within space-time does not really affect or raise the number of proper Homothetic Vector Fields and it turned out to be similar to GR, Teleparallel theory demonstrates the existence of just one proper HVF under observation [35]. Masoom Ali Shahani *et al.* in the year of 2017, explored the non-static cylindrically symmetric space-times to explore proper THVFs using direct integration and diagonal tetrads. The space-times like static cylindrically symmetric, Bianchi type I, non-static, and static plane symmetric were also covered in this article. It was shown that the special classes of these space-times produce six, seven, and eight THVFs with non-zero torsion [36]. In another study of 2018, Amjad Ali sought out teleparallel proper HVFs on Lorentzian manifolds characterizing special axially symmetric static space-times. Using the teleparallel Lie derivative on the metric for homothetic equations, a set of ten differential equations exhibiting non-linearity and with interrelations is obtained. Following this, the equations are tackled one by one to determine potential solutions for each metric function. It came to light that solely in a single case, teleparallel proper HVFs exists for particular metric function scenario. Within the context of space-times, there exist eight dimensional Teleparallel Homothetic Vector Fields, among them only single vector satisfies the criteria for being a proper teleparallel HVF while the other seven vectors identified as teleparallel Killing Vector Fields (KVFs) [37].

Apart from Teleparallel Homothetic Vector Fields, authors tried to research in static SS space-times. Within the paradigm of GR, a term "static" often employed to depict different forms of SS space-times line elements. For static SS space-times they find their analytical solutions. In one of the studies [38], Takeno present the notion of "staticness" inherent in a spherically symmetric space-time as an essential characteristic of the space-times, this characteristic is based on the concept that the mathematical aspect of GR is conceptualized as a theory of analytical invariants. Subsequently, various characteristics of SS symmetric space-times are described. An approach utilized by the author in his research is in accordance with the framework of characteristic systems applied to spherically symmetric space-time models, as formulated by an author. Examples were also given in the work. In another study, an approach known as “Noether symmetry approach” was used by Paliathanasis *et al.* and SS solutions were obtained for $f(T)$ gravity models. After generating solutions, it was concluded that the solutions accomplished are of greater generality than those acquired through typical solution techniques [39]. In the year 2015, Ali *et al.* presented a categorization of static SS space-times based on their Noether symmetries. Firstly, equations defining the Noether symmetries were derived from the usual Lagrangian of a static space-times with spherical symmetry which were then integrated for each specific case. The research revealed that spherically symmetric static space–times can be classified into six distinct categories each category corresponds to the dimensions of the associated Noether algebras: five, six, seven, nine, eleven, and seventeen. By utilizing Noether's theorem, the first integrals were determined associated with each symmetry. And some new spherically symmetric static solutions were obtained [40]. In another article which was published in 2016, Zubair *et al.* investigated the wormhole solutions within the context of $f(R,T)$ theory of gravity. Here, R symbolizes the scalar curvature while T represents the sum of the diagonal elements of the stress-energy tensor associated with matter. Three different scenarios of static spherically symmetric (SS) geometry were considered alongside the inclusion of matter contents in the form of anisotropic, isotropic, and barotropic fluids. It was analyzed how energy conditions manifest across different fluid types, gave their solutions and present a graphical representation of the acquired results. In summary, it was determined that wormhole solutions featuring anisotropic matter are both realistic and stable in $f(R,T)$. To wrap up, the investigation revealed that wormhole solutions sustained by anisotropic matter exhibit both realism and stability within this gravitational framework [41]. Another analysis is being persued by Khan *et al.* to explore teleparallel CKVFs in plane symmetric non-static space-times. Through calculations, ten equations were derived governing the behavior of teleparallel

Conformal Killing Vectors which exhibit linearity with respect to the elements of teleparallel CKVFs. Given certain integrability requirements, a general solution to these equations was given by utilizing the CKVFs and conformal factor components. To achieve the ultimate expression of teleparallel CKVFs and conformal factor, the integrability conditions were fully investigated for seven distinct metric functions selection. It was demonstrated that for three cases, teleparallel CKVFs have been reduced to THVFs or TKVFs. While in other four cases, proper CKVFs were obtained [42]. In 2018, Shabbir *et al.* categorized static SS space-times considering $f(R)$ gravity theory based on their CVFs. Direct integration technique was employed and revealed that within the paradigm of $f(R)$ gravity, static SS space-times exhibit that CVFs are just KVFs or HVFs. Among the six scenarios deliberated, it was found that only one led to the transformation of CVFs into HVFs and in remaining cases, CVFs transformed into KVFs [43]. In 2021, a paper was published in which Bokhari *et al.* examined HVFs of Bianchi type I space-times by taking help of Rif tree method. Rather than opting for direct integration, homothetic symmetry equations were transformed into reduced involutive form by means of a computer algorithm. It divided the integration problem into multiple cases, each of which was presented as a tree with constraints on the metric functions. The metrics explicit expressions and their corresponding Homothetic Vector Fields were determined by solving a set of homothetic symmetry equations subject to these constraints. New physically realistic metrics were obtained, differing from those found using direct integration [44]. In another study attempt had been made by Khan *et al.* with the hope of identifying all the static, cylindrically symmetric space-time metrics that exhibit homothetic symmetries. Bearing that in mind, authors utilized a Maple-developed algorithm to analyze the homothetic symmetry equations which results in all possible static, cylindrically symmetric metrics that could exhibit proper homothetic symmetry.

In each case, solutions were derived for the homothetic symmetry equations, resulting in the ultimate expression of homothetic symmetry vector fields. Then the derived results underwent a comparison process with those derived directly through integration techniques. Moreover, it has come to attention that the Rif tree method they applied yields not only replicates the metrics derived from direct integration techniques, but the method also uncovers new metrics [45]. In another article of 2022, Hussain *et al.* discusses the classification of Lie symmetries for SS symmetric space-times using Rif tree methodology. An analysis of Lie symmetry equations is conducted using Maple algorithm, with the goal of uncovering possible static spherically symmetric admitting Lie symmetries. Lie symmetries were presented for all of the obtained metrics, which were given by Killing, Homothetic, and Conformal Vector Fields respectively. Through this approach, it has been determined that all previously acquired metrics from direct integration techniques are reproduced as well as several new physically realistic metrics have been attained [46]. In another article, Hussain *et al.* examines how Conformal Vector Fields (CVFs) were utilized to categorize static SS perfect fluid space-times within *f(T)* gravity. Their initial focus lied in examining static spherically symmetric solutions achieved by tackling the EFEs within *f(T)* gravity's context. Afterward, a direct integration approach was adopted to categorize the solutions. In the procedure of categorization, 20 cases were observed. They study those cases individually and found that in three scenarios, when space-times is analyzed within *f(T)* gravity, it permits proper CVFs. Meanwhile, in the other sixteen instances, the space-times either conform to flatness or exhibit the presence of Killing Vector Fields, the space-time in remaining one case admits proper Homothetic Vector Fields [47].

Chapter 2

Basic Concepts and Definitions

2.1 Inertial Frame of Reference

This term particularly refers to reference frame where an object continues in a state of rest or follows a straight path at a steady speed unless influenced by an external force. If an object is in inertial frame then it is necessary for an object to follow the principles of Newton's first law. It is a non-accelerating frame of reference which means if we ever observe some acceleration on an object in this frame, it will be due to the forces acting on an object, not due to the frame itself. Furthermore, if one frame is inertial then any reference frame moving at a uniform velocity relative to it is also inertial. Thus, we cannot deny the fact that inertial frames are relative to each other.

A person standing still on an escalator that is moving at a steady speed and a ball placed on a flat table inside a room without external disturbances can be used to demonstrate this frame of reference.

2.2 Non-Inertial Frame of Reference

It refers to the reference frame which is either accelerating or rotating. In this frame, objects are influenced by fictitious or pseudo-forces (the forces experienced by observers in non-inertial frames that do not exist in inertial frames) due to the acceleration of the frame itself. In order to describe the motion of objects correctly in non-inertial frames, Newton laws of motion must be modified to include these pseudo-forces. The frame undergoes a change in velocity which implies that the non-inertial frames are accelerating, either linearly or rotationally.

A feeling of a person being pushed back into their seat as the car suddenly accelerates forward (this feeling is actually due to the 'pseudo-force' that seems to push them in the opposite direction of the car's acceleration) can be used to demonstrate this frame of reference.

2.3 Terrestrial Gravity

Terrestrial gravity refers to the Earth's gravitational force acting on objects. It is the gravitational force that is responsible for drawing objects towards the center of the Earth. Terrestrial gravity determines the weight of the objects and cause them to fall when dropped. It is undeniable that this gravity varies with an altitude and latitude.

The phenomena of free falling of objects, swinging motion of pendulum, projectile motions and weight measurement are few of the many effects of a terrestrial gravity.

2.4 Calestial Gravity

Celestial gravity refers to the gravitational forces exerted by celestial bodies such as planets, stars, moon and other astronomical objects. These gravitational forces dictate the movements and interactions of bodies in space, i.e., from the orbits of planets around stars to the behavior of galaxies. We experience calestial gravity by noticing the moon's gravitational attraction toward the Earth, as it causes the sea levels to rise and fall in a periodic pattern known as 'Tides.'

By considering the space-time we notice calestial gravity by observing that the Sun's gravitational pull causes the Earth to orbit it. The nearly circular orbit results from the balance between the earth's forward motion and the pull of the sun's gravity. Apart from planetry orbits, black holes also exert an extremely strong gravitational force due to their massive mass concentrated in a small volume. They have a very strong gravity which warps the space-time and create a region from which not even light can escape.

2.5 Speed of Light

The speed at which electromagnetic waves, such as, visible light, move through a vacuum is referred to as the 'speed of light.' It serves as a core principle of nature, represented by (c) and its value is around 299,792,458 ms^{-1} [48]. For the sake of simplicity and ease in calculations and scientific investigations it is a common practice to round this value to $3*10^8$ ms^{-1}. According to Einstein's relativity theory, c is a crucial element which represents the ultimate speed limit for the transmission of information and the movement of objects with some mass and does not change for any observer, no matter how they move in relation to the light source.

2.6 Newtonian Gravity

In 1686, Sir Isaac Newton proposed his classical theory of gravitation (Newtonian Gravity) in his work *Philosophiæ Naturalis Principia Mathematica* [49]. It is one of the first comprehensive theory which is able to explain how objects with certain masses attract each other. According to Newton, all mass-containing bodies exert attractive forces on each other due to gravity. He explained his concepts by giving his 'Law of Universal Gravitational Attraction' which is discussed in section (1.1.2). Newtonian Gravity had certain limitations in it like; it failed to explain the precession of Mercury's orbit and was inadequate in describing strong gravitational fields, like those around black holes.

2.7 Special Theory of Relativity

In 1905, Albert Einstein put forward this Theory of Special Relativity. At the heart of this fundamental theory are two crucial postulates. The first one is the Principle of Relativity according to which all non-accelerating reference frames observe the same physics laws. The second postulate asserts that the speed of light remains constant for every observer within an inertial reference frame. These two postulates lead to several non-intuitive but experimentally verified consequences like time dilation, length contraction and mass-energy equivalence relation.

2.7.1 Time Dilation:

This phenomenon occurs when time passes more slowly for an object traveling at a considerable fraction of the speed of light, as observed by a stationary observer. This phenomenon intensifies as the object's velocity gets closer to the speed of light. This is not just a theoretical concept as it has real-world applications too. As an example, the high speeds of GPS satellites orbiting the Earth cause them to experience time dilation.

2.7.2 Length Contraction:

This refers to the phenomenon where a fast moving object, will appear shortened in the direction of motion from the perspective of a stationary observer. This phenomenon only occurs along the axis of the object's motion and shows significant effects as objects reach the speed of light.

2.7.3 Mass-Energy Equivalence Relation:

This relation is one of the most famous results of relativity theory. It is expressed by a mathematical equation:

$$E = mc^2. \tag{2.1}$$

The equation (2.1) reveals the essential correlation between mass and energy by showing that they are interchangeable and any object with mass (m) has an equivalent amount of energy (E) , where 'c' refers to light's speed in a vaccum discussed in section (2.5). Furthermore, a small fraction of mass can be changed into an enormous quantity of energy because of the c^2 factor, which is a very large number (i.e. $c = 3 * 10^8 ms^{-1}$).

2.8 Equivalence Principle

This principle was proposed by Albert Einstein in 1907, which marked the beginning of his development of General Relativity. According to this principle, observations taken by an observer in an inertial frame having gravitational field and observations taken by an observer in an accelerated reference frame in the absence of gravity are equivalent. This principle can be separated into two distinct parts, commonly called the *'Weak Equivalence Principle'* and the *'Strong Equivalence Principle.'*

2.8.1 Weak Equivalence Principle

This demonstrates that:

"A test particle's motion in the field of gravity does not rely on its mass or what it is made of."

The term 'test particle' refers to the particle that moves through a gravitational field but does not modify or contribute to the field in any way. Hence, if the only acting force on a free-falling object is a force of gravity, then both the composition and body's mass will not have any kind of effect on the motion of a body.

In the 17th century, Galileo Galilei conducted an experiment in which he dropped two distinct things with different compositions and masses at the same time from the roof of Pisa's Tower. According to his weak principle of equivalence, both things must get to the ground at the same time, despite the fact that they have different masses. He saw that a body's composition and mass have no effect on its motion when it is subjected to the force of gravity (assuming that no other external forces are present outside gravity). Because of this experiment, this idea is also called as "Galilean Equivalence Principle" [50].

2.8.2 Strong Equivalence Principle

This states that:

"A particle's movement in a gravitational field is equivalent to the behavior of a particle at rest in an accelerating system."

We can also take it as, an observer in a gravitational field will experience the same physical effects as an observer accelerating at the same rate in a gravity-free scenario. We can demonstrate this principle with the help of an experiment known as the 'elevator experiment,' in which an observer is placed in an elevator which is either stationary on the earth's surface or accelerating through empty space. For the observer, the physical effects being inside a stationary elevator are the same as those felt in an elevator accelerating at the same rate [50].

2.9 General Theory of Relativity

In 1915, Albert Einstein generalized his Special Theory of Relativity to include the effects of acceleration and introduced a generalized version of the theory which we called as 'General Relativity (GR) theory.' This GR theory was completely dependent on gravity and in this theory, gravity was not seen as just a force but it was a result of the distortion of space-time. Question arises here is that what is curvature? According to GR, Anything that influences the energy-momentum tensor and produces a gravitational field is defined as 'matter.' Einstein in his theory posits that space-time bends under the influence of enormous objects (matter) which we called 'curvature.' Hence, this curvature is owing to the existence of energy and mass. It was described that massive objects (like stars and planets) causes the space-time to curve, and this curvature impacts how objects move and how time flows. Different objects which moves in this curved space-time follow the paths called 'geodesics.' The curvature also explains why planets orbit stars, what causes light to bend in the presence of massive objects *(gravitational lensing)* and why time flows differently in intense gravitational fields *(time dilation)*.

2.9.1 Gravitational Lensing

Einstein's GR theory predicted the phenomenon of Gravitational Lensing, where light from a distant source, such as a star or galaxy, bent as it travels near a massive object. The actual reason behind this bending of light is that the gravity of an object with a dense mass bends the space-time around it, and light follows the curved paths of space-time. This phenomenon of Gravitational Lensing can be illustrated in Figure 2:

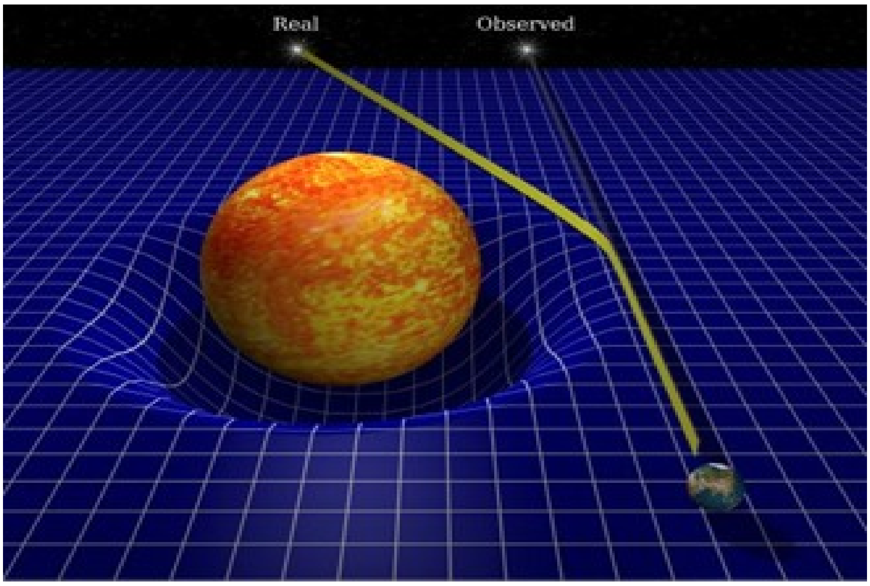

Figure 2: An observed bending of light due to the effect of gravitational lensing.

2.10 Space-time

In the era before Relativity Theory, time was regarded as a universal quantity, which is progressing at a steady rate in every frame of reference and is not influenced by the motion of an observer. Hence, time remains uniform regardless of whether an object is moving or stationary. In a pre-relativity framework, three numbers (spatial coordinates) are used to specify the point's location. For instance, in cartesian coordinate system, these are 'x,y,z', while in the polar coordinate system, these are 'r,θ,ϕ.' The formulation of General Relativity (GR) relied heavily on Hermann Minkowski's mathematical formulation. With time added as a fourth dimension, Minkowski was able to merge the three spatial dimensions into a 4-dimensional manifold to represent space-time. A manifold is a topological space where the area around each point is approximately flat and this region can be represented by Euclidean geometry. This means that a neighbourhood of surrounding points that is essentially flat and this neighbourhood exists around every point. This concept is illustrated clearly when we observe Earth. Despite knowing that Earth is spherical, it looks flat from the perspective of the average human eye. We define events in the space-time as points on the manifold. As we are working in a space-time which is four-dimensional, each of these points will need four coordinates so that it can be represented in a unique way. In spherical coordinates, we refer these points in coordinate form as (t,r,θ,ϕ), the standard notation is (x^0,x^1,x^2,x^3), where x^0 corresponds to the time coordinate. On the other hand, we can be able to use either one of them whenever necessary.

Mathematically, a space-time (M,g) is defined as a four-dimensional, smooth, Hausdorff manifold M together with a Lorentz metric g, of signature $(+,-,-,-)$ or $(-,+,+,+)$ [51]. Space-time M is composed of elemnets known as events while the Lorentz metric g determines the geometry of M. In GR, the presence of energy leads to the curvature of space-time, as defined by the Riemann curvature tensor. This tensor vanishes in Special Relativity, making space-time flat and defining it as 'Minkowski Space.' The metric tensor within Minkowski space is referred to as the Minkowski metric, usually denoted by η.

2.11 Curvature and some other important Tensors

The term curvature represents the extent to which a geometric object deviates from a flat shape. Here, the word 'flat' does not always means that it's a straight line. It can have different meanings based on different situations. Likewise, in the case of curves, the term flat represent a straight line, while for surfaces, it may correspond to the Euclidean plane. Curvature is defined in two distinct forms which are extrinsic curvature and intrinsic curvature. The concept of extrinsic curvature is restricted to objects situated in higher-dimensional spaces. On the other hand, intrinsic curvature can be defined for any manifold, regardless of whether it is situated within a higher-dimensional space. The curvature of a manifold can be described by a tensor. Below some of the tensors like Metric Tensor, Riemann Curvature Tensor and Ricci Tensor are defined:

2.11.1 Metric Tensor

If we want to determine angles on curves and distances between points in space-times, it is necessary to define the term metric tensor. This tensor, which is labeled as g_{pq} is a rank-2 symmetric tensor that is defined on a smooth manifold. To effectively describe space-time, the metric must have a Lorentzian signature. Based on the convention, this could mean that the signature is either $(+,-,-,-)$ or $(-,+,+,+)$. This implies that the metric should have the property of non-degeneracy.

The metric tensor is commonly denoted by ds^2 and in relation to its components g_{pq} and coordinates, one can define the local line element on the space-time as follows:

$$ds^2 = g_{pq} dx^p dx^q, \tag{2.2}$$

where p and q are the Einstein summations and can take the values 0,1,2 and 3. The metric tensor on an n-dimensional manifold with the signature of $(1, n-1)$ or $(n-1, 1)$ is called Lorentzian. Additionally, the metric of space-time is inherently Lorentzian [52].

2.11.2 Riemann Curvature Tensor

On a Riemannian manifold, the Riemann curvature tensor (RCT) defines a tensor at each point to describe the curvature of the manifold. Or we can say that it is used to describe the extent to which the manifold is curved. The Levi-Civita connection is used to derive this connection. This is defined using Christoffel symbols as [53]:

$$R^b{}_{acd} = \bar{\Gamma}^b_{ad,c} - \bar{\Gamma}^b_{cd,a} + \bar{\Gamma}^\nu_{ad}\,\bar{\Gamma}^b_{c\nu} - \bar{\Gamma}^\nu_{cd}\,\bar{\Gamma}^b_{a\nu}, \tag{2.3}$$

where comma "," represents the operator for ordinary or partial derivatives. The Riemann tensor specifically quantifies how much a vector deviates from its original position in tangent space when it is moved from one point to another around a closed loop in the manifold. Remember that in last two indices the Riemann tensor is antisymmetric, with

$$R^b{}_{acd} = -R^b{}_{adc}. \tag{2.4}$$

2.11.3 Ricci Tensor

Ricci tensor is the component of space-time curvature that indicates how matter tends to focus or converge over time. This tensor, which is expressed as R_{ab}, is a Rank-2 symmetric tensor, i.e. R_{ab}=R_{ba}. In the absence of a metric, the first and third indices of Riemann curvature tensor given in equation (2.3) are contracted:

$$R_{ad} = R^{b}_{abd}. \tag{2.5}$$

Thus, we can say that no metric was necessary in order to execute this contraction. If we make an assumption that metric is defined on manifold and then contract it with the Ricci tensor, we derive the Ricci scalar, expressed as:

$$g^{ab}R_{ab} = R. \tag{2.6}$$

2.11.4 Energy-Momentum Tensor

This EMT (often denoted as the Stress-Energy Tensor) is a mathematical key object in GR. This tensor is of second order and exhibits symmetry, typically denoted as U_{pq}, which describes the stress, density, and flux of energy and momentum in space-time. This describes how energy and momentum are distributed and flow throughout the space-time, helping to comprehend how energy and matter affect the space-time's curvature. Energy-Momentum Tensor also satisfies law of conservation, which implies that energy and momentum are conserved. Just like matter, EFEs use this tensor as a source to associate the geometry of space-time (curvature) to the matter it contains. The terms p and q in the U_{pq} are the indices of space-time that may take on values 0, 1, 2 and 3 only.

2.11.5 Torsion Tensor

This tensor is mostly used in theories like Teleparallel Gravity. It can help us to measure how much a space-time is 'twisted' or how much the parallel transport of vectors fails to be symmetric, which implies that this tensor captures how much the symmetry is broken. It is represented by the symbol $T^{\alpha}_{\nu\lambda}$, and can be expressed as [54]:

$$T^{\alpha}_{\nu\lambda} = \Gamma^{\alpha}{}_{\lambda\nu} - \Gamma^{\alpha}{}_{\nu\lambda} = e_i{}^{\alpha}(\partial_{\nu} e^i{}_{\lambda} - \partial_{\lambda} e^i{}_{\nu}), \tag{2.7}$$

which is comprised of 24 independent components. The $\Gamma^{\alpha}{}_{\lambda\nu}$ in equation (2.7) is the symbol used to represent the Weitzenb$\ddot{o}$ck Connection. Furthermore, the last two indices of the torsion tensor are antisymmetric.

2.12 Manifold

Manifold is a fundamental concept which is used to generalize and extend the idea of surfaces and spaces. Through manifolds, we gain insights into the dynamics of matter, energy, and gravitational interactions. In GR, space-time is modeled as a four-dimensional manifold. This means that at every point in space-time, we can locally describe the space-time using the familiar coordinates of Euclidean space, but globally, the structure is be more complex and curved. The term "curve" refers to a one-dimensional manifold, even though it need not to be curved in the traditional sense. Key instances of one-dimensional manifolds include the real line, circle, and parabola. Surfaces are manifolds of dimension two. These manifolds include the plane, sphere, cylinder, paraboloid, and ellipsoid. Similarly, a 3-dimensional manifold's example is the set of points, called the unit 3-sphere, defined by $(x_1, x_2, x_3, x_4) \in R^4$ and satisfying the expression $x_1{}^2 + x_2{}^2 + x_3{}^2 + x_4{}^2 = 1$. And, if we define M as a four-dimensional compact, smooth and connected Hausdorff manifold and g as a symmetric (0,2)-type tensor with non-degeneracy, called the Lorentzian metric on M then an ordered pair of the form (M, g) is labeled as space-time.

2.13 Einstein Field Equations for General Relativity

Issac Newton in his Newton's model of the universe believed that space is always equivalent to itself, and inside it, particles move and interact with one another. Furthermore, the effects of gravity may spread at an infinitely high rate of speed. On the other hand, Einstein thought that the existing idea of space did not seem realistic, thus he formulated the remarkable concept that space is a gravitational field [55]. To address this issue, a more precise version of the field equations was needed. After going through a process of trial and error, he arrived at the conclusion that the Riemann geometry could clearly explain the curvature of space that was a cause of diverse distribution of matter.

According to Riemann's geometry, the Riemann curvature tensor $R_{\alpha\beta\gamma\nu}$ quantifies the degree at which two vectors move in parallel. The reason that this tensor is named as 'curvature' is because it disappears if the space is flat [56]. The year 1915 marks the year when Einstein developed the field equations given by [57]:

$$R_{pq} - \frac{1}{2} R g_{pq} = \kappa^2 U_{pq}. \tag{2.8}$$

The equations shown above are collectively called as "Einstein Field Equations" or EFEs. On the LHS of equation (2.8), R represents the Ricci scalar, R_{pq} denotes Ricci tensor and the gravitational field is shown by the metric tensor g_{pq} whereas U_{pq} is the energy-momentum tensor which is responsible for identifying the form of energy and momentum that is existing at every single point in space-time. In order to meet the conditions of the Newtonian limit, the value of coupling constant κ^2 should be equal to $8\pi G$.

Ten non-linear partial differential equations make up the system of EFEs. Due to their complex non-linear form, determining exact solutions for these equations is exceptionally challenging. In the vacuum case, i.e. when $U_{pq} = 0$, equation (2.8) becomes $R_{pq} = 0$, which may be very difficult to solve. Thus, the equations, in their most basic form, are challenging to solve, but they may be managed if certain space-time symmetries are assumed to exist. Or if we assume specific geometric constraints on the metric and Ricci or energy-momentum tensors, The field equations can be reduced to provide exact solutions. Within the literature, various exact solutions to EFEs have been investigated by presuming the existence of specific symmetries [58, 59].

2.14 Cosmological Constant

The equations of Einstein showed that the gravity would cause the universe to collapse under its own weight. So, he offered the first modification of his EFEs in 1917. And in order to achieve his goal of a 'static universe,' he suggested that one extra term must be added to equation (2.8). He believed that the universe is static in size, so an additional element is required in his EFE in order to prevent the gravitational collapse. This additional element is now known as 'Cosmological Constant.' The variation of cosmological constant with EFEs is given [60] as:

$$R_{pq} - \frac{1}{2}Rg_{pq} + \Lambda g_{pq} = \kappa^2 U_{pq} \tag{2.9}$$

where Λ is a cosmological constant. Not long after Einstein proposed the cosmological constant, astronomical measurements confirmed that the universe is expanding (not static) [61].

Later on, he considered introducing this factor into his equation as the greatest mistake of his professional life.

2.15 Geodesics

In General Relativity, we have another concept of Geodesics which is of notable interest. Geodesics are known to be the paths which is followed by the free-falling objects through space-time. These paths are determined by space-times curvature, which in turn is influenced by the distribution of mass and energy. It can also be explained as the shortest path connecting two points in a warped space or space-time. To illustrate, a geodesic is essentially a straight line in flat and two-dimensional Euclidean geometry. If we imagine a flat piece of paper, then shortest distance between the two points on that paper will be a straight line. And in curved spaces (i.e. Riemannian Geometry), geodesics are more complex. If we imagine a surface of a sphere (like the Earth), the geodesics will be the great circles and the shortest path connecting two points on a sphere lies along the arc of a great circle.

2.16 Black Holes

Through the concept of space-time curvature and EFEs, GR postulates the presence of black holes in the universe. As stated earlier in the section (2.9), that massive objects causes the space-time to curve. So imagine, when a massive object (i.e. star) die and collapses under its own gravity, it can become so dense that it creates a black hole. Black hole is basically a region where space-time is curved so steeply that not even light can escape from it. The EFEs predicted that due to an excessive concentration of mass in a small enough space, the space-time's curvature becomes infinite at a point known as 'singularity.' This point of singularity lies at the center of the black hole. The area surrounding the singularity is referred to as the 'event horizon' where the gravitational pull is so strong that escape is impossible and anything which crosses this boundary is permanently lost.

2.17 Reasons to Modify GR (or TEGR)

Considering that TEGR (or GR) successfully explains an enormous variety of experimentally known phenomena, then why would we find the need to modify it? Some of the review articles [1, 62–64] provide more explanations of modified gravity. Let's talk about some of the cosmological concerns. GR requires a cosmological constant Λ that behaves as a fluid under negative pressure $p_\Lambda = -p_\Lambda$ in order to understand the present accelerating nature of universe. This scenario can only be generated by GR, by incorporating additional scalar fields, without resorting to a cosmological constant. A significant mismatch exists between the value of cosmological constant calculated from observations and value predicted when considering both quantum and classical components of vacuum energy. This issue is often referred to as 'cosmological constant problem,' which few scientists believe can't be solved without making modifications to GR, introducing new scalar fields, or revising the Standard Model [61].

The presence of cosmic singularities poses yet another challenge in the field of cosmology. Cosmic singularities such as those at the centers of black holes and the Big Bang, where space time curvature becomes infinite, and the laws of physics as we know them break down. Avoiding these singularities in GR is an impossible thing but according to certain modified theories, bounce solutions can be constructed to avoid certain singularities that are linked to the beginning of universe [1].

During the early stages of the universe's expansion, it went through a phase of rapid acceleration. As inflation [65] is followed by a radiation period and the cosmological constant cannot stop the inflation (acceleration), it is impossible to utilize one to characterise this epoch. Using a scalar field called the inflaton, GR can explain this accelerated epoch. GR does not explain the underlying cause of inflation and neither the cause of the inflation nor its characteristics are explained by GR. Modified gravity has the potential to thoroughly describe the inflationary era [66, 67].

At present, the order of magnitude of total matter energy density (dark matter+baryonic) is same as the dark energy density. But why they are identical now is not yet understood. Is there a physical explanation that could justify why such quantities should possess the same magnitude? This issue has been referred to as the "coincidence problem" [68, 69]. Until now, there is not a theoretical explanation for this. Even according to some physicists it is not really an issue and is only a coincidence. This issue might be resolved by making certain modifications to GR.

2.18 Teleparallel Theory of Gravity or TEGR

Teleparallel Equivalent of General Relativity or TEGR framework provides an alternative perspective Einstein's GR theory. In TEGR, the term 'torsion' is used instead of using the term 'curvature' to describe the gravity. Torsion basically measures how much space-time "twists" rather than how it "bends" like in GR . A detailed analysis comparing torsion and curvature has been presented in [70] .The TEGR presents an alternative formulation that is based on space that is universally flat in which torsion replaces curvature as the defining feature of gravity. TEGR is dynamically equivalent to GR which implies that the equations of motion and physical predictions in TEGR are the same as those in GR. From the motion of planets to the bending of light to the expansion of the universe, both theories describe the same gravitational phenomena. Although these two theories yield the same results but their underlying mathematical framework and interpretation of gravity are different. In the framework of GR, the curvature of space-time is described by RCT and in TEGR theory, the connection named Weitzenb$\ddot{o}$ck is used which has zero curvature but non-zero torsion. TEGR provides a foundation for extending and modifying General Relativity, such as in the domain of $f(T)$ gravity, the torsion scalar T is modified to function $f(T)$. These modifications aim to address unresolved issues in cosmology and gravity, such as dark energy and quantum gravity.

2.19 *f(T)* Gravity

Another variation of GR is the theory of $f(T)$ gravity, which serves as an extension to TEGR. This finest modified theory was introduced by Bengochea [71] in which T represents the torsion scalar. It is believed by some scientists that quantam theory of gravity may contain $f(T)$ gravity as its theoretical foundation. The purpose of $f(T)$ gravity is that it may be able to explain wide range of phenomenon such as the presence of dark matter and energy and universe's rapid acceleration. Apart from studying accelerated expansion, $f(T)$ gravity further aids in reconstructing various cosmological models, addressing perturbations in both vacuum and matter [72]. Following this, Linder and Myrzakulov [73, 74] recommended several $f(T)$ gravity models to study various physical phenomena. Capozziello *et al.* also explained the influence of $f(T)$ gravity to the development of cosmography by shedding light on the large-scale behavior of the universe [75]. Moreover, $f(T)$ gravity, under reasonable constraints, agrees with GR and is consistent with studies involving the solar system and binary pulsars

[76].

The EFEs of $f(T)$ gravity are given as [47]:

$$S_{\mu}^{\nu\beta}\partial_{\nu}TF_{T}+\left[e^{-1}\partial_{\nu}(eS_{\mu}^{\nu\beta})-e_{\mu}^{\lambda}T_{\nu\lambda}^{\alpha}S_{\alpha}^{\nu\beta}\right]F+\frac{1}{4}e_{\mu}^{\beta}f=ke_{\mu}^{i}T_{i}^{\beta}, \tag{2.10}$$

where $S_{\mu}^{\nu\beta}$ is the spin tensor, $f=f(T)$, $F(T)=\dfrac{df(T)}{dT}$, e denotes the determinant of tetrad field e_{μ}^{λ}, $T_{\nu\lambda}^{\alpha}$ has been used for the torsion tensor, $k=4\pi G$, where G is the gravitational constant and T_{i}^{β} indicates the Energy-Momentum Tensor.

2.20 Fundamentals of Teleparallel Theory of Gravity

2.20.1 Tetrad Field

In the context of TEGR, the tetrad is the main object of dynamical interest. Tetrad is said to be the most dynamic and basic gravitational field which is also called vierbein meaning ‘four legs.’ Tetrad field replaces the metric as the main object describing space-time geometry. At each point p on the manifold M, the tetrad field is represented by a set of four linearly independent vectors. These vectors form a basis for the tangent space denoted by T_pM. The tetrad field at each point in space-time signifies a local inertial frame where the space-time looks flat. By space-time appears ‘flat’ means that the space-time will be Minkowski space-time and the laws of physics will resemble those in special relativity.

The tetrad field consists of four vectors e_{p}^{μ} where p represents the local frame indices and μ are the space-time indices (i.e. $\mu=0,1,2,3$). Mathematically, the representation of tetrad is, $(e_{p}^{0},e_{p}^{1},e_{p}^{2},e_{p}^{3})$. The first component of the vector is a timelike vector and the remaining three components are spacelike vectors. The field e_{p}^{μ} can also connects the space-time metric $g_{\mu\nu}$ to the metric on tangent-space $\eta = \eta_{pq} dx^a dx^b$ by the relation [77]:

$$\eta_{pq} = g_{\mu\nu}\, e_p{}^{\mu}\, e_q{}^{\nu}, \tag{2.11}$$

where $\eta_{pq} = diag(-1,1,1,1)$ signifies the Minkowski metric. The orthonormality constraint makes it possible to get the metric from the tetrad. The metric can be derived from the tetrad in the manner as follows:

$$g_{\mu\nu} = \eta_{pq}\, e^p{}_{\mu}\, e^q{}_{\nu} \text{ or } \quad g^{\mu\nu} = \eta^{pq}\, e_p{}^{\mu}\, e_q{}^{\nu}. \tag{2.12}$$

Additionally, $e = det(e^p_\mu) = \sqrt{-g}$ relates tetrad determinant to the metric determinant.

2.20.2 Weitzenb$\ddot{o}$ck Connection

General Relativity operates within Riemannian space, where the RCT is determined through the Levi-Civita connection, while the metric is set by the EFEs. The gravitational field in Teleparallel Theory (TT) is defined by the torsion tensor and in order to define this tensor, TT employs the Weitzenb$\ddot{o}$ck connection which is an alternate link to the Levi-Civita connection. This Weitzenb$\ddot{o}$ck connection illustrates that how vectors change as they are parallel transported over a curved space-time. The curvature of space-time can affect both the direction and length of vectors being transported in parallel. However, one of the crucial feature of the Weitzenb$\ddot{o}$ck connection is that the length of vectors parallel transported across it is preserved. Torsion can be defined as the degree to which the transport of vectors along a curve deviates from the vector's direction, therefore, a torsion-free connection means 'the parallel transport of vectors maintains the vector's direction.' Thus, Weitzenb$\ddot{o}$ck connection is characterized by the condition that the parallel transport of vectors preserves the vector norm and that the connection is torsion-free, i.e. the length and direction of the vector are retained.

2.21 Teleparallel Lie Derivative

In geometry, Lie derivative helps compare a geometric object's value at one point on a curve to its value as it moved to another point along the same curve. So basically, it determines how the tensor field changes from one point on the manifold M to another. Lie derivative is useful when it comes to identifying and studying the symmetries in space-time. Teleparallel Lie derivative (TLD) was introduced by [78]. And the TLD 'L' of a second-rank covariant tensor along a vector field Y can be written as:

$$L_Y^T M_{pq} \equiv M_{pq,c} Y^c + M_{cq} Y^c_{,p} + M_{pc} Y^c_{,q} + Y^c (M_{\sigma q} T^\sigma_{pc} + M_{p\sigma} T^\sigma_{qc}). \tag{2.13}$$

Where, the Lie derivative of the metric tensor g_{pq} along a vector field Y is expressed as:

$$L_Y^T g_{pq} \equiv g_{pq,c} Y^c + g_{cq} Y^c_{,p} + g_{pc} Y^c_{,q} + Y^c (g_{\sigma q} T^\sigma_{pc} + g_{p\sigma} T^\sigma_{qc}) = 2\psi g_{pq}. \tag{2.14}$$

2.22 Space-time Symmetries

The GR theory, as described in section (2.13), is derived by EFEs that are markedly non-linear. Given their significant non-linearity, determining their exact solutions is a tough challenge. Hence, the incorporation of space-time symmetries is essential for various reasons. One major advantage of employing symmetries is their ability to transform PDEs into ODEs through specific constraints which are therefore easier to manage. Secondly, symmetries often help in categorizing and classifying the exact solutions to EFEs. Although there are various space-time symmetries, but we will restrict our attention to three essential ones, specifically; Killing, Homothetic and Conformal symmetries.

A vector field X is called a TKVF if the Lie derivative of the metric tensor g_{pq} with respect to Y vanishes and equation (2.14) becomes:

$$L_Y^T g_{pq} \equiv g_{pq,c} Y^c + g_{cq} Y^c_{,p} + g_{pc} Y^c_{,q} + Y^c (g_{\sigma q} T^{\sigma}_{pc} + g_{p\sigma} T^{\sigma}_{qc}) = 0, \tag{2.15}$$

The equation (2.15) implies that the metric is invariant under the flow generated by Y or that the space-time has a symmetry associated with Y.

Similarly, a vector field Y will be THVF when if it satisfies:

$$L_Y^T g_{pq} \equiv g_{pq,c} Y^c + g_{cq} Y^c_{,p} + g_{pc} Y^c_{,q} + Y^c (g_{\sigma q} T^{\sigma}_{pc} + g_{p\sigma} T^{\sigma}_{qc}) = 2\psi g_{pq}, \quad \psi = constant \tag{2.16}$$

In other words, a Teleparallel vector field Y is called proper THVF when $\psi \neq 0$. The equation (2.16) implies that the metric tensor g_{pq} is scaled by a constant factor under the flow of Y, rather than being fully invariant.

And the vector field Y will be TCFV when:

$$L_Y^T g_{pq} \equiv g_{pq,c} Y^c + g_{cq} Y^c_{,p} + g_{pc} Y^c_{,q} + Y^c (g_{\sigma q} T^{\sigma}_{pc} + g_{p\sigma} T^{\sigma}_{qc}) = 2\psi g_{pq}, \tag{2.17}$$

where ψ(y) is a scalar function depending on the coordinates.

2.23 Energy Conditions

Energy conditions play a crucial rule in order to address the momentous cosmological issues. They assist in determining whether the matter within stellar compact objects is normal or exotic in nature. These energy conditions are basically the number of conditions which must be satisfied for energy-momentum tensor U_{pq} to properly reflect any known matter fields. The fulfillment of these requirements ensures that the U_{pq} accurately reflects genuine sources of energy and momentum, and the tensor becomes less arbitrary.

Mathematically, energy conditions serve as boundary constraints that ensure the energy density remains positive. Despite this, the model is not bound by the physical restrictions imposed by the energy conditions. Nevertheless, these conditions offer important insights into the energy-matter content in real-world models. The energy conditions namely weak energy conditions (WECs), strong energy conditions (SECs), dominant energy conditions (DECs) and null energy conditions (NECs) are defined as:

1. Weak Energy Condition (WEC): $0 \leq p+\rho$ and $\rho \geq 0$

2. Strong Energy Condition (SEC): $0 \leq p+\rho$ and $\rho+3p \geq 0$

3. Dominant Energy Condition (DEC): $\rho \geq 0$ and $\rho \geq |p|$.

4. Null Energy Condition (NEC): $0 \leq p+\rho$.

Chapter 3

Classification of Static Spherically Symmetric Perfect Fluid Space-Times via Conformal Vector Fields in *f(T)* gravity

3.1 Introduction

In this chapter, the paper of Fiaz Hussain *et al*. [47] in which static SS perfect fluid space-times have been classified via conformal vector fields (CVFs) in $f(T)$ gravity have been reviewed. The study of static SS space-times plays a pivotal role in the field of GR and its modifications, particularly when considering physical systems such as stars, black holes, and cosmological models. In the case of $f(T)$ gravity, CVFs help in the classification of static SS perfect fluid space-times by determining the conditions under which such space-times possess enhanced symmetry. In this chapter, firstly static SS solutions are explored by solving the Einstein Field Equations in $f(T)$ gravity. Consequently, the resulting solutions are classified in which 20 cases arise. Each case is thoroughly analyzed to check the existence of CVFs. The outcomes of this investigation are presented through tables.

3.2 Mathematical Formulation

Consider a static SS space-times in the usual coordinates (t, r, θ, ϕ) labeled $(\eta^0, \eta^1, \eta^2, \eta^3)$ respectively with the line element [58]

$$ds^2 = -e^{u(r)}dt^2 + e^{v(r)}dr^2 + P^2(r)\left[d\theta^2 + \sin^2\theta d\phi^2\right], \tag{3.1}$$

where $u = u(r)$, $v = v(r)$and $P = P(r)$ are unknown functions of r (radial coordinate) and are non-zero everywhere. The minimum number KVFs admitted by the above space-times (3.1) are [79]

$$\eta_1 = \partial_t,\ \eta_2 = \sin\phi\,\partial_\theta + \cos\phi\cot\theta\,\partial_\phi,\ \eta_3 = \cos\phi\,\partial_\theta - \sin\phi\cot\theta\,\partial_\phi,\ \eta_4 = \partial_\phi. \tag{3.2}$$

Assuming a diagonal tetrad, the torsion scalar T for the above space-times (3.1) are:

$$T = \frac{2e^{-v}P'}{P}\left(u' + \frac{P'}{P}\right), \tag{3.3}$$

where u' and P' represents derivative of the functions $(i.e.\ \frac{d}{dr})$. To examine the CVFs of static SS space-times, EFEs of $f(T)$ gravity have been used which are [71]:

$$S_\mu^{\nu\beta}\partial_\nu T F_T + \left[e^{-1}\partial_\nu(eS_\mu^{\nu\beta}) - e_\mu^\lambda T_{\nu\lambda}^\alpha S_\alpha^{\nu\beta}\right]F + \frac{1}{4}e_\mu^\beta f = ke_\mu^i T_i^\beta, \tag{3.4}$$

where $S_\mu^{\nu\beta}$ is the spin tensor, $f = f(T)$, $F(T) = \dfrac{df(T)}{dT}$, e represents the determinant of tetrad field e_μ^λ, $T_{\nu\lambda}^\alpha$ has been used for the torsion tensor, $k = 4\pi G$, where G is the gravitational constant and T_i^β is the Energy-Momentum Tensor. By assuming that matter content is perfect fluid as defined by the EMT, a set of equations is obtained by making use of a diagonal tetrad that is formed by putting equation (3.1) into equation (3.4)

$$\frac{f}{4}-\left[\frac{e^{-v}}{P}\left(u'P'-v'P'+\frac{2P'^2}{P}+2P''\right)-\frac{1}{P^2}\right]\cdot\frac{F}{2}-e^{-v}\frac{P'T'}{P}f_{TT}=4\pi\rho, \tag{3.5}$$

$$\left[\frac{e^{-v}}{P}\left(2u'P'+\frac{2P'^2}{P}\right)-\frac{1}{P^2}\right]-\frac{F}{4}=4\pi\rho, \tag{3.6}$$

$$e^{-v}\left(\frac{3u'P'}{2P}+\frac{u''}{2}+\frac{u'^2}{4}-\frac{u'h'}{4}-\frac{v'P'}{2P}+\frac{P'^2}{P^2}+\frac{P''}{P}\right)\cdot\frac{F}{2}-\frac{f}{4}+\frac{T'e^{-v}}{2}\left(\frac{u'}{2}+\frac{P'}{P}\right)f_{TT}=4\pi\rho, \tag{3.7}$$

where the p and ρ in the above equations denotes the pressure and energy density (ED) of the fluid's distribution respectively. The equations of motion in $f(T)$ gravity corresponding to a diagonal tetrad field in static SS space-times (3.1) involve an additional equation that emerges from the (r,θ) component [90]

$$e^{-\frac{3v}{2}}\cot\theta f_{TT}T'=0, \tag{3.8}$$

where $T'=\left[\frac{-u'P''}{P}+(u'+v')\frac{P'^2}{P^2}+\frac{2P'^3}{P^3}+\frac{P'}{P}\left(u'v'-u''-\frac{2P''}{P}\right)\right]$. Then equations (3.5) to (3.8) are solved which is comprised of unknowns u,v,P,f,ρ and p. According to the equation (3.8), it follows that either f_{TT} is zero or $T'=0$. The first scenario results in linear $f(T)$ gravity, while second suggests that the equations of motion above admit solutions where the torsion scalar remains constant. Each case is examined individually. When f_{TT}, we have $f(T)=d_1T+d_2$, where $d_1,d_2\in\Re$. Undoubtedly, linear $f(T)$ gravity simplifies the EFEs to great extent. However, due to the presence of non-linearity in the space-time components, system of equations (3.5) to (3.7) is difficult to solve. Hence, the solution is seek by classification procedure. Using the fact $f_{TT}=0$ which indicates $f(T)=d_1T+d_2$, in equations (3.5) to (3.7). After substitution, the subtraction of equations (3.6) and (3.7) from equation (3.5) is done which provides a set of two equations, from which subtracting one from the other leads to

$$e^{-v}\left[\frac{u''}{2}+\frac{u'^2}{4}-\frac{u'P'}{2P}-\frac{u'v'}{4}-\frac{v'P'}{2P}-\frac{P'2}{P^2}+\frac{P''}{P}\right]+\frac{1}{P^2}=0. \tag{3.9}$$

To find a solution to equation (3.9), specific constraints on the space-time components are imposed which are given as

i. $u=u(r), \quad v=\text{constant} \quad \text{and} \quad P=P(r).$

ii. $u=\text{constant}, \quad v=v(r) \quad \text{and} \quad P=P(r).$

iii. $u=u(r), \quad v=v(r) \quad \text{and} \quad P=\text{constant}.$

iv. $u=u(r) \quad \text{and} \quad v(r)=P(r).$

v. $u(r)=v(r) \quad \text{and} \quad P=P(r).$

vi. $u(r)=P(r) \quad \text{and} \quad v=v(r).$

vii. $u=\text{constant} \quad \text{and} \quad v(r)=P(r).$

viii. $u(r)=P(r) \quad \text{and} \quad v=\text{constant}.$

ix. $u(r)=v(r) \quad \text{and} \quad P=\text{constant}.$

x. $u=u(r) \quad \text{and} \quad v=P=\text{constant}.$

xi. $u=P=\text{constant} \quad \text{and} \quad v=v(r).$

xii. $u=v=\text{constant} \quad \text{and} \quad P=P(r).$

xiii. $u(r)=v(r)=P(r).$

xiv. $u=v=P=\text{constant}.$

By using the above mentioned possibilities, the solutions of equation (3.9) are represented by the following cases in which torsion scalar is also deduced by employing the obtained solutions in equation (3.3)

1. $u = \text{constant}$, $v = v(r)$, $e^{-v}v'r + 2e^{-v} - 2 = 0$ implies $v = \ln\left(\frac{1}{1+k_1r^2}\right)$, $P = r$ and $T = \left(\frac{2}{r^2} + 2k_1\right)$, where $k_1 \in \Re \setminus \{0\}$.

2. $u = u(r)$, $v = \text{constant}$, $2ru'' + ru'^2 - 2u' = 0 \implies$ u = $\ln r^4$, $e^v = 1$, $P = r$ and $T = 10r^{-2}$.

3. $u = u(r)$, $v = v(r)$, $u = v^{-1}$, $r^2\left(v'' - v'^2\right) + 2(1 - e^v) = 0$ implies $v = \ln\left(1 - \frac{k_1}{r} + \frac{k_2r^2}{3}\right)^{-1}$, $u = \ln\left(1 - \frac{k_1}{r} + \frac{k_2r^2}{3}\right)$, $P = r$ and $T = \left(\frac{2}{r^2} + 2k_2\right)$, where $k_1, k_2 \in \Re \setminus \{0\}$.

4. $u = u(r), v = v(r)$, $u = h^{-1}$, $e^u\left(\frac{u''}{2} + \frac{u'^2}{2} - \frac{1}{r^2}\right) + \frac{1}{r^2} = 0 \implies$ u = $\ln\left(1 - \frac{2M}{r}\right)$, $v = \ln\left(1 - \frac{2M}{r}\right)^{-1}$, $P = r$ and $T = \left(\frac{2}{r^2}\right)$, where M represents the Arnowitt-Deser-Misner mass.

5. $u = u(r)$, $v = v(r)$, $u = v^{-1}$, $r^2\left(u'' + u'^2\right) - 2\left(1 - e^{-u}\right) = 0$ implies $u = \ln\left(1 - \frac{\Lambda r^2}{3}\right)$, $v = \ln\left(1 - \frac{\Lambda r^2}{3}\right)^{-1}$, $P = r$ and $T = \left(\frac{2}{r^2} - 2\Lambda\right)$, where Λ is the cosmological constant.

6. $u = u(r)$, $v = v(r)$, $ru' + 1 = 0$ which gives $u = \ln\left(\frac{k_1}{r}\right)$, $4e^v - rv' + 1 = 0$ implies $v = \ln\left(\frac{r}{k_2 - 4r}\right)$, $P = r$ and $T = 0$, where $k_1, k_2 \in \Re \setminus \{0\}$.

7. $u = u(r)$, $v = v(r)$, $ru' - 2 = 0$ implies $u = \ln\left(k_1r^2\right)$, $e^v - rv' - 2 = 0 \implies$ v = $\ln\left(\frac{2}{1+2k_2r^2}\right)$, $P = r$ and $T = \frac{3(1+2k_2r^2)}{r^2}$, where $k_1, k_2 \in \Re \setminus \{0\}$.

8. $u = u(r), v = v(r)$, $\left(\dfrac{u''}{2} + \dfrac{u'^2}{4}\right) = 0 \implies$ u =$\ln\left(\frac{k_1r+k_2}{2}\right)^2$, $-k_1v'e^{-v} + 2(k_1r + k_2) = 0$ which implies $v = \ln\left(\frac{k_1}{k_1k_3 - k_1r^2 - 2k_2r}\right)$, $P = 1$ and $T = 0$, where $k_1, k_2, k_3 \in \Re \setminus \{0\}$.

9. $u = u(r)$, $v = v(r)$, $u = v^{-1}$, $u'' + u'^2 + 2e^{-u} = 0$ implies $u = \ln\left(k_2 - k_1r - r^2\right)$, $v = \ln\left(k_2 - k_1r - r^2\right)^{-1}$, $P = 1$ and $T = 0$, where $k_1, k_2 \in \Re \setminus \{0\}$.

10. $u = \text{constant} = k_1 \neq 0$, $v = \text{constant} = k_2 \neq 0$, $u = \ln(k_1)$, $v = \ln(k_2)$, $PP'' - P'^2 + k_2 = 0$ implies $P = r\sqrt{k_2}$ and $T = \frac{2}{k_2 r^2}$, where $k_1, k_2 \in \Re \setminus \{0\}$ with $k_1 \neq k_2$.

Now, the second possibility which arises from equation (3.8), i.e. $T' = 0$, gives

$$\left[\frac{-u'P''}{P} + (u'+v')\frac{P'^2}{P^2} + \frac{2P'^3}{P^3} + \frac{P'}{P}\left(u'v' - u'' - \frac{2P''}{P}\right)\right] = 0. \tag{3.10}$$

Equation (3.10) yields solutions with a torsion scalar that is constant, as dictated by the condition $T' = 0$.

For the sake of finding the solutions for equation (3.10), a similar classification procedure is performed which is done for finding the solutions of equation (3.9). We have identified the following cases

11. $u = u(r)$, $v = v(r)$, $ru'' - u' = 0$ implies $u = \left(\frac{k_1 r^2}{2} + k_2\right)$, $rv'(ru'+1) + 2 = 0 \implies$ v = $\ln\left(\frac{k_3\sqrt{k_1 r^2+1}}{r}\right)^2$, $P = r$ and $T = \frac{2}{k_3^2}$, where $k_1, k_2, k_3 \in \Re$ $(k_1, k_3 \neq 0)$.

12. $u = u(r)$, $v = v(r)$, $1 + rv' = 0$ implies $v = \ln\left(\frac{k_1}{r}\right)$, $r^2u'' - rv' - 2 = 0 \implies$ u = $\ln\left(\frac{k_3 e^{k_2 r}}{r}\right)$, $P = r$ and $T = \frac{2k_2}{k_3}$, where $k_1, k_2, k_3 \in \Re (k_1, k_3 \neq 0)$.

13. $u = u(r)$, $v = v(r)$, $2 + rv' = 0$ implies $v = \ln\left(\frac{k_1}{r^2}\right)$, $ru'' - u'(1+rv') = 0 \implies u = \ln(k_3 r^{k_2})$, $P = r$ and $T = \frac{2(k_2+1)}{k_1}$, where $k_1, k_2, k_3 \in \Re(k_1, k_2, k_3 \neq 0)$.

14. $u = u(r)$, $v = v(r)$, $2 + ru' = 0$ implies $u = \ln\left(\frac{k_1}{r^2}\right)$, $ru'' - v'(1+ru') = 0 \implies$ v = $\ln\left(\frac{k_2}{r^2}\right)$, $P = r$ and $T = \frac{-2}{k_2}$, where $k_1, k_2 \in \Re$ $(k_1, k_2 \neq 0)$.

15. $u = u(r)$, $v = v(r)$, $r^2u'' - 2 = 0$ implies $u = \ln\left(\frac{e^{k_1 r + k_2}}{r^2}\right)$, $u' + v'(1+ru') = 0 \implies$ v = $\ln\left[\frac{k_3(k_1 r - 1)}{r^2}\right]$, $P = r$ and $T = \frac{2}{k_3}$, where $k_1, k_2, k_3 \in \Re$ $(k_1, k_3 \neq 0)$.

16. $u = \text{constant}$, $v = v(r)$, $rv' + 2 = 0 \implies v = \ln\left(\frac{k_1}{r^2}\right)$, $e^u = 1$, $P = r$, $T = \frac{2}{k_1}$, where $k_1 \in \Re \setminus \{0\}$.

17. $u=u(r)$, $v=\text{constant}=k_1\neq 0$, $r^2u''-ru'-2=0 \Longrightarrow u=\left(\frac{k_2r^2}{2}-\ln r+k_3\right)$, $P=r$, $T=\frac{2k_2}{e^{k_1}}$, where $k_1,k_2,k_3\in\Re\setminus\{0\}$.

18. $u=u(r)$, $v=v(r)$, $u''=0$ implies $u=(k_1r+k_2)$, $ru'(1+rv')+rv'+2=0 \Longrightarrow$ v $=\ln\left[\frac{k_3(k_1r+1)}{r^2}\right]$, $P=r$ and $T=\frac{2}{k_3}$, where $k_1,k_2,k_3\in\Re(k_1,k_3\neq 0)$.

19. $u=\text{constant}=k_1\neq 0$, $v=v(r)$, $P''=0 \Longrightarrow$ P = k_2r+k_3, $v'P+2P'=0 \Longrightarrow$ v = $\ln\left(\frac{k_4}{(k_2r+k_3)^2}\right)$, $T=\frac{2k_2^2}{k_4}$, where $k_1,k_2,k_3,k_4\in\Re\setminus\{0\}$.

20. $u=\text{constant}=k_1\neq 0, v=\text{constant}=k_2\neq 0, P''-P'^2=0 \Longrightarrow P=e^{k_3r+k_4}, T=\frac{2k_3^2}{e^{k_2}}$, where

    $k_1,k_2,k_3,k_4\in\Re\setminus\{0\}$, with $k_1\neq k_2$.

The above 20 solutions are further utilize to identify the CVFs which are in accordance with the following equation

$$L_Y^T g_{pq}\equiv g_{pq,c}Y^c+g_{cq}Y^c_{,p}+g_{pc}Y^c_{,q}+Y^c(g_{\sigma q}T^\sigma_{pc}+g_{p\sigma}T^\sigma_{qc})=2\alpha g_{pq}, \tag{3.11}$$

where $L, g_{pq}, \alpha=\alpha(t,r,\theta,\phi)$ and comma represents the Lie derivative, metric tensor, conformal factor and partial derivative respectively. The CVFs for the cases 1,2 and 3 are determined in [80]. Accordingly, these three cases are overlooked with the remaining cases which are solved here. The results of all the calculations are shown in the table below

Table 1: CVFs of the obtained Static Spherically symmetric metrics.

Case No	Metric components	CVFs	Conformal Factor	Description
4.	$u = ln\left(1-\frac{2M}{r}\right)$, $v = ln\left(1-\frac{2M}{r}\right)^{-1}$ and $P = r$.	$\eta_1,\ \eta_2,\ \eta_3$ and η_4.	$\alpha = 0$.	KVFs
5.	$u = ln\left(1-\frac{\Lambda r^2}{3}\right)$, $v = ln\left(1-\frac{\Lambda r^2}{3}\right)^{-1}$ and $P = r$.	$\eta_1,\ \eta_2,\ \eta_3$ and η_4.	$\alpha = 0$.	KVFs
6.	$u = ln\left(\frac{k_1}{r}\right)$, $v = ln\left(\frac{r}{k_2-4r}\right)$ and $P = r$.	$\eta_1,\ \eta_2,\ \eta_3$ and η_4.	$\alpha = 0$.	KVFs
7.	$u = ln\left(k_1 r^2\right)$, $v = ln\left(\frac{2}{1+2k_2r^2}\right)$ and $P = r$.	$\eta_1,\ \eta_2,\ \eta_3,\ \eta_4,\ \eta_5^{**} = rt\sqrt{1+2k_2r^2}\frac{\partial}{\partial r}+\frac{2}{k_1}\ln\left(\frac{\sqrt{2k_2}r}{1+\sqrt{1+2k_2r^2}}\right)\frac{\partial}{\partial t}$ and $\eta_6^{**} = r\sqrt{1+2k_2r^2}\frac{\partial}{\partial r}$.	$\alpha = \sqrt{1+2k_2r^2}\left(c_1t+c_2\right)$, where, $c_1, c_2 \in \Re\,(c_1 \neq 0)$.	CVFs
8.	$u = ln\left(\frac{k_1r+k_2}{2}\right)^2$, $v = ln\left(\frac{k_1}{k_1k_3-k_1r^2-2k_2r}\right)$ and $P = 1$.	$\eta_1,\ \eta_2,\ \eta_3,\ \eta_4,\ \eta_5 = -[\Omega_1\cos\psi]\frac{\partial}{\partial t}-[\Omega_2\sin\psi]\frac{\partial}{\partial r}$ and $\eta_6 = [\Omega_1\sin\psi]\frac{\partial}{\partial t}-[\Omega_2\cos\psi]\frac{\partial}{\partial r}$.	$\alpha = 0$.	KVFs
9.	$u = ln\left(k_2-k_1r-r^2\right)$, $v = \ln\left(k_2-k_1r-r^2\right)^{-1}$ and $P = 1$.	$\eta_1,\ \eta_2,\ \eta_3,\ \eta_4,\ \eta_5 = -[\Omega_3\cos\psi_1]\frac{\partial}{\partial t}-[\Omega_4\sin\psi_1]\frac{\partial}{\partial r}$ and $\eta_6 = [\Omega_3\sin\psi_1]\frac{\partial}{\partial t}-[\Omega_4\cos\psi_1]\frac{\partial}{\partial r}$.	$\alpha = 0$.	KVFs
10.	$u = ln\left(k_1\right)$, $v = ln\left(k_2\right)$ and $P = r\sqrt{k_2}$.	$\eta_1,\ \eta_2,\ \eta_3,\ \eta_4,\ \eta_5^{*} = t\frac{\partial}{\partial t}+r\frac{\partial}{\partial r}$.	$\alpha = c_1$, where $c_1 \in \Re$.	HVFs
11.	$u = \left(\frac{k_1r^2}{2}+k_2\right)$, $v = ln\left(\frac{k_3\sqrt{k_1r^2+1}}{r}\right)^2$ and $P = r$.	$\eta_1,\ \eta_2,\ \eta_3$ and η_4.	$\alpha = 0$.	KVFs
12.	$u = \ln\left(\frac{k_3e^{k_2r}}{r}\right)$, $v = \ln\left(\frac{k_1}{r}\right)$ and $P = r$.	$\eta_1,\ \eta_2,\ \eta_3$ and η_4.	$\alpha = 0$.	KVFs
13.	$u = \ln\left(k_3r^{k_2}\right)$, $v = \ln\left(\frac{k_1}{r^2}\right)$ and $P = r$.	$\eta_1,\ \eta_2,\ \eta_3$ and η_4.	$\alpha = 0$.	KVFs
14.	$u =$ constant, $v = \ln\left(\frac{k_2}{r^2}\right)$ and $P = r$.	$\eta_1,\ \eta_2,\ \eta_3$ and η_4.	$\alpha = 0$.	KVFs
15.	$u = \ln\left(\frac{e^{k_1r+k_2}}{r^2}\right)$, $v = \ln\left[\frac{k_3(k_1r-1)}{r^2}\right]$ and $P = r$.	$\eta_1,\ \eta_2,\ \eta_3$ and η_4.	$\alpha = 0$.	KVFs
16.	$u =$ constant, $v = \ln\left(\frac{k_1}{r^2}\right)$ and $P = r$.	$\eta_1,\ \eta_2,\ \eta_3,\ \eta_4,\ \eta_5^{**} = e^{\frac{t}{\sqrt{k_1}}}\left(\frac{r}{\sqrt{k_1}}\frac{\partial}{\partial t}+\frac{r^2}{\sqrt{k_1}}\frac{\partial}{\partial r}\right)$ and $\eta_6^{**} = e^{\frac{-t}{\sqrt{k_1}}}\left(\frac{-r}{\sqrt{k_1}}\frac{\partial}{\partial t}+\frac{r^2}{\sqrt{k_1}}\frac{\partial}{\partial r}\right)$.	$\alpha = \frac{r\psi}{k_1}$, where $\psi = \left[c_1e^{\frac{t}{k_1}}+c_2e^{\frac{-t}{k_1}}\right]$ with $c_1, c_2 \in \Re \neq 0$.	CVFs
17.	$u = \left(\frac{k_2r^2}{2}-\ln r+k_3\right)$, $v =$ constant $= k_1 \neq 0$ and $P = r$.	$\eta_1,\ \eta_2,\ \eta_3$ and η_4.	$\alpha = 0$.	KVFs
18.	$u = \left(k_1r+k_2\right)$, $v = \ln\left[\frac{k_3(k_1r+1)}{r^2}\right]$ and $P = r$.	$\eta_1,\ \eta_2,\ \eta_3$ and η_4.	$\alpha = 0$.	KVFs
19.	$u =$ constant $= k_1 \neq 0$, $v = \ln\left[\frac{k_4}{(k_2r+k_3)^2}\right]$ and $P = (k_2r+k_3)$.	$\eta_1,\ \eta_2,\ \eta_3$ and η_4.	$\alpha = 0$.	KVFs
20.	$u = \ln\left(k_1\right)$, $v = \ln\left(k_2\right)$ and $P = e^{k_3r+k_4}$.	$\eta_1,\ \eta_2,\ \eta_3,\ \eta_4,\ \eta_5^{**} = e^{k_3\Omega_5}\left(\frac{1}{\sqrt{k_1k_2}}\frac{\partial}{\partial t}+\frac{1}{\sqrt{k_2}}\frac{\partial}{\partial r}\right)$ and $\eta_6^{**} = e^{k_3\Omega_6}\left(\frac{-1}{\sqrt{k_1k_2}}\frac{\partial}{\partial t}+\frac{1}{\sqrt{k_2}}\frac{\partial}{\partial r}\right)$.	$\alpha = \frac{k_3\psi}{k_2}$, where $\psi = \left[c_1e^{k_3\Omega_5}+c_2e^{k_3\Omega_6}\right]$ with $c_1, c_2 \in \Re \neq 0$.	CVFs

In the above table, η_1, η_2, η_3 and η_4 represent the CVFs given in equation (3.2) whereas η_5^{*}, η_5^{**} and η_6^{**} are the proper CVFs.

3.3 Results and Discussion

From the very beginning of General Relativity (GR) and its modified forms, the methods of investigating the solutions of the EFEs and using them to address problems in related areas have been a topic of discussion. In the context of GR, extensive work has been carried out in [58], focusing on different background structures known as space-time geometries. At points where General Relativity (GR) must be modified or extended, the task of finding solutions is made considerably easier by applying symmetries, especially Noether symmetries present in a given space-time [81, 82]. Symmetries have been shown to play a pivotal role in exploring and classifying the known solutions to the EFEs. Conformal symmetry is used not just to generate solutions but also offers characteristics that enable the discussion of numerous physical phenomena in contemporary cosmology and astrophysics. In astrophysical studies, compact stars, dense stars, and gravstars require conformal motion to examine their internal structure.

HVFs represent another significant discovery in this study. These vector fields are useful for analyzing constants of motion, which, in turn, help in tracking particle trajectories in space-time. The singularity issues in General Relativity have been tackled by utilizing homothetic symmetry. HVFs are crucial in the analysis of self-similar solutions of the EFEs [80]. The solutions derived here can be classified into several categories. They may be conformally flat or may not be flat. In cases 1 and 2, the space-times are found to be conformally flat, achieving the maximum dimension of 15 and the CVFs for case 3 are explored in [80] where the space-time resulting from this contains four KVFs. For the remaining cases, following results have been discovered as a consequence of categorization.

- The CVFs in cases 4, 5, 6, 8, 9, 11, 12, 13, 14, 15, 17, 18, and 19 become KVFs. In physical terms, KVFs are responsible for conservation laws i.e. the conservation of linear momentum and energy are associated with the translational KVFs ∂_t and ∂_ϕ respectively, while the conservation of angular momentum is represented by the rotational isometries η_2 and η_3.

- For cases 7, 16, and 20, the space-times possess proper CVFs. The CVFs in the given cases have been found to possess a dimension of six, with four of them being KVFs, as expressed in the equation (3.2), and the other two being proper CVFs, namely η_5^{**} and η_6^{**}.

- In case 10, the space-time admits proper HVFs as a result of the conformal factor α being a non-zero constant. HVFs are of great importance for several reasons. First, they have proven useful in analyzing constants of motion, which are essential for studying particle trajectories in space-time. Second, homothetic motion helps tackle the singularity problem in GR. Moreover, the study of self-similar solutions to the EFEs has highlighted the significant role of HVFs.

Chapter 4

Teleparallel Homothetic Vector Fields of Static Spherically Symmetric Space-Times in *f(T)* gravity

4.1 Introduction

The main aim of this chapter is the classification of static spherically symmetric (SS) space-times as per their THVFs in $f(T)$ theory of gravity. It extends the work of Fiaz Hussain *et al.* [47] in which static SS perfect fluid space-times have been classified via conformal vector fields (CVFs) in $f(T)$ gravity. Static SS space-times refers to the type of space-time which is "static" and "spherically symmetric." The term "static" implies that the geometry of a space-time does not vary with time, which implies that the properties of the space-time remain constant as time progresses or it represent a space-time which is time-independent that is, gravitational field and the space-time metric (which describes distances and times between events in space-time) remain constant in time. The term "spherically symmetric" implies that the geometry of a space-time is invariant under rotations about a central point, which means that properties of the space-time remain unchanged if you rotate or twist it around the central point. Also in SS space-time, the space-time metric is dependent on the radial coordinate r.

4.2 Mathematical Formulation

Consider a static SS space-times in the usual coordinates (t,r,θ,ϕ) labeled (y^0,y^1,y^2,y^3) respectively with the line element [58]

$$ds^2 = -e^{u(r)}dt^2 + e^{v(r)}dr^2 + P^2(r)\left[d\theta^2 + \sin^2\theta d\phi^2\right], \tag{4.1}$$

where $u = u(r)$, $v = v(r)$and $P = P(r)$ are unknown functions of r (radial coordinate) alone and are non-zero everywhere. The least number of KVFS which above space-times (4.1) admit are [79]

$$Y_1 = \partial_t,\, Y_2 = \sin\phi\,\partial_\theta + \cos\phi\cot\theta\,\partial_\phi,\, Y_3 = \cos\phi\,\partial_\theta - \sin\phi\cot\theta\,\partial_\phi,\, Y_4 = \partial_\phi. \tag{4.2}$$

Using equation (2.7), the corresponding non-vanishing torsion components are:

$$T^0_{10} = -T^0_{01} = \frac{\dot{u}}{2}, \quad T^2_{12} = -T^2_{21} = \frac{\dot{P}}{P}, \quad T^3_{13} = -T^3_{31} = \frac{\dot{P}}{P}, \tag{4.3}$$

where $\dot{u}$ and $\dot{P}$ represents derivative of the functions u and P with respect to r. Using equations (4.1) and (4.3) in equation (2.16), we get the following ten set of equations:

$$Y^0_{,0} = \alpha, \tag{4.4}$$

$$-2e^{u(r)}Y^0_{,1} + 2e^{v(r)}Y^1_{,0} - \dot{u}e^{u(r)}Y^0 = 0, \tag{4.5}$$

$$-e^{u(r)}Y^0_{,2} + P^2(r)Y^2_{,0} = 0, \tag{4.6}$$

$$-e^{u(r)}Y^{0}_{,3}+P^{2}(r)\sin^{2}\theta Y^{3}_{,0}=0, \tag{4.7}$$

$$\dot{v}Y^{1}+2Y^{1}_{,1}=2\alpha, \tag{4.8}$$

$$e^{v(r)}Y^{1}_{,2}+P^{2}(r)Y^{2}_{,1}+P\dot{P}Y^{2}=0, \tag{4.9}$$

$$e^{v(r)}Y^{1}_{,3}+P^{2}(r)\sin^{2}\theta Y^{3}_{,1}+P\dot{P}sin^{2}\theta Y^{3}=0, \tag{4.10}$$

$$Y^{2}_{,2}=\alpha, \tag{4.11}$$

$$Y^{2}_{,3}+\sin^{2}\theta Y^{3}_{,2}=0, \tag{4.12}$$

$$\cot\theta Y^{2}+Y^{3}_{,3}=\alpha. \tag{4.13}$$

By integrating equations (4.4), (4.8), (4.11) and (4.12) and doing some simplifications on them. The following equations are obtained:

$$\begin{cases} Y^{0} &= \alpha t+M^{1}(r,\theta,\phi), \\ Y^{1} &= \alpha e^{-\frac{v}{2}}\int e^{\frac{v}{2}}dr+e^{-\frac{v}{2}}M^{4}(t,\theta,\phi), \\ Y^{2} &= \alpha\theta+M^{2}(t,r,\phi), \\ Y^{3} &= \cot\theta M^{2}_{\phi}(t,r,\phi)+M^{3}(t,r,\phi), \end{cases} \tag{4.14}$$

where $M^{1}(r,\theta,\phi)$, $M^{2}(t,r,\phi)$, $M^{3}(t,r,\phi)$ and $M^{4}(t,\theta,\phi)$ are functions of integration which are to be determined. In order to find solution for equations (4.4) to (4.13), Each possible form of the metric for static SS space-times will be considered and then addressed individually.

The solution of the EFEs of static SS space-times in $f(T)$ gravity has already been found in [47]. Here we have the following 20 cases followed from [47]:

1. $u=$ constant, $\quad v=v(r), \quad e^{-v}v'r+2e^{-v}-2=0$ implies $v=\ln\left(\frac{1}{1+k_1r^2}\right), \quad P=r$ and $T=\left(\frac{2}{r^2}+2k_1\right)$, where $k_1\in\Re\setminus\{0\}$.

2. $u=u(r)$, $v=$ constant, $2ru''+ru'^2-2u'=0 \implies$ u = $\ln r^4$, $e^v=1$, $P=r$ and $T=10r^{-2}$.

3. $u=u(r),\ v=v(r),\ u=v^{-1},\ r^2\left(v''-v'^2\right)+2(1-e^v)=0$ implies $v=\ln\left(1-\frac{k_1}{r}+\frac{k_2r^2}{3}\right)^{-1}$, $u=\ln\left(1-\frac{k_1}{r}+\frac{k_2r^2}{3}\right)$, $P=r$ and $T=\left(\frac{2}{r^2}+2k_2\right)$, where $k_1,k_2\in\Re\setminus\{0\}$.

4. $u=u(r), v=v(r),\ u=h^{-1},\ e^u\left(\frac{u''}{2}+\frac{u'^2}{2}-\frac{1}{r^2}\right)+\frac{1}{r^2}=0 \implies$ u = $\ln\left(1-\frac{2M}{r}\right)$, $v=\ln\left(1-\frac{2M}{r}\right)^{-1}$, $P=r$ and $T=\left(\frac{2}{r^2}\right)$, where M represents the Arnowitt-Deser-Misner mass.

5. $u=u(r),\ v=v(r),\ u=v^{-1},\ r^2\left(u''+u'^2\right)-2\left(1-e^{-u}\right)=0$ implies $u=\ln\left(1-\frac{\Lambda r^2}{3}\right)$, $v=\ln\left(1-\frac{\Lambda r^2}{3}\right)^{-1}$, $P=r$ and $T=\left(\frac{2}{r^2}-2\Lambda\right)$, where Λ is the cosmological constant.

6. $u=u(r),\ v=v(r),\ ru'+1=0$ which gives $u=\ln\left(\frac{k_1}{r}\right)$, $4e^v-rv'+1=0$ implies $v=\ln\left(\frac{r}{k_2-4r}\right)$, $P=r$ and $T=0$, where $k_1,k_2\in\Re\setminus\{0\}$.

7. $u=u(r),\ v=v(r),\ ru'-2=0$ implies $u=\ln\left(k_1r^2\right)$, $e^v-rv'-2=0 \implies$ v = $\ln\left(\frac{2}{1+2k_2r^2}\right)$, $P=r$ and $T=\frac{3(1+2k_2r^2)}{r^2}$, where $k_1,k_2\in\Re\setminus\{0\}$.

8. $u=u(r), v=v(r), \left(\dfrac{u''}{2}+\dfrac{u'^2}{4}\right)=0 \implies$ u =ln $\left(\frac{k_1r+k_2}{2}\right)^2$, $-k_1v'e^{-v}+2(k_1r+k_2)=0$ which implies $v=\ln\left(\frac{k_1}{k_1k_3-k_1r^2-2k_2r}\right)$, $P=1$ and $T=0$, where $k_1,k_2,k_3\in\Re\setminus\{0\}$.

9. $u=u(r),\ v=v(r),\ u=v^{-1},\ u''+u'^2+2e^{-u}=0$ implies $u=\ln\left(k_2-k_1r-r^2\right)$, $v=\ln\left(k_2-k_1r-r^2\right)^{-1}$, $P=1$ and $T=0$, where $k_1,k_2\in\Re\setminus\{0\}$.

10. $u=\text{constant}=k_1\neq 0,\ v=\text{constant}=k_2\neq 0,\ u=\ln(k_1),\ v=\ln(k_2),\ PP''-P'^2+k_2=0$ implies $P=r\sqrt{k_2}$ and $T=\frac{2}{k_2r^2}$, where $k_1,k_2\in\Re\setminus\{0\}$ with $k_1\neq k_2$.

11. $u=u(r),\ v=v(r), ru''-u'=0$ implies $u=\left(\frac{k_1r^2}{2}+k_2\right)$, $rv'(ru'+1)+2=0 \implies$ v = $\ln\left(\frac{k_3\sqrt{k_1r^2+1}}{r}\right)^2$, $P=r$ and $T=\frac{2}{k_3^2}$, where $k_1,k_2,k_3\in\Re\ (k_1,k_3\neq 0)$.

12. $u=u(r)$, $v=v(r)$, $1+rv'=0$ implies $v=\ln\left(\frac{k_1}{r}\right)$, $r^2u''-rv'-2=0 \implies$ u = $\ln\left(\frac{k_3e^{k_2r}}{r}\right)$, $P=r$ and $T=\frac{2k_2}{k_3}$, where $k_1,k_2,k_3\in\Re(k_1,k_3\neq 0)$.

13. $u=u(r)$, $v=v(r)$, $2+rv'=0$ implies $v=\ln\left(\frac{k_1}{r^2}\right)$, $ru''-u'(1+rv')=0 \implies u=$ $\ln(k_3r^{k_2})$, $P=r$ and $T=\frac{2(k_2+1)}{k_1}$, where $k_1,k_2,k_3\in\Re(k_1,k_2,k_3\neq 0)$.

14. $u=u(r)$, $v=v(r)$, $2+ru'=0$ implies $u=\ln\left(\frac{k_1}{r^2}\right)$, $ru''-v'(1+ru')=0 \implies$ v = $\ln\left(\frac{k_2}{r^2}\right)$, $P=r$ and $T=\frac{-2}{k_2}$, where $k_1,k_2\in\Re$ $(k_1,k_2\neq 0)$.

15. $u=u(r)$, $v=v(r)$, $r^2u''-2=0$ implies $u=\ln\left(\frac{e^{k_1r+k_2}}{r^2}\right)$, $u'+v'(1+ru')=0 \implies$ v = $\ln\left[\frac{k_3(k_1r-1)}{r^2}\right]$, $P=r$ and $T=\frac{2}{k_3}$, where $k_1,k_2,k_3\in\Re$ $(k_1,k_3\neq 0)$.

16. $u=$ constant, $v=v(r)$, $rv'+2=0 \implies v=\ln\left(\frac{k_1}{r^2}\right)$, $e^u=1$, $P=r$, $T=\frac{2}{k_1}$, where $k_1\in\Re\setminus\{0\}$.

17. $u=u(r)$, $v=\text{constant}=k_1\neq 0$, $r^2u''-ru'-2=0 \implies u=\left(\frac{k_2r^2}{2}-\ln r+k_3\right)$, $P=$ r, $T=\frac{2k_2}{e^{k_1}}$, where $k_1,k_2,k_3\in\Re\setminus\{0\}$.

18. $u=u(r)$, $v=v(r)$, $u''=0$ implies $u=(k_1r+k_2)$, $ru'(1+rv')+rv'+2=0 \implies$ v = $\ln\left[\frac{k_3(k_1r+1)}{r^2}\right]$, $P=r$ and $T=\frac{2}{k_3}$, where $k_1,k_2,k_3\in\Re(k_1,k_3\neq 0)$.

19. $u=\text{constant}=k_1\neq 0$, $v=v(r)$, $P''=0 \implies$ P = k_2r+k_3, $v'P+2P'=0 \implies$ v = $\ln\left(\frac{k_4}{(k_2r+k_3)^2}\right)$, $T=\frac{2k_2^2}{k_4}$, where $k_1,k_2,k_3,k_4\in\Re\setminus\{0\}$.

20. $u=\text{constant}=k_1\neq 0, v=\text{constant}=k_2\neq 0, P''-P'^2=0 \implies P=e^{k_3r+k_4}, T=$ $\frac{2k_3^2}{e^{k_2}}$, where

$k_1,k_2,k_3,k_4\in\Re\setminus\{0\}$, with $k_1\neq k_2$.

In this section, we shall discuss only four cases in detail and in the remaining cases, main results are shown. The four detailed cases are given below:

Case 1

In the case 1, we are given $u = \text{constant}$, $v = \ln\left(\frac{1}{1+k_1 r^2}\right)$ and $P = r$ where $k_1 \in \Re \setminus \{0\}$. For this case, the equation of space-time (4.1) will take the form:

$$ds^2 = -dt^2 + \frac{1}{(1+k_1 r^2)} dr^2 + r^2 \left[d\theta^2 + \sin^2\theta d\phi^2\right]. \tag{4.15}$$

For Case 1, the ten set of equations from (4.4) to (4.13) will take the form:

$$Y^0_{,0} = \alpha, \tag{4.16}$$

$$-Y^0_{,1} + \frac{1}{(1+k_1 r^2)} Y^1_{,0} = 0, \tag{4.17}$$

$$-Y^0_{,2} + r^2 Y^2_{,0} = 0, \tag{4.18}$$

$$-Y^0_{,3} + r^2 \sin^2\theta Y^3_{,0} = 0, \tag{4.19}$$

$$\frac{-k_1 r}{(1+k_1 r^2)} Y^1 + Y^1_{,1} = \alpha, \tag{4.20}$$

$$\frac{1}{(1+k_1 r^2)} Y^1_{,2} + r^2 Y^2_{,1} + r Y^2 = 0, \tag{4.21}$$

$$\frac{1}{(1+k_1 r^2)} Y^1_{,3} + r^2 \sin^2\theta Y^3_{,1} + r \sin^2\theta Y^3 = 0, \tag{4.22}$$

$$Y^2_{,2} = \alpha, \tag{4.23}$$

$$Y^2_{,3} + \sin^2\theta Y^3_{,2} = 0, \tag{4.24}$$

$$\cot\theta Y^2 + Y^3_{,3} = \alpha. \tag{4.25}$$

Moreover, the system of equations (4.14) will take the form:

$$\begin{cases} Y^0 &= \alpha t + M^1(r,\theta,\phi), \\ Y^1 &= \alpha\sqrt{1+k_1r^2}\left[\frac{1}{\sqrt{k_1}}sinh^{-1}(\sqrt{k_1}r)\right] + c_1 + \left[\sqrt{1+k_1r^2}\right]M^4(t,\theta,\phi), \\ Y^2 &= \alpha\theta + M^2(t,r,\phi), \\ Y^3 &= \cot\theta\, M^2_\phi(t,r,\phi) + M^3(t,r,\phi). \end{cases} \quad (4.26)$$

where $M^1(r,\theta,\phi)$, $M^2(t,r,\phi)$, $M^3(t,r,\phi)$ and $M^4(t,\theta,\phi)$ are integration functions that need to be determined.

Considering the system of equations (4.26) and equation (4.17) , we get the following form

$$\frac{1}{\sqrt{1+k_1r^2}}M^4_t(t,\theta,\phi) - M^1_r(r,\theta,\phi) = 0. \quad (4.27)$$

By differentiating equation (4.27) with respect to t, we obtain

$$\frac{1}{\sqrt{1+k_1r^2}}M^4_{tt}(t,\theta,\phi) = 0. \quad (4.28)$$

After simplifying an equation (4.28), we obtain

$$M^4(t,\theta,\phi) = tN^1(\theta,\phi) + N^2(\theta,\phi), \quad (4.29)$$

where $N^1(\theta,\phi)$ and $N^2(\theta,\phi)$ are integration functions. By replacing the obtained value into the equation (4.27) we get the following form

$$\frac{1}{\sqrt{1+k_1r^2}}\left[N^1(\theta,\phi)\right] - M^1_r(r,\theta,\phi) = 0. \quad (4.30)$$

After simplifying an equation (4.30) we get

$$M^1(r,\theta,\phi) = \frac{1}{\sqrt{k_1}}\left[\sinh^{-1}(\sqrt{k_1}r)\right]N^3(\theta,\phi) + N^4(\theta,\phi), \quad (4.31)$$

where $N^3(\theta,\phi)$ and $N^4(\theta,\phi)$ are integration functions. By replacing the obtained value in system (4.26), we get the following system of equations:

$$\begin{cases} Y^0 &= \alpha t + \dfrac{1}{\sqrt{k_1}}\left[\sinh^{-1}(\sqrt{k_1}r)\right] N^3(\theta,\phi) + N^4(\theta,\phi), \\ Y^1 &= \alpha\sqrt{1+k_1r^2}\left[\dfrac{1}{\sqrt{k_1}}\sinh^{-1}(\sqrt{k_1}r) + c_1\right] + \sqrt{1+k_1r^2}\left[tN^1(\theta,\phi) + N^2(\theta,\phi)\right], \\ Y^2 &= \alpha\theta + M^2(t,r,\phi), \\ Y^3 &= \cot\theta\, M^2_\phi(t,r,\phi) + M^3(t,r,\phi). \end{cases} \tag{4.32}$$

Considering equation (4.18) and using system of equations (4.32), we get

$$-\frac{1}{\sqrt{k_1}}\sinh^{-1}(\sqrt{k_1}r)N^3_\theta(\theta,\phi) - N^4_\theta(\theta,\phi) + r^2M^2_t(t,r,\phi) = 0. \tag{4.33}$$

Differentiating (4.33) in terms of t, we get $r^2M^2_{tt}(t,r,\phi)=0$, $\Longrightarrow M^2_{tt}(t,r,\phi)=0$, $\Longrightarrow M^2_t(t,r,\phi)=N^5(r,\phi)$ and after some simplifications, the equation will take the form

$$M^2(t,r,\phi) = tN^5(r,\phi) + N^6(r,\phi), \tag{4.34}$$

where $N^5(r,\phi)$ and $N^6(r,\phi)$ are integration functions. Substituting back the above value in (4.33) we get

$$-\frac{1}{\sqrt{k_1}}\sinh^{-1}(\sqrt{k_1}r)N^3_\theta(\theta,\phi) - N^4_\theta(\theta,\phi) + r^2N^5(r,\phi) = 0. \tag{4.35}$$

By firstly differentiating (4.35) with respect to θ, we get $-\frac{1}{\sqrt{k_1}}.\sinh^{-1}(\sqrt{k_1}r).N^3_{\theta\theta}(\theta,\phi) - N^4_{\theta\theta}(\theta,\phi) = 0$, then differentiating with respect to r leads to $-\frac{1}{\sqrt{k_1}}\left[\frac{1}{\sqrt{1+k_1r^2}}\right].\sqrt{k_1}.\left[N^3_{\theta\theta}(\theta,\phi)\right] = 0$, and with the help of some simplifications, we get the value of the function of integration $N^3(\theta,\phi)$ which is,

$$N^3(\theta,\phi) = \theta D^1(\phi) + D^2(\phi), \tag{4.36}$$

where $D^1(\phi)$ and $D^2(\phi)$ are integration functions. Now, using the above calculated values in equation (4.35), we obtain

$$-\frac{1}{\sqrt{k_1}}\sinh^{-1}(\sqrt{k_1}r)\left[D^1(\phi)\right]-N^4_\theta(\theta,\phi)+r^2N^5(r,\phi)=0. \tag{4.37}$$

Differentiating above equation with respect to θ, we get $-N^4_{\theta\theta}(\theta,\phi)=0$ and further simplifications leads to

$$N^4(\theta,\phi)=\theta D^3(\phi)+D^4(\phi), \tag{4.38}$$

where $D^3(\phi)$ and $D^4(\phi)$ are functions of integration. By making use of the above value in equation (4.37), we get

$$N^5(r,\phi)=\frac{1}{r^2\sqrt{k_1}}\sinh^{-1}(\sqrt{k_1}r)\left[D^1(\phi)\right]+\frac{1}{r^2}\left[D^3(\phi)\right]. \tag{4.39}$$

Now, refreshing the system of equation (4.32) with the help of above information

$$\begin{cases} Y^0 &= \alpha t+\dfrac{1}{\sqrt{k_1}}\sinh^{-1}(\sqrt{k_1}r)\left[\theta D^1(\phi)+D^2(\phi)\right]+\theta D^3(\phi)+D^4(\phi),\\ Y^1 &= \alpha\sqrt{1+k_1r^2}.\dfrac{1}{\sqrt{k_1}}\sinh^{-1}(\sqrt{k_1}r)+c_1+\sqrt{1+k_1r^2}\left[t\theta D^1(\phi)+tD^2(\phi)\right]+N^2(\theta,\phi),\\ Y^2 &= \alpha\theta+\left[\dfrac{t}{r^2\sqrt{k_1}}\sinh^{-1}(\sqrt{k_1}r)\right]\left[D^1(\phi)\right]+\dfrac{t}{r^2}\left[D^3(\phi)\right]+N^6(r,\phi),\\ Y^3 &= \cot\theta\left[tN^5_\phi(r,\phi)+N^6_\phi(r,\phi)\right]+M^3(t,r,\phi). \end{cases} \tag{4.40}$$

Considering equation (4.19) and set of equations (4.40) we get

$$\begin{aligned} -\frac{1}{\sqrt{k_1}}\sinh^{-1}(\sqrt{k_1}r).\theta.D^1_\phi(\phi)-\frac{1}{\sqrt{k_1}}\sinh^{-1}(\sqrt{k_1}r)D^2_\phi(\phi)-\theta.D^3_\phi(\phi)-D^4_\phi(\phi)\\ +r^2\sin^2\theta\cot\theta N^5_\phi(r,\phi)+r^2\sin^2\theta M^3_t(t,r,\phi)=0. \end{aligned} \tag{4.41}$$

Differentiating (4.41) with respect to t, we get $r^2\sin^2\theta M^3_{tt}(t,r,\phi)=0$, $\Longrightarrow$ $M^3_{tt}(t,r,\phi)=0$, $\Longrightarrow$ $M^3_t(t,r,\phi)=N^7(r,\phi)$ and after some simplifications, the equation will take the form

$$M^3(t,r,\phi)=tN^7(r,\phi)+N^8(r,\phi), \tag{4.42}$$

where $N^7(r,\phi)$ and $N^8(r,\phi)$ are integration functions. Substituting the value obtained above into the equation (4.41) and after simplifying it, we get

$$-\frac{1}{\sqrt{k_1}}\sinh^{-1}(\sqrt{k_1}r)\left[\theta D^1_\phi(\phi)-D^2_\phi(\phi)\right]-\theta D^3_\phi(\phi)-D^4_\phi(\phi)+\frac{r^2}{2}\left[2\sin\theta\cos\theta\right]N^5_\phi(r,\phi)$$
$$+r^2\sin^2\theta N^7(r,\phi)=0. \tag{4.43}$$

Differentiating equation (4.43) twice with respect to θ , we obtain

$$r^2\left[-2\sin 2\theta\right].N^5_\phi(r,\phi)+r^2\left[2\cos 2\theta\right].N^6(r,\phi)=0. \tag{4.44}$$

The simplifications in equation (4.44) yields $N^6(r,\phi)=0$ and $N^5_\phi(r,\phi)=0 \implies \mathrm{N}^5(r,\phi)=D^5(r)$ where $D^5(r)$ are the integration functions. Refreshing the system of equation results in equation (4.45) given below:

$$\begin{cases} Y^0 &= \alpha t+\frac{1}{\sqrt{k_1}}\sinh^{-1}(\sqrt{k_1}r)\left[\theta D^1(\phi)+D^2(\phi)\right]+\theta D^3(\phi)+D^4(\phi),\\ Y^1 &= \alpha\sqrt{1+k_1r^2}.\frac{1}{\sqrt{k_1}}\sinh^{-1}(\sqrt{k_1}r)+c_1+\sqrt{1+k_1r^2}\left[t\theta D^1(\phi)+tD^2(\phi)\right]+N^2(\theta,\phi),\\ Y^2 &= \alpha\theta+\left[\frac{t}{r^2\sqrt{k_1}}\sinh^{-1}(\sqrt{k_1}r)\right]\left[D^1(\phi)\right]+\frac{t}{r^2}\left[D^3(\phi)\right]+N^6(r,\phi),\\ Y^3 &= \cot\theta\left[N^6_\phi(r,\phi)\right]+N^8(r,\phi). \end{cases} \tag{4.45}$$

Considering equation (4.25) and set of equations (4.45) we get

$$\cot\theta\left[\alpha\theta+\left(\frac{t}{r^2\sqrt{k_1}}.\sinh^{-1}(\sqrt{k_1}r)\right).D^1(\phi)+\frac{t}{r^2}.D^3(\phi)+N^6(r,\phi)\right]$$
$$+\cot\theta\left[N^6_{\phi\phi}(r,\phi)\right]+N^8_\phi(r,\phi)=\alpha. \tag{4.46}$$

Simplification and differentiation of equation (4.46) with respect to t brings out

$$\frac{1}{\sqrt{k_1}}.\sinh^{-1}\sqrt{k_1}r.D^1(\phi)+D^3(\phi)=0. \tag{4.47}$$

Differentiating equation (4.47) with respect to r yields

$$\frac{1}{\sqrt{k_1}}.\frac{1}{\sqrt{1+k_1r^2}}.\sqrt{k_1}.D^1(\phi)=0. \tag{4.48}$$

By simplifying (4.48) we get $D^1(\phi) = 0$. Substitution of the value $D^1(\phi) = 0$ in equation (4.47) gives $D^3(\phi) = 0$. Now, using these values in equation (4.46) we get an equation of form:

$$\cot\theta.\alpha\,\theta + \cot\theta.N^6(r,\phi) + \cot\theta.N^6_{\phi\phi}(r,\phi) + N^8_{\phi}(r,\phi) = \alpha. \tag{4.49}$$

Dividing both sides of the above equation with $\cot\theta$ and then taking its derivative with respect to θ, we derive an equation that takes the form

$$N^8_{\phi}(r,\phi) = \alpha.\tan^2\theta. \tag{4.50}$$

Again differentiating it with θ will give:

$$2\alpha\tan\theta\sec^2\theta = 0. \tag{4.51}$$

$$\implies \alpha = 0. \tag{4.52}$$

In equation (4.52), $\alpha = 0$ implies that we will have Teleparallel Killing Vector Fields for this case and not the Teleparallel Homothetic Vector Fields which we wanted to find. That being the scenario, we end our Case 1 by stating the fact that no such THVFs exists for this case.

Case 2

For the given case, we have $u = \ln\left[1 - \frac{2M}{r}\right]$, $v = \ln\left[1 - \frac{2M}{r}\right]^{-1}$, and $P = r$ where M represents the Arnowitt-Deser-Misner mass. For this case, the equation of space-time (4.1) will take the form:

$$ds^2 = \left[\frac{2M}{r} - 1\right]dt^2 + \left[1 - \frac{2M}{r}\right]^{-1} dr^2 + r^2\left[d\theta^2 + \sin^2\theta\, d\phi^2\right]. \tag{4.53}$$

For Case 2, the ten set of equations from (4.4) to (4.13) will take the following form:

$$Y^0_{,0} = \alpha, \tag{4.54}$$

$$\left(\frac{2M-r}{r}\right)Y^0_{,1}+\left(\frac{r}{r-2M}\right)Y^1_{,0}-\frac{M}{r^2}Y^0=0, \tag{4.55}$$

$$\left(\frac{2M-r}{r}\right)Y^0_{,2}+r^2Y^2_{,0}=0, \tag{4.56}$$

$$\left(\frac{2M-r}{r}\right)Y^0_{,3}+r^2\sin^2\theta Y^3_{,0}=0, \tag{4.57}$$

$$\left(\frac{-M}{r^2-2Mr}\right)Y^1+Y^1_{,1}=\alpha, \tag{4.58}$$

$$\left(\frac{r}{r-2M}\right)Y^1_{,2}+r^2Y^2_{,1}+rY^2=0, \tag{4.59}$$

$$\left(\frac{r}{r-2M}\right)Y^1_{,3}+r^2\sin^2\theta Y^3_{,1}+r\sin^2\theta Y^3=0, \tag{4.60}$$

$$Y^2_{,2}=\alpha, \tag{4.61}$$

$$Y^2_{,3}+\sin^2\theta Y^3_{,2}=0, \tag{4.62}$$

$$\cot\theta Y^2+Y^3_{,3}=\alpha. \tag{4.63}$$

Moreover, the system of equations (4.14) will take the form:

$$\begin{cases} Y^0 &= \alpha t+M^1(r,\theta,\phi), \\ Y^1 &= \alpha\sqrt{\frac{r-2M}{r}}\int\frac{1}{\sqrt{\frac{r-2M}{r}}}dr+\sqrt{\frac{r-2M}{r}}M^4(t,\theta,\phi), \\ Y^2 &= \alpha\theta+M^2(t,r,\phi), \\ Y^3 &= \cot\theta M^2_\phi(t,r,\phi)+M^3(t,r,\phi). \end{cases} \tag{4.64}$$

where $M^1(r,\theta,\phi)$, $M^2(t,r,\phi)$, $M^3(t,r,\phi)$ and $M^4(t,\theta,\phi)$ are integration functions that need to be determined.

Considering the system of equations (4.64) and equation (4.56) , we get the following form

$$\left[\frac{2M-r}{r}\right]\left[M^1_\theta(r,\theta,\phi)\right]+r^2\left[M^2_t(t,r,\phi)\right]=0. \tag{4.65}$$

By differentiating equation (4.65) with respect to t, we get

$$r^2 M^2_{tt}(t,r,\phi) = 0. \implies M^2_t(t,r,\phi) = N^1(r,\phi). \tag{4.66}$$

After simplification, we obtain

$$M^2(t,r,\phi) = tN^1(r,\phi) + N^2(r,\phi), \tag{4.67}$$

where $N^1(r,\phi)$ and $N^2(r,\phi)$ are integration functions. Placing the aforementioned value back into the equation (4.65) we get the following form

$$\left[\frac{2M-r}{r}\right]\left[M^1_\theta(r,\theta,\phi)\right] + r^2 N^1(r,\phi) = 0. \tag{4.68}$$

Differentiating an equation (4.68) with respect to θ we get

$$\left[\frac{2M-r}{r}\right] M^1_{\theta\theta}(r,\theta,\phi) = 0. \tag{4.69}$$

Simplifying the above equation gives the value of function of integration $M^1(r,\theta,\phi)$

$$M^1(r,\theta,\phi) = \theta N^3(r,\phi) + N^4(r,\phi), \tag{4.70}$$

where $N^3(r,\phi)$ and $N^4(r,\phi)$ are integration functions. Inserting the values from the above step in system (4.64), we obtain the following system of equations:

$$\begin{cases} Y^0 &= \alpha t + \theta N^3(r,\phi) + N^4(r,\phi), \\ Y^1 &= \alpha\sqrt{\frac{r-2M}{r}} \int \frac{1}{\sqrt{\frac{r-2M}{r}}} dr + \sqrt{\frac{r-2M}{r}} M^4(t,\theta,\phi), \\ Y^2 &= \alpha\theta + tN^1(r,\phi) + N^2(r,\phi), \\ Y^3 &= \cot\theta\left[tN^1_\phi(r,\phi) + N^2_\phi(r,\phi)\right] + M^3(t,r,\phi). \end{cases} \tag{4.71}$$

Considering equation (4.57) and using system of equations (4.71), we get

$$\left[\frac{2M-r}{r}\right] \cdot \left[\theta N^3_\phi(r,\phi) + N^4_\phi(r,\phi)\right] + r^2\sin^2\theta\left[\cot\theta . N^1_\phi(r,\phi) + M^3_t(t,r,\phi)\right] = 0. \tag{4.72}$$

Differentiating (4.72) with respect to t, we get $r^2.\sin^2\theta.M^3_{tt}(t,r,\phi)=0, \implies M^3_t(t,r,\phi)=N^5(r,\phi)$, and after some simplifications, the equation will take the form

$$M^3(t,r,\phi)=tN^5(r,\phi)+N^6(r,\phi), \tag{4.73}$$

where $N^5(r,\phi)$ and $N^6(r,\phi)$ are integration functions. Now, substituting the above calculated value into the equation (4.72) and doing some simplifications in it, we obtain

$$\left[\frac{2M-r}{r}\right].\theta.N^3_\phi(r,\phi)+\left[\frac{2M-r}{r}\right].N^4_\phi(r,\phi)+\frac{r^2}{2}.\sin 2\theta.N^1_\phi(r,\phi)+r^2.\sin^2\theta.N^5(r,\phi)=0. \tag{4.74}$$

Differentiating equation (4.74) twice with respect to θ, we get an equation of form

$$2r^2\left[-\sin 2\theta.N^1_\phi(r,\phi)+\cos 2\theta.N^5(r,\phi)\right]=0. \tag{4.75}$$

Further simplification of equation (4.75) gives $N^1_\phi(r,\phi)=0 \implies \mathrm{N}^1(r,\phi)=D^1(r)$, which also leads to $N^5(r,\phi)=0$. Using these values in equation (4.74) we get:

$$\left[\frac{2M-r}{r}\right].\theta.N^3_\phi(r,\phi)+\left[\frac{2M-r}{r}\right].N^4_\phi(r,\phi)=0. \tag{4.76}$$

Some simplifications of (4.76) and taking its derivative with respect to θ yields the value of $N^3(r,\phi)=D^2(r)$ and $N^4_\phi(r,\phi)=0 \implies \mathrm{N}^4(r,\phi)=D^3(r)$. Now, using all the values we find till now to update our system (4.71).

$$\begin{cases} Y^0 &= \alpha t+\theta D^2(r)+D^3(r), \\ Y^1 &= \alpha\sqrt{\dfrac{r-2M}{r}}\displaystyle\int\sqrt{\frac{r}{r-2M}}dr+\sqrt{\frac{r-2M}{r}}M^4(t,\theta,\phi), \\ Y^2 &= \alpha\theta+tD^1(r)+N^2(r,\phi), \\ Y^3 &= \cot\theta N^2_\phi(r,\phi)+N^6(r,\phi). \end{cases} \tag{4.77}$$

Considering equation (4.59) and set of equations (4.77) we get

$$\sqrt{\frac{r}{r-2M}}.M^4_\theta(t,\theta,\phi)+r^2\left[tD^1_r(r)+N^2_r(r,\phi)\right]+r\left[\alpha\theta+tD^1(r)+N^2(r,\phi)\right]=0. \tag{4.78}$$

Differentiating (4.78) with respect to θ gives

$$\sqrt{\frac{r}{r-2M}}M^4_{\theta\theta}(t,\theta,\phi)+r\alpha=0. \tag{4.79}$$

Again differentiating the above equation with respect to θ lead to $M^4_{\theta\theta\theta}(t,\theta,\phi)=0$, $\Longrightarrow$ $M^4_{\theta\theta}(t,\theta,\phi)=N^7(t,\phi)$, $\Longrightarrow$ $M^4_{\theta}(t,\theta,\phi)=\theta N^7(t,\phi)+N^8(t,\phi)$ which after simplification gives

$$M^4(t,\theta,\phi)=\frac{\theta^2}{2}.N^7(t,\phi)+\theta N^8(t,\phi)+N^9(t,\phi), \tag{4.80}$$

where $N^7(t,\phi)$ and $N^8(t,\phi)$ and $N^9(t,\phi)$ are integration functions. By replacing the above value in (4.79), we get

$$\sqrt{\frac{r}{r-2M}}.N^7(t,\phi)+r\alpha=0. \tag{4.81}$$

Differentiation of equation (4.81) with respect to t gives $N^7_t(t,\phi)=0$, $\Longrightarrow \mathrm{N}^7(t,\phi)=D^4(\phi)$, where $D^4(\phi)$ is an integration function. Using the aforementioned value in equation (4.81), we get

$$\sqrt{\frac{r}{r-2M}}.D^4(\phi)+r\alpha=0. \tag{4.82}$$

Differentiation of equation (4.82) with respect to ϕ gives $D^4_{\phi}(\phi)=0 \Longrightarrow \mathrm{D}^4(\phi)=c_1$, where c_1 is the constant of integration. Putting the calculated value in equation (4.82), we get

$$\sqrt{\frac{r}{r-2M}}.c_1+r\alpha=0. \tag{4.83}$$

Differentiation of equation (4.83) with respect to r gives

$$\frac{1}{2\sqrt{\frac{r}{r-2M}}}.c_1.\frac{d}{dr}\left[\frac{r}{r-2M}\right]+\alpha=0. \tag{4.84}$$

Simplification of equation (4.84) lead us to equation (4.85) given below

$$-Mc_1\left[\frac{1}{\sqrt{r}}(r-2M)^{-\frac{3}{2}}\right]+\alpha=0. \tag{4.85}$$

Differentiation of equation (4.85) with respect to r gives an equation of the form

$$-Mc_1\left[-\frac{1}{2r^{\frac{3}{2}}}.(r-2M)^{-\frac{3}{2}}+(r^{-\frac{1}{2}}).\left(\frac{-3}{2(r-2M)^{\frac{5}{2}}}\right)\right]=0. \tag{4.86}$$

Simplifying an equation (4.86), we get a form

$$\frac{Mc_1(2r-M)}{r^{\frac{3}{2}}(r-2M)^{\frac{5}{2}}}=0, \tag{4.87}$$

$$\implies Mc_1(2r-M)=0. \tag{4.88}$$

By differentiating equation (4.88) with respect to r, we get

$$2Mc_1=0. \tag{4.89}$$

In equation (4.89), it has been already given that M represents the Arnowitt-Deser-Misner mass. And mass cannot be zero. Hence, $c_1=0$ and equation (4.89) will take the form

$$c_1=0. \tag{4.90}$$

Putting value of c_1 in equation (4.85) gives

$$\alpha=0. \tag{4.91}$$

The equation above, with $\alpha=0$, indicates that we will obtain Teleparallel Killing Vector Fields instead of the Teleparallel Homothetic Vector Fields. We conclude Case 2 by affirming that no such THVFs are found in this case.

Case 3

In this case, we have $u=\ln(k_1),\ v=\ln(k_2)$, and $P=r\sqrt{k_2}$ where $k_1,k_2\in\Re\setminus\{0\}$ with $k_1\neq k_2$. For this case, the equation of space-time (4.1) will take the following form:

$$ds^2=-k_1dt^2+k_2dr^2+r^2k_2\left[d\theta^2+\sin^2\theta\,d\phi^2\right]. \tag{4.92}$$

For Case 3, the ten set of equations from (4.4) to (4.13) will take the following form:

$$Y^0_{,0} = \alpha, \tag{4.93}$$

$$-k_1 Y^0_{,1} + k_2 Y^1_{,0} = 0, \tag{4.94}$$

$$-k_1 Y^0_{,2} + r^2 k_2 Y^2_{,0} = 0, \tag{4.95}$$

$$-k_1 Y^0_{,3} + r^2 k_2 \sin^2\theta Y^3_{,0} = 0, \tag{4.96}$$

$$Y^1_{,1} = \alpha, \tag{4.97}$$

$$Y^1_{,2} + r^2 Y^2_{,1} + r Y^2 = 0, \tag{4.98}$$

$$Y^1_{,3} + r^2 \sin^2\theta Y^3_{,1} + r\sin^2\theta Y^3 = 0, \tag{4.99}$$

$$Y^2_{,2} = \alpha, \tag{4.100}$$

$$Y^2_{,3} + \sin^2\theta Y^3_{,2} = 0, \tag{4.101}$$

$$\cot\theta Y^2 + Y^3_{,3} = \alpha. \tag{4.102}$$

As well, the system of equations (4.14) will take the form:

$$\begin{cases} Y^0 &= \alpha t + M^1(r,\theta,\phi), \\ Y^1 &= \alpha r + c_1 + \frac{1}{\sqrt{k_2}}.M^4(t,\theta,\phi), \\ Y^2 &= \alpha\theta + M^2(t,r,\phi), \\ Y^3 &= \cot\theta\, M^2_\phi(t,r,\phi) + M^3(t,r,\phi). \end{cases} \tag{4.103}$$

where $M^1(r,\theta,\phi)$, $M^2(t,r,\phi)$, $M^3(t,r,\phi)$ and $M^4(t,\theta,\phi)$ are the functions arising from integration that are yet to be found.

Considering the system of equations (4.103) and equation (4.95) , we get the following form

$$-k_1\left[M^1_\theta(r,\theta,\phi)\right] + r^2 k_2\left[M^2_t(t,r,\phi)\right] = 0. \tag{4.104}$$

By differentiating equation (4.104) with respect to θ, we get

$$-k_1 M^1_{\theta\theta}(r,\theta,\phi) = 0. \tag{4.105}$$

After simplifying an equation (4.105), we obtain

$$M^1(r,\theta,\phi) = \theta N^1(r,\phi) + N^2(r,\phi), \tag{4.106}$$

where $N^1(r,\phi)$ and $N^2(r,\phi)$ are integration functions. Substituting the value of $M^1(r,\theta,\phi)$ in equation (4.104) we get the following form

$$-k_1\left[N^1(r,\phi)\right] + r^2 k_2\left[M^2_t(t,r,\phi)\right] = 0. \tag{4.107}$$

After differentiating an equation (4.107) with respect to t we get

$$k_2.M^2_{tt}(t,r,\phi) = 0. \tag{4.108}$$

Simplification of the above equation (4.108) gives with respect to t gives $M^2_t(t,r,\phi) = N^3(r,\phi)$ which lead to

$$M^2(t,r,\phi) = tN^3(r,\phi) + N^4(r,\phi), \tag{4.109}$$

where $N^3(r,\phi)$ and $N^4(r,\phi)$ are integration functions. Using the above information in system (4.103), we derive the following set of equations:

$$\begin{cases} Y^0 &= \alpha t + \theta N^1(r,\phi) + N^2(r,\phi), \\ Y^1 &= \alpha r + \dfrac{1}{\sqrt{k_2}} M^4(t,\theta,\phi) + c_1, \\ Y^2 &= \alpha\theta + tN^3(r,\phi) + N^4(r,\phi), \\ Y^3 &= \cot\theta\left[tN^3_\phi(r,\phi) + N^4_\phi(r,\phi)\right] + M^3(t,r,\phi). \end{cases} \tag{4.110}$$

Considering equation (4.96) and using system of equations (4.110), we get

$$-k_1\left[\theta N^1_\phi(r,\phi) + N^2_\phi(r,\phi)\right] + r^2 k_2 \sin^2\theta\left[\cot\theta\, N^3_\phi(r,\phi) + M^3_t(t,r,\phi)\right] = 0. \tag{4.111}$$

Differentiating (4.111) with respect to t we get

$$r^2 k_2 \sin^2\theta M^3_{tt}(t,r,\phi) = 0. \tag{4.112}$$

After some simplifications, the equation (4.112) will take the form

$$M^3(t,r,\phi) = tN^5(r,\phi) + N^6(r,\phi), \tag{4.113}$$

where $N^5(r,\phi)$ and $N^6(r,\phi)$ are integration functions. By replacing the above value into the equation (4.111) and doing some simplifications in it, we obtain

$$-k_1\left[\theta N^1_\phi(r,\phi) + N^2_\phi(r,\phi)\right] + r^2 k_2 \sin^2\theta\left[\cot\theta\, N^3_\phi(r,\phi) + N^5(r,\phi)\right] = 0. \tag{4.114}$$

After doing some simplifications in equation (4.114) we get

$$-k_1.\theta.N^1_\phi(r,\phi) - k_1.N^2_\phi(r,\phi) + \frac{r^2}{2}.k_2.2\sin\theta\cos\theta.N^3_\phi(r,\phi) + r^2.k_2.\sin^2\theta.N^5(r,\phi) = 0. \tag{4.115}$$

Differentiating (4.115) with respect to θ we acquire

$$-k_1.N^1_\phi(r,\phi) + r^2.k_2.\cos 2\theta.N^3_\phi(r,\phi) + r^2.k_2.\sin 2\theta.N^5(r,\phi) = 0. \tag{4.116}$$

Again differentiating the above equation with respect to θ we get $N^3_\phi(r,\phi) = 0 \implies \mathrm{N}^3(r,\phi) = D^1(r)$ which also leads to $N^5(r,\phi) = 0$. Using the value of $N^5(r,\phi) = 0$ we also get $N^1_\phi(r,\phi) = 0 \implies \mathrm{N}^1(r,\phi) = D^2(r) \implies \mathrm{N}^2(r,\phi) = D^3(r)$. Now, using all the values we find till now to update our system (4.110).

$$\begin{cases} Y^0 = \alpha t + \theta D^2(r) + D^3(r), \\ Y^1 = \alpha r + \frac{1}{\sqrt{k_2}} M^4(t,\theta,\phi) + c_1, \\ Y^2 = \alpha\theta + tD^1(r) + N^4(r,\phi), \\ Y^3 = \cot\theta\left[N^4_\phi(r,\phi)\right] + N^6(r,\phi). \end{cases} \tag{4.117}$$

Considering equation (4.94) and set of equations (4.117) we get

$$-k_1\left[\theta D_r^2(r)+D_r^3(r)\right]+\sqrt{k_2}.M_t^4\left(t,\theta,\phi\right)=0. \tag{4.118}$$

Differentiation of (4.118) with respect to t gives $M_{tt}^4(t,\theta,\phi)=0,\implies M_t^4(t,\theta,\phi)=N^7(\theta,\phi)$ which lead to the value of $M^4(t,\theta,\phi)$ given below

$$M^4(t,\theta,\phi)=tN^7(\theta,\phi)+N^8(\theta,\phi), \tag{4.119}$$

where $N^7(\theta,\phi)$ and $N^8(\theta,\phi)$ are integration functions. At this point system of equations will take the form:

$$\begin{cases} Y^0 &= \alpha t+\theta D^2(r)+D^3(r),\\ Y^1 &= \alpha r+\dfrac{1}{\sqrt{k_2}}\left[tN^7(\theta,\phi)+N^8(\theta,\phi)\right]+c_1,\\ Y^2 &= \alpha\,\theta+tD^1(r)+N^4(r,\phi),\\ Y^3 &= \cot\theta\left[N_\phi^4(r,\phi)\right]+N^6(r,\phi). \end{cases} \tag{4.120}$$

Using the equation (4.98) and system of equations (4.119) we get

$$\frac{1}{\sqrt{k_2}}\left[tN_\theta^7(\theta,\phi)\right]+\frac{1}{\sqrt{k_2}}\left[N_\theta^8(\theta,\phi)\right]+r^2\left[tD_r^1(r)+N_r^4(r,\phi)\right]+r\left[\alpha\,\theta+tD^1(r)s+N^4(r,\phi)\right]=0. \tag{4.121}$$

Differentiation of equation (4.120) with respect to θ gives

$$\frac{1}{\sqrt{k_2}}\left[tN_{\theta\theta}^7(\theta,\phi)\right]+\frac{1}{\sqrt{k_2}}\left[N_{\theta\theta}^8(\theta,\phi)\right]+r\alpha=0. \tag{4.122}$$

Differentiation of equation (4.122) with respect to r will turn the above equation into form

$$\alpha=0. \tag{4.123}$$

In the aforementioned equation, $\alpha=0$ suggests that no THVFs exists for this case.

Case 4

In this case, given that $u = \ln(k_1)$, $v = \ln(k_2)$, and $P = e^{k_3 r + k_4}$ where $k_1, k_2, k_3, k_4 \in \Re$ ($k_3 \neq 0$) with $k_1 \neq k_2$. For this case, the equation of space-time (4.1) will take the form:

$$ds^2 = -k_1\, dt^2 + k_2\, dr^2 + e^{2(k_3 r + k_4)} \left[d\theta^2 + \sin^2\theta\, d\phi^2\right]. \tag{4.124}$$

For Case 5, the ten set of equations from (4.4) to (4.13) will take the following form:

$$Y^0_{,0} = \alpha, \tag{4.125}$$

$$-k_1\, Y^0_{,1} + k_2\, Y^1_{,0} = 0, \tag{4.126}$$

$$-k_1\, Y^0_{,2} + e^{2[k_3 r + k_4]}\, Y^2_{,0} = 0, \tag{4.127}$$

$$-k_1\, Y^0_{,3} + e^{2[k_3 r + k_4]} \sin^2\theta\, Y^3_{,0} = 0, \tag{4.128}$$

$$Y^1_{,1} = \alpha, \tag{4.129}$$

$$k_2\, Y^1_{,2} + e^{2[k_3 r + k_4]}\, Y^2_{,1} + k_3\, e^{2[k_3 r + k_4]}\, Y^2 = 0, \tag{4.130}$$

$$k_2\, Y^1_{,3} + e^{2[k_3 r + k_4]} \sin^2\theta\, Y^3_{,1} + k_3\, e^{2[k_3 r + k_4]} \sin^2\theta Y^3 = 0, \tag{4.131}$$

$$Y^2_{,2} = \alpha, \tag{4.132}$$

$$Y^2_{,3} + \sin^2\theta Y^3_{,2} = 0, \tag{4.133}$$

$$\cot\theta Y^2 + Y^3_{,3} = \alpha. \tag{4.134}$$

Moreover, the system of equations (4.14) will take the form:

$$\begin{cases} Y^0 &= \alpha t + M^1\left(r,\theta,\phi\right), \\ Y^1 &= \alpha r + c_1 + \frac{1}{\sqrt{k_2}} M^4\left(t,\theta,\phi\right), \\ Y^2 &= \alpha\,\theta + M^2\left(t,r,\phi\right), \\ Y^3 &= \cot\theta\, M^2_{\phi}\left(t,r,\phi\right) + M^3\left(t,r,\phi\right). \end{cases} \tag{4.135}$$

where $M^1\left(r,\theta,\phi\right)$, $M^2\left(t,r,\phi\right)$, $M^3\left(t,r,\phi\right)$ and $M^4\left(t,\theta,\phi\right)$ are functions which derived from the integration and are yet to be determined.

Considering the system of equations (4.134) and equation (4.126) , we get the following form

$$-k_1\,\left[M^1_r(r,\theta,\phi)\right] + \sqrt{k_2}\left[M^4_t(t,\theta,\phi)\right] = 0. \tag{4.136}$$

By differentiating equation (4.135) with respect to t, we get

$$\sqrt{k_2}\,M^4_{tt}(t,\theta,\phi) = 0. \tag{4.137}$$

After simplifying an equation (4.137), we obtain

$$M^4(t,\theta,\phi) = tN^1(\theta,\phi) + N^2(\theta,\phi), \tag{4.138}$$

where $N^1(\theta,\phi)$ and $N^2(\theta,\phi)$ are integration functions. By replacing the above value into the equation (4.136) we get the following form

$$-k_1\,\left[M^1_r(r,\theta,\phi)\right] + \sqrt{k_2}\,N^1(\theta,\phi) = 0. \tag{4.139}$$

Differentiating an equation (4.139) with respect to r we get

$$-k_1\,M^1_{rr}(r,\theta,\phi). \tag{4.140}$$

Simplifying the above equation gives the value of function of integration $M^1(r,\theta,\phi)$ which is

$$M^1(r,\theta,\phi) = rN^3(\theta,\phi) + N^4(\theta,\phi), \tag{4.141}$$

where $N^3(\theta,\phi)$ and $N^4(\theta,\phi)$ are integration functions. Placing the aforementioned value back in system (4.135), we get the following system of equations:

$$\begin{cases} Y^0 &= \alpha t + rN^3(\theta,\phi) + N^4(\theta,\phi), \\ Y^1 &= \alpha r + c_1 + \dfrac{1}{\sqrt{k_2}}\left[tN^1(\theta,\phi) + N^2(\theta,\phi)\right], \\ Y^2 &= \alpha\,\theta + M^2(t,r,\phi), \\ Y^3 &= \cot\theta\, M^2_{\phi}(t,r,\phi) + M^3(t,r,\phi). \end{cases} \tag{4.142}$$

Considering equation (4.127) and using system of equations (4.142), we get

$$-k_1\left[rN^3_{\theta}(\theta,\phi) + N^4_{\theta}(\theta,\phi)\right] + e^{2[k_3 r + k_4]}\left[M^2_t(t,r,\phi)\right] = 0. \tag{4.143}$$

Differentiating (4.143) with respect to t, we get $e^{2[k_3 r+k_4]}\left[M^2_{tt}(t,r,\phi)\right] = 0, \implies M^2_{tt}(t,r,\phi) = 0, \implies M^2_t(t,r,\phi) = N^5(r,\phi)$, and this leads to

$$M^2(t,r,\phi) = tN^5(r,\phi) + N^6(r,\phi), \tag{4.144}$$

where $N^5(r,\phi)$ and $N^6(r,\phi)$ are integration functions. Substituting the above value back in (4.143) and doing some simplifications in it, we obtain

$$-k_1\left[rN^3_{\theta}(\theta,\phi)\right] - k_1\left[N^4_{\theta}(\theta,\phi)\right] + e^{2[k_3 r+k_4]}\left[N^5(r,\phi)\right] = 0. \tag{4.145}$$

Differentiating equation (4.145) with respect to θ, we get an equation of form

$$-k_1\left[rN^3_{\theta\theta}(\theta,\phi)\right] - k_1\left[N^4_{\theta\theta}(\theta,\phi)\right] = 0. \tag{4.146}$$

Now, differentiating the above equation (4.145) with respect to r, we get $-k_1 N^3_{\theta\theta}(\theta,\phi) = 0$, $\implies \mathrm{N}^3_{\theta\theta}(\theta,\phi) = 0$ which further gives,

$$N^3(\theta,\phi) = \theta D^1(\phi) + D^2(\phi), \tag{4.147}$$

where $D^1(\phi)$ and $D^2(\phi)$ are functions of integrations. Now, using the value of $N^3(\theta,\phi)$ in equation (4.146) we get the value of $N^4(\theta,\phi)$ which is

$$N^4(\theta,\phi) = \theta D^3(\phi) + D^4(\phi), \tag{4.148}$$

where $D^3(\phi)$ and $D^4(\phi)$ are integration functions. Now, taking use of all the values we found till now to update our system (4.142).

$$\begin{cases} Y^0 &= \alpha t + r\left[\theta D^1(\phi) + D^2(\phi)\right] + \theta D^3(\phi) + D^4(\phi), \\ Y^1 &= \alpha r + c_1 + \dfrac{1}{\sqrt{k_2}}\left[tN^1(\theta,\phi) + N^2(\theta,\phi)\right], \\ Y^2 &= \alpha\,\theta + tN^5(r,\phi) + N^6(r,\phi), \\ Y^3 &= \cot\theta\,\left[tN^5_\phi(r,\phi) + N^6_\phi(r,\phi)\right] + M^3(t,r,\phi). \end{cases} \tag{4.149}$$

Considering equation (4.128) and set of equations (4.149) we get

$$-k_1\left[r\theta D^1_\phi(\phi) + rD^2_\phi(\phi) + \theta D^3_\phi(\phi) + D^4_\phi(\phi)\right] + e^{2[k_3 r + k_4]}\sin^2\theta \\ \left[\cot\theta\, N^5_\phi(r,\phi) + M^3_t(t,r,\phi)\right] = 0. \tag{4.150}$$

Differentiation of equation (4.151) with respect to t gives

$$M^3_{tt}(t,r,\phi) = 0. \tag{4.151}$$

Simplification of above equation gives,

$$M^3(t,r,\phi) = tN^7(r,\phi) + N^8(r,\phi), \tag{4.152}$$

where $N^7(r,\phi)$ and $N^8(r,\phi)$ are integration functions. Substituting the above value back in (4.151) implies

$$-k_1 r\theta D^1_\phi(\phi) - k_1 rD^2_\phi(\phi) - k_1\theta D^3_\phi(\phi) - k_1 D^4_\phi(\phi) + \frac{1}{2}.e^{2[k_3 r + k_4]}.\sin^2\theta\,\cot\theta\,.N^5_\phi(r,\phi) \\ + e^{2[k_3 r + k_4]}.\sin^2(\theta).N^7(r,\phi) = 0. \tag{4.153}$$

By differentiating the above equation with respect to θ, we get

$$-k_1 r\theta D^1_\phi(\phi) - k_1 D^3_\phi(\phi) + \frac{1}{2}.2\cos 2\theta\,.e^{2[k_3 r + k_4]}\,.N^5_\phi(r,\phi) + 2\sin\theta\cos\theta\,.e^{2[k_3 r + k_4]}\,.N^7(r,\phi) = 0. \tag{4.154}$$

Again differentiation of equation (4.154) with respect to θ gives

$$2e^{2[k_3 r + k_4]}\left[-\sin 2\theta\, N^5_\phi(r,\phi) + \cos 2\theta\, N^7(r,\phi)\right] = 0. \tag{4.155}$$

By differentiating once again (4.155) with respect to θ we get $N^5(r,\phi) = D^5(r)$ and from equation (4.155) we get $N^7(r,\phi) = 0.$, where $D^5(r)$ is an integration function. Putting the above values in equation (4.154), we get

$$-k_1\left[rD^1_\phi(\phi) + D^3_\phi(\phi)\right] = 0. \tag{4.156}$$

Differentiation of equation (4.156) with respect to r gives $D^1_\phi(\phi) = 0 \implies \mathrm{D}^1(\phi) = c_1$, and $D^3_\phi(\phi) = 0 \implies \mathrm{D}^3(\phi) = c_2$, where c_1 and c_2 are the constants of integration. Putting the above values in equation (4.153), we get

$$-k_1\left[r\theta D^2_\phi(\phi) + D^4_\phi(\phi)\right] = 0. \tag{4.157}$$

Differentiation of equation (4.157) with respect to r gives $D^2_\phi(\phi) = 0 \implies \mathrm{D}^2(\phi) = c_3$, and $D^4_\phi(\phi) = 0 \implies \mathrm{D}^4(\phi) = c_4$, where c_3 and c_4 are the constants of integration. Using the found information we update of system of equation which is

$$\begin{cases} Y^0 &= \alpha t + r\theta c_1 + r c_3 + \theta c_2 + c_4, \\ Y^1 &= \alpha r + c_1 + \frac{1}{\sqrt{k_2}}\left[tN^1(\theta,\phi) + N^2(\theta,\phi)\right], \\ Y^2 &= \alpha\theta + tD^5(r) + N^6(r,\phi), \\ Y^3 &= \cot\theta\, N^6_\phi(r,\phi) + N^8(r,\phi). \end{cases} \tag{4.158}$$

Using an equation (4.130) and considering the system of equation (4.158) we get

$$k_2 \left[\frac{1}{\sqrt{k_2}} t N^1_\theta(\theta,\phi) + \frac{1}{\sqrt{k_2}} N^2_\theta(\theta,\phi) \right] + e^{2[k_3 r + k_4]} \left[t D^5_r(r) + N^6_r(r,\phi) \right] + k_3 e^{2[k_3 r + k_4]} \left[\alpha\theta + t D^5(r) + N^6(r,\phi) \right] = 0. \quad (4.159)$$

Differentiation of equation (4.159) with respect to θ gives an equation of the form

$$\sqrt{k_2}\, t N^1_{\theta\theta}(\theta,\phi) + \sqrt{k_2} N^2_{\theta\theta}(\theta,\phi) + k_3\, e^{2[k_3 r + k_4]} \alpha = 0. \quad (4.160)$$

Differentiation of above equation with respect to t gives $\sqrt{k_2} N^1_{\theta\theta}(\theta,\phi) = 0$ implies $N^1_{\theta\theta}(\theta,\phi) = 0$ and $N^1_\theta(\theta,\phi) = D^6(\phi)$ implies $N^1(\theta,\phi) = \theta D^6(\phi) + D^7(\phi)$, where $D^6(\phi)$ and $D^7(\phi)$ are the functions of integration. Using the value of $N^1_{\theta\theta}(\theta,\phi)$ in equation (4.160) we get

$$\sqrt{k_2} N^2_{\theta\theta}(\theta,\phi) + k_3\, e^{2[k_3 r + k_4]} \alpha = 0. \quad (4.161)$$

Differentiation of equation (4.161) with respect to r lead to

$$2 {k_3}^2 . e^{2[k_3 r + k_4]} . \alpha = 0, \quad (4.162)$$

$$\Longrightarrow \alpha = 0. \quad (4.163)$$

In the above equation, $\alpha = 0$ shows that Teleparallel Killing Vector Fields will arise in this situation. Therefore, we end our Case 4 by acknowledging that no THVFs exists for this case.

Table 2: The classification of Static Spherically symmetric space-times using THVFs.

Case No	Metric components	THVFs	Teleparallel Homothetic Factor	Description
2.	$u = lnr^4,\ v = \text{constant and } P = r$.	$Y_1,\ Y_2,\ Y_3,\ Y_4$.	$\alpha = 0$.	TKVFs
3.	$u = ln\left(1-\frac{k_1}{r}+\frac{k_2r^2}{3}\right),\ v = ln\left(1-\frac{k_1}{r}+\frac{k_2r^2}{3}\right)^{-1}$ and $P = r$.	$Y_1,\ Y_2,\ Y_3,\ Y_4$.	$\alpha = 0$.	TKVFs
5.	$u = ln\left(1-\frac{\Lambda r^2}{3}\right),\ v = ln\left(1-\frac{\Lambda r^2}{3}\right)^{-1}$ and $P = r$.	$Y_1,\ Y_2,\ Y_3,\ Y_4$.	$\alpha = 0$.	TKVFs
6.	$u = ln\left(\frac{k_1}{r}\right),\ v = ln\left(\frac{r}{k_2-4r}\right)$ and $P = r$.	$Y_1,\ Y_2,\ Y_3,\ Y_4$.	$\alpha = 0$.	TKVFs
7.	$u = ln\left(k_1r^2\right),\ v = ln\left(\frac{2}{1+2k_2r^2}\right)$ and $P = r$.	$Y_1,\ Y_2,\ Y_3,\ Y_4$.	$\alpha = 0$.	TKVFs
8.	$u = ln\left(\frac{k_1r+k_2}{2}\right)^2,\ v = ln\left(\frac{k_1}{k_1k_3-k_1r^2-2k_2r}\right)$ and $P = 1$.	$Y_1,\ Y_2,\ Y_3,\ Y_4$.	$\alpha = 0$.	TKVFs
9.	$u = ln\left(k_2-k_1r-r^2\right),\ v = \ln\left(k_2-k_1r-r^2\right)^{-1}$ and $P = 1$.	$Y_1,\ Y_2,\ Y_3,\ Y_4$.	$\alpha = 0$.	TKVFs
11.	$u = \left(\frac{k_1r^2}{2}+k_2\right),\ v = ln\left(\frac{k_3\sqrt{k_1r^2+1}}{r}\right)^2$ and $P = r$.	$Y_1,\ Y_2,\ Y_3,\ Y_4$.	$\alpha = 0$.	TKVFs
12.	$u = \ln\left(\frac{k_3e^{k_2r}}{r}\right),\ v = \ln\left(\frac{k_1}{r}\right)$ and $P = r$.	$Y_1,\ Y_2,\ Y_3,\ Y_4$.	$\alpha = 0$.	TKVFs
13.	$u = \ln(k_3r^{k_2}),\ v = \ln\left(\frac{k_1}{r^2}\right)$ and $P = r$.	$Y_1,\ Y_2,\ Y_3,\ Y_4$.	$\alpha = 0$.	TKVFs
14.	$u = \text{constant},\ v = \ln\left(\frac{k_1}{r^2}\right)$ and $P = r$.	$Y_1,\ Y_2,\ Y_3,\ Y_4$.	$\alpha = 0$.	TKVFs
15.	$u = \ln\left(\frac{e^{k_1r+k_2}}{r^2}\right),\ v = \ln\left[\frac{k_3(k_1r-1)}{r^2}\right]$ and $P = r$.	$Y_1,\ Y_2,\ Y_3,\ Y_4$.	$\alpha = 0$.	TKVFs
16.	$u = \text{constant},\ v = \ln\left(\frac{k_1}{r^2}\right)$ and $P = r$.	$Y_1,\ Y_2,\ Y_3,\ Y_4$.	$\alpha = 0$.	TKVFs
17.	$u = \left(\frac{k_2r^2}{2}-\ln r+k_3\right),\ v = \text{constant} = k_1 \neq 0$ and $P = r$.	$Y_1,\ Y_2,\ Y_3,\ Y_4$.	$\alpha = 0$.	TKVFs
18.	$u = (k_1r+k_2),\ v = \ln\left[\frac{k_3(k_1r+1)}{r^2}\right]$ and $P = r$.	$Y_1,\ Y_2,\ Y_3,\ Y_4$.	$\alpha = 0$.	TKVFs
19.	$u = \text{constant} = k_1 \neq 0,\ v = \ln\left[\frac{k_4}{(k_2r+k_3)^2}\right]$ and $P = (k_2r+k_3)$.	$Y_1,\ Y_2,\ Y_3,\ Y_4$.	$\alpha = 0$.	TKVFs

The Y_1, Y_2, Y_3 and Y_4 of Table (2) represents the KVFs mentioned in equation (4.2). This table shows that no THVFs exists for Static Spherically symmetric space-times in $f(T)$ gravity.

4.3 Physical Parameters of Solutions

By plugging in the values of the metric components and torsion scalar T in equation (3.5) to (3.7), we get the values of dynamical parameters like Energy Density (ρ) and pressure components for the aforementioned cases listed below:

Table 3: Physical parameters of obtained solutions

Case No	Metric components	Density (ρ)	Pressure Components
1.	$u=\text{constant},\ v=\ln\left(\frac{1}{1+k_1r^2}\right)$ and $P=r$.	$\rho=\frac{1}{16\pi}\left(d_2-6k_1d_1\right)$.	$p=-\frac{1}{16\pi}\left(d_2-2k_1d_1\right)$.
2.	$u=lnr^4,\ v=\text{constant},$ and $P=r$.	$\rho=\frac{d_2}{16\pi}$.	$p=\frac{1}{16\pi}\left[\frac{8d_1}{r^2}-d_2\right]$.
3.	$u=ln\left(1-\frac{k_1}{r}+\frac{k_2r^2}{3}\right),\ v=ln\left(1-\frac{k_1}{r}+\frac{k_2r^2}{3}\right)^{-1}$ and $P=r$.	$\rho=-\frac{1}{16\pi}\left(2k_2d_1-d_2\right)$	$p=\frac{1}{16\pi}\left(2k_2d_1-d_2\right)$.
4.	$u=\ln\left(1-\frac{2M}{r}\right),\ v=\ln\left(1-\frac{2M}{r}\right)^{-1}$ and $P=r$	$\rho=\frac{d_2}{16\pi}$.	$p=-\frac{d_2}{16\pi}$.
5.	$u=ln\left(1-\frac{\Lambda r^2}{3}\right),\ v=ln\left(1-\frac{\Lambda r^2}{3}\right)^{-1}$ and $P=r$.	$\rho=\frac{1}{16\pi}\left(2\Lambda d_1+d_2\right)$.	$p=-\frac{1}{16\pi}\left(2\Lambda d_1+d_2\right)$.
6.	$u=ln\left(\frac{k_1}{r}\right),\ v=ln\left(\frac{r}{k_2-4r}\right)$ and $P=r$.	$\rho=\frac{d_2}{16\pi}$.	$p=-\frac{d_2}{16\pi}$.
7.	$u=ln\left(k_1r^2\right),\ v=ln\left(\frac{2}{1+2k_2r^2}\right)$ and $P=r$.	$\rho=\frac{1}{16\pi}\left[\left(\frac{1-6k_2r^2}{r^2}\right)d_1+d_2\right]$	$p=\frac{1}{16\pi}\left[\left(\frac{1+6k_2r^2}{r^2}\right)d_1-d_2\right]$.
8.	$u=ln\left(\frac{k_1r+k_2}{2}\right)^2,\ v=ln\left(\frac{k_1}{k_1k_3-k_1r^2-2k_2r}\right)$ and $P=1$.	$\rho=\frac{d_2}{16\pi}$.	$p=-\frac{d_2}{16\pi}$.
9.	$u=ln\left(k_2-k_1r-r^2\right),\ v=\ln\left(k_2-k_1r-r^2\right)^{-1}$ and $P=1$.	$\rho=\frac{d_2}{16\pi}$.	$p=-\frac{d_2}{16\pi}$
10.	$u=\ln(k_1),\ v=\ln(k_2)\ and\ P=r\sqrt{k_2}$.	$\rho=\frac{d_2}{16\pi}$.	$p=-\frac{d_2}{16\pi}$.
11.	u=$\left(\frac{k_1r^2}{2}+k_2\right),\ v=ln\left(\frac{k_3\sqrt{k_1r^2+1}}{r}\right)^2$ and $P=r$.	$\rho=\frac{1}{16\pi}\left[\frac{2}{k_3^2}d_1+d_2\right]$.	$p=-\frac{1}{16\pi}\left[\frac{2}{k_3^2}d_1+d_2\right]$.
12.	$u=\ln\left(\frac{k_3e^{k_2r}}{r}\right),\ v=\ln\left(\frac{k_1}{r}\right)$ and $P=r$.	$\rho=\frac{1}{16\pi}\left[\frac{2k_2}{k_3}d_1+d_2\right]$.	$p=-\frac{1}{16\pi}\left[\frac{2k_2}{k_3}d_1+d_2\right]$.
13.	$u=\ln\left[k_3r^{k_2}\right],\ v=\ln\left[\frac{k_1}{r^2}\right]$, and $P=r$.	$\rho=\frac{1}{16\pi}\left[\frac{2(k_2+1)}{k_1}d_1+d_2\right]$.	$p=-\frac{1}{16\pi}\left[\frac{2(k_2+1)}{k_1}d_1+d_2\right]$.
14.	$u=\text{constant},\ v=\ln\left(\frac{k_1}{r^2}\right)$ and $P=r$.	$\rho=\frac{1}{16\pi}\left[\frac{-2}{k_2}d_1+d_2\right]$.	$p=-\frac{1}{16\pi}\left[\frac{-2}{k_2}d_1+d_2\right]$.
15.	$u=\ln\left(\frac{e^{k_1r+k_2}}{r^2}\right),\ v=\ln\left[\frac{k_3(k_1r-1)}{r^2}\right]$ and $P=r$.	$\rho=\frac{1}{16\pi}\left[\frac{2}{k_3}d_1+d_2\right]$.	$p=-\frac{1}{16\pi}\left[\frac{-2}{k_3}d_1+d_2\right]$.

Case No	Metric components	Density (ρ)	Pressure Components
16.	$u = \text{constant},\ v = \ln\left(\frac{k_1}{r^2}\right)$ and $P = r.$	$\rho = \frac{1}{16\pi}\left[\frac{2}{k_1}d_1 + d_2\right].$	$p = -\frac{1}{16\pi}\left[\frac{2}{k_1}d_1 + d_2\right].$
17.	$u = \left(\frac{k_2 r^2}{2} - \ln r + k_3\right),\ v = \text{constant} = k_1 \neq 0$ and $P = r.$	$\rho = \frac{1}{16\pi}\left[\frac{2k_2}{e^{k_1}}d_1 + d_2\right].$	$p = -\frac{1}{16\pi}\left[\frac{2k_2}{e^{k_1}}d_1 + d_2\right].$
18.	$u = (k_1 r + k_2),\ v = \ln\left[\frac{k_3(k_1 r+1)}{r^2}\right]$ and $P = r.$	$\rho = \frac{1}{16\pi}\left[\frac{2}{k_3}d_1 + d_2\right].$	$p = -\frac{1}{16\pi}\left[\frac{2}{k_3}d_1 + d_2\right].$
19.	$u = \text{constant} = k_1 \neq 0,\ v = \ln\left[\frac{k_4}{(k_2 r+k_3)^2}\right]$ and $P = (k_2 r + k_3).$	$\rho = \frac{1}{16\pi}\left[\frac{2k_2^2}{k_4}d_1 + d_2\right].$	$p = -\frac{1}{16\pi}\left[\frac{2k_2^2}{k_4}d_1 + d_2\right].$
20.	$u = \ln(k_1),\ v = \ln(k_2)$ and $P = e^{k_3 r + k_4}.$	$\rho = \frac{1}{16\pi}\left[\frac{2k_3^2}{e^{k_2}}d_1 + d_2\right].$	$p = -\frac{1}{16\pi}\left[\frac{2k_3^2}{e^{k_2}}d_1 + d_2\right].$

4.4 Classification of Solutions via Energy Conditions

Considering the physical aspects, the energy bounds for the current solutions provided in the Cases 1 to 20 are displayed in the table provided below:

Table 4: Classification of solutions via energy conditions.

Cases	WEC: $\rho \geq 0, 0 \leq p+\rho$.	SEC: $0 \leq p+\rho, 0 \leq \rho+3p$.	DEC: $\lvert p\rvert \leq \rho, 0 \leq \rho$.	NEC: $0 \leq p+\rho$.
1.	Satisfied if $d_1 \leq 0, d_2 \geq 6k_1d_1$.	Satisfied if $d_1 \leq 0, d_2 \leq 0$.	Satisfied if $d_2 \geq 6k_1d_1, d_2 \geq \frac{8k_1d_1}{2}$.	Satisfied if $d_1 \leq 0$.
2.	Satisfied if $d_1 \geq 0, d_2 \geq 0$.	Satisfied if $d_1 \geq 0, d_2 \leq \frac{12d_1}{r^2}$.	Satisfied if $d_2 \geq 0, d_1 \leq \frac{r^2d_2}{4}$.	Satisfied if $d_1 \geq 0$.
3.	Satisfied if $d_2 \geq 2k_2d_1$.	Satisfied if $d_2 \leq 2k_2d_1, d_1 \leq 0$.	Satisfied if $d_2 \geq 2k_2d_1$.	Satisfied.
4.	Satisfied if $d_2 \geq 0$.	Satisfied if $d_2 \leq 0$.	Satisfied if $d_2 \geq 0$.	Satisfied.
5.	Satisfied if $d_2 \geq 2\Lambda d_1$.	Satisfied if $d_2 \leq 2\Lambda d_1$.	Satisfied if $d_2 \geq 2\Lambda d_1$.	Satisfied.
6.	Satisfied if $d_2 \geq 0$.	Satisfied if $d_2 \leq 0$.	Satisfied if $d_2 \geq 0$.	Satisfied.
7.	Satisfied if $d_1 \geq 0, d_2 \geq \left[\frac{1-6k_2r^2}{r^2}\right] d_1$.	Satisfied if $d_1 \geq 0,\ d_2 \leq \left[\frac{1+6k_2r^2}{r^2}\right] d_1$.	Satisfied if $d_2 \geq \left[\frac{1-6k_2r^2}{r^2}\right] d_1, d_2 \geq 6k_2d_1$.	Satisfied if $d_1 \geq 0$.
8.	Satisfied if $d_2 \geq 0$.	Satisfied if $d_2 \leq 0$.	Satisfied if $d_2 \geq 0$.	Satisfied.
9.	Satisfied if $d_2 \geq 0$.	Satisfied if $d_2 \leq 0$.	Satisfied if $d_2 \geq 0$.	Satisfied.
10.	Satisfied if $d_2 \geq 0$.	Satisfied if $d_2 \leq 0$.	Satisfied if $d_2 \geq 0$.	Satisfied.
11.	Satisfied if $d_2 \geq \frac{-2}{k_3^2}d_1$.	Satisfied if $d_2 \leq \frac{-2}{k_3^2}d_1$.	$d_2 \geq \frac{-2}{k_3^2}d_1$.	Satisfied.
12.	Satisfied if $d_2 \geq \frac{-2k_2}{k_3}d_1$.	Satisfied if $d_2 \leq \frac{-2k_2}{k_3}d_1$.	Satisfied if $d_2 \geq \frac{-2k_2}{k_3}d_1$.	Satisfied.
13.	Satisfied if $d_2 \geq \frac{-2(k_2+1)}{k_1}d_1$.	Satisfied if $d_2 \leq \frac{-2(k_2+1)}{k_1}d_1$.	Satisfied if $d_2 \geq \frac{-2(k_2+1)}{k_1}d_1$.	Satisfied.
14.	Satisfied if $d_2 \geq \frac{2}{k_2}d_1$.	Satisfied if $d_2 \leq \frac{2}{k_2}d_1$.	Satisfied if $d_2 \geq \frac{2}{k_2}d_1$.	Satisfied.
15.	Satisfied if $d_2 \geq -\frac{2}{k_3}d_1$.	Satisfied if $d_2 \leq -\frac{2}{k_3}d_1$.	Satisfied if $d_2 \geq -\frac{2}{k_3}d_1$.	Satisfied.

Cases	WEC: $\rho \geq 0, 0 \leq p+\rho$.	SEC: $0 \leq p+\rho, 0 \leq \rho+3p$.	DEC: $\lvert p \rvert \leq \rho, 0 \leq \rho$.	NEC: $0 \leq p+\rho$.
16.	Satisfied if $d_2 \geq -\frac{2}{k_1}d_1$.	Satisfied if $d_2 \leq -\frac{2}{k_1}d_1$.	Satisfied if $d_2 \geq -\frac{2}{k_1}d_1$.	Satisfied.
17.	Satisfied if $d_2 \geq -\frac{2k_2}{e^{k_1}}d_1$.	Satisfied if $d_2 \leq -\frac{2k_2}{e^{k_1}}d_1$.	Satisfied if $d_2 \geq -\frac{2k_2}{e^{k_1}}d_1$.	Satisfied.
18.	Satisfied if $d_2 \geq -\frac{2}{k_3}d_1$.	Satisfied if $d_2 \leq -\frac{2}{k_3}d_1$.	Satisfied if $d_2 \geq -\frac{2}{k_3}d_1$.	Satisfied.
19.	Satisfied if $d_2 \geq -\frac{2k_2^2}{k_4}d_1$.	Satisfied if $d_2 \leq -\frac{2k_2^2}{k_4}d_1$.	Satisfied if $d_2 \geq -\frac{2k_2^2}{k_4}d_1$.	Satisfied.
20.	Satisfied if $d_2 \geq -\frac{2k_3^2}{e^{k_2}}d_1$.	Satisfied if $d_2 \leq -\frac{2k_3^2}{e^{k_2}}d_1$.	Satisfied if $d_2 \geq -\frac{2k_3^2}{e^{k_2}}d_1$.	Satisfied.

4.5 Results and Discussion

By examining the exact solutions of the EFEs through symmetry properties, one can easily analyze the geometry and uncover the underlying physical nature of the space-time structure. Undoubtedly, symmetries create necessary constraints that function as a guide to simplify the physical problems being addressed. In this chapter, we classify static SS space-times via THVFs in the scope of $f(T)$ gravity. For this investigation, we observe the EFEs given in paper [47], in which direct integration technique has been used to categorize the resulting solutions. The classification revealed a total of 20 cases. We solved all those 20 cases individually for the sake of classifying the static SS space-times as per their THVFs in $f(T)$ gravity. By utilizing space-time (4.1) and non-zero torsion components (4.3), we obtain ten set of equations (i.e. from (4.4) to (4.13)) and after doing simplifications on some of them we get equation (4.14). To determine the solutions of equations (4.4) to (4.13), we consider every single case from those 20 cases and solve each possibility in turn. Out of the 20 cases, 4 cases have been discussed briefly in section (4.2). In all the cases, we obtain that THVFs are the TKVFs. The results of all cases is shown in Table (2). Additionally, the solutions are also classified via Energy Conditions. The density ρ and the pressure components of all cases have been found and shown in Table (3). And the detailed classification of solutions via energy conditions is in Table (4).

Chapter 5

Conclusion

This research investigates the Teleparallel Homothetic Vector Fields of static SS space-times within the framework of $f(T)$ gravity. The primary objective of this work was to extend our understanding of extended theories of gravity, specifically the TEGR and its extensions, such as $f(T)$ gravitational theory, by analyzing the solutions of the field equations under specific conditions. Different cases are solved to see whether the THVFs exists for the particular space-time or not. The cases yields that for the given space-time, the TKVFs exist and no such THVFs are found. The absence of THVFs for the given space-time indicates that the space-time does not exhibit any scaling symmetry and our results will coincide with those derived from General Relativity. Dynamical parameters like energy density ρ and pressure p are also evaluated. For every static SS model corresponding to the cases (1)-(10), the ρ and p have different values. While for the cases (11)-(20), the relationship between the non-zero constants ED and pressure is given by $\rho = -p$. The energy density and p for cases (1),(3),(4),(5),(6),(8),(9) and (10)-(20) are linked as $\rho = -p$, which demonstrates that the dominant universes resemble those filled with dark energy or energy density associated with a vacuum or cosmological constant.

In cases (2) and (7), the ED and pressure are positive if the constants d_1 and d_2 are found to be positive. Instances like these generate an attractive gravitational effect on the expansion of our universe, while dark energy exerts negative pressure is often understood as an antigravity effect. Furthermore, the solutions are classified based on energy conditions.

5.1 Future Work

We can extend this research by finding the THVFs in other modified theories of gravity. Likewise, we can explore the THVFs for static SS space-times in $f(R)$ theory of gravity. Besides, one can extend the study of THVFs to different symmetry classes, such as cylinderically symmetric space-times or plane symmetric space-times. As gravity is influenced by matter and energy, thus by introducing specific matter models (i.e. scalar fields, electromagnetic fields or perfect fluids), we can study how these matter sources affect the structure of THVFs. The interaction of HVFs with black hole solutions can also be investigated. These extensions will provide new insights and solutions to current cosmological and astrophysical problems.

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