Hankel Determinant of Logarithmic coefficient for a new class of q-Starlike functions associated with Lune

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Candidate of Master of Science in Mathematics at the National University of Modern Languages do hereby declare that the thesis Hankel Determinant of Logarithmic coefficient for a new class of q-Starlike functions associated with Lune submitted by me in partial fulfillment of MSMA degree, is my original work and has not been submitted or published earlier. I also solemnly declare that it shall not, in the future, be submitted by me for obtaining any other degree from this or any other university or institution. I also understand that if evidence of plagiarism is found in my thesis/dissertation at any stage, even after the award of a degree, the work may be canceled and the degree revoked.

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ABSTRACT

Title: Hankel Determinant of Logarithmic coefficient for a new class of q-Starlike functions associated with Lune

This thesis aims to introduce and characterize novel subclasses of univalent functions within the open unit disk. The utilization of q-calculus will be employed to establish the q-extension of starlike and convex functions of logrithmic coefficient for Starlike functions associated with lune. Additionally, we will investigate notable properties, including bounds on the coefficients of analytic functions, and the Fekete–Szegő inequality. Furthermore, we will explore Second Hankel Determinants for functions belonging to these newly defined classes. It will be shown that newly obtained results are advanced as compare to the already derived results by numerous researchers in the field of Geometric Function Theory. The special cases of newly derived results will be presented in the form of corollaries.

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LIST OF SYMBOLS

Ω	-	Open unit disk
$\mathscr{A}$	-	Class of Analytic functions
$\mathscr{S}$	-	Class of Univalent functions
$\mathscr{P}$	-	Class of Carathéodory functions
S^{*}	-	Class of Starlike functions
C	-	Class of Convex functions
$\prec$	-	Subordination symbol
S^{*}	-	Class of Starlike functions with respect to origin
C	-	Class of Convex functions with respect to origin
$S^{*}_{\leftmoon}(q)$	-	Class of $\hat{q}$-Starlike functions Subordinated with $\hat{q}$-lune function
$C_{\leftmoon}(q)$	-	Class of $\hat{q}$-Convex functions Subordinated with $\hat{q}$-lune function
$\mathscr{D}_{\hat{q}}$	-	q-Derivative operator symbol
$\mathscr{H}$	-	Hankel Determinant symbol

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In the name of Allah, the Most Gracious and Most Merciful, I begin this acknowledgment with the verse, " You Alone we Worship; You Alone we ask for Help." I am deeply grateful to Allah, the Most Wise, for His countless blessings that have guided me through this academic journey. I am reminded of the Hadith of the Prophet Muhammad (peace be upon him) who said, "Seek knowledge from the cradle to the grave."

I would like to express my deep appreciation to my family for their unwavering support, to my teachers for their invaluable guidance, for it is through your collective encouragement and inspiration that I stand here today, humbly acknowledging the blessings and opportunities Allah has bestowed upon me.

May Allah accept our efforts and guide us on the path of righteousness and wisdom.

DEDICATION

I dedicate this thesis to my parents, whose boundless love and sacrifices, and to my teachers, whose guidance and wisdom, have illuminated my path and made this achievement possible.

CHAPTER 1

INTRODUCTION AND LITERATURE REVIEW

The main proponents of Geometric Function Theory, especially Cauchy, Riemann, and Weierstrass, are frequently contrasted in modern works on the history of mathematics, as seen in [1]. According to Remmert, these three mathematicians had a significant influence on the development of complex analysis and, more broadly, the current theory of complex functions during the 1800s. The things that set their contributions apart are the different ways in which they have approached the problem of explaining the notion of holomorphic functions. They have all approached this basic area of mathematical theory from different angles and with different techniques.

Riemann's geometric viewpoint enabled him to investigate holomorphic functions by mapping them across various domains in the complex plane departing from this viewpoint. Lastly, Remmert narrate that the theory of locally evolved homomorphic functions into convergent power series is known as Weierstrass theory of complex functions.

At the age of 25, Cauchy, the first of the three function theory pioneers previously mentioned, made contributions to complex theory and went on to publish over 200 works in this field; for more information, see [2]. The Cauchy Integral Theorem and the idea of the definite integral with complex bounds were both developed by him.He established the Cauchy Integral Formulas and investigated the expansion of an analytic function in power series in this study. Fellow French mathematicians began to assist Cauchy in 1840 as he laid the groundwork for function theory. Liouville developed a number of theorems pertaining to elliptic functions, and Puiseux examined these functions' behaviour at their branch points in his seminal work on algebraic functions; for further information, see [3]. For the first time, Briot and Bouquet methodically collected the cumulative research of these mathematicians in a number of articles; for more information, see [4, 5, 6].

A few decades later, in Göttingen, was where the second major actor Riemann made significant contributions to the advancement of function theory. For more information, refer to his well-known dissertation and In his esteemed works on Abelian functions [7], Riemann starting in 1851, followed Cauchy's example and defined an analytic function using the Cauchy-Riemann differential equations.Riemann investigated what minimal criteria must be met in order to define such a function. He developed the well-known Riemann Mapping Theorem as a result of this research; for more information, see [8].Riemann surfaces and the Dirichlet Principle are important components of Riemann's methodology.Gauss is the mathematician who had the biggest impact on Riemann's development of his function theory. Gauss had previously made substantial contributions to the theory of conformal mapping and understood fundamental ideas in function theory, such as complex integration and the Cauchy Integral Theorem. It is among the most active fields of research today, As a consequence of this discovery open the unit disc. $\mathbb{E} = \{|\hat{y}| < 1; \hat{y} \in \mathbb{C}\}$ can be utilised as a domain rather than a complex random domain.This theorem holds great importance as it forms the basis of the science of geometric functions.

The basis for Weierstrass's later function theory was established in three works. Weierstrass is the third founder of function theory. An important part of his later advances is hinted at in these pieces. Weierstrass is noteworthy for having independently demonstrated the Laurent Theorem before Laurent's discovery.The notion of identical convergence, analytic functions via power series, and the formulation of the Cauchy Estimates are some of the other significant contributions made in these publications. The roots of Weierstrassian function theory have been explored by a number of writers; see [9].

While there are many classes and subclasses under Geometric Function Theory [10], establishing coefficient boundaries is a key area of study. Functions belonging to various subfamilies of the normalised analytic functions of class $\mathscr{A}$ are classified within this framework. The Bieberbach theorem, which was first proposed by German mathematician Ludwig Bieberbach's 1916 work is significant in this perspective. The sole class of functions with a single value that this theorem applies to is $\mathscr{S}$. His second coefficient, $\hat{\alpha}_2$, was found for functions in the class $\mathscr{S}$, which is a class of univalent functions.

Bieberbach's conjecture, which greatly advanced the science and was often pursued in attempts to prove, originated from this theorem. For many mathematicians, the Bieberbach conjecture has long been a hurdle, despite its simplicity in formulation. Instead of giving up on solving it, a lot of people have created other approaches that are now commonly used in the industry.

Despite the many attempts made by mathematicians to substantiate this theory, it has proven to be an elusive problem. $|\check{c}_3| \leq 3$ was proved by mathematician Karl Loewner in 1923; see [11]. Others were able to verify this outcome for the general situation thanks to this proof. For over thirty years, there was no advancement until 1955, when Gangadharan *et al.* [12] verified the Bieberbach conjecture for $m = 4$, i.e. $|\check{c}_4| \leq 4$.

In 1985, as mentioned in[13], mathematician Louis de Branges was able to successfully verify the Bieber-bach conjecture's general form and he created an extensive, intricate, and precise justification for this hypothesis. The accomplishments of De Branges were emphasised, and several new research problems and approaches were proposed, during an international symposium that took place at Purdue in March 1985.

The Fekete–Szego inequality, which is associated Regarding the coefficients of a polynomial with particular properties, the Bieberbach conjecture is mostly used in complex analysis.Fekete and Szego established this inequality in 1933; further information may be found in see [14]. Numerous significant ramifications and applications arise from the Fekete-Szegő inequality in complicated analysis. It can be used, for instance, to obtain constraints on the coefficients of functions in the starlike or convex function classes, two subclasses of analytic functions.It's crucial to keep in mind that the Fekete-Szegő inequality is sharp, meaning that it equals itself for some functions.These types of functions are taken into consideration important for comprehending the behaviour of analytic functions in the unit disk.

Based just on the forms of their images to the main and other geometrical qualities, Geometric Function Theory is divided into several classes and even smaller sub-classes. The class of geometric function theory represented by $\mathscr{A}$ is the category of normalized analytical function. Within this classification are analytical functions in disc $\mathbb{E}$ that are normalised using the following axioms: $\tau(0)=0, \tau'(0)=1$. When τ and υ belong to the class $\mathscr{A}$ of functions written as, $\tau \prec \upsilon$, if $\tau(\hat{y}) = \upsilon(\mathbb{E}(\hat{y}))$ where the analytic function $\mathbb{E}(\hat{y})$ is located within the open unit disc meeting both criteria of $\mathbb{E}(0)=0$ with $|\mathbb{E}(\hat{y})|<1$, for more detail look[15] .
Class $\mathscr{S}$ function contains the univalent Functions that are analytical within a unit disk that is open $\mathbb{E}$ and normalied by the constraints $\tau(0)=0, \tau'(0)=1$.A popular the Koebe function

serves as a representative example within the class .The equivalence relation between function τ and function υ is as follows: Function τ is expressed as being subordinate to function υ is $\tau \prec \upsilon \iff \tau(0) = \upsilon(0), \tau(\mathbb{E}) \subseteq \upsilon(\mathbb{E})$. If υ is univalent in disk $\mathbb{E}$ and τ is analytical in disk $\mathbb{E}$. Koebe (1907) explored univalent functions, which included univalent analytic functions in the disk $\mathbb{E}$, refer to [16] for details.

The family $\mathscr{S}$ function has four major subdivisions: the set of convex functions, represented by C, and the set of starlike functions, represented by S^*. Further information can be found in reference [17]. Beginning with the attempts to prove the Bieberbach conjecture, this classification was initiated. Through a connection known as the Alexander relation [18], which may be expressed as follows, Alexander connected two categories: S^* denoting starlike univalent functions, while in 1915, C stood for convex univalent functions. Given $\tau \in \mathscr{A}$, $\tau \in C \iff \hat{y}\tau' \in S^*$ occurs.

Subordination was used by Ma and Minda [19] to define the class of starlike functions, and they looked at classes of starlike functions including,

$$S^* = \left\{ \tau \in \mathscr{A} : \frac{\hat{y}\tau'(\hat{y})}{\tau(\hat{y})} \prec \delta(\hat{y}), \quad \hat{y} \in \mathbb{E} \right\}.$$

Additionally, the category of convex functions known as,

$$C = \left\{ \tau \in \mathscr{A} : \frac{(\hat{y}\tau'(\hat{y}))'}{\tau'(\hat{y})} \prec \delta(\hat{y}), \quad \hat{y} \in \mathbb{E} \right\},$$

where $\delta(\hat{y})$ is

$$\delta(\hat{y}) = \left(-1 + \frac{2}{1 - \hat{y}} \right),$$

satisfy Schwa function in disk $\mathbb{E}$.

Several scholars have made important contributions in this area by identifying several useful properties.These features included the radius of starlikness, radius of convexity, coefficient estimations, sufficiency characteristics, and distortion limitations.

H is the category of analytical functions in the format of $\{ \ \hat{y} \in \mathbb{C} : |\hat{y}| < 1 \ \}$ in $\mathbb{E}$.

$$f(\hat{y}) = \sum_{n=1}^{\infty} a_n \hat{y}^n \qquad \hat{y} \in \ \mathbb{E}. \tag{1.1}$$

In such case, H constitutes a topological vector space that is locally convex and has the consistent topology of convergence over small subsets of D. Let the category of functions be represented by $\mathscr{A}$.Where $f'(0) \ = \ 1$ and $f(0) \ = \ 0$, $f \in H$ Stated differently, the functions f of the type

$$f(\hat{y}) = \hat{y} + \sum_{n=2}^{\infty} a_n \hat{y}^n \quad \hat{y} \in \ \mathbb{E}. \tag{1.2}$$

Let $\mathscr{S}$ denote the subset comprising every univalent function in $\mathscr{A}$. We consult the classic texts for a univalent function theory in general.

$$F_f(\hat{y}) := \log\frac{f(\hat{y})}{\hat{y}} = 2\sum_{n=1}^{\infty}\gamma_n(f)\hat{y}^n, \ \hat{y}\in\mathbb{E}, \ log\ 1 := 0, \tag{1.3}$$

$f \in \mathscr{S}$ is connected to a logarithmic function. The terms "logarithmic coefficients of f" refer to the values $\gamma_n := \gamma_n(f)$. The logarithmic coefficients are known to be important in Milin's conjecture (refer to [20]; also consult [21], page 155] for additional details), the unexpected revelation is that only two logarithmic coefficients are necessary for the class $\mathscr{S}$.

γ_1 and γ_2 have sharp estimates known.

$$|\gamma_1|\leq 1, \qquad |\gamma_2|\leq\frac{1}{2}+\frac{1}{e}=0.635\dots$$

Many writers have recently thought about assessing the magnitude regarding logarithmic coefficients for $f \in \mathscr{S}$ and different subcategories we cite the following articles: [22, 23, 24] and their references.

Definition 1.1 Assume two analytic functions, f and g. If there is a self-map w under the condition w(0)= 0 such that $f(\hat{y}) = g(w(\hat{y}))$, then f is subordinated by g and expressed as $f(\hat{y}) \prec g(\hat{y})$. Furthermore, $f(D) \subseteq g(D)$ given that $f(0) = g(0)$ and g is univalent.

The class $S^*_{\Cap}$ developed by Raina and Sokol [25], is provided by

$$S^*_{\Cap} = \left\{ f\in \mathrm{S} : \left| \left(\frac{\hat{y}f'(\hat{y})}{f(\hat{y})}\right)^2 - 1 \right| \leq 2\left|\frac{\hat{y}f'(\hat{y})}{f(\hat{y})}\right|, \quad \hat{y}\in\mathbb{E} \right\}.$$

Geometrically, a function $f \in S^*_{\Cap}$ is characteried in the following manner: for every $\hat{y} \in \mathbb{E}$, the region enclosed by the lune is included in the ratio $\frac{\hat{y}f'(\hat{y})}{f(\hat{y})}$. The relation $\{w \in C : |w^2 - 1| \leq 2|w|\}$ provides the answer. Utiliing the concept of subordination, $S^*_{\Cap}$ is delineated as

$$S^*_{\Cap} = \left\{ f\in \mathrm{S} : \frac{\hat{y}f'(\hat{y})}{f(\hat{y})} \prec \hat{y} + \sqrt{1+\hat{y}^2} = q(\hat{y}), \quad \hat{y}\in\mathbb{E} \right\},$$

where $q(0) = 1$ is the result of selecting the square root's branch. Class $C_{\Cap}$ is a convex function whose characterisation is

$$C_{\Cap} = \left\{ f\in \mathrm{S} : 1 + \frac{\hat{y}f''(\hat{y})}{f'(\hat{y})} \prec q(\hat{y}), \quad \hat{y}\in \mathrm{D} \right\},$$

A number of scholars have done a great deal of research on the class $S^{*}_{\leftmoon}$. Raina and Sokol [26, 27] explored the coefficient estimates within the class$S^{*}_{\leftmoon}$, where as Gandhi and Ravi [28] looked into the class's radius-related problems. Sharma and colleagues [29] examined a few differential subordinations associated with the class $S^{*}_{\leftmoon}$. Sufficient conditions and integral representation are suplied by Raina et al. [30] for the functions in the class $S^{*}_{\leftmoon}$. A supposition concerning this class's coefficients were put up in a recent contribution by Cho et al. [31].
The evaluation of the limits of Hankel determinants, consisting of the coefficients of analytic function of characteristics in $\mathbb{E}$ type (2), has received a lot of attention in the realm of geometric function theory Hankel matrices and determinants hold significance across various mathematical domains and find extensive applications [32]. The objective of this study is to provide the precise limit for the logarithmic coefficients that make up the second Hankel determinant. We begin by defining Hankel determinants for the scenario in which $f \in \mathscr{A}$.

For $q,\ n \in N$, the following is the definition of the Hankel determinant of Taylor coefficients for functions $f \in \mathscr{A}$ is $\mathscr{H}_{n,q}(f)$ as seen below

$$\mathscr{H}_{n,q}(f) = \begin{vmatrix} a_n & a_{n+1} \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} \cdots & a_{n+q} \\ \vdots & \vdots \ddots & \vdots \\ a_{n+q-1} & a_{n+q} \quad \cdots & a_{n+2(q-1)} \end{vmatrix}.$$

The logarithmic coefficients of $f \in \mathscr{S}$ are the elements of a Hankel determinant that Kowalcyk and Lecko [33] recently suggested, recognizing the wide application of these coefficients. The expression for this determinant is as follows:

$$\mathscr{H}_{n,q}(F_f/2) = \begin{vmatrix} \gamma_n & \gamma_{n+1} \cdots & \gamma_{n+q-1} \\ \gamma_{n+1} & \gamma_{n+2} \cdots & \gamma_{n+q} \\ \vdots & \vdots \ddots & \vdots \\ \gamma_{n+q-1} & \gamma_{n+q} \quad \cdots & \gamma_{n+2(q-1)} \end{vmatrix}.$$

Extensive research has been conducted on Hankel determinants for various function classes such as starlike, convex, and others, resulting in the identification of their precise constraints (refer to [33, 34, 35, 36, 37]) . Recently, specific subclasses functions that are starlike, convex, univalent, highly starlike, and highly convex. have been examined for their Hankel determinants with

logarithmic coefficients, alongside their associated literature (see [38, 33, 39] and the citations there in). However, further investigation is required to determine the exact boundaries of Hankel determinants of logarithmic coefficients across a broader spectrum of function classes.
Using (1.2) and differentiating (1.3),

$$F_f'(\hat{y}) = \frac{d}{d\hat{y}}\left(log\,\frac{f(\hat{y})}{\hat{y}}\right) = \frac{d}{d\hat{y}}\,logf(\hat{y}) - \frac{d}{d\hat{y}}\,log(\hat{y})\; = \frac{f'(\hat{y})}{f(\hat{y})} - \frac{1}{\hat{y}}$$

$$f(\hat{y}) = \hat{y} + a_2\hat{y}^2 + a_3\hat{y}^3 + a_4\hat{y}^4 + a_5\hat{y}^5 + a_6\hat{y}^6 \dots$$

$$f'(\hat{y}) = 1 + 2a_2\hat{y} + 3a_3\hat{y}^2 + 4a_4\hat{y}^3 + 5a_5\hat{y}^4 + 6a_6\hat{y}^5 \dots$$

$$f''(\hat{y}) = 0 + 2a_2 + 6a_3\hat{y} + 12a_4\hat{y}^2 + 20a_5\hat{y}^3 \dots,$$

$$F_f'(\hat{y}) = \frac{d}{d\hat{y}}\left(log\,\frac{f(\hat{y})}{\hat{y}}\right) = \frac{\hat{y}f'(\hat{y}) - f(\hat{y})}{\hat{y}f(\hat{y})} \tag{1.4}$$

$$\frac{\hat{y}f'(\hat{y}) - f(\hat{y})}{\hat{y}f(\hat{y})} = \frac{a_2\hat{y}^2 + 2a_3\hat{y}^3 + 3a_4\hat{y}^4 + 4a_5\hat{y}^5 + 5a_6\hat{y}^6}{\hat{y}^2 + a_2\hat{y}^3 + a_3\hat{y}^4 + a_4\hat{y}^5 + a_5\hat{y}^6}$$

$$\frac{d}{d\hat{y}}\left(log\frac{f(\hat{y})}{\hat{y}}\right) = a_2 + (2a_3 - a_2^2)\hat{y} + (3a_4 - 3a_2a_3 - a_2^3)\hat{y}^2 + (4a_5 - 2a_2^3 - 4a_4a_2 + 4a_2^2a_3 - a_2^4)\hat{y}^3$$
$$+ (5a_6 - 5a_2a_5 - 5a_3a_4 + 5a_2^2a_4 + 5a_3^2a_2 - 5a_2^3a_3 + a_2^3)\hat{y}^4 \dots, \tag{1.5}$$

differentiating again (1.3)

$$\frac{d}{d\hat{y}}\left(2\sum_{n=1}^{\infty}\gamma_n(f)\,\hat{y}^n\right) = \frac{d}{d\hat{y}}\,[\,2\gamma_1(f)\hat{y} + 2\gamma_2(f)\,\hat{y}^2 + 2\gamma_3(f)\,\hat{y}^3 + 2\gamma_4(f)\,\hat{y}^4 + 2\gamma_5(f)\,\hat{y}^5 + \dots]$$
$$\frac{d}{d\hat{y}}\left(2\sum_{n=1}^{\infty}\gamma_n(f)\,\hat{y}^n\right) = 2\gamma_1 + 4\gamma_2\hat{y} + 6\gamma_3\hat{y}^2 + 8\gamma_4\hat{y}^4 + 10\gamma_5\hat{y}^5 + \dots, \tag{1.6}$$

comparatively (1.5) with (1.6)

1. $\hat{y}^o$:

$$2\gamma_1 = a_2,$$
$$\gamma_1 = \frac{1}{2}a_2$$

2. $\hat{y}^1$:

$$4\gamma_2 = (2a_3 - a_2^2),$$
$$\gamma_2 = \frac{1}{2}(a_3 - \frac{1}{2}a_2^2)$$

3. $\hat{y}^2$:

$$6\gamma_3 = (3a_4 - 3a_2a_3 - a_2^3),$$
$$\gamma = \frac{1}{2}(a_4 - a_2a_3 + \frac{1}{3}a_2^3)$$

4. $\hat{y}^3$:

$$8\gamma_4 = (4a_5 - 2a_2^3 - 4a_4a_2 + 4a_2^2a_3 - a_2^4)$$
$$\gamma_4 = \frac{1}{8}\ (4a_5 - 2a_2^3 - 4a_4a_2 + 4a_2^2a_3 - a_2^4)$$
$$\gamma_4 = \frac{1}{2}(a_5 - a_4a_2 + a_2^2a_3 - \frac{1}{2}a_3^2 - \frac{1}{4}a_2^4)$$

5. $\hat{y}^4$:

$$10\gamma_5 = 5a_6 - 5a_2a_5 - 5a_3a_4 + 5a_2^2a_4 + 5a_3^2a_2 - 5a_2^3a_3 + a_2^5)$$
$$\gamma_5 = \frac{1}{10}\ (5a_6 - 5a_2a_5 - 5a_3a_4 + 5a_2^2a_4 + 5a_3^2a_2 - 5a_2^3a_3 + a_2^5)$$
$$\gamma_5 = \frac{1}{2}(a_6 - a_2a_5 - a_3a_4 + a_2^2a_4 + a_2a_3^2 + a_2^3a_3 + \frac{1}{5}a_2^5).$$

The quick calculation demonstrates that

$$\gamma_1 = \frac{1}{2}a_2$$
$$\gamma_2 = \frac{1}{2}(a_3 - \frac{1}{2}a_2^2)$$
$$\gamma_3 = \frac{1}{2}(a_4 - a_2a_3 + \frac{1}{3}a_2^3)$$
$$\gamma_4 = \frac{1}{2}(a_5 - a_2a_4 + a_2^2a_3 - \frac{1}{2}a_3^2 - \frac{1}{4}a_2^4)$$
$$\gamma_5 = \frac{1}{2}(a_6 - a_2a_5 - a_3a_4 + a_2a_3^2 + a_2^2a_4 - a_2^3a_3 + \frac{1}{5}a_2^5).$$

Calculating the Hankel determinant, where the entries consist of logarithmic coefficients, is particularly intriguing and pertinent as a result of logarithmic coefficients' growing importance in current research. Specifically, defining the second Hankel determinant involves notable attention $F_f/2$ stated as

$$\mathscr{H}_{2,1}(F_f/2) = \gamma_1\gamma_3 - \gamma_2^2$$

$$= \left(\frac{1}{2}a_2\right)\left(\frac{1}{2}(a_4 - a_2a_3 + \frac{1}{3}a_2^3\right) - \left(\frac{1}{2}\,(a_3 - \frac{1}{2}a_2^2)\right)^2$$

$$= \frac{a_2a_4}{4} - \frac{a_2^2a_3}{4} + \frac{a_2^4}{12} - \frac{a_3^2}{4} - \frac{a_2^4}{16} + \frac{2a_2^2a_3}{8}$$

$$= \frac{12a_2a_4 + a_2^4 - 12a_3^2}{48}$$

$$\mathscr{H}_{2,1}(F_f/2) = \gamma_1\gamma_3 - \gamma_2^2 = \frac{1}{48}(a_2^4 - 12a_3^2 + 12a_2a_4). \tag{1.7}$$

In this study, the precise limit of the Hankel determinant will be investigated $\mathscr{H}_{1,2}\left(\frac{F_f}{2}\right)$ for two categories of functions, specifically star-like and convex functions related to lune.

The logarithmic coefficients associated with the Koebe function $f(\hat{y}) = \hat{y}/(1-\hat{y})^2$ are known to be $\gamma_n = \frac{1}{n}$ or each positive integer n. Given the common occurrence of the Koebe function as an external function in the geometric theory of analytic functions issues, it stands to reason that $\gamma_n = \frac{1}{n}$ holds true for functions in S. But, even in terms of order of magnitude, this is not generally true. A number of writers have recently examined the issue of calculating the bound of the logarithmic coefficients in various circumstances; for example, see [40, 41, 22, 35, 24].
In the realm of specific subgroups of Starlike and Convex, Shi [42] revealed the limitations of the third-order Hankel determinant within an open unit disc for univalent functions related to exponential functions.
Significant aspect in delineating the limits of the Hankel determinant for functions exhibiting positive real components lies in the Carathéodory class $\mathscr{P}$ and its coefficient constraints within the domain of geometric function theory. Functions with positive real parts significantly influence the criteria for both C convex and S^* starlike univalent functions, alongside C convex functions. Every function conforming to this form is encompassed within this class

$$p(\hat{y}) = 1 + p_1\hat{y} + p_2\hat{y}^2 + p_3\hat{y}^3 \ldots$$

A function belonging to p is termed a Carathéodory function. It's established that $c_n \leq 2$, n ≥ 1 in a function $p \in \mathscr{P}$ (see [21])

Satisfying these requirements and being analytic in open unit disc $\mathbb{E}$

$$p(0)=1 \; with \;\; \Re[p(\hat{y})]>0$$

Claimed to be functions of class $\mathscr{P}$; see [43] for further information. A function with a positive real portion in $\mathscr{S}$ the disc $\mathbb{E}$ is defined as any function in λ accordingly. The Möbius function stands out as the most well known illustration of a function within this category, defined as follows: the Koebe function is extremely important in the class $\mathscr{S}$ function.

$$\hat{M}(\hat{y}) \; = \; 1+2\hat{y}+2\hat{y}^2+2\hat{y}^3 \ldots \; = 1+2\sum_{j=1}^{\infty}\hat{y}^j$$

Quantum theory is a vital tool for dealing with difficult and complex information. The term ordinary calculus refers to it without the concept of limits. This field of mathematics is fascinating. Furthermore, it is essential to a great deal of physics, such as black holes and cosmic strings; for further details, see [44]. There are two distinct forms of quantum calculus: as well as the q- and h-calculus. Here, q and h represent quantum and Planck's constant, respectively. Researchers are interested within the realm of q-calculus and its applications across various domains.

Using the operator, numerous of subclasses within the category of analytic functions are thoroughly investigated . To advance the preliminary groundwork of q-calculus within geometric function theory, Ismail [45] studying the well-known Fekete-Szegö Inequality, and established the family of functions that resemble q-stars and are associated with a specific trigonometric function, such sine functions. Subsequently, several known convolution results were applied to show that the specified class has both necessary and sufficient conditions. Starlikeness radii, growth and distortion limitations, and the extreme point theorem were among the other subjects covered.

In his presentation of q-calculus applications in geometric function theory, Srivastava [46] made use of q-analogues of hyper geometric functions. Moreover, two general subclasses of complex order, negative coefficient normalised analytic functions were examined using various q-calculus operators and fractional q-calculus operators, in addition to the extreme points, growth and distortion theorems, and concepts related to starlikeness and convexity, are explored and coefficient estimations were discovered for every one of the newly specified classes.

Let $\mathscr{S}$ denote the subset comprising every univalent function in $\mathscr{A}$. let

$$F_f(\hat{y}) := log\frac{f(\hat{y})}{\hat{y}} = 2\sum_{n=1}^{\infty}\gamma_n(f)\hat{y}^n, log\ 1 := 0,$$

$f \in \mathscr{S}$ is connected to a logarithmic function and now we solve this logrithmic function into q-logrithmic function with the help of q-derivative definition and Taylor series expansion.

$$F_{f,q}(\hat{y}) := log_q\frac{f(\hat{y})}{\hat{y}} = 2\sum_{n=1}^{\infty}\gamma_n^q(f)\hat{y}^n, \qquad \hat{y}\in\mathbb{E},$$

Let q-derivative of $\left(log\ \frac{f(\hat{y})}{\hat{y}}\right)$ So,first we have

$$g(\hat{y}) = log\left(\frac{f(\hat{y})}{\hat{y}}\right)$$

$$g(q(\hat{y})) = log\left(\frac{f(q\hat{y})}{q\hat{y}}\right).$$

Using the definition of the q-derivative, we obtain

$$\mathscr{D}_q log_q\left(\frac{f(\hat{y})}{\hat{y}}\right) = \frac{log_q\left(\frac{f(q\hat{y})}{q\hat{y}}\right) - log_q\left(\frac{f(\hat{y})}{\hat{y}}\right)}{q\hat{y}-\hat{y}},$$

simplify q-logarithmic terms

$$log_q\left(\frac{f(q\hat{y})}{q\hat{y}}\right) = log_q(f(q\hat{y}) - log_q(q\hat{y})) = log_q\left(f(q\hat{y}) - log_q(q) - \mathrm{log_q}(\hat{y})\right)$$

$$log_q\left(\frac{f(\hat{y})}{\hat{y}}\right) = log_q\ (f(\hat{y}) - log_q\ (\hat{y})),$$

in above phrase, enter these two q-logrithmic equations

$$\mathscr{D}_q log_q\left(\frac{f(\hat{y})}{\hat{y}}\right) = \frac{log_q\left(f(q\hat{y}) - log_q\ (q) - log_q(\hat{y})\right) - (log_q f(\hat{y}) - log_q(\hat{y}))}{q\hat{y}-\hat{y}}$$

$$\mathscr{D}_q log_q\left(\frac{f(\hat{y})}{\hat{y}}\right) = \frac{log_q\ (f(q\hat{y})) - log_q\ (f(\hat{y}))}{q\hat{y}\ -\ \hat{y}} - \frac{log_q\ (q)}{q\hat{y}\ -\ \hat{y}},$$

since we are aware

$$\mathscr{D}_q log_q(f(\hat{y})) = \frac{log_q\ (f(q\hat{y})) - log_q\ (f(\hat{y}))}{q\hat{y}\ -\ \hat{y}} = \frac{log_q\left(\frac{f(q\hat{y})}{f(\hat{y})}\right)}{q\hat{y}-\hat{y}}$$

$$\mathscr{D}_q log_q\left(\frac{f(\hat{y})}{\hat{y}}\right) = \frac{log_q\left(\frac{f(q\hat{y})}{f(\hat{y})}\right)}{q\hat{y}-\hat{y}} - \frac{log_q\ (q)}{q\hat{y}-\hat{y}},$$

now we examine that $log_q\left(\frac{f(q\hat{y})}{f(\hat{y})}\right)$

$$f(\hat{y}) = 1 + a_1 q\hat{y} + a_2 q^2 \hat{y}^2 + a_3 q^3 \hat{y}^3 + \ldots.$$

$$f(q\hat{y}) = 1 + a_1 \hat{y} + a_2 \hat{y}^2 + a_3 \hat{y}^3 + \ldots.$$

$$\frac{f(q\hat{y})}{f(\hat{y})} = \frac{1 + a_1 q\hat{y} + a_2 q^2 \hat{y}^2 + a_3 q^3 \hat{y}^3 + ..}{1 + a_1 \hat{y} + a_2 \hat{y}^2 + a_3 \hat{y}^3 + \ldots}$$

$$f(q\hat{y}) = f(\hat{y} + (q-1)\hat{y}) = f(\hat{y}) + \mathscr{D}_q log_q(f(\hat{y}))(q-1)\hat{y} + \ldots$$

$$log_q\left(\frac{f(q\hat{y})}{f(\hat{y})}\right) = log_q\left(\frac{f(\hat{y})}{f(\hat{y})} + \frac{\mathscr{D}_q(f(\hat{y}))(q-1)\hat{y}}{f(\hat{y})} + \frac{\mathscr{D}_q^2(f(\hat{y}))((q-1)\hat{y})^2}{2!f(\hat{y})} + \ldots\right)$$

$$log_q\left(\frac{f(q\hat{y})}{f(\hat{y})}\right) = log_q\left(1 + \frac{\mathscr{D}_q(f(\hat{y}))(\mathrm{q}-1)\hat{y}}{f(\hat{y})} + \frac{\mathscr{D}_q^2(f(\hat{y}))((\mathrm{q}-1)\hat{y})^2}{2!f(\hat{y})} + \ldots\right)$$

Implementing log_q as well as the fact that

$$log_q(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3}$$

$$t = \frac{\mathscr{D}_q(f(\hat{y}))(q-1)\hat{y}}{f(\hat{y})}$$

So,

$$log_q\left(1 + \frac{\mathscr{D}_q(f(\hat{y}))(q-1)\hat{y}}{f(\hat{y})}\right) = \frac{\mathscr{D}_q(f(\hat{y}))(q-1)\hat{y}}{f(\hat{y})} - \frac{\left(\frac{\mathscr{D}_q(f(\hat{y}))(q-1)\hat{y}}{f(\hat{y})}\right)^2}{2} + \ldots$$

$$log_q\left(\frac{f(q\hat{y})}{f(\hat{y})}\right) = \frac{\mathscr{D}_q(f(\hat{y}))(q-1)\hat{y}}{f(\hat{y})},$$

$$\mathscr{D}_q log_q\left(\frac{f(\hat{y})}{\hat{y}}\right) = \frac{\frac{\mathscr{D}_q(f(\hat{y}))(\mathrm{q}-1)\hat{y}}{f(\hat{y})}}{q\hat{y} - \hat{y}} - \frac{log_q(q)}{q\hat{y} - \hat{y}},$$

by Taylor series expansion

$$log_q(q) = (q-1) - \frac{(q-1)^2}{2} + \dots$$

now we obtain

$$\mathscr{D}_q log_q\left(\frac{f(\hat{y})}{\hat{y}}\right) = \frac{\frac{\mathscr{D}_q(f(\hat{y}))(\mathrm{q}-1)\hat{y}}{f(\hat{y})} - (q-1)}{\hat{y}(q\ -1)}$$

$$\mathscr{D}_q log_q\left(\frac{f(\hat{y})}{\hat{y}}\right) = (q\ -1)\ \frac{\left(\frac{\mathscr{D}_q(f(\hat{y}))\hat{y}}{f(\hat{y})} - 1\right)}{\hat{y}(q\ -1)}$$

$$\mathscr{D}_q log_q\left(\frac{f(\hat{y})}{\hat{y}}\right) = \frac{\left(\frac{\mathscr{D}_q(f(\hat{y}))\hat{y}}{f(\hat{y})} - 1\right)}{\hat{y}} = \left(\frac{\mathscr{D}_q(f(\hat{y}))\hat{y}}{f(\hat{y})\hat{y}} - \frac{1}{\hat{y}}\right)$$

$$\mathscr{D}_q log_q\left(\frac{f(\hat{y})}{\hat{y}}\right) = \left(\frac{\mathscr{D}_q f(\hat{y})}{f(\hat{y})} - \frac{1}{\hat{y}}\right) \tag{1.8}$$

$$f(\hat{y}) = \hat{y} + a_2\hat{y}^2 + a_3\hat{y}^3 + a_4\hat{y}^4 + a_5\hat{y}^5 + a_6\hat{y}^6$$

$$\mathscr{D}_q f(\hat{y}) = 1 + [2]_q a_2\hat{y} + [3]_q a_3\hat{y}^2 + [4]_q a_4\hat{y}^3 + [5]_q a_5\hat{y}^4 + [6]_q a_6\hat{y}^5$$

$$\mathscr{D}_q^2 f(\hat{y}) = 0 + [2]_q a_2 + [6]_q a_3\hat{y} + [12]_q a_4\hat{y}^2 + [20]a_5\hat{y}^3$$

Simplification of (1.8) then we achieve

$$\mathscr{D}_q log_q\left(\frac{f(\hat{y})}{\hat{y}}\right) = \frac{\hat{y}\mathscr{D}_q f(\hat{y}) - f(\hat{y})}{\hat{y}f(\hat{y})} \tag{1.9}$$

Putting value of $f(\hat{y}), \mathscr{D}_q f(\hat{y})$ in (1.9) we get

$$\frac{\hat{y}\mathscr{D}_q f(\hat{y}) - f(\hat{y})}{\hat{y}f(\hat{y})} = \frac{a_2\hat{y}^2 + [2a]_{q3}\hat{y}^3 + [3]_q a_4\hat{y}^4 + [4]_q a_5\hat{y}^5 + [5]_q a_6\hat{y}^6}{\hat{y}^2 + a_2\hat{y}^3 + a_3\hat{y}^4 + a_4\hat{y}^5 + a_5\hat{y}^6}$$

$$\begin{aligned} \mathscr{D}_q(log_q\frac{f(\hat{y})}{\hat{y}}) &= a_2 + \left([2]_q a_3 - a_2^2\right)\hat{y} + \left([3]_q a_4 - [3]_q a_2 a_3 - a_2^3\right)\hat{y}^2 \\ &+ \left([4]_q a_5 - [2]_q a_2^3 - [4]_q a_4 a_2 + [4]_q a_2^2 a_3 - a_2^4\right)\hat{y}^3 \\ &+ \left([5]_q a_6 - [5]_q a_2 a_5 - [5]_q a_3 a_4 + [5]_q a_2^2 a_4 + [5]_q a_3^2 a_2 - [5]_q a_2^3 a_3 + a_2^3\right)\hat{y}^4. \end{aligned} \tag{1.10}$$

$$\mathscr{D}_q(2\sum_{n=1}^{\infty}\gamma_n^q(f)\hat{y}^n) = \mathscr{D}_q\ [2\gamma_1^q(f)\hat{y} + 2\gamma_2^q(f)\ \hat{y}^2 + 2\gamma_3^q(f)\ \hat{y}^3 + 2\gamma_4^q(f)\ \hat{y}^4 + 2\gamma_5^q(f)\ \hat{y}^5 + \dots]$$

$$\mathscr{D}_q(2\sum_{n=1}^{\infty}\gamma_n^q\hat{y}^n) = 2\gamma_1^q + [4]_q\gamma_2^q\hat{y} + [6]_q\gamma_3^q\hat{y}^2 + [8]_q\gamma_4^q\hat{y}^3 + [10]_q\gamma_5^q\hat{y}^5 + \dots \tag{1.11}$$

Examine (1.10) and (1.11).

$\hat{y}^o$:

$$[2]_q\gamma_1^q = a_2$$
$$\gamma_1^q = \frac{1}{[2]_q}a_2$$

$\hat{y}^1$:

$$[4]_q\gamma_2^q = ([2]_q a_3 - a_2^2)$$

$$\gamma_2^q = \frac{1}{[4]_q}([2]_q a_3 - a_2^2)$$

$\hat{y}^2$:

$$[6]_q\gamma_3^q = ([3]_q a_4 - [3]_q a_2 a_3 - a_2^3)$$

$$\gamma_3^q = \frac{1}{[6]_q}([3]_q a_4 - [3]_q a_2 a_3 - a_2^3)$$

$\hat{y}^3$:

$$[8]_q\gamma_4^q = ([4]_q a_5 - [2]_q a_2^3 - [4]_q a_4 a_2 + [4]_q a_2^2 a_3 - a_2^4)$$

$$\gamma_4^q = \frac{1}{[8]_q}\,([4]_q a_5 - [2]_q a_2^3 - [4]_q a_4 a_2 + [4]_q a_2^2 a_3 - a_2^4)$$

Similarly,

$\hat{y}^4$:

$$[10]_q\gamma_5^q = ([5]_q a_6 - [5]_q a_2 a_5 - [5]_q a_3 a_4 + [5]_q a_2^2 a_4 + [5]_q a_3^2 a_2 - [5]_q a_2^3 a_3 + a_2^3)$$

$$\gamma_5^q = \frac{1}{[10]_q}([5]_q a_6 - [5]_q a_2 a_5 - [5]_q a_3 a_4 + [5]_q a_2^2 a_4 + [5]_q a_3^2 a_2 - [5]_q a_2^3 a_3 + a_2^3)$$

Second Hankel Determinant of $F_f/2$

$$\mathscr{H}_{2,1}(F_f/2) = (\gamma_1^q \gamma_3^q - (\gamma_2^q)^2)$$

$$= \left[\left(\frac{1}{2}a_2\right)\left(\frac{1}{[6]_q}([3]_q a_4 - [3]_q a_2 a_3 - a_2^3)\right) - \left(\frac{1}{[4]_q}([2]_q a_3 - a_2^2)\right)^2 \right]$$

$$= \frac{[3]_q a_2 a_4}{2[6]_q} - \frac{[3]_q a_2^2 a_3}{2[6]_q} + \frac{a_2^4}{2[6]_q} - \frac{[4]_q a_3^2}{[16]_q} - \frac{a_2^4}{[16]_q} + \frac{[4]_q a_2^2 a_3}{[16]_q}$$

$$= \frac{[12]_q a_2 a_4 + a_2^4 - [12]_q a_3^2}{[48]_q}$$

$$\mathscr{H}_{2,1}(F_f/2) = (\gamma_1^q \gamma_3^q - (\gamma_2^q)^2) = \frac{1}{[48]_q}(a_2^4 - [12]_q a_3^2 + [12]_q a_2 a_4) \tag{1.12}$$

Despite having many applications in mathematics, mechanics, and physics, q-calculus is basically just basic classical calculus without limit ideas. As a result, it is rapidly evolving.

1.1 Preface

"The purpose regarding this thesis is to utilise the subordination notion to assess and elaborate a few subclasses of analytic functions. It is organised into five chapters, each of which has the following brief introduction:"

"In **Chapter 1**, a thorough literature survey is provided, emphasising important ideas from the Geometric Function Theory lectures. The class of analytic functions, the class of Carathéodory functions, and the class of univalent functions are all discussed in this exploration, along with pertinent subclasses. These ideas form the core of this thesis."

Chapter 2 "mostly focusses on fundamental concepts of Geometric Function Theory, providing an crucial framework for the subsequent chapters. It begins by discussing the concepts of normalised univalent functions and analytic functions in the context of the open unit disc. Afterward, a number of basic subclasses of univalent functions are defined. Preliminary lemmas that will be used in later chapters are presented at the end of this chapter. It should be noted that this chapter completely references and recognises existing theories in the field instead than offering any novel findings."

Chapter 3"discusses the category of convex functions associated with the lune function and examines the category of starlike functions with regard to the lune function. Furthermore, several of the primary findings are examined. Making sure the review work is correctly cited is very important."

Chapter 4 "focusses on the category of starlike functions with respect to the lune function associated with the q-starlike function, which is a particular subclass of univalent functions. Established results for functions in this class are also inferred in this chapter. The newly derived results are shown to be consistent with those earlier affirmed by other researchers through corollaries."

CHAPTER 2

DEFINITIONS AND PRELIMINARY CONCEPTS

2.1 Overview

This section objective is to go over some crucial traditional findings that will form the basis of further investigation. A detailed discussion of the normalised analytic univalent functions and Carathéodory functions will be provided. There will be consideration of several exclusive functions, a renowned linear operator, and introductory lemmas. Perhaps the most intriguing feature of highly complex function theory is the relationship between geometry and analysis.

Definition 2.1.1. *[47] If a function is differentiable at every point within a complex field, it is said to be holomorphic in that domain. At point $\hat{y}_0$, a complex valued function $\xi(\hat{y})$ that possesses a derivative is differentiable,*

$$\xi'(\hat{y}) = \lim_{\hat{y}\to\hat{y}_0} \frac{\xi(\hat{y}) - \xi(\hat{y}_0)}{\hat{y} - \hat{y}_0},$$

such a function ξ is analytic at $\hat{y}_0$ if it is differentiable at all points in its neighbourhood at $\hat{y}_0$.

The fact that $\hat{y}_0$ must have derivatives of all orders and that ξ has a Taylor series expression is one of the marvels of complex analysis,

$$\xi(\hat{y}) = \sum_{k=0}^{\infty} \frac{\xi^k(\hat{y}_0)}{k!}(\hat{y} - \hat{y}_0)^k$$

2.2 Domain

Within the theory of geometric functions, we are always pointing to a certain domain. An open, linked set is called a domain. In terms of geometry, the open unit disc is equivalent to a disc with a radius of 1 that is centred at the origin and does not include the disk's boundary. Stated differently, it encompasses all complex numbers within the disc but excludes the points situated on its perimeter.

A key idea in complex analysis, the open unit disc is useful in many different mathematical and analytical situations, including complex integration mapping functions as well as conformal mappings.

Definition 2.2.1. *[47, 48] An open unit disk in the complex plane, refers to a set of complex numbers that lie within a specific region in the complex plane. It is defined as the set of all complex numbers whose distance from the origin is less than 1. In mathematical notation, the open unit disk is represented as,*

$$\mathbb{E} = \{|\hat{y}| < 1; \hat{y} \in \mathbb{C}\}.$$

Here, $|\hat{y}|$ indicates the modulus or absolute value of $\hat{y}$, and $\hat{y}$ represents a complex number. The distance between $\hat{y}$ and the origin indicated by the condition $|\hat{y}| < 1$.

2.3 Analytic and Univalent functions

Geometric functions and analytical structures are related, and this relationship forms the basis of the theory of univalent functions. For Analytic and Univalent functions we define categories in this context.

Definition 2.3.1. *[49] An analytic function, sometimes referred to as a holomorphic function, is a function with complicated values that is differentiable and defined at each point in a certain area of the complex plane. A function ξ is considered analytical if it is differentiable at all points within the region, according to more rigorous definitions.*

One significant implication is that a power series representation exists for an analytic function. The variable $\hat{y}$ can be expressed as an infinite sum of its powers, which indicates that the function is analytic within the region

$$\xi = \hat{y} + \sum_{m=2}^{\infty} \hat{a}_m \hat{y}^m$$

where one can find the coefficients.

Definition 2.3.2. *[50] If a function is analytic in open unit disc $\mathbb{E}$ and normalised by these requirements $\xi(0) = 0$, then it belongs to class $\mathscr{A}$, the class of Normalised Analytic Function.Assuming $\xi'(0) = 1$.*

$$\xi = \hat{y} + \sum_{m=2}^{\infty} \hat{a}_m \hat{y}^m, \quad \hat{y} \in \mathbb{E}.$$

Definition 2.3.3. *[16] Univalent functions, sometimes referred to as one-to-one analytic functions or univalent mappings, are a particular kind of analytic function that maintains injectivity. In particular, if a function ξ defined on an area in the complex plane maps various complex numbers to separate images, it is said to be univalent. This means that it does not have two different inputs that map to the same output.*

Assume that ξ is an analytic function defined on a domain in the complex plane. If the condition $\xi(\hat{y}_1) \neq \xi(\hat{y}_2)$ holds for any separate complex numbers $\hat{y}_1$ and z_2 in domain, then the function ξ is considered univalent in domain.

Stated differently, an injective or one-to-one function within its domain is referred to as a univalent function.The mapping of points from its domain to its range does not result in any self-intersections or overlaps.

Geometric function theory and complex analysis are two areas of great interest for uniform functions. Their significance lies in their numerous applications in fields like complex dynamics, conformal mapping, and Riemann surface theory. Univalent functions provide useful geometric qualities and can be studied to better understand the complex characteristics of transformations and mappings.

Definition 2.3.4. *[16] If a function is analytic and univalent within the open unit disc $\mathbb{E}$, it belongs to the class $\mathscr{S}$, which is made up of Univalent Functions. Two normalisation requirements must be met by the function: $\xi(0) = 0$ and $\xi'(0) = 1$.*

A function that belongs to the class $\mathscr{S}$ is exemplified by the Koebe Function.

$$\xi = \hat{y} + \sum_{m=2}^{\infty} m\hat{y}^m, \quad \hat{y} \in \mathbb{E}.$$

2.4 Carathéodory function

Other functions were found to have image domains that were limited to the open half plane after many complex valued functions were found to have image domains that encompassed the full complex plane. This class of functions, represented by $\mathscr{P}$, is known as the Carathéodory function class.

Definition 2.4.1. *[50] The expression for a function $p \in \mathscr{P}$ that is analytic in $\mathbb{E}$ is*

$$p(\hat{y}) = 1 + \sum_{i=1}^{\infty} c_i \hat{y}^i,$$

In this case, $\mathrm{Re}[p(\hat{y})] \geq 0$ *and* $p(0) = 1$.

The most well-known illustration of a function in this class is the mobius function, which is defined as,

$$\mathbf{M} = \left(-1 + \frac{2}{1-\hat{y}}\right).$$

2.5 Subordination

The behaviour of analytic functions is examined using the word "subordination" in geometric function theory. Lindelof was the first to propose the subordination principle in 1909. There were further developments by Littlewood and Rogosinski [15]. To define the subordination principle, the Schwarz function is employed.

Definition 2.5.1. *[49] Assume two analytic functions, f and g. If there is a self-map w under the condition w(0)= 0 such that $f(\hat{y}) = g(w(\hat{y}))$, then f is subordinated by g and expressed as $f(\hat{y}) \prec g(\hat{y})$. Furthermore, $f(D) \subseteq g(D)$ given that $f(0) = g(0)$ and g is univalent.*

2.6 Logarithmic Function

The expression defines $F_f(\hat{y})$ as the natural logarithm of the function.$f(\hat{y})$ divided by $\hat{y}$ and it's represented as a power series expansion in terms of $\hat{y}$ with coefficient $\gamma_n(f)$ in [51].

$$F_f(\hat{y}) \ := \log \frac{f(\hat{y})}{\hat{y}} = 2\sum_{n=1}^{\infty} \gamma_n(f)\,\hat{y}^n, \hat{y} \in \mathbb{E}, \ \ log\ 1 \ := \ 0$$

Where,

1. $\log \frac{f(\hat{y})}{\hat{y}}$: The logarithm of the function f divided by $\hat{y}$.
2. $2\sum_{n=1}^{\infty} \gamma_n(f)\,\hat{y}^n$: A power series expansion of the logarithmic function $F_f(\hat{y})$.
3. It's represented as a sum of terms involving powers of $\hat{y}$, with coefficients given by the values of $\gamma_n(f)$ for n $\geq$ 1.
4. here n $\geq$ 1 indicates that the summation is performed over all terms starting from $n = 1$.

2.7 Lune Function

In complex plane, the crescent shape formed by two intersecting circles is called a lune. If we denote a lune by (L(a,b)), where (a) and (b) Radii of circles bounding the lune. Lune is a non-trivial geometric region because its often characterized by complex boundaries.

Definition 2.7.1. *The inequality* $\{w \in C: |w^2 - 1| \leq 2|w|\}$ *define a lune-shaped region [52]. Due to the fact that it symbolises the collection of all complex numbers w whose transformed values are under the map* $w \to w^2 - 1$ *lie within the range specified by* $2|w|$. *This set represents a geometric region that resembles a crescent or "lune" or the space formed by two circles that cross.*

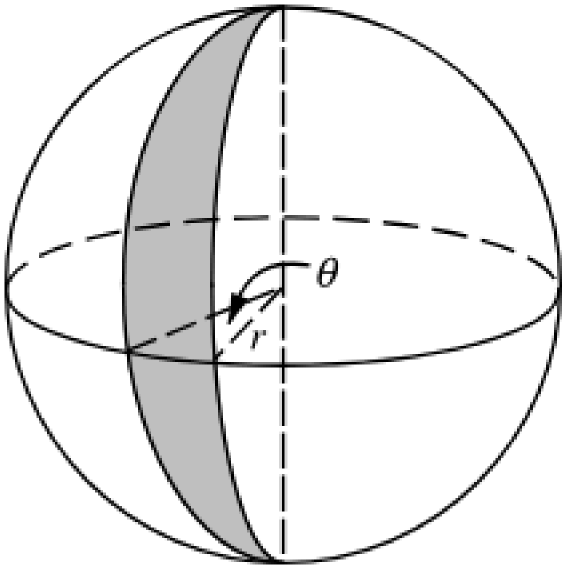

Figure 2.1: Geometry of Lune Function

2.8 Certain sub-classes of univalent functions

Univalent function analysis is a field that is both old and constantly changing. The previous ten to fifteen years have seen a number of noteworthy breakthroughs. Different subclasses of the class of univalent functions have been presented, mostly motivated by the geometric characteristics of their image domains. Among other things, this context has defined the classes of Starlike and Convex functions.

Definition 2.8.1. *[50, 49] An "starlike function" is a function that projects the disc $\mathbb{E}$ onto a domain $\mathbf{B}$ that, with respect to the origin, resembles a starlike domain shown in Figure 2.2. S^* represents the subclass of which all starlike functions are included. According to the origin, the domain is starlike if $\hat{y} = 0$ and the linear segment that connects 0 to any other point of the domain $\mathbf{B}$ is entirely contained within the complex plane. That is,*

$$\forall \hat{y} \in \mathbf{B},\ \lambda\ \hat{y} \in \mathbf{B}$$

In the case of $0 \leq \psi \leq 1$, if $\hat{y} \in \mathbf{B}$, it is imperative that every point of domain be observable from

$\hat{y}$.

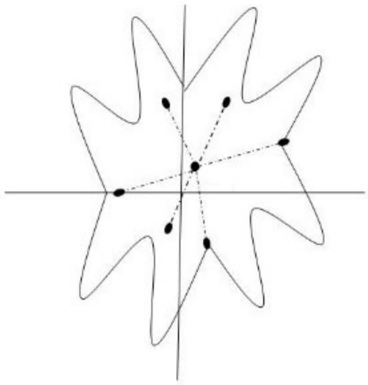

Figure 2.2: Starlike domain

Definition 2.8.2. *[50, 49] Using a convex function, the disc $\mathbb{E}$ is moved onto a convex domain* **B** *with respect to the origin shown in Figure 2.3. C represents the $\mathscr{S}$ subclass that includes all convex functions. A domain is considered convex if a line segment that connects any two points of a domain* **B** *in the complex plane lies entirely within that domain. That is,*

$$[\psi\hat{y}_1 + (1-\psi)\hat{y}_2] \in \mathbf{B},$$

where $\hat{y}_1$ and $\hat{y}_2$ both are in **B** *with $0 \leq \psi \leq 1$.*

Definition 2.8.3. *[25] A function is considered to be starlike accociated with lune function $S^*_{☾}$ if,*

$$S^*_{☾} = \left\{ f \in \mathrm{S} : \left| \left(\frac{\hat{y}f'(\hat{y})}{f(\hat{y})} \right)^2 - 1 \right| \leq 2 \left| \frac{\hat{y}f'(\hat{y})}{f(\hat{y})} \right|, \quad \hat{y} \in \mathbb{E} \right\}$$

Definition 2.8.4. *[25] A function is considered to be convex accociated with lune function $C_{☾}$ if,*

$$C_{☾} = \left\{ f \in \mathrm{S} : 1 + \frac{\hat{y}f''(\hat{y})}{f'(\hat{y})} \prec q(\hat{y}), \quad \hat{y} \in \mathbb{E} \right\}$$

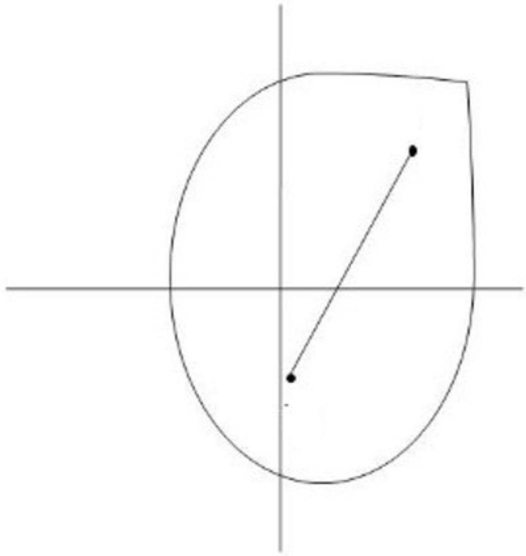

Figure 2.3: Convex domain

2.9 Quantum Calculus

Early in the 20th century, American mathematician Jackson created the first version of quantum calculus. His definition about the q-analog of the integral operator and derivative was the first.

Definition 2.9.1. *[53] A branch of mathematics called quantum calculus, sometimes referred to as q-calculus or Jackson's q-calculus, introduces a parameter q and generalises several ideas from classical calculus.*

A key idea in q-calculus, a field of mathematics that extends classical calculus by adding a parameter q, is the q-derivative operator, which is frequently represented as $\mathscr{D}_q$.

Definition 2.9.2. *[53] The definition of the q-derivative for a differentiable function $\tau(\hat{y})$ is*

$$\mathscr{D}_{\mathbf{q}}\tau(\hat{y}) = \frac{\tau(\hat{y}) - \tau(\mathbf{q}\hat{y})}{(1-\mathbf{q})\hat{y}}, \quad \hat{y} \neq 0$$

where

$0 < \mathbf{q} < 1$. *Its Maclaurins series is*

$$\mathscr{D}_{\mathbf{q}}\tau(\hat{y}) = \sum_{\hat{n}=0}^{\infty} [\hat{n}]_{\mathbf{q}} \check{c}_n \hat{y}^{\hat{n}-1},$$

where

$$[\hat{n}]_{\mathbf{q}} = \begin{cases} \frac{1-\mathbf{q}^{\hat{n}}}{1-\mathbf{q}}, & \hat{n} \in \mathbb{C} \\ \sum_{\hat{n}=0}^{\hat{n}-1} \mathbf{q}^{\hat{n}}, & \hat{n} \in \mathbb{N}. \end{cases}$$

Definition 2.9.3. *The q-derivative of the natural logarithm function $ln(x)$ is defined as:*

$$\mathscr{D}_q[ln(x)] = \frac{ln(q)}{(q-1)x}$$

Definition 2.9.4. *[53] q-series, or power series containing q-analogs of the common calculus operations, are introduced by quantum calculus. Numerous fields use these series for example, the q-binomial theorem in combinatorics and the q-analog of the partition function in number theory.*

2.10 Hankel Determinant

Definition 2.10.1. *The determinant of the related Hankel matrix is known as the Hankel determinant. The Hankel determinant was defined by Pommerenke [54] for the class of univalent functions for integers that are positive. n,s that are defined below,*

$$|\mathscr{H}_n(s)| = \begin{vmatrix} \breve{\gamma}_s & \breve{\gamma}_{s+1} & \breve{\gamma}_{s+2} & \cdots & \breve{\gamma}_{s+n-1} \\ \breve{\gamma}_{s+1} & \breve{\gamma}_{s+2} & \breve{\gamma}_{s+3} & \cdots & \breve{\gamma}_{s+n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \breve{\gamma}_{s+n-1} & \breve{\gamma}_{s+n} & \breve{\gamma}_{s+n+1} & \cdots & \breve{\gamma}_{s+2n-2} \end{vmatrix}.$$

2.11 Preliminary Lemmas

Here are a few lemmas which will be essential to advancing our findings in the chapters that follow.

Lemma 2.11.1. *[55, 56] If $p \in \mathscr{P}$ is of the form (2.1) with $c_1 \geq 0$, then*

$$c_1 = 2\tau_1 \tag{2.1}$$

$$c_2 = 2\tau_1^2 + 2(1-\tau_1^2)\tau_2 \tag{2.2}$$

$$c_3 = 2\tau_1^3 + 4(1-\tau_1^2)\tau_1\tau_2 - 2(1-\tau_1^2)\tau_1\tau_2^2 + 2(1-\tau_1^2)(1-|\tau_2|^2)\tau_3 \tag{2.3}$$

For some τ_1 [0,1] and $\tau_2, \tau_3 \in \overline{\mathbb{E}} := \{\hat{y} \in \mathbb{C} : |\hat{y}| \leq 1\}$.
For $\tau_1 \in \mathbb{T} := \hat{y} \in \mathbb{C} : |\hat{y}| = 1$, there is a distinct function $p \in \mathscr{P}$ with c_1 as in (2.1) namely

$$P(\hat{y}) = \frac{1+\tau_1\hat{y}}{1-\tau_1\hat{y}}, \ \hat{y} \in \mathbb{E}.$$

For $\tau_1 \in \mathbb{D} := \tau_2 \in \mathbb{T}$, there is a distinct function $p \in \mathscr{P}$ with c_1 and as in (2.1) and (2.2), namely

$$p(\hat{y}) = \frac{1+(\overline{\tau_1}\tau_2+\tau_1)\hat{y}+\tau_2 z^2}{1+((\overline{\tau_1}\tau_2-\tau_1)\hat{y}-\tau_2 z^2}, \ \hat{y} \in \mathbb{E}.$$

For $\tau_1, \tau_2 \in \mathbb{E}$ and $\tau_3 \in \mathbb{T}$, there is a unique funtion $p \in \mathscr{P}$ with c_1, c_2 and c_3 as in (2.1)-(2.3), namely

$$p(\hat{y}) = \frac{1+(\overline{\tau_2}\tau_3+\overline{\tau_1}\tau_2+\tau_1)\hat{y}+(\overline{\tau_1}\tau_3+\tau_1\overline{\tau_2}\tau_3+\tau_2)\hat{y}^2+\tau_3\hat{y}^3}{1+(\overline{\tau_2}\tau_3+\overline{\tau_1}\tau_2-\tau_1)\hat{y}+(\overline{\tau_1}\tau_3-\tau_1\overline{\tau_2}\tau_3-\tau_2)\hat{y}^2-\tau_3\hat{y}^3}, \ \hat{y} \in \mathbb{E}.$$

Lemma 2.11.2. *Assume that [57] A, B, and C are real numbers and*

$$Y(A,B,C) := max\ |A+B\hat{y}+C\hat{y}^2| + 1 - |\hat{y}|^2 : \hat{y} \in \overline{\mathbb{E}}.$$

(i) If $AC \geq 0$, then

$$Y(A,B,C) = \begin{cases} |A|+|B|+|C|, & |\mathrm{B}| \geq 2(1-|C|), \\ \\ 1+|A|+\frac{B^2}{4(1-|C|)}, & |B| < 2(1-|C|). \end{cases}$$

$$Y(A,B,C) = \begin{cases} 1-|A|+\frac{B^2}{4(1+|C|)} \ , \ -4AC(C^{-2}-1) \leq B^2 \ \wedge \ \ |B| < 2(1-|C|), \\ \\ 1+|A| + \frac{B^2}{4(1-|C|)}, \quad B^2 < min\ [4(1+|C|)^2, -4AC(C^{-2}-1)], \\ \\ R(A,B,C), \qquad\qquad\qquad\qquad Otherwise, \end{cases}$$

Where

$$R(A,B,C) := \begin{cases} |A|+|B|-|C|, \quad |C|(|B|+4|A|) \leq |AB|, \\ \\ -|A|+|B|+|C|, \quad |AB| \leq |C|(|B|-4|A|), \\ \\ (|C|+|A|)\sqrt{1-\frac{B^2}{4AC}}, \qquad Otherwise. \end{cases}$$

We split our thesis into two sections to make it more clear: one for starlike functions and the other for convex functions related to lune. These sections contain several function families belonging to class $\mathscr{A}$, and they support our primary findings.

CHAPTER 3

CLASS OF STARLIKE AND CONVEX FUNCTIONS WITH SECOND HANKEL DETERMINANT OF LOGRITHMIC COEFFICIENTS

3.1 Introduction

A number of essential and conventional findings that form the basis of additionally investigation are intended to be examined in this chapter. Reviewing Starlike functions and introducing a new class called Convex functions are the first two topics covered in this part. The establishment of these categories is related to lune function. A number of important discoveries will also be looked at, such as the second Hankel Determinants, the well-known Fekete–Szegő inequality, logrithmic coefficient bounds, and the subordination approach.

Definition 3.1 Class $S^*_{\leftmoon}$ given by Raina and Sokol [25].

$$S^*_{\leftmoon} = \left\{ f \in \mathrm{S} : \left| \left(\frac{\hat{y} f'(\hat{y})}{f(\hat{y})} \right)^2 - 1 \right| \leq 2 \left| \frac{\hat{y} f'(\hat{y})}{f(\hat{y})} \right|, \hat{y} \in \mathbb{E} \right\}$$

According to the function $f \in S^*_{\leftmoon}$, the region bounded by the lune is included in the ratio $\frac{\hat{y} f'(\hat{y})}{f(\hat{y})}$ for any $\hat{y} \in \mathbb{E}$. For example, $\{w \in C : |w^2 - 1| \leq 2|w|\}$ gives it.

Class $S^*_{\rightmoon}$ is established by applying the subordination concept in [51].

$$S^*_{\rightmoon} = \left\{ f \in S : \frac{\hat{y} f'(\hat{y})}{f(\hat{y})} \prec \hat{y} + \sqrt{1+\hat{y}^2} = q(\hat{y}), \ \hat{y} \in \mathbb{E} \right\}$$

where $q(0) = 1$ is the branch of the square root thus $\sqrt{1+0^2}$, or $\sqrt{1}$, should equal 1.

Definition 3.2 A class is considered to be convex in [25] acociated with lune function $C_{\rightmoon}$ if,

$$C_{\rightmoon} = \left\{ f \in \mathrm{S} : 1 + \frac{\hat{y} f''(\hat{y})}{f'(\hat{y})} \prec q(\hat{y}), \quad \hat{y} \in \mathbb{E} \right\}$$

3.2 The Logarithmic Coefficients of Class $S^*_{\rightmoon}$ Functions and their Second Hankel Determinant

Theorem 3.2.1. *Identifying the sharp bound of* $|\mathscr{H}_{2,1}(F_f/2)|$ *for function in the class* $S^*_{\rightmoon}$.

Let $f \in S^*_{\rightmoon}$. *Then*

$$|\mathscr{H}_{2,1}(F_f/2)| \leq \frac{1}{16}, \tag{3.1}$$

given the function $g \in S^*_{\rightmoon}$, *the inequality is sharp*

$$g(\hat{y}) = \hat{y}\, exp\left(\int_0^{\hat{y}_0} \frac{x^2 + \sqrt{1+x^4} - 1}{x} dx \right) = \hat{y} + \frac{\hat{y}^3}{2} + \frac{\hat{y}^5}{4} + \ldots$$

Proof. Let $f \in S^*_{\rightmoon}$. Therefore, under definition 1.1

$$\frac{\hat{y} f'(\hat{y})}{f(\hat{y})} = w(\hat{y}) + \sqrt{1 + w^2(\hat{y})}, \tag{3.2}$$

where w is a Schwarz function with $w(0) = 0$ and $|w(\hat{y})| \leq 1$ in $\mathbb{E}$.

let $h \in \mathscr{P}$, then write it as

$$w(\hat{y}) = \frac{h(\hat{y}) - 1}{h(\hat{y}) + 1} \tag{3.3}$$

$$h(\hat{y}) = 1 + c_1\hat{y} + c_2\hat{y}^2 + c_3\hat{y}^3 + c_4\hat{y}^4 + \ldots,$$

then (3.3) is

$$\frac{h(\hat{y}) - 1}{h(\hat{y}) + 1} = \frac{1 + c_1\hat{y} + c_2\hat{y}^2 + c_3\hat{y}^3 + c_4\hat{y}^4 - 1}{1 + c_1\hat{y} + c_2\hat{y}^2 + c_3\hat{y}^3 + c_4\hat{y}^4 + 1} = \frac{c_1\hat{y} + c_2\hat{y}^2 + c_3\hat{y}^3 + c_4\hat{y}^4}{2 + c_1\hat{y} + c_2\hat{y}^2 + c_3\hat{y}^3 + c_4\hat{y}^4}$$

$$w(\hat{y}) = \frac{1}{2}c_1(\hat{y}) + \frac{1}{2}(c_2 - \frac{1}{2}c_1^2)(\hat{y})^2 + \frac{1}{2}(c_3 - c_1c_2 + \frac{1}{4}c_1^3)(\hat{y})^3 + \ldots, \tag{3.4}$$

simplification of (3.2) is

$$\frac{\hat{y}f'(\hat{y})}{f(\hat{y})} = \frac{\hat{y} + 2a_2\hat{y}^2 + 3a_3\hat{y}^3 + 4\text{a}_4\hat{y}^4}{\hat{y} + a_2\hat{y}^2 + a_3\hat{y}^3 + a_4\hat{y}^4}$$

$$\frac{\hat{y}f'(\hat{y})}{f(\hat{y})} = 1 + a_2\hat{y} + (2a_3 - a_2^2)\hat{y}^2 + (3a_4 - 3a_2a_3 + a_2^3)\hat{y}^3 + \ldots, \tag{3.5}$$

from (3.2) and (3.4), a simple computation shows that

$$w(\hat{y}) + \sqrt{1 + w^2(\hat{y})} = \frac{1}{2}c_1\hat{y} + \frac{1}{2}(c_2 - \frac{1}{2}c_1^2)\hat{y}^2 + \sqrt{1 + \left(\frac{1}{2}c_1\hat{y} + \frac{1}{2}(c_2 - \frac{1}{2}c_1^2)\hat{y}^2\right)^2}$$

$$\sqrt{1 + w^2(\hat{y})} = \sqrt{1 + \frac{1}{4}c_1^2\hat{y}^2 + \frac{1}{4}\left(c_2 - \frac{1}{2}c_1^2\right)^2\hat{y}^4 + \frac{1}{2}c_1(c_2 - \frac{1}{2}c_1^2)\hat{y}^3},$$

using bionomial expansion:

$$\sqrt{1+x} = 1 + \frac{1}{2}x + \frac{1}{8}x^2 + \frac{1}{16}x^3 + \ldots$$

$$x = \frac{1}{4}c_1^2\hat{y}^2 + \frac{1}{4}\left(c_2 - \frac{1}{2}c_1^2\right)^2\hat{y}^4 + \frac{1}{2}c_1(c_2 - \frac{1}{2}c_1^2)\hat{y}^3,$$

thus we have

$$\sqrt{1 + w^2(\hat{y})} = \sqrt{1+x} = 1 + \frac{1}{2}x + \frac{1}{8}x^2$$

$$= 1 + \frac{1}{2}\left(\frac{1}{4}c_1^2\hat{y}^2 + \frac{1}{4}(c_2 - \frac{1}{2}c_1^2)^2\hat{y}^4 + \frac{1}{2}c_1(c_2 - \frac{1}{2}c_1^2)\hat{y}^3\right)$$

$$+ \frac{1}{8}\left(\frac{1}{4}c_1^2\hat{y}^2 + \frac{1}{4}(c_2 - \frac{1}{2}c_1^2)\hat{y}^4 + \frac{1}{2}(c_2 - \frac{1}{2}c_1^2)\hat{y}^3\right)^2 \ldots$$

$$\sqrt{1 + w^2(\hat{y})} = 1 + \frac{1}{8}c_1^2\hat{y}^2 + \frac{1}{4}c_1(c_2 - \frac{1}{2}c_1^2)\hat{y}^3 + \frac{1}{8}(c_2 - \frac{1}{2}c_1^2)^2\hat{y}^4, \tag{3.6}$$

now we obtain

$$w(\hat{y}) + \sqrt{1 + w^2(\hat{y})} = \frac{1}{2}c_1\hat{y} + \frac{1}{2}(c_2 - \frac{1}{2}c_1^2)\hat{y}^2 + \frac{1}{2}(c_3 - c_1c_2 + \frac{1}{4}c_1^3)\hat{y}^3 + 1 + \frac{1}{8}c_1^2\hat{y}^2 + \frac{1}{4}c_1(c_2 - \frac{1}{2}c_1^2)\hat{y}^3,$$

by adding similar terms of $\hat{y}^2$ and$\hat{y}^3$

$$w(\hat{y}) + \sqrt{1 + w^2(\hat{y})} = 1 + \frac{1}{2}c_1\hat{y} - \left(\frac{1}{8}c_1^2 + \frac{1}{2}c_2\right)\hat{y}^2 + \left(\frac{1}{2}c_3 - \frac{1}{4}c_1c_2\right)\hat{y}^3, \tag{3.7}$$

so (3.2) is

$$\frac{\hat{y}f'(\hat{y})}{f(\hat{y})} = w(\hat{y}) + \sqrt{1+w^2(\hat{y})},$$

after comparing (3.5) and (3.7) we have

1. Oder of $\hat{y}^0$:

$$\hat{y}^0 = 1$$

2. Oder of $\hat{y}^1$:

$$a_2 = \frac{1}{2}c_1$$

3. Oder of $\hat{y}^2$:

$$\begin{aligned} 2a_3 - a_2^2 &= -\frac{1}{8}c_1^2 + \frac{1}{2}c_2 \\ 2a_3 &= \frac{1}{2}c_2 - \frac{1}{8}c_1^2 + \frac{1}{4}{c_1}^2 \\ a_3 &= \frac{1}{4}c_2 + \frac{1}{16}c_1^2 , \end{aligned}$$

similarly

4. Oder of $\hat{y}^3$:

$$\begin{aligned} (3a_4 - 3a_2a_3 + a_2^3) &= \left(\frac{1}{2}c_3 - \frac{1}{4}c_1c_2\right) \\ 3a_4 - 3\left(\frac{1}{2}c_1\right)\left(\frac{1}{4}c_2 + \frac{1}{16}c_1^2\right) + \left(\frac{1}{2}c_1\right)^3 &= \frac{1}{2}c_3 - \frac{1}{4}c_1c_2 \\ 3a_4 - \left(\frac{3}{8}c_1c_2 + \frac{1}{32}c_1^3\right) &= \frac{1}{2}c_3 - \frac{1}{4}c_1c_2 \\ 3a_4 &= \frac{3}{8}c_1c_2 - \frac{1}{32}c_1^3 + \frac{1}{2}c_3 - \frac{1}{4}c_1c_2 \\ 3a_4 &= \frac{1}{8}c_1c_2 - \frac{1}{32}c_1^3 + \frac{1}{2}c_3 \\ a_4 &= \frac{1}{24}c_1c_2 + \frac{1}{6}c_3 + \frac{1}{96}c_1^3 \end{aligned}$$

$$\begin{cases} a_2 = \frac{1}{2}c_1 \\ a_3 = \frac{1}{4}c_2 + \frac{1}{16}c_1^2 \\ a_4 = \frac{1}{24}c_1c_2 + \frac{1}{6}c_3 + \frac{1}{96}c_1^3 \end{cases} \tag{3.8}$$

Since the class $\mathscr{P}$ and $\mathscr{H}_{2,1}(F_f/2)$ is invariant under rotation, and we assume that $c_1 \in [0,2]$ that is in view of (1.13) that $\tau \in [0,1]$. Using (3.8) in (1.7) We possess

$$\mathscr{H}_{2,1}(F_f/2) = \frac{1}{48}\left(\frac{1}{2}\,c_1\right)^4 - 12\left(\frac{1}{4}c_2 + \frac{1}{16}c_1^2\right)^2 + 12\left(\frac{1}{2}\,c_1\right)\left(\frac{1}{24}c_1c_2 + \frac{1}{6}c_3 + \frac{1}{96}c_1^3\right)$$

$$\mathscr{H}_{2,1}(F_f/2) = \frac{1}{48}\left(\frac{1}{16}c_1^4 - \frac{12}{16}c_2^2 - \frac{12}{256}c_1^4 - \frac{12}{32}c_1^2c_2 + \frac{12}{48}c_1^2c_2 + \frac{12}{12}c_1c_3 - \frac{12}{192}c_1^4\right),$$

after further simplification

$$\mathscr{H}_{2,1}(F_f/2) = \frac{1}{48}\left(-\frac{3}{4}c_2^2 - \frac{3}{64}c_1^4 - \frac{1}{8}c_1^2c_2 + c_1c_3\right)$$

$$\mathscr{H}_{2,1}(F_f/2) = \frac{1}{48\times 64}\left(-48c_2^2 - 3c_1^4 - 8c_1^2c_2 + 64c_1c_3\right)$$

$$\mathscr{H}_{2,1}(F_f/2) = \frac{1}{3072}\left(-3c_1^4 - 8c_1^2c_2 - 48c_2^2 + 64c_1c_3\right), \tag{3.9}$$

by the Lemma (2.12.1) we have value of c_1, c_2 and c_3 use in (3.9)

$$\mathscr{H}_{2,1}(F_f/2) = \frac{1}{3072}\,[-3(2\tau_1)^4 - 8(2\tau_1)^2\left(2\left(1-\tau_1^2\right)\tau_2\right) - 48(2\tau_1^2 + 2(1-\tau_1^2)\tau_2)^2$$
$$+ 64(2\tau_1)(2\tau_1^3 + 4(1-\tau_1^2)\tau_1\tau_2 - 2(1-\tau_1^2)\tau_1\tau_2^2 + 2(1-\tau_1^2)(1-|\tau_2|^2)\tau_3)\,]$$

$$\mathscr{H}_{2,1}(F_f/2) = \frac{1}{3072}[-48\tau_1^4 - 64\tau_1^4 - 64\tau_1^2\tau_2 + 64\tau_1^4\tau_2 - 192\tau_1^4 - 384\tau_1^2\tau_2 + 384\tau_1^4\tau_2 + 384\tau_1^2\tau_2^2 - 192\tau_2^2$$
$$-192\tau_1^4\tau_2^2 + 256\tau_1^4 + 512\tau_1^2(1-\tau_1^2)\tau_2 - 256\,(1-\tau_1^2)\tau_1^2\tau_2^2 + 256\tau_1(1-\tau_1^2)(1-|\tau_2|^2)\tau_3]$$

$$\mathscr{H}_{2,1}(F_f/2) = \frac{1}{3072}\,[-304\tau_1^4 - 448\tau_1^2\tau_2 + 448\tau_1^4\tau_2 - 192\tau_2^2 - 384\tau_1^2\tau_2^2 - 192\tau_1^4\tau_2^2 + 256\tau_1^4 + 512\tau_1^2\tau_2$$
$$-512\tau_1^4\tau_2 - 256\,\tau_1^2\tau_2^2 + 256\,\tau_1^4\tau_2^2 + 256\tau_1\tau_3 - 256\tau_1^3\tau_3 - 256\tau_1\tau_3|\tau_2|^2 + 256\tau_1^3|\tau_2|^2\tau_3]$$

$$\mathscr{H}_{2,1}(F_f/2) = \frac{1}{16\times 192}\,[-48\tau_1^4 + 64\tau_1^2\tau_2 - 64\tau_1^4\tau_2 + 64\tau_1^4\tau_2^2 + 128\tau_1^2\tau_2^2 - 192\tau_2^2 + 256\tau_1\tau_3 - 256\tau_1^3\tau_3$$
$$-256\tau_1\tau_3|\tau_2|^2 + 256\tau_1^3|\tau_2|^2\tau_3\,]$$

$$\mathscr{H}_{2,1}(F_f/2) = \frac{1}{192}\,[-3\tau_1^4 + 4\tau_1^2\tau_2 - 4\tau_1^4\tau_2 + 4\tau_1^4\tau_2^2 + 8\tau_1^2\tau_2^2 - 12\tau_2^2 + 16\tau_1\tau_3 - 16\tau_1^3\tau_3 - 16\tau_1\tau_3|\tau_2|^2$$
$$+ 16\tau_1^3|\tau_2|^2\tau_3],$$

as we know that

1. $-4(1-\tau_1^2)(3+\tau_1^2)\ \tau_2^2 = 4\tau_2^2\tau_1^4+8\tau_1^2\tau_2^2-12\tau_2^2$

2. $16\tau_1\tau_3(1-\tau_1^2)(1-|\tau_2^2|) = +16\tau_1\tau_3-16\tau_1^3\tau_3-16\tau_1\tau_3|\tau_2|^2+16\tau_1^3|\tau_2|^2\tau_3$

$$\mathscr{H}_{2,1}(F_f/2) = \frac{1}{192}\ [-3\tau_1^4+4(1-\tau_1^2)\tau_1^2\tau_2-4(1-\tau_1^2)(3+\tau_1^2)\ \tau_2^2\ +16\tau_1\tau_3(1-\tau_1^2)(1-|\tau_2^2|)\ ], \tag{3.10}$$

Examine the subsequent possible cases on τ_1:

Case 1.

if $\tau_1 = 1$ then from (3.10) we easily obtain

$$|\mathscr{H}_{2,1}(F_f/2)| = \frac{1}{192}[-3+0]$$
$$|\mathscr{H}_{2,1}(F_f/2)| = \frac{1}{64}$$

Case 2.

if $\tau_1 = 0$ then from (3.10) we see that

$$|\mathscr{H}_{2,1}(F_f/2)| = \frac{1}{192}[-4(3)\tau_2^2\ ]$$
$$|\mathscr{H}_{2,1}(\frac{F_f}{2})| = \frac{1}{16}|\tau_2^2\ | \leq \frac{1}{16}$$

Case 3.

Assume $\tau \in (0,1)$. Utilising the triangle inequality in (3.10) and by using the fact that $|\tau_3| \leq 1$, we obtain

$$\mathscr{H}_{2,1}(F_f/2) = \frac{1}{192}\ [-3\tau_1^4+4(1-\tau_1^2)\tau_1^2\tau_2-4(1-\tau_1^2)(3+\tau_1^2)\ \tau_2^2\ +16\tau_1\ (1-\tau_1^2)(1-|\tau_2^2|)\ ],$$

if $\frac{1}{12}\tau_1\ (1-\tau_1^2)$ common from inside we have

$$\mathscr{H}_{2,1}(F_f/2) = \frac{1}{12}\tau_1\ (1-\tau_1^2)\left\{\frac{-3\tau_1^3}{16\left(1-\tau_1^2\right)}+\frac{4\tau_1\tau_2}{16}-\frac{\left(3+\tau_1^2\right)\ \tau_2^2}{4\tau_1}+(1-|\tau_2^2|)\right\}, \tag{3.11}$$

where

$$A = \frac{-3\tau_1^3}{16\left(1-\tau_1^2\right)}, \qquad B = \frac{\tau_1}{4}, \qquad C = \frac{-\ (3-\tau_1^2)}{4\tau_1},$$

so, we have

$$= \frac{1}{12}\tau_1\ \left(1-\tau_1^2\right)(|A+B\tau_2+C\tau_2^2\ |+1-|\tau_2^2|), \tag{3.12}$$

Note that $AC > 0$, allowing us to use Lemma (2.12.2) case (i). We now examine every circumstance in case (i).

3(a) The inequality is observed:

$$|B|-2\left(1-|C|\right)=\frac{\tau_1}{4}-2\left(1-\frac{-(3-\tau_1^2)}{4\tau_1}\right)$$

$$=\frac{\tau_1}{4}\ -2+\frac{2\ (3-\ \tau_1^2)}{4\tau_1}$$

$$=\frac{2\tau_1\left(\tau_1-8\right)+4(3+\tau_1^2)}{(2\tau_1)(4)}$$

$$=\frac{(6\tau_1^2+16\tau_1+12)}{8\tau_1}$$

$$|B|-2\left(1-|C|\right)=\frac{(3\tau_1^2-8\tau_1+6)}{4\tau_1}>0.$$

Which is true for all $\tau_1\in(0,1)$. Consequently, Lemma (2.12.2) leads to furthermore, the inequality (3.12) that

$$|\mathscr{H}_{2,1}(F_f/2)|\leq\frac{1}{12}\tau_1\left(1-\tau_1^2\right)\left(|A|+|B|+|C|\right)$$

$$=\frac{1}{12}\tau_1\left(1-\ \tau_1^2\right)\left(\left|\frac{-3\tau_1^3}{16\left(1-\ \tau_1^2\right)}\right|+\left|\frac{\tau_1}{4}\right|+\left|\frac{-(3-\ \tau_1^2)}{4\tau_1}\right|\right)$$

$$=\frac{-3\tau_1^4}{192}+\frac{\tau_1^2\left(1-\tau_1^2\right)}{48}+\frac{\tau_1\left(1-\tau_1^2\right)-\ (3-\ \tau_1^2)}{48\tau_1}$$

$$=\frac{-3\tau_1^4+4\tau_1^2-4\tau_1^4+12\tau_1^2+12\tau_1^4-12-12\tau_1^2}{192}$$

$$=\frac{4\tau_1^2-12+5\tau_1^4}{192}$$

$$|\mathscr{H}_{2,1}(F_f/2)|=\frac{1}{192}(12-4\tau_1^2-5\tau_1^4),$$

if τ_1= 0 then

$$\begin{aligned}|\mathscr{H}_{2,1}(F_f/2)| &= \frac{1}{192}[-12\ -4(0)-5(0)]\\ &= \frac{1}{192}[-12]\\ &= \left|\frac{1}{192}\ [-12]\right| \leq \frac{1}{16}\\ |\mathscr{H}_{2,1}(F_f/2)| &\leq \frac{1}{16}\end{aligned}$$

3(b) Next, it's simple to verify that

$$\begin{aligned}|B|-2(1-|C|) &= \frac{\tau_1}{4}\ -2\left(1-\frac{-(3-\ \tau_1^2)}{4\tau_1}\right)\\ &= \frac{(3\tau_1^2-8\tau_1+6)}{4\tau_1}<0,\end{aligned}$$

which is not true for all $\tau_1 \in (0,1)$.

Summarizing case The inequality (3.1) is established for 1, 2, and 3.It is enough to demonstrate that the bound is crisp to finish the evidence. The function $g \in S^*_{\complement}$ is examined as follows to demonstrate that

$$g(\hat{y}) = zexp\left(\int_0^z \frac{x^2+\sqrt{1+x^4}-1}{x}dx\right) = z+\frac{\hat{y}^3}{2}+\frac{\hat{y}^5}{4}+\ldots,$$

with a_2= $a_4 = 0$ and $a_3 = \frac{1}{12}$ use in (1.7) by straightforward calculation, it is readily seen that $|\mathscr{H}_{2,1}(F_g/2)| = \frac{1}{16}$. This complete the proof. □

3.3 The Logarithmic Coefficients of Class $C_{\complement}$ Functions and their Second Hankel Determinant

Theorem 3.3.1. *Let $f\ \in\ C_{\complement}$ .Then*

$$|\mathscr{H}_{2,1}(F_f/2)| \leq \frac{23}{3264}, \tag{3.13}$$

given the function $h\ \in\ C_{\complement}$, the inequality is sharp

$$h(\hat{y}) = \int_0^{\hat{y}} \frac{h_0(x)}{x}dx = \hat{y}+\frac{\sqrt{69}}{12\sqrt{17}}\hat{y}^3+\frac{1}{20}\left(\frac{69}{136}+\frac{\sqrt{69}}{4\sqrt{17}}\right)\hat{y}^5+\ldots,$$

where $h_0(\hat{y})$ is given by (3.20).

Proof. Suppose $f \in C_{\mathbb{C}}$. Then, as defined by (1.1), we observe that

$$1+\frac{\hat{y}f''(\hat{y})}{f'(\hat{y})}=w(\hat{y})+\sqrt{1+w^2(\hat{y})}, \tag{3.14}$$

where w is a Schwarz function with $w(0)=0$ and $|w(\hat{y})|\leq 1$ in $\mathbb{E}$. let $h\in\mathscr{P}$.Then we can write

$$w(\hat{y})=\frac{h(\hat{y})-1}{h(\hat{y})+1} \tag{3.15}$$

$$h(\hat{y})=1+c_1\hat{y}+c_2\hat{y}^2+c_3\hat{y}^3+c_4\hat{y}^4+\ldots,$$

after simplification (3.15) is

$$\frac{h(\hat{y})\ -\ 1}{h(\hat{y})\ +\ 1}=\frac{1+c_1\hat{y}+c_2\hat{y}^2+c_3\hat{y}^3+c_4\hat{y}^4-1}{1+c_1z+c_2\hat{y}^2+c_3\hat{y}^3+c_4\hat{y}^4+1}=\frac{c_1\hat{y}+c_2\hat{y}^2+c_3\hat{y}^3+c_4\hat{y}^4}{2+c_1z+c_2\hat{y}^2+c_3\hat{y}^3+c_4\hat{y}^4} \tag{3.16}$$

$$w(\hat{y})=\frac{1}{2}c_1\hat{y}+\frac{1}{2}(c_2-\frac{1}{2}c_1^2)\hat{y}^2+\frac{1}{2}(c_3-c_1c_2+\frac{1}{4}c_1^3)\hat{y}^3+\ldots,$$

computation of (3.14) is

$$1+\frac{\hat{y}f''(\hat{y})}{f'(\hat{y})}=\frac{1+4a_2\hat{y}+9a_3\hat{y}^2+16a_4\hat{y}^3+25a_5\hat{y}^4}{\hat{y}+a_2\hat{y}^2+a_3\hat{y}^3+a_4\hat{y}^4}$$

$$1+\frac{\hat{y}f''(\hat{y})}{f'(\hat{y})}=1+2a_2\hat{y}+(6a_3-4a_2^2)\hat{y}^2+(12a_4-18a_2a_3+8a_2^3)\hat{y}^3\ldots, \tag{3.17}$$

simple calculation from (3.14) and (3.16), reveals that

$$w(\hat{y})+\sqrt{1+w^2(\hat{y})}=\frac{1}{2}c_1\hat{y}+\frac{1}{2}(c_2-\frac{1}{2}c_1^2)\hat{y}^2+\sqrt{1+\left(\frac{1}{2}c_1z+\frac{1}{2}(c_2-\frac{1}{2}c_1^2)\hat{y}^2\right)^2}$$

$$w(\hat{y})+\sqrt{1+w^2(\hat{y})}=\sqrt{1+\frac{1}{4}c_1^2\hat{y}^2+\frac{1}{4}(c_2-\frac{1}{2}c_1^2)\hat{y}^4+\frac{1}{2}c_1(c_2-\frac{1}{2}c_1^2)\hat{y}^3},$$

by using Bionomial expansion that is

$$\sqrt{1+x}=1+\frac{1}{2}x+\frac{1}{8}x^2+\frac{1}{16}x^3+\ldots$$
$$x=\frac{1}{4}c_1^2\hat{y}^2+\frac{1}{4}(c_2-\frac{1}{2}c_1^2)\hat{y}^4+\frac{1}{2}c_1(c_2-\frac{1}{2}c_1^2)\hat{y}^3,$$

$$\sqrt{1+w^2(\hat{y})} = \sqrt{1+x} = 1+\frac{1}{2}x+\frac{1}{8}x^2$$

$$1+\frac{1}{2}x-\frac{1}{8}x^2 = 1+\frac{1}{2}\left(\frac{1}{4}c_1^2\hat{y}^2+\frac{1}{4}(c_2-\frac{1}{2}c_1^2)\hat{y}^4+\frac{1}{2}c_1(c_2-\frac{1}{2}c_1^2)\hat{y}^3\right)$$
$$+\frac{1}{8}\left(\frac{1}{4}c_1^2\hat{y}^2+\frac{1}{4}(c_2-\frac{1}{2}c_1^2)\hat{y}^4+\frac{1}{2}c_1(c_2-\frac{1}{2}c_1^2)\hat{y}^3\right)^2$$

$$\sqrt{1+w^2(\hat{y})} = 1+\frac{1}{8}c_1^2\hat{y}^2+\frac{1}{4}c_1(c_2-\frac{1}{2}c_1^2)\hat{y}^3+\frac{1}{8}(c_2-\frac{1}{2}c_1^2)\hat{y}^4, \tag{3.18}$$

now we obtain

$$w(\hat{y})+\sqrt{1+w^2(\hat{y})} = \frac{1}{2}c_1\hat{y}+\frac{1}{2}(c_2-\frac{1}{2}c_1^2)\hat{y}^2+\frac{1}{2}(c_3-c_1c_2+\frac{1}{4}c_1^3)\hat{y}^3+1+\frac{1}{8}c_1^2\hat{y}^2$$
$$+\frac{1}{4}c_1(c_2-\frac{1}{2}c_1^2)\hat{y}^3,$$

after adding similar terms of $\hat{y}^2$ and z^3

$$w(\hat{y})+\sqrt{1+w^2(\hat{y})} = 1+\frac{1}{2}c_1\hat{y}-\left(\frac{1}{8}c_1^2+\frac{1}{2}c_2\right)\hat{y}^2+\left(\frac{1}{2}c_3-\frac{1}{4}c_1c_2\right)\hat{y}^3 \tag{3.19}$$

So, (3.14) will become,

$$1+\frac{\hat{y}f''(\hat{y})}{f'(\hat{y})} = w(\hat{y})+\sqrt{1+w^2(\hat{y})},$$

by comparing (3.17) and (3.19) we have,

Oder of $\hat{y}^0$:-

$$\hat{y}^0 = 1$$

Oder of $\hat{y}^1$:-

$$a_2 = \frac{1}{4}c_1$$

Oder of $\hat{y}^2$:-

$$6a_3-4a_2^2 = -\frac{1}{8}c_1^2+\frac{1}{2}c_2$$
$$6a_3-4(\frac{1}{4}c_1)^2 = -\frac{1}{8}c_1^2+\frac{1}{2}$$
$$6a_3 = \frac{1}{2}c_2+\frac{1}{8}c_1^2$$
$$a_3 = \frac{1}{12}c_2+\frac{1}{48}c_1^2,$$

similarly,

Oder of $\hat{y}^3$:-

$$a_4 = \frac{1}{96}c_1c_2+\frac{1}{24}c_3-\frac{1}{384}c_1^3$$

$$\begin{cases} a_2 = \frac{1}{4}c_1 \\ a_3 = \frac{1}{12}c_2 + \frac{1}{48}c_1^2 \\ a_4 = \frac{1}{96}c_1c_2 + \frac{1}{24}c_3 + \frac{1}{384}c_1^3 \end{cases} \tag{3.20}$$

Since the class $\mathscr{P}$ and $\mathscr{H}_{2,1}(\mathrm{F_f}/2)$ is invariant under rotation, and we assume that $c_1 \in [0,2]$ that is in view of (1.13) that $\tau \in [0,1]$. Using (3.20) in (1.7) we have

$$\mathscr{H}_{2,1}(F_f/2) = \frac{1}{48}\left(\frac{1}{4}c_1\right)^4 - 12\left(\frac{1}{12}c_2 + \frac{1}{48}c_1^2\right)^2 + 12\left(\frac{1}{4}c_1\right)\left(\frac{1}{96}c_1c_2 + \frac{1}{24}c_3 - \frac{1}{384}c_1^3\right)$$

$$= \frac{1}{48}\left(\frac{1}{256}c_1^4 - \frac{12}{144}c_2^2 - \frac{12}{2304}c_1^4 - \frac{24}{576}c_1^2c_2 + \frac{12}{384}c_1^2c_2 + \frac{12}{96}c_1c_3 - \frac{12}{1536}c_1^4\right)$$

$$= \frac{1}{48}\left(-\frac{448}{49152}c_1^4 - \frac{8}{768}c_1^2c_2 - \frac{12}{144}c_2^2 + \frac{12}{96}c_1c_3\right)$$

$$= \frac{1}{48}\left(-\frac{7}{768}c_1^4 - \frac{1}{96}c_1^2c_2 - \frac{1}{12}c_2^2 + \frac{1}{8}c_1c_3\right)$$

$$= \frac{1}{48 \times 768}\left(-7c_1^4 - 8c_1^2c_2 - 64c_2^2 + 96c_1c_3\right)$$

$$\mathscr{H}_{2,1}(F_f/2) = \frac{1}{36864}\left(-7c_1^4 - 8c_1^2c_2 - 64c_2^2 + 96c_1c_3\right), \tag{3.21}$$

by the Lemma (2.12.1) we have value of c_1, c_2 and c_3 use in (3.21)

$$\mathscr{H}_{2,1}(F_f/2) = \frac{1}{36864}[-7(2\tau_1)^4 - 8(2\tau_1)^2\left(2\left(1-\tau_1^2\right)\tau_2\right) - 64(2\tau_1^2+2(1-\tau_1^2)\tau_2)^2$$
$$+96(2\tau_1)(2\tau_1^3+4(1-\tau_1^2)\tau_1\tau_2 - 2(1-\tau_1^2) + 2(1-\tau_1^2)(1-|\tau_2|^2)\tau_3)]$$

$$\mathscr{H}_{2,1}(F_f/2) = \frac{1}{36864}[-112\tau_1^4 - 64\tau_1^4 - 64\tau_1^2\tau_2 + 64\tau_1^4\tau_2 - 256\tau_1^4 - 512\tau_1^2\tau_2 + 512\tau_1^4\tau_2 + 512\tau_1^2\tau_2^2$$
$$-256\tau_2^2 - 256\tau_1^4\tau_2^2 + 384\tau_1^4 + 768\tau_1^2(1-\tau_1^2)\tau_2 - 384(1-\tau_1^2)\tau_1^2\tau_2^2$$
$$+384\tau_1(1-\tau_1^2)(1-|\tau_2|^2)\tau_3],$$

after more simplification and adding similar terms

$$\mathscr{H}_{2,1}(F_f/2) = \frac{1}{36864}[-432\tau_1^4 - 576\tau_1^2\tau_2 + 576\tau_1^4\tau_2 - 256\tau_2^2 - 512\tau_1^2\tau_2^2 - 256\tau_1^4\tau_2^2 + 384\tau_1^4 + 768\tau_1^2\tau_2$$
$$-768\tau_1^4\tau_2 - 384\tau_1^2\tau_2^2 + 384\tau_1^4\tau_2^2 + 384\tau_1(1-\tau_1^2)(1-|\tau_2|^2)\tau_3]$$

$$\mathscr{H}_{2,1}(F_f/2) = \frac{1}{2304 \times 16}[-48\tau_1^4 - 192\tau_1^2\tau_2 - 192\tau_1^4\tau_2 + 128\tau_1^4\tau_2^2 + 128\tau_1^2\tau_2^2 - 256\tau_2^2$$
$$+384\tau_1(1-\tau_1^2)(1-|\tau_2|^2)\tau_3]$$

$$\mathscr{H}_{2,1}(F_f/2) = \frac{1}{2304}[-3\tau_1^4 - 12\tau_1^2\tau_2 - 12\tau_1^4\tau_2 - 16\tau_2^2 + 8\tau_1^4\tau_2^2 + 8\tau_1^2\tau_2^2 + 24\tau_1\tau_3\ (1-\tau_1^2)(\ 1-|\tau_2|^{\ 2})\ ],$$

as we know

1. $12\tau_1^2\tau_2 - 12\tau_1^4\tau_2 = 12(1-\ \tau_1^2)\tau_1^2\tau_2^2$
2. $-16\tau_2^2 + 8\tau_1^4\tau_2^2 + 8\tau_1^2\tau_2^2 = -8(1-\tau_1^2)(2+\tau_1^2)\tau_2^2$

$$\mathscr{H}_{2,1}(F_f/2) = \frac{1}{2304}[-3\tau_1^4 + 12(1-\ \tau_1^2)\tau_1^2\ \tau_2^2\ - 8(1-\ \tau_1^2)\ (2+\ \tau_1^2)\ \tau_2^2\ + 24\tau_1\tau_3(1-\ \tau_1^2)(1-|\tau_2^2|)]. \tag{3.22}$$

Examine the subsequent possible cases on τ_1:

Case 1.

if $\tau_1 = 1$ then from (3.22), we obtain

$$|\mathscr{H}_{2,1}(F_f/2)| = \frac{1}{2304}[-3+0]$$
$$|\mathscr{H}_{2,1}(F_f/2)| = \frac{1}{768}$$

Case 2.

if $\tau_1 = 0$ then from (3.22), we see that is

$$|\mathscr{H}_{2,1}(F_f/2)| = \frac{1}{2304}[-8(2)\tau_2^2\ + 0]$$
$$= |\frac{-16}{2304}\tau_2^2\ | = \frac{1}{144}\tau_2^2$$
$$|\mathscr{H}_{2,1}(F_f/2)| = \frac{1}{144}\tau_2^2$$

Case 3.

Suppose $\tau\ \in\ (0,1)$. Utilising (3.22), and the triangle inequality, as well as the fact that $|\tau_3| \leq 1$, we obtain:

$$\mathscr{H}_{2,1}(F_f/2) = \frac{1}{2304}[-3\tau_1^4 - 12\tau_1^2\tau_2 - 12\tau_1^4\tau_2 - 16\tau_2^2 + 8\tau_1^4\tau_2^2 + 8\tau_1^2\tau_2^2 + 24\tau_1\tau_3\ (1-\tau_1^2)(\ 1-|\tau_2|^{\ 2})],$$

taking $\frac{1}{96}\tau_1\ (1-\ \tau_1^2)$ common from inside the term

$$\mathscr{H}_{2,1}(F_f/2) = \frac{1}{96}\tau_1\ (1-\ \tau_1^2)\left\{\frac{-3\tau_1^3}{8\left(1-\ \tau_1^2\right)} + \frac{\tau_1\tau_2}{2}\ -\ \frac{\left(2-\tau_1^2\right)\ \tau_2^2}{3\tau_1} + (1-\ |\tau_2^2|)\right\}, \tag{3.23}$$

where

$$A = \frac{-\tau_1^3}{8\left(1-\ \tau_1^2\right)}, \qquad B = \frac{\tau_1}{2}, \qquad C = \frac{-\ (2+\tau_1^2)}{3\tau_1},$$

now we obtain

$$\mathscr{H}_{2,1}(F_f/2) = \frac{1}{96}\tau_1 \ \left(1- \ \tau_1^2\right)\left(|A+B\tau_2+C\tau_2^2\ |+1- |\tau_2^2|\right). \tag{3.24}$$

Since we can see that $AC > 0$, we can use Lemma (2.12.2) case (i). We now examine every condition of case (i).

3(a) The inequality is seen

$$\begin{aligned}|B|-2(1-|C|) &= \frac{\tau_1}{2}\ -2\left(1-\frac{-\ (2+\tau_1^2)}{3\tau_1}\right)\\ &= \frac{\tau_1}{2}\ -2+\frac{2\ (2+\tau_1^2)}{3\tau_1}\\ &= \frac{\tau_1-4}{2}\ +\frac{4+2\tau_1^2}{3\tau_1}\\ &= \frac{(3\tau_1\,(\tau_1-4)+2(4+2\tau_1^2)}{(6\tau_1)}\\ &= \frac{7\tau_1^2-12\tau_1+8}{6\tau_1}\\ |B|-2(1-|C|) &= \frac{(7\tau_1^2-12\tau_1+8)}{6\tau_1} > 0,\end{aligned}$$

which is true for all $\tau_1 \in (0,1)$. Lemma (2.12.2) and the inequality (3.24) therefore imply that

$$\begin{aligned}|\mathscr{H}_{2,1}(F_f/2)| &\le \frac{1}{96}\tau_1\ \left(1-\ \tau_1^2\right)(|A|+|B|+|C|)\\ &= \frac{1}{96}\tau_1\ \left(1-\ \tau_1^2\right)\left(\left|\frac{-\tau_1^3}{8\left(1-\ \tau_1^2\right)}\right|+\left|\frac{\tau_1}{2}\right|+\left|\frac{-\ (2+\tau_1^2)}{3\tau_1}\right|\right)\\ &= \frac{\tau_1^4}{768}+\frac{\tau_1^2\left(1-\ \tau_1^2\right)}{192}+\frac{\tau_1\ \left(1-\ \tau_1^2\right)\left(2+\tau_1^2\right)}{288\tau_1}\\ &= \frac{\tau_1^4}{768}+\frac{\tau_1^2-\ \tau_1^4}{192}+\frac{-2-2\tau_1^2+\tau_1^2-\tau_1^4}{288}\\ &= \frac{3\tau_1^4+12\tau_1^2-12\tau_1^4-16-16\tau_1^2+8\tau_1^4-8\tau_1^4}{2304}\\ |\mathscr{H}_{2,1}(F_f/2) &= \frac{4\tau_1^2+16-17\tau_1^4}{2304},\end{aligned}$$

if $\tau_1 = 0$ then

$$\begin{aligned}|\mathscr{H}_{2,1}(F_f/2)| &= \frac{1}{2304}[16\,]\\ |\mathscr{H}_{2,1}(F_f/2)| &= \left|\frac{1}{2304}\ [16\,]\ \right| \le\ \frac{1}{144}\\ |\mathscr{H}_{2,1}(F_f/2)| &= \le\ \frac{1}{144}\end{aligned}$$

3(**b**) Next, it is easy to check that

$$|B|-2(1-|C|)=\frac{\tau_1}{2}-2\left(1-\frac{(2+\tau_1^2)}{4\tau_1}\right)$$
$$|B|-2(1-|C|)=\frac{(7\tau_1^2-12\tau_1+8)}{6\tau_1}<0,$$

which is not true for all $\tau_1 \in (0,1)$.Summarizing the inequality (3.13) is proven in cases 1, 2, and 3.It is enough to demonstrate that the bound is sharp to finish the evidence.To demonstrate that we take into account the function $h \in C_{☾}$, as follows:

$$h(\hat{y})=\int_0^{\hat{y}}\frac{h_o(x)}{x}dx=\hat{y}+\frac{\sqrt{69}}{12\sqrt{17}}\hat{y}^3+\frac{1}{20}\left(\frac{69}{136}+\frac{\sqrt{69}}{4\sqrt{17}}\right)\hat{y}^5+\ldots,$$

where

$$h_o(x)=x+\frac{\sqrt{69}}{4\sqrt{17}}x^3+\frac{1}{4}\left(\frac{69}{136}+\frac{\sqrt{69}}{4\sqrt{17}}\right)x^5$$
$$\frac{h_o(x)}{x}=1+\frac{\sqrt{69}}{4\sqrt{17}}x^2+\frac{1}{4}\left(\frac{69}{136}+\frac{\sqrt{69}}{4\sqrt{17}}\right)x^4$$
$$\int_0^{\hat{y}}\frac{h_o(x)}{x}dx=\int_0^{\hat{y}}1dx+\int_o^{\hat{y}}\frac{\sqrt{69}}{4\sqrt{17}}x^2+\int_o^{\hat{y}}\frac{1}{4}\left(\frac{69}{136}+\frac{\sqrt{69}}{4\sqrt{17}}\right)x^4$$
$$h(\hat{y})=\int_0^{\hat{y}}\frac{h_0(x)}{x}dx=\hat{y}+\frac{\sqrt{69}}{12\sqrt{17}}\hat{y}^3+\frac{1}{20}\left(\frac{69}{136}+\frac{\sqrt{69}}{4\sqrt{17}}\right)\hat{y}^5+\ldots$$

With $a_2=a_4=0$ and $a_3=\frac{\sqrt{69}}{12\sqrt{17}}$ in (1.7). Then we have

$$|\mathscr{H}_{2,1}(F_h/2)|=\frac{23}{3264}$$

which depicts that the bound is sharp.This complete the proof. □

3.4 Summary

This chapter examined the category of star-like functions connected to lune function, as defined by Mandal and Ahamed [51]. Furthermore, the class of convex functions related to lune functions was proposed as a new category. Several findings pertaining to the Fekete–Szegő inequality, logrithmic coefficient bounds, and the second Hankel determinants were examined for both of these classes.

CHAPTER 4

q-EXTENSION OF STARLIKE AND CONVEX FUNCTIONS WITH SECOND HANKEL DETERMINANT OF LOGRITHMIC COEFFICIENTS

4.1 Introduction

This chapter's goal is to define a few new types of univalent functions. These classes correspond to the logrithmic coefficient associated with the lune function and are q-extensions of convex and starlike functions. This is $q \in (0,1)$. In this chapter, some significant findings are presented. Logarithmic coefficients of lune functions in the classes $S^*_{☾}$ and $C_{☾}$ have the q-Version of the Second Hankel Determinant defined as follows.

Definition 4.1.1. *A function associated with the lune function $S^*_{☾}(q)$ is regarded as q-starlike if,*

$$S^*_{☾} = \left\{ f \in \mathrm{S} : \left| \left(\frac{\hat{y}\mathscr{D}_q f(\hat{y})}{f(\hat{y})} \right)^2 - 1 \right| \leq 2 \left| \frac{\hat{y}\mathscr{D}_q f(\hat{y})}{f(\hat{y})} \right|, \quad \hat{y} \in \mathbb{E} \right\}$$

Definition 4.1.2. *A function associated with a lune function $C_{☾}(q)$ is regarded as q-convex if,*

$$C_{☾} = \left\{ f \in \mathrm{S} : 1 + \frac{\hat{y}\mathscr{D}_q^2 f(\hat{y})}{\mathscr{D}_q f(\hat{y})} \prec q(\hat{y}), \quad \hat{y} \in \mathbb{E} \right\}$$

4.2 q-Extension of Logarithmic Coefficients of Class $S^*_{☾}$ Functions and their Second Hankel Determinant is New Class $S^*_{☾}(q)$

Theorem 4.2.1. *Finding the sharp limit of* $|\mathscr{H}_{2,1}(F_{f,q}/2)|$ *for the class function* $S^*_{☾}(q)$.

Let $f \in S^*_{☾}(q)$. *Then*

$$|\mathscr{H}_{2,1}(F_{f,q}/2)| \leq \frac{12}{\bar{q_o}} \tag{4.1}$$

where $\bar{q_o} = 1+q+q^2+q^3...+q^{191}$.

given the function $g \in S^*_{☾}(q)$, *the inequality is sharp*

$$g(\hat{y}) = \hat{y}\ exp_q\left(\int_0^{\hat{y}} \frac{x^2+\sqrt{1+x^4}-1}{x} d_q x\right) = \hat{y} + \frac{\hat{y}^3}{q_2} + \frac{\hat{y}^5}{q_4} + \ldots,$$

Proof. Let $f \in S^*_{☾}(q)$. Considering definition 1.1, it is evident that

$$\frac{\hat{y}\mathscr{D}_q f(\hat{y})}{f(\hat{y})} = w(\hat{y}) + \sqrt{1+w^2(\hat{y})}. \tag{4.2}$$

In $\mathbb{E}$, let $h \in \mathscr{P}$ and let w is a Schwarz function with $w(0) = 0$ and $|w(\hat{y})| \leq 1$. After that, we can write

$$w(\hat{y}) = \frac{h(\hat{y})\ -\ 1}{h(\hat{y})+1} \tag{4.3}$$

$$h(\hat{y}) = 1 + c_1\hat{y} + c_2\hat{y}^2 + c_3\hat{y}^3 + c_4\hat{y}^4 + \ldots,$$

so, (4.3) computation is

$$\frac{h(\hat{y})\ -\ 1}{h(\hat{y})\ +\ 1} = \frac{1\ +\ c_1\hat{y}\ +\ c_2\hat{y}^2\ +\ c_3\hat{y}^3 + c_4\hat{y}^4 - 1}{1\ +\ c_1\hat{y}\ +\ c_2\hat{y}^2\ +\ c_3\hat{y}^3 + c_4\hat{y}^4 + 1} = \frac{c_1\hat{y}\ +\ c_2\hat{y}^2\ +\ c_3\hat{y}^3 + c_4\hat{y}^4}{2\ +\ c_1\hat{y}\ +\ c_2\hat{y}^2\ +\ c_3\hat{y}^3 + c_4\hat{y}^4}$$

$$w(\hat{y}) = \frac{1}{2}c_1\hat{y} + \frac{1}{2}(c_2 - \frac{1}{2}c_1^2)\hat{y}^2 + \frac{1}{2}(c_3 - c_1c_2 + \frac{1}{4}c_1^3)\hat{y}^3 + \ldots, \tag{4.4}$$

simplification of (4.2) is

$$\frac{\hat{y}\mathscr{D}_q f(\hat{y})}{f(\hat{y})} = \frac{\hat{y}(1 + [2]_q a_2\hat{y} + [3]_q a_3\hat{y}^2 + [4]_q a_4\hat{y}^3)}{(\hat{y} + a_2\hat{y}^2 + a_3\hat{y}^3 + a_4\hat{y}^4)} = \frac{\hat{y} +\ [2]_q a_2\hat{y}^2\ +\ [3]_q a_3\hat{y}^3\ +\ [4]_q a_4\hat{y}^4}{\hat{y} + a_2\hat{y}^2 + a_3\hat{y}^3 + a_4\hat{y}^4}$$

$$\frac{\hat{y}\mathscr{D}_q f(\hat{y})}{f(\hat{y})} = 1 + ([2]_q a_2 - a_2)\hat{y} + (\ [3]_q a_3 - a_3 - [2]_q a_2 + a_2^2)\hat{y}^2 + \ldots, \tag{4.5}$$

from (4.2) and (4.4), a simple computation show that

$$w(\hat{y})+\sqrt{1+w^2(\hat{y})}=\frac{1}{2}c_1\hat{y}+\frac{1}{2}(c_2-\frac{1}{2}c_1^2)\hat{y}^2+\sqrt{1+\left(\frac{1}{2}c_1\hat{y}+\frac{1}{2}(c_2-\frac{1}{2}c_1^2)\hat{y}^2\right)^2}$$

$$\sqrt{1+w^2(\hat{y})}=\sqrt{1+\frac{1}{4}c_1^2\hat{y}^2+\frac{1}{4}\left(c_2-\frac{1}{2}c_1^2\right)^2\hat{y}^4+\frac{1}{2}c_1(c_2-\frac{1}{2}c_1^2)\hat{y}^3},$$

using Bionomial expansion

$$\sqrt{1+x}=1+\frac{1}{2}x+\frac{1}{8}x^2+\frac{1}{16}x^3+\dots$$

$$x=\frac{1}{4}c_1^2\hat{y}^2+\frac{1}{4}\left(c_2-\frac{1}{2}c_1^2\right)^2\hat{y}^4+\frac{1}{2}c_1(c_2-\frac{1}{2}c_1^2)\hat{y}^3,$$

thus we have,

$$\sqrt{1+w^2(\hat{y})}=\sqrt{1+x}=1+\frac{1}{2}x+\frac{1}{8}x^2 \tag{4.6}$$

$$1+\frac{1}{2}x-\frac{1}{8}x^2=1+\frac{1}{2}\left(\frac{1}{4}c_1^2\hat{y}^2+\frac{1}{4}(c_2-\frac{1}{2}c_1^2)^2\hat{y}^4+\frac{1}{2}c_1(c_2-\frac{1}{2}c_1^2)\hat{y}^3\right)$$

$$\sqrt{1+w^2(\hat{y})}=1+\frac{1}{8}c_1^2\hat{y}^2+\frac{1}{8}(c_2-\frac{1}{2}c_1^2)^2\hat{y}^4+\frac{1}{4}c_1(c_2-\frac{1}{2}c_1^2)\hat{y}^3,$$

with (4.6) we have

$$\sqrt{1+w^2(\hat{y})}=1+\frac{1}{8}c_1^2\hat{y}^2+\frac{1}{4}c_1(c_2-\frac{1}{2}c_1^2)\hat{y}^3+\frac{1}{8}(c_2-\frac{1}{2}c_1^2)^2\hat{y}^4, \tag{4.7}$$

now we obtain

$$w(\hat{y})+\sqrt{1+w^2(\hat{y})}=\frac{1}{2}c_1\hat{y}+\frac{1}{2}(c_2-\frac{1}{2}c_1^2)\hat{y}^2+\frac{1}{2}(c_3-c_1c_2+\frac{1}{4}c_1^3)\hat{y}^3+1+\frac{1}{8}c_1^2\hat{y}^2$$
$$+\frac{1}{4}c_1(c_2-\frac{1}{2}c_1^2)\hat{y}^3,$$

by adding similar terms of $\hat{y}^2$ and$\hat{y}^3$

$$w(\hat{y})+\sqrt{1+w^2(\hat{y})}=1+\frac{1}{2}c_1\hat{y}-\left(\frac{1}{8}c_1^2+\frac{1}{2}c_2\right)\hat{y}^2+\left(\frac{1}{2}c_3-\frac{1}{4}c_1c_2\right)\hat{y}^3, \tag{4.8}$$

so (4.2) will become

$$\frac{z\mathcal{D}_qf(\hat{y})}{f(\hat{y})}=w(\hat{y})+\sqrt{1+w^2(\hat{y})}$$

$$\frac{\hat{y}\mathcal{D}_qf(\hat{y})}{f(\hat{y})}=1+([2]_qa_2-a_2)\hat{y}+([3]_qa_3-a_3-[2]_qa_2-a_2^2)\hat{y}^2,$$

by comparing (4.5) and (4.8) we have

Oder of $\hat{y}^1$:

$$[2]_q a_2 - a_2 = \frac{1}{2}c_1$$

$$a_2([2]_q - 1) = \frac{1}{2}c_1$$
$$a_2 = \frac{c_1}{2([2]_q - 1)}$$

Oder of $\hat{y}^2$:

$$([3]_q a_3 - a_3 - [2]_q a_2 - a_2^2) = -\frac{1}{8}c_1^2 + \frac{1}{2}c_2$$

$$a_3([3]_q - 1) = -\frac{1}{8}c_1^2 + \frac{1}{2}c_2 + a_2^2\ ([2]_q - 1)$$

$$a_3([3]_q - 1) = -\frac{1}{8}c_1^2 + \frac{1}{2}c_2 + \left(\frac{c_1}{2([2]_q - 1)}\right)^2 ([2]_q - 1)$$

$$a_3([3]_q - 1) = -\frac{1}{8}c_1^2 + \frac{1}{2}c_2 + \frac{{c_1}^2}{\left(4([2]_q - 1)\right)}$$

$$a_3 = \frac{{c_1}^2}{8([2]_q - 1)([3]_q - 1)} + \frac{c_2}{2([3]_q - 1)}$$

Oder of $\hat{y}^3$:

$$([4]_q a_4 - a_4 - [2]_q a_2 a_3 - [3]_q a_2 a_3 + 2a_2 a_3 + [2]_q a_2^3 + a_2^3) = \left(\frac{1}{2}c_3 - \frac{1}{4}c_1 c_2\right)$$

$$([4]_q - 1)a_4 = \left(\frac{c_1}{2([2]_q - 1)}\right)\left(\frac{{c_1}^2}{8([2]_{\mathbf{q}} - 1)([3]_q - 1)} + \frac{c_2}{2([3]_q - 1)}\right)\left([5]_q - 2\right)$$

$$-\left(\frac{c_1}{2([2]_q - 1)}\right)^3 ([2]_q + 1) + \frac{c_3}{2} - \frac{c_1 c_2}{4}$$

$$a_4 = \frac{{c_1}^3\left([-5]_q - 2\right) - 2{c_1}^3([3]_q - 1)}{([80]_q + 16)} + \frac{c_1c_2\left([5]_q - 2\right) - c_1c_2([1]_q + 1)}{([28]_q - 4)} + \frac{c_3}{2([4]_q - 1)}$$

$$a_4 = \frac{-{c_1}^3\left([-5]_q + 2 + [6]_q - 2\right)}{([80]_q + 16)} + \frac{c_1c_2\left([5]_q - 2 - [1]_q - 1\right)}{([28]_q - 4)} + \frac{c_3}{([8]_q - 2)}$$

$$a_4 = \frac{-{c_1}^3\left([1]_q\right)}{([80]_q + 16)} + \frac{c_1c_2\left([4]_q - 3\right)}{([28]_q - 4)} + \frac{c_3}{([8]_q - 2)}$$

$$\begin{cases} a_2 = \frac{c_1}{2([2]_q - 1)} \\ a_3 = \frac{{c_1}^2}{8([2]_q - 1)([3]_q - 1)} + \frac{c_2}{2([3]_q - 1)} \\ a_4 = \frac{-{c_1}^3\left([1]_q\right)}{([80]_q + 16)} + \frac{c_1c_2\left([4]_q - 3\right)}{([28]_q - 4)} + \frac{c_3}{([8]_q - 2)} \end{cases} \tag{4.9}$$

With the assumption that the class $\mathscr{P}$ *and* $\mathscr{H}_{2,1}(F_{f,q}/2)$ is invariant under rotation, $c_1 \in [0,2]$ that is in view of (1.13) that $\tau \in [0,1]$. Using (4.9) in (1.12) we have

$$\mathscr{H}_{2,1}(F_{f,q}/2) = \frac{1}{[48]_q}\left(\left(\frac{c_1}{2([2]_q - 1)}\right)^4 - [12]_q\left(\frac{{c_1}^2}{8([2]_q - 1)([3]_q - 1)} + \frac{c_2}{2([3]_q - 1)}\right)^2\right)$$

$$\left(+[12[_q\left(\frac{c_1}{2([2]_q - 1)}\right)\left(\frac{-{c_1}^3\left([1]_q\right)}{([80]_q + 16)} + \frac{c_1c_2\left([4]_q - 3\right)}{([28]_q - 4)} + \frac{c_3}{([8]_q - 2)}\right)\right)$$

$$= \frac{1}{[48]_q}\left(\frac{{c_1}^4}{16([2]_q - 1)^4} - \frac{4[3]_q{c_1}^4}{64([3]_q - 1)^2([2]_q - 1)^2} - \frac{4[3]_q{c_2}^2}{4([3]_q - 1)^2} - \frac{4[6]_q{c_1}^2c_2}{16([3]_q - 1)^2([2]_q - 1)^2}\right)$$

$$\left(-\frac{[6]_q{c_1}^4\left([1]_q\right)}{([80]_q + 16)} + \frac{[6]_q{c_1}^2c_2\left([4]_q - 3\right)}{([20]_q + 4)} + \frac{[6]_q c_1 c_3}{([8]_q - 2)}\right)$$

$$= \frac{1}{[48]_q}\left(\frac{\mathrm{c}_1{}^4}{16([2]_q-1)^4} - \frac{[3]_q c_1{}^4}{16([3]_q+1)} - \frac{[3]_q c_2{}^2}{([3]_q+1)} - \frac{[3]_q c_1{}^2 c_2}{2([3]_q+1)}\right)$$

$$\left(-\frac{[6]_q c_1{}^4\left([1]_{\mathbf{q}}\right)}{16([5]_q+1)} + \frac{[6]_q c_1{}^2 c_2\left([4]_q-3\right)}{([20]_q+4)} + \frac{[6]_q c_1 c_3}{([4]_q+2)}\right)$$

$$\mathscr{H}_{2,1}(F_{f,q}/2) = \frac{1}{[48]_q}\left(-\frac{[3]_q c_1{}^4}{16\left([3]_q+1\right)} - \frac{[3]_q c_2{}^2}{\left([[3]_{\mathrm{q}}]_q+1\right)} - \frac{[3]_q c_1{}^2 c_2[-8+16]_q}{\left([184]_q+8\right)} + \frac{[6]_q c_1 c_3}{[4]_q+2}\right),$$

after further simplification

$$= \frac{1}{[48]_q}\left(-\frac{[3]_q c_1{}^4}{16\left([3]_q+1\right)} - \frac{16\times[3]_q c_2{}^2}{\left([3]_q+1\right)\times 16} - \frac{[3]_q c_1{}^2 c_2([-8]_q+16)}{\left([184]_q+8\right)} + \frac{[6]_q c_1 c_3\times 16\left([3]_q+1\right)}{([4]_q+2)\times 16\left([3]_q+1\right)}\right)$$

$$= \frac{1}{[48]_q}\left(-\frac{[3]_q c_1{}^4}{16\left([3]_q+1\right)} - \frac{16\times[3]_q c_2{}^2}{\left([3]_q+1\right)\times 16} - \frac{[6]_q c_1{}^2 c_2([-8]_q+16)}{\left([368]_q+16\right)} + \frac{[6]_q c_1 c_3\times 16\left([3]_q+1\right)}{([4]_q+2)\times 16\left([3]_q+1\right)}\right)$$

$$= \frac{1}{[48]_q 16\left([3]_q+1\right)}\left(-[3]_q c_1{}^4 - [48]_q c_2{}^2 - \frac{[6]_q c_1{}^2 c_2([-8]_q+16)}{\left([5]_q+1\right)} + \frac{[6]_q c_1 c_3\times 16\left([3]_q+1\right)}{([4]_q+2)}\right)$$

$$= \frac{1}{[48]_q 16\left([3]_q+1\right)}\left(\frac{\begin{array}{c}-[3]_q c_1{}^4([4]_q+2)\left([5]_q+1\right) - [48]_q c_2{}^2([4]_q+2)\left([5]_q+1\right)\\ -[6]_q c_1{}^2 c_2([-8]_q+16)([4]_q+2) + [6]_q c_1 c_3\times 16\left([3]_q+1\right)\left([5]_q+1\right)\end{array}}{([4]_q+2)\left([5]_q+1\right)}\right)$$

$$= \frac{1}{[48]_q 16\left([3]_q+1\right)}\left(\frac{\begin{array}{c}-[3]_q c_1{}^4([34]_q+2) - [48]_q c_2{}^2([34]_q+2) - c_1{}^2 c_2([-8]_q+16)([24]_q+[12]_q)\\ +c_1 c_3\times 16\left([3]_q+1\right)\left([30]_q+[6]_q\right)\end{array}}{([34]_q+2)}\right),$$

as we know when $q \to 1^-$ so,

$$([34]_q+2) = ([24]_q+[12]_q) = \left([30]_q+[6]_q\right)$$

$$\mathscr{H}_{2,1}(F_{f,q}/2) = \frac{([34]_q+2)}{[48]_q 16\left([3]_q+1\right)}\left(\frac{-[3]_q c_1{}^4 - [48]_q c_2{}^2 - 6c_1{}^2 c_2([-8]_q+16) + c_1 c_3\times 16\left([3]_q+1\right)}{([34]_q+2)}\right)$$

$$= \frac{1}{[48]_q 16([3]_q+1)}(-[3]_q {c_1}^4 - [48]_q {c_2}^2 - {c_1}^2 c_2([-8]_q+16) + c_1 c_3 \times 16([3]_q+1))$$

$$\mathscr{H}_{2,1}(F_{f,q}/2) = \frac{\left(-[3]_q {c_1}^4 - [48]_q {c_2}^2 - {c_1}^2 c_2([-8]_q+16) + c_1 c_3 \times 16\left([3]_q+1\right)\right)}{[48]_q 16\left([3]_q+1\right)}, \tag{4.10}$$

by Lemma (3.1.1) we have value of c_1, c_2 and c_3 use in (4.10)

$$\mathscr{H}_{2,1}(F_{f,q}/2) = \frac{1}{[48]_q 16\left([3]_q+1\right)} \left[\begin{array}{c} -[3]_q(2\tau_1)^4 - [48]_q(2\tau_1^2+2(\ 1-\ \tau_1^2\ )\tau_2)^2 \\ -([-8]_q+16)(4\tau_1)^2(2\tau_1^2+2\left(\ 1-\ \tau_1^2\ \right)\tau_2) \\ +\left(4\tau_1^4+8(\ 1-\ \tau_1^2\ )\tau_1^2\tau_2 - 4(\ 1-\ \tau_1^2\ )\tau_1^2\tau_2^2\right)16([3]_q+1) \\ +\left(2(\ 1-\ \tau_1^2\ )(\ 1-|\tau_2|^{\ 2})\tau_3\right)16([3]_q+1) \end{array} \right]$$

$$= \frac{1}{[48]_q 16\left([3]_q+1\right)} \left[\begin{array}{c} -48{\tau_1}^4 - 192\tau_1^4 - 384\tau_1^2\tau_2 + 384\tau_1^4\tau_2 \\ +384\tau_1^2\tau_2^2 - 192\tau_2^2 - 192\tau_1^4\tau_2^2 - 8\tau_1^4([-8]_q+16) \\ -8\tau_1^2\tau_2([-8]_q+16) + 8\tau_1^4\tau_2([-8]_q+16) \\ +\left(4\tau_1^4+8\tau_1^2\tau_2-8\tau_1^4\tau_2-4\ \tau_1^2\tau_2^2+4\ \tau_1^4\tau_2^2\right)16([3]_q+1) \\ +\left((\ 4\tau_1-\ 4\tau_1^3\ )(\ 1-|\tau_2|^2\tau_3)\right)16([3]_q+1) \\ \end{array} \right]$$

$$= \frac{16}{[48]_q 16\left([3]_q+1\right)} \left[\begin{array}{c} \frac{-240{\tau_1}^4}{16} - \frac{384\tau_1^2\tau_2}{16} + \frac{384\tau_1^4\tau_2}{16} + \frac{384\tau_1^2\tau_2^2}{16} - \frac{192\tau_2^2}{16} - \frac{192\tau_1^4\tau_2^2}{16} \\ + \frac{(-8\tau_1^4([-8]_q+16) - 8\tau_1^2\tau_2([-8]_q+16) + 8\tau_1^4\tau_2([-8]_q+16))}{16} \\ + \frac{16\left([3]_q+1\right) \times \left[+4\tau_1^4+8\tau_1^2\tau_2-8\tau_1^4\tau_2-4\tau_1^2\tau_2^2+4\tau_1^4\tau_2^2+\ \left(\ 4\tau_1-\ 4\tau_1^3\ \right)\left(\ 1-|\tau_2|^2\tau_3\right)\right]}{16} \end{array} \right]$$

$$= \frac{1}{[48]_q\left([3]_q+1\right)}\left[\begin{array}{c} -15\tau_1^4-24\tau_1^2\tau_2+24\tau_1^4\tau_2+24\tau_1^2\tau_2^2-12\tau_2^2-12\tau_1^4\tau_2^2 \\ +\left[-\frac{\tau_1^4([-8]_q+16)}{2}-\frac{\tau_1^2\tau_2([-8]_q+16)}{2}+\frac{\tau_1^4\tau_2([-8]_q+16)}{2}\right] \\ +\left[\begin{array}{c} 4\tau_1^4\left([3]_q+1\right)+8\tau_1^2\tau_2\left([3]_q+1\right)-8\tau_1^4\tau_2\left([3]_q+1\right) \\ -4\,\tau_1^2\tau_2^2\left([3]_q+1\right)+4\,\tau_1^4\tau_2^2\left([3]_q+1\right) \\ +\left([3]_q+1\right)\left(4\tau_1-4\tau_1^3\right)\left(1-|\tau_2|^2\tau_3\right) \end{array}\right] \end{array}\right]$$

$$= \frac{1}{[48]_q\left([3]_q+1\right)}\left[\begin{array}{c} -15\tau_1^4-24\tau_1^2\tau_2+24\tau_1^4\tau_2+24\tau_1^2\tau_2^2-12\tau_2^2-12\tau_1^4\tau_2^2 \\ +\left[-\frac{\tau_1^4([-8]_q+16)}{2}-\frac{\tau_1^2\tau_2([-8]_q+16)}{2}+\frac{\tau_1^4\tau_2([-8]_q+16)}{2}\right] \\ \left[\begin{array}{c} [12]_q\tau_1^4+4\tau_1^4+[24]_q\tau_1^2\tau_2+8\tau_1^2\tau_2-[24]_q\tau_1^4\tau_2-8\tau_1^4\tau_2 \\ -[12]_q\tau_1^2\tau_2^2+4\tau_1^2\tau_2^2+[12]_q\tau_1^4\tau_2^2+4\tau_1^4\tau_2^2 \\ +\left([3]_q+1\right)\left(4\tau_1-4\tau_1^3\right)\left(1-|\tau_2|^2\tau_3\right) \end{array}\right] \end{array}\right]$$

$$
=\frac{1}{[48]_q\left([3]_q+1\right)}\left[\begin{array}{c}
-15\tau_1^4+4\tau_1^4-24\tau_1^2\tau_2+8\tau_1^2\tau_2+24\tau_1^4\tau_2-8\tau_1^4\tau_2 \\
+24\tau_1^2\tau_2^2+4\tau_1^2\tau_2^2-12\tau_2^2-12\tau_1^4\tau_2^2+4\tau_1^4\tau_2^2 \\
\left[\frac{-16\tau_1^4}{2}+\frac{[8]_q\tau_1^4}{2}-\frac{16\tau_1^2\tau_2}{2}+\frac{[8]_q\tau_1^2\tau_2}{2}-\frac{[8]_q\tau_1^4\tau_2}{2}+\frac{16\tau_1^4\tau_2}{2}\right] \\
\left[\begin{array}{c}
[12]_q\tau_1^4+[24]_q\tau_1^2\tau_2-[24]_q\tau_1^4\tau_2 \\
-\tau_1^2\tau_2^2[12]_q+\tau_1^4\tau_2^2[12]_q \\
+\left([3]_q+1\right)\left(4\tau_1-4\tau_1^3\right)\left(1-|\tau_2|^2\tau_3\right)
\end{array}\right]
\end{array}\right]
$$

$$
=\frac{1}{[48]_q\left([3]_q+1\right)}\left[\begin{array}{c}
-15\tau_1^4+4\tau_1^4-8\tau_1^4-24\tau_1^2\tau_2+8\tau_1^2\tau_2-8\tau_1^2\tau_2 \\
+24\tau_1^2\tau_2^2+4\tau_1^2\tau_2^2+24\tau_1^4\tau_2-8\tau_1^4\tau_2+8\tau_1^4\tau_2 \\
-12\tau_2^2-12\tau_1^4\tau_2^2+4\tau_1^4\tau_2^2 \\
\left[\begin{array}{c}
\tau_1^4[12]_q+\frac{[8]_q\tau_1^4}{2}+[24]_q\tau_1^2\tau_2+\frac{[8]_q\tau_1^2\tau_2}{2}-[24]_q\tau_1^4\tau_2-\frac{[8]_q\tau_1^4\tau_2}{2} \\
-[12]_q\tau_1^2\tau_2^2+[12]_q\tau_1^4\tau_2^2 \\
+\left([3]_q+1\right)\left(4\tau_1-4\tau_1^3\right)\left(1-|\tau_2|^2\tau_3\right)
\end{array}\right]
\end{array}\right]
$$

$$= \frac{1}{[48]_q\left([3]_q+1\right)}\left[\begin{array}{c} -19\tau_1^4-24\tau_1^2\tau_2+20\tau_1^2\tau_2^2+24\tau_1^4\tau_2-12\tau_2^2-8\tau_1^4\tau_2^2 \\ \left[\begin{array}{c} +\frac{[32]_q\tau_1^4}{2}+\frac{[56]_q\tau_1^2\tau_2}{2}-\frac{[56]_q\tau_1^4\tau_2}{2} \\ -[12]_q\tau_1^2\tau_2^2+[12]_q\tau_1^4\tau_2^2 \\ +\left([3]_q+1\right)\left(4\tau_1-4\tau_1^3\right)\left(1-|\tau_2|^2\tau_3\right) \end{array}\right] \end{array}\right]$$

$$= \frac{1}{[48]_q\left([3]_q+1\right)}\left[\begin{array}{c} -19\tau_1^4-24\tau_1^2\tau_2+20\tau_1^2\tau_2^2+24\tau_1^4\tau_2-12\tau_2^2-8\tau_1^4\tau_2^2 \\ \left[\begin{array}{c} +\frac{[32]_q\tau_1^4}{2}+\frac{[56]_q\tau_1^2\tau_2}{2}-\frac{[56]_q\tau_1^4\tau_2}{2} \\ -[12]_q\tau_1^2\tau_2^2+[12]_q\tau_1^4\tau_2^2 \\ +\left([12]_q+4\right)\tau_1(1-\tau_1^2)\left(1-|\tau_2|^2\tau_3\right) \end{array}\right] \end{array}\right]$$

$$\mathscr{H}_{2,1}(F_{f,q}/2) = \frac{1}{[48]_q\left([3]_q+1\right)}\left[\begin{array}{c} -19\tau_1^4+\frac{[32]_q\tau_1^4}{2}-24\tau_1^2\tau_2+\frac{[56]_q\tau_1^2\tau_2}{2}+24\tau_1^4\tau_2-\frac{[56]_q\tau_1^4\tau_2}{2} \\ -8\tau_1^4\tau_2^2+[12]_q\tau_1^4\tau_2^2-[12]_q\tau_1^2\tau_2^2+20\tau_1^2\tau_2^2 \\ -12\tau_2^2+\left([12]_q+4\right)\tau_1(1-\tau_1^2)\left(1-|\tau_2|^2\tau_3\right) \end{array}\right]. \tag{4.11}$$

The subsequent situations could now arise on τ_1 :

Case 1.

if $\tau_1 = 1$. Then from (4.11) we obtain

$$|\mathscr{H}_{2,1}(F_{f,q}/2)| = \frac{1}{[48]_q\left([3]_q+1\right)}\left[\begin{array}{c} -19+\frac{[32]_q}{2}-24\tau_2+\frac{[56]_q\tau_2}{2}+24\tau_2-\frac{[56]_q\tau_2}{2} \\ -8\tau_2^2+[12]_q\tau_2^2-[12]_q\tau_2^2+20\tau_2^2 \\ -12\tau_2^2+\left([12]_q+4\right)\tau_1(1-1)\left(1-|\tau_2|^2\tau_3\right) \end{array}\right]$$

$$|\mathscr{H}_{2,1}(F_{f,q}/2)| = \left| \frac{1}{[48]_q\left([3]_q+1\right)} \left(-19+\frac{[32]_q}{2}\right) \right|$$

If $q \to 1^-$ then we have

$$|\mathscr{H}_{2,1}(F_{f,q}/2)| = \frac{1}{64}$$

Case 2.

if $\tau_1 = 0$ then from (4.11), we see that

$$|\mathscr{H}_{2,1}(F_{f,q}/2)| = \frac{1}{[48]_q\left([3]_q+1\right)}(-4(3)\tau_2^2\ )$$

$$|\mathscr{H}_{2,1}(F_{f,q}/2)| = \left| \frac{-12}{[48]_q\left([3]_q+1\right)} \right| |\tau_2|^2 \ \leq \frac{12}{[48]_q\left([3]_q+1\right)}$$

If $q \to 1^-$ the we have

$$|\mathscr{H}_{2,1}(F_{f,q}/2)| = \frac{1}{16}|\tau_2|^2 \ \leq \ \frac{1}{16}$$

Case 3.

Consider $\tau \ \in \ (0,1)$. Utilising the triangle inequality in (4.11) and by using the fact that $|\tau_3| \leq 1$, we obtain

$$= \frac{1}{[144]_q + [48]_q} \left\{ \begin{array}{c} \frac{(-38+[32]_q)\tau_1^4}{2} + \frac{-48\tau_1^2\tau_2+48\tau_1^4\tau_2-[56]_q\tau_1^4\tau_2+[56]_q\tau_1^2\tau_2}{2} \\ \\ \left(-8\tau_1^4+[12]_q\tau_1^4-[12]_q\tau_1^2+20\tau_1^2-12\right)\ \tau_2^2 \\ \\ +\left([12]_q+4\right)\tau_1(1-\tau_1^2)(1-\ |\tau_2^2|) \end{array} \right\}$$

$$= \frac{1}{[12]_q([12]_q+[4]_q)} \left\{ \begin{array}{c} \frac{(-38+[32]_q)\tau_1^4}{2} + \frac{-48\tau_1^2\tau_2+48\tau_1^4\tau_2-[56]_q\tau_1^4\tau_2+[56]_q\tau_1^2\tau_2}{2} \\ \\ \left(+[12]_q\tau_1^4-[12]_q\tau_1^2-8\tau_1^4+20\tau_1^2-12\right)\ \tau_2^2 \\ \\ +\left([12]_q+4\right)\tau_1(1-\tau_1^2)(1-\ |\tau_2^2|) \end{array} \right\}$$

$$= \frac{1}{[12]_q([12]_q+[4]_q)} \left\{ \begin{array}{c} \frac{2(-19+[16]_q)\tau_1^4}{2} + \frac{-48\tau_1^2\tau_2(1-\tau_1^2)+[56]_q\tau_1^2\tau_2(1-\tau_1^2)}{2} \\ \\ (-[12]_q\tau_1^2\ (-\tau_1^2+1) - 4\left(2\tau_1^4-5\tau_1^2+3\right))\tau_2^2 \\ \\ +\left([12]_q+4\right)\tau_1(1-\tau_1^2)(1-|\tau_2^2|) \end{array} \right\}$$

$$= \frac{1}{[12]_q([16]_q)} \left\{ \begin{array}{c} \frac{2(-19+[16]_q)\tau_1^4}{2} + \frac{-48\tau_1^2\tau_2(1-\tau_1^2)+[56]_q\tau_1^2\tau_2(1-\tau_1^2)}{2} \\ \\ (-[12]_q\tau_1^2\ (-\tau_1^2+1) + 4(1-\tau_1^2)(2\tau_1^2-3))\tau_2^2 \\ \\ +\left([12]_q+4\right)\tau_1(1-\tau_1^2)(1-|\tau_2^2|) \end{array} \right\}$$

$$= \frac{1}{[12]_q}\tau_1\ (1-\tau_1^2) \left\{ \begin{array}{c} \frac{(-19+[16]_q)\tau_1^3}{[16]_q(1-\tau_1^2)} + \frac{(-24+[28]_q)\tau_1\tau_2}{[16]_q} \\ \\ +\frac{(-[12]_q\tau_1^2+4(2\tau_1^2-3)}{[16]_q\tau_1})\tau_2^2 \\ \\ \frac{+\left([12]_q+4\right)\ (1-|\tau_2^2|)}{[16]_q} \end{array} \right\}$$

$$\mathscr{H}_{2,1}(F_{f,q}/2) = \frac{1}{[12]_q}\tau_1\ (1-\tau_1^2) \left\{ \begin{array}{c} \frac{(-19+[16]_q)\tau_1^3}{[16]_q(1-\tau_1^2)} + \frac{(-24+[28]_q)\tau_1\tau_2}{[16]_q} \\ \\ +\frac{\tau_1^2(-[12]_q+8)\ -12)}{[16]_q\tau_1})\tau_2^2 \\ \\ \frac{+\left([12]_q+4\right)\ (1-|\tau_2^2|)}{[16]_q} \end{array} \right\} \tag{4.12}$$

$$A = \frac{(-19+[16]_q)\tau_1^3}{[16]_q(1-\tau_1^2)}, \qquad B = \frac{(-24+[28]_q)\tau_1}{[16]_q} \qquad C = \frac{\tau_1^2(-[12]_q+8)\ -12)}{[16]_q\tau_1}.$$

So, above term (4.12) will becomes

$$\mathscr{H}_{2,1}(F_{f,q}/2) = \frac{1}{[12]_q}\tau_1\ (1-\ \tau_1^2){=}(|A+B\tau_2+C\tau_2|+1-|\tau_2|^2). \tag{4.13}$$

Since we can see that $AC > 0$, we may use Lemma (3.1.2) case (i). We now examine every circumstance in case (i).

3(a) The inequality is seen.

$$|B|-2\,(1-|C|) = \left(\frac{(-24+[28]_q)\tau_1}{[16]_q}\right) - 2\left(1\ -\ \frac{\tau_1^2([12]_q-8)+12)}{[16]_q\tau_1}\right)$$

$$= \frac{(-24+[28]_q)\tau_1}{[16]_q}\ -2+\frac{\tau_1^2([24]_q-16)+24)}{[16]_q\tau_1}$$

$$= \frac{(-24+[28]_q)\tau_1^2-2[16]_q\tau_1+\tau_1^2([24]_q-16)+24}{[16]_q\tau_1}$$

$$= \frac{-24\tau_1^2+[28]_q\tau_1^2-2[16]_q\tau_1+\tau_1^2\,[24]_q-16\tau_1^2+24}{[16]_q\tau_1}$$

$$|B|-2\,(1-|C|) = \frac{-40\tau_1^2+[52]_q\tau_1^2-[32]_q\tau_1+24)}{[16]_q\tau_1}$$

If $q \to 1^-$ then we have

$$|B|-2\,(1-|C|) = \frac{12\tau_1^2-32\tau_1+24}{16\tau_1}$$

$$|B|-2\,(1-|C|) = \frac{(3\tau_1^2-8\tau_1+6)}{(4\tau_1)} > 0.$$

which is true for all $\tau_1 \in (0,1)$. Therefore, Lemma (3.1.2). implies it. Additionally, the inequality (4.13) that

$$|\mathscr{H}_{2,1}(F_{f,q}/2)| \le \frac{1}{[12]_q}\tau_1\ \left(1-\ \tau_1^2\right)(|A|+|B|+|C|)$$

$$= \frac{1}{[12]_q}\tau_1\ \left(1-\ \tau_1^2\right)\left(\left|\frac{(-19+[16]_q)\tau_1^3}{[16]_q(1-\ \tau_1^2)}\right| + \left|\frac{(+[28]_q-24)\tau_1}{[16]_q}\right| + \left|\frac{\tau_1^2(-[12]_q+8)\ -12)}{[16]_q\tau_1}\right|\right)$$

$$= \frac{1}{[12]_q}\tau_1 \ (1- \tau_1^2)\left(\frac{(19-[16]_q)\tau_1^3}{[16]_q(1- \tau_1^2)} + \frac{([28]_q-24)\tau_1}{[16]_q} + \frac{\tau_1^2([12]_q-8)+12)}{[16]_q\tau_1}\right)$$

$$= \frac{\tau_1 \ (1- \tau_1^2)}{[12]_q}\left(\frac{\tau_1^2([12]_q-8)+12)(1- \tau_1^2)+(19-[16]_q)\tau_1^4([28]_q-24)\tau_1^2(1- \tau_1^2)}{[16]_q(1- \tau_1^2)\tau_1}\right)$$

$$= \frac{1}{[192]_q}\left(\tau_1^2([12]_q-8)+12)(1- \tau_1^2)+(19-[16]_q)\tau_1^4([28]_q-24)\tau_1^2(1- \tau_1^2)\right)$$

$$= \frac{1}{[192]_q}\left(\begin{array}{c} [12]_q\tau_1^2-8\tau_1^2+12-[12]_q\tau_1^4+8\tau_1^4-12\tau_1^2+19\tau_1^4-[16]_q\tau_1^4 \\ +[28]_q\tau_1^2-24\tau_1^2-[28]_q\tau_1^4+-24\tau_1^4) \end{array}\right)$$

$$|\mathscr{H}_{2,1}(F_{f,q}/2)| = \frac{1}{[192]_q}\left([40]_q\tau_1^2-44\tau_1^2+12-[56]_q\tau_1^4+51\tau_1^4\right)$$

If $q \to 1^-$ then we have

$$|\mathscr{H}_{2,1}(F_{f,q}/2)| = \frac{1}{192}(12-4\tau_1^2-5\tau_1^4)$$

If $\tau_1 = 0$ then

$$\begin{aligned} |\mathscr{H}_{2,1}(F_{f,q}/2)| &= \frac{1}{192}[-12 \ -4(0)-5(0)] \\ &= \frac{1}{192}[-12\,] \\ &= \left|\frac{1}{192}\,[-12\,]\right| \leq \ \frac{1}{16} \\ |\mathscr{H}_{2,1}(F_{f,q}/2)| &\leq \ \frac{1}{16}. \end{aligned}$$

3(b) Next, it's simple to verify that

$$|B|-2(1-|C|) = \frac{(-24+[28]_q)\tau_1}{[16]_q} \ -2\left(1 \ - \ \frac{\tau_1^2([12]_q-8)+12)}{[16]_q\tau_1}\right)$$

If $q \to 1^-$ then we have

$$|B|-2(1-|C|) = \frac{(3\tau_1^2-8\tau_1+6)}{\tau_1} < 0$$

which is not true for all $\tau_1 \in (0,1)$.

After a summary of cases 1,2 and 3, the inequality (4.1) is proven.Demonstrating that the bound is sharp is sufficient to finish the proof. To demonstrate that we consider the function $g \in S^*_{☾}(q)$ like this.

$$g(\hat{y}) = \hat{y}\, exp_q\left(\int_0^{\hat{y}} \frac{x^2 + \sqrt{1+x^4} - 1}{x} d_q x\right) = \hat{y} + \frac{\hat{y}^3}{[2]_q} + \frac{\hat{y}^5}{[4]_q} + \ldots,$$

with $a_2 = a_4 = 0$ and $a_3 = \frac{1}{[2]_q}$ in (1.12). Then we have

$$|\mathscr{H}_{2,1}(F_g/2)| = \frac{12}{\bar{q}_o}.$$

This complete the proof. □

4.3 q-Extension of Logarithmic Coefficients of Class $C_{☾}$ Functions and their Second Hankel Determinant is the New Class $C_{☾}(q)$

Theorem 4.3.1. *Let $f \in C_{☾}(q)$. Then*

$$|\mathscr{H}_{2,1}(F_{f,q}/2)| \leq \frac{1}{\bar{q}_1} \tag{4.14}$$

where $\bar{q}_1 = 1 + q + q^2 + q^3 + \ldots q^{143}$.
given the function $h \in C_{☾}(q)$ the inequality is sharp

$$h(\hat{y}) = \int_0^{\hat{y}} \frac{h_0(x)}{x} d_q x = \hat{y} + \frac{\sqrt{69}}{6q_2\sqrt{17}}\hat{y}^3 + \left(\frac{\sqrt{69}}{10q_4\sqrt{17}}\right)\hat{y}^5 + \ldots,$$

where $h_0(\hat{y})$ is given by (4.21) .

Proof. Let $f \in C_{☾}(q)$. Considering definition 1.1, it is evident that

$$1 + \frac{z\mathscr{D}_q{}^2 f(\hat{y})}{\mathscr{D}_q f(\hat{y})} = w(\hat{y}) + \sqrt{1 + w^2(\hat{y})} \tag{4.15}$$

In $\mathbb{E}$, let $h \in \mathscr{P}$ and let w is a Schwarz function with $w(0) = 0$ and $|w(\hat{y})| \leq 1$. After that, we can write

$$w(\hat{y}) = \frac{h(\hat{y}) - 1}{h(\hat{y}) + 1} \tag{4.16}$$

$$h(\hat{y}) = 1 + c_1\hat{y} + c_2\hat{y}^2 + c_3\hat{y}^3 + c_4\hat{y}^4 + \ldots$$

Then (4.16) is

$$\frac{h(\hat{y})-1}{h(\hat{y})+1}=\frac{1+c_1\hat{y}+c_2\hat{y}^2+c_3\hat{y}^3+c_4\hat{y}^4-1}{1+c_1\hat{y}+c_2\hat{y}^2+c_3\hat{y}^3+c_4\hat{y}^4+1}=\frac{c_1\hat{y}+c_2\hat{y}^2+c_3\hat{\hat{y}}^3+c_4\hat{y}^4}{2+c_1\hat{y}+c_2\hat{y}^2+c_3\hat{y}^3+c_4\hat{y}^4}$$

$$w(\hat{y})=\frac{1}{2}c_1\hat{y}+\frac{1}{2}(c_2-\frac{1}{2}c_1^2)\hat{y}^2+\frac{1}{2}(c_3-c_1c_2+\frac{1}{4}c_1^3)\hat{y}^3+\dots \tag{4.17}$$

And the term (4.15) is

$$1+\frac{\hat{y}\mathscr{D}_q{}^2f(\hat{y})}{\mathscr{D}_q(\hat{y})}=1+\frac{\hat{y}([2]_qa_2+[6]_qa_3\hat{y}+[12]_qa_4\hat{y}^2+[20]_qa_5\hat{y}^3)}{1+[2]_qa_2\hat{y}+[3]_qa_3\hat{y}^2+[4]_qa_4\hat{y}^3+[5]_qa_5\hat{y}^4}=\frac{1+[4]_qa_2\hat{y}+[9]_qa_3\hat{y}^2+[16]_qa_4\hat{y}^3}{1+[2]_qa_2\hat{y}+[3]_qa_3\hat{y}^2+[4]_qa_4\hat{y}^3}$$

$$1+\frac{\hat{y}\mathscr{D}_q{}^2f(\hat{y})}{\mathscr{D}_q(\hat{y})}=1+[2]_qa_2\hat{y}+([6]_qa_3-[4]_qa_2^2)\hat{y}^2+([12]_qa_4-[6]_qa_2a_3+[8]_qa_2^3)\hat{y}^3 \tag{4.18}$$

(4.15) using with (4.17) then we have

$$w(\hat{y})+\sqrt{1+w^2(\hat{y})}=\frac{1}{2}c_1\hat{y}+\frac{1}{2}(c_2-\frac{1}{2}c_1^2)\hat{y}^2+\sqrt{1+\left(\frac{1}{2}c_1\hat{y}+\frac{1}{2}(c_2-\frac{1}{2}c_1^2)\hat{y}^2\right)^2}$$

$$=\sqrt{1+\frac{1}{4}c_1^2\hat{y}^2+\frac{1}{4}(c_2-\frac{1}{2}c_1^2)\hat{y}^4+\frac{1}{2}c_1(c_2-\frac{1}{2}c_1^2)\hat{y}^3}$$

using bionomial expansion:

$$\sqrt{1+x}=1+\frac{1}{2}x+\frac{1}{8}x^2+\frac{1}{16}x^3+\dots$$
$$x=\frac{1}{4}c_1^2\hat{y}^2+\frac{1}{4}(c_2-\frac{1}{2}c_1^2)\hat{y}^4+\frac{1}{2}c_1(c_2-\frac{1}{2}c_1^2)\hat{y}^3,$$

thus we have

$$\sqrt{1+w^2(\hat{y})}=1+\frac{1}{2}x+\frac{1}{8}x^2$$
$$=1+\frac{1}{2}\left(\frac{1}{4}c_1^2\hat{y}^2+\frac{1}{4}(c_2-\frac{1}{2}c_1^2)\hat{y}^4+\frac{1}{2}c_1(c_2-\frac{1}{2}c_1^2)\hat{y}^3\right)$$
$$=1+\frac{1}{8}c_1^2\hat{y}^2+\frac{1}{8}(c_2-\frac{1}{2}c_1^2)\hat{y}^4+\frac{1}{4}(c_1c_2-\frac{1}{2}c_1^2)\hat{y}^3,$$
$$\sqrt{1+w^2(\hat{y})}=1+\frac{1}{8}c_1^2\hat{y}^2+\frac{1}{4}c_1(c_2-\frac{1}{2}c_1^2)\hat{y}^3+\frac{1}{8}(c_2-\frac{1}{2}c_1^2)\hat{y}^4. \tag{4.19}$$

Now by (4.17) and (4.19) we have

$$w(\hat{y})+\sqrt{1+w^2(\hat{y})}=\frac{1}{2}c_1\hat{y}+\frac{1}{2}(c_2-\frac{1}{2}c_1^2)\hat{y}^2+\frac{1}{2}(c_3-c_1c_2+\frac{1}{4}c_1^3)\hat{y}^3+1+\frac{1}{8}c_1^2\hat{y}^2$$
$$+\frac{1}{4}c_1(c_2-\frac{1}{2}c_1^2)\hat{y}^3,$$

by adding similar terms of $\hat{y}^2$ and $\hat{y}^3$

$$w(\hat{y})+\sqrt{1+w^2(\hat{y})}=1+\frac{1}{2}c_1\hat{y}-\left(\frac{1}{8}c_1^2+\frac{1}{2}c_2\right)\hat{y}^2+\left(\frac{1}{2}c_3-\frac{1}{4}c_{\ 1}c_2\right)\hat{y}^3. \tag{4.20}$$

So (4.15) is,

$$1+\frac{z\mathscr{D}_q{}^2 f(\hat{y})}{\mathscr{D}_q(\hat{y})}=w(\hat{y})+\sqrt{1+w^2(\hat{y})},$$

by comparing (4.18) and (4.21) we have

Oder of $\hat{y}^1$:

$$a_2([2]_q-1)=\frac{1}{2}c_1$$
$$a_2=\frac{c_1}{2([2]_q)}$$

Oder of $\hat{y}^2$:

$$([6]_q a_3-[4]_q a_2^2)=-\frac{1}{8}c_1^2+\frac{1}{2}c_2$$
$$[6]_q a_3=-\frac{1}{8}c_1^2+\frac{1}{2}c_2+a_2{}^2([4]_q)$$
$$[6]_q a_3=-\frac{1}{8}c_1^2+\frac{1}{2}c_2+\left(\frac{c_1}{2([2]_q)}\right)^2([4]_q)$$
$$[6]_q a_3=-\frac{1}{8}c_1^2+\frac{1}{2}c_2+\frac{c_1{}^2}{\left(4([4]_q)\right)}([4]_q)$$
$$a_3=\frac{-c_1{}^2}{8([6]_q)}+\frac{c_2}{2\,[6]_q}+\frac{c_1{}^2}{(4)\,[6]_q}$$
$$a_3=\frac{c_1{}^2}{8([6]_q)}+\frac{c_2}{2\,[6]_q}$$

Oder of $\hat{y}^3$:

$$([12]_q a_4 - [18]_q a_2 a_3 + [8]_q a_2^3) = \left(\frac{1}{2}c_3 - \frac{1}{4}c_{_1}c_2\right)$$

$$[12]_q a_4 = \left(\frac{[18]_q c_1}{2([2]_q)}\right)\left(\frac{[18]_q {c_1}^2}{8([6]_q)} + \frac{[18]_q c_2}{2([6]_q)}\right) - [8]_q \left(\frac{c_1}{2([2]_q)}\right)^3 + \frac{c_3}{2} - \frac{c_1 c_2}{4}$$

$$[12]_q a_4 = \left(\frac{[18]_q {c_1}^3}{16([12]_q)} + \frac{[18]_q c_1^2 c_2}{4([12]_q)}\right) - [8]_q \left(\frac{{c_1}^3}{8([8]_q}\right) + \frac{c_3}{2} - \frac{c_1 c_2}{4}$$

$$[12]_q a_4 = \frac{[18]_q {c_1}^3}{16([12]_q)} + \frac{[18]_q c_1 c_2}{4([12]_q)} - \frac{{c_1}^3}{8} + \frac{c_3}{2} - \frac{c_1 c_2}{4}$$

$$[12]_q a_4 = \frac{[18]_q {c_1}^3 - 2[12]_q {c_1}^3}{16([12]_q)} + \frac{[18]_q c_1 c_2 - [12]_q c_1 c_2}{4[12]_q} + \frac{c_3}{2}$$

$$[12]_q a_4 = \frac{-[6]_q {c_1}^3}{16([12]_q)} + \frac{+[6]_q c_1 c_2}{4[12]_q} + \frac{c_3}{2}$$

$$a_4 = \frac{-{c_1}^3}{16[24]_q} + \frac{c_1 c_2}{4[24]_q} + \frac{c_3}{2[12]_q}$$

$$\begin{cases} a_2 = \frac{c_1}{2([2]_q)} \\ a_3 = \frac{{c_1}^2}{8([6]_q)} + \frac{c_2}{2\,[6]_q} \\ a_4 = \frac{-{c_1}^3}{16[24]_q} + \frac{c_1 c_2}{4[24]_q} + \frac{c_3}{2[12]_q} \end{cases} \tag{4.21}$$

Since the class P and $\mathscr{H}_{2,1}(F_{f,q}/2)$ is invariant under rotation, and we assume that $c_1 \in [0,2]$

that is in view of (1.13) that $\tau \in [0,1]$. Using (4.21) in (1.12) we have

$$\mathscr{H}_{2,1}(F_{f,q}/2) = (\gamma_1^q\gamma_3^q - (\gamma_2^q)^2) = \frac{1}{[48]_q}(a_2^4 - [12]_q a_3^2 + [12]_q a_2 a_4)$$

$$= \frac{1}{[48]_q}\left[\left(\frac{c_1}{2([2]_q)}\right)^4 - 3[4]_q\left(\frac{{c_1}^2}{8([6]_q)} + \frac{c_2}{(2[6]_q)}\right)^2\right] + \left[\left(\frac{3[4]_q c_1}{2([2]_q)}\right)\left(\frac{-{c_1}^3}{16[24]_q} + \frac{c_1c_2}{4[24]_q} + \frac{c_3}{2[12]_q}\right)\right]$$

$$= \frac{1}{[48]_q}\left[\left(\frac{{c_1}^4}{16[16]_q}\right) - \left(\frac{3[4]_q {c_1}^4}{64[36]_q} + \frac{3[4]_q {c_2}^2}{4[36]_q} + \frac{3[8]_q {c_1}^2 c_2}{16[36]_q}\right)\right] + \left[\left(\frac{-3{c_1}^4}{8[48]_q} + \frac{3{c_1}^2c_2}{2[48]_q} + \frac{3c_1c_3}{[24]_q}\right)\right]$$

$$= \frac{1}{[48]_q}\left[\frac{{c_1}^4}{16[16]_q} - \frac{3{c_1}^4}{16[36]_q} - \frac{3{c_2}^2}{[36]_q} - \frac{3{c_1}^2c_2}{2[36]_q} - \frac{3{c_1}^4}{8[48]_q} + \frac{3{c_1}^2c_2}{2[48]_q} + \frac{3c_1c_3}{[24]_q}\right]$$

$$= \frac{1}{[48]_q}\left[\left(\frac{{c_1}^4}{16[16]_q} - \frac{3{c_1}^4}{16[36]_q} - \frac{3{c_1}^4}{8[48]_q}\right)\left(-\frac{3{c_1}^2c_2}{2[36]_q} + \frac{3{c_1}^2c_2}{2[48]_q}\right) + \left(-\frac{3{c_2}^2}{[36]_q} + \frac{3c_1c_3}{[24]_q}\right)\right]$$

$$= \frac{1}{[48]_q}\left[\left(-\frac{[7]_q {c_1}^4}{16[48]_q} - \frac{{c_1}^2c_2}{2[48]_q} - \frac{3{c_2}^2}{[36]_q} + \frac{3c_1c_3}{[24]_q}\right)\right]$$

$$= \left[\left(-\frac{[7]_q {c_1}^4}{[36864]_q} - \frac{{c_1}^2c_2}{[4608]_q} - \frac{3{c_2}^2}{[1728]_q} + \frac{3c_1c_3}{[1152]_q}\right)\right]$$

$$= \left[-\frac{[7]_q {c_1}^4}{[36864]_q} - \frac{[8]_q \times {c_1}^2c_2}{[4608]_q \times [8]_q} - \frac{3{c_2}^2 \times \frac{[64]_q}{3}}{[1728]_q \times \frac{[64]_q}{3}} + \frac{3c_1c_3 \times [32]_q}{[1152]_q \times [32]_q}\right]$$

$$\mathscr{H}_{2,1}(F_{f,q}/2) = \frac{\left[-[7]_q {c_1}^4 - [8]_q {c_1}^2 c_2 - [64]_q {c_2}^2 + [96]_q c_1 c_3\right]}{[36864]_q}. \tag{4.22}$$

By the Lemma (3.1.1) we have value of c_1, c_2 and c_3 use in (4.22).

$$\mathscr{H}_{2,1}(F_{f,q}/2) = \frac{1}{[36864]_q}\left[\begin{array}{c} -[7]_q(2\tau_1)^4 - [64]_q(2\tau_1^2+2(\ 1-\ \tau_1^2\ )\tau_2)^2 \\ -[8]_q(2\tau_1)^2(2\tau_1^2+2\left(\ 1-\ \tau_1^2\ \right)\tau_2)+[96]_q(2\tau_1)(2\tau_1^3+4(\ 1-\ \tau_1^2\ )\tau_1\tau_2 \\ -\ 2(\ 1-\ \tau_1^2\ )\tau_1\tau_2^2+\ 2(\ 1-\ \tau_1^2\ )(\ 1-|\tau_2|^{\ 2})\tau_3) \end{array}\right]$$

$$= \frac{1}{[36864]_q}\left[\begin{array}{c} -[7]_q 16\tau_1^4 - [64]_q(4\tau_1^2+4\tau_2^2(\ 1-\tau_1^2)^2+8\tau_1^2(\ 1-\ \tau_1^2)\tau_2) \\ -[8]_q(4\tau_1^2)\ (2\tau_1^2+2\tau_2-2\tau_2\tau_1^2)+[96]_q(2\tau_1)(2\tau_1^3+4\tau_1\tau_2-4\tau_1^3\ \ \tau_2 \\ -2\tau_1\tau_2^2+2\tau_2^2\tau_1^3(\ 2-2\tau_1^2\ )(\ 1-|\tau_2|^{\ 2})\tau_3) \end{array}\right]$$

$$= \frac{1}{[36864]_q}\left[\begin{array}{c} -[7]_q 16\tau_1^4 - [64]_q[4\tau_1^4+4\tau_2^2(1-\tau_1^4+2\tau_1^2)+8\tau_1^2\tau_2-8\tau_1^4\tau_2 \\ -[8]_q(8\tau_1^4+8\tau_1^2\tau_2-8\tau_2\tau_1^4)+[96]_q(4\tau_1^4+8\tau_1^2\tau_2-8\tau_2\tau_1^4-4\tau_1^2\tau_2^2 \\ -4\tau_1^4\tau_2^2+(4\tau_1-4\tau_1^3\ )(\ 1-|\tau_2|^{\ 2})\tau_3) \end{array}\right]$$

$$= \frac{1}{[36864]_q}\left[\begin{array}{c} -[7]_q 16\tau_1^4 - [64]_q[4\tau_1^4+4\tau_2^2+4\tau_2^2\tau_1^4-8\tau_2^2\tau_1^2+8\tau_1^2\tau_2-8\tau_1^4\tau_2 \\ -[8]_q(8\tau_1^4+8\tau_1^2\tau_2-8\tau_2\tau_1^4)+[96]_q(4\tau_1^4+8\tau_1^2\tau_2-8\tau_2\tau_1^4-4\tau_1^2\tau_2^2 \\ -4\tau_1^4\tau_2^2+(4\tau_1-4\tau_1^3\ )(\ 1-|\tau_2|^{\ 2})\tau_3) \end{array}\right]$$

$$= \frac{1}{[36864]_q} \left[\begin{array}{c} -[112]_q \tau_1^4 - [256]_q \tau_1^4 - [256]_q \tau_2^2 - [256]_q \tau_2^2 \tau_1^4 + [512]_q \tau_2^2 \tau_1^2 \\ \\ -[512]_q \tau_1^2 \tau_2 + [512]_q \tau_1^4 \tau_2 - [64]_q \tau_1^4 - [64]_q \tau_1^2 \tau_2 + [64]_q \tau_2 \tau_1^4) \\ \\ +[384]_q \tau_1^4 + [768]_q \tau_1^2 \tau_2 - [768]_q \tau_2 \tau_1^4 - [384]_q \tau_1^2 \tau_2^2 \\ \\ +[384]_q \tau_1^4 \tau_2^2 + [96]_q (4\tau_1 - 4\tau_1^3)(1 - |\tau_2|^2)\tau_3) \end{array} \right]$$

$$= \frac{1}{[36864]_q} \left[\begin{array}{c} -[112]_q \tau_1^4 - [256]_q \tau_1^4 - [256]_q \tau_2^2 - [256]_q \tau_2^2 \tau_1^4 + [512]_q \tau_2^2 \tau_1^2 \\ \\ -[512]_q \tau_1^2 \tau_2 + [512]_q \tau_1^4 \tau_2 - [64]_q \tau_1^4 - [64]_q \tau_1^2 \tau_2 + [64]_q \tau_2 \tau_1^4) \\ \\ +[384]_q \tau_1^4 + [768]_q \tau_1^2 \tau_2 - [768]_q \tau_2 \tau_1^4 - [384]_q \tau_1^2 \tau_2^2 \\ \\ +[384]_q \tau_1^4 \tau_2^2 + [96]_q (4\tau_1 - 4\tau_1^3)(1 - |\tau_2|^2)\tau_3) \end{array} \right]$$

$$= \frac{1}{[36864]_q} \left[\begin{array}{c} -[48]_q \tau_1^4 - [256]_q \tau_2^2 + [128]_q \tau_2^2 \tau_1^4 + [128]_q \tau_2^2 \tau_1^2 + [192]_q \tau_1^2 \tau_2 \\ \\ -[192]_q \tau_1^4 \tau_2 + [96]_q (4\tau_1 - 4\tau_1^3)(1 - |\tau_2|^2)\tau_3) \end{array} \right]$$

$$= \frac{1}{[2304]_q \times [16]_q} \left[\begin{array}{c} -[48]_q \tau_1^4 - [256]_q \tau_2^2 + [128]_q \tau_2^2 \tau_1^4 + [128]_q \tau_2^2 \tau_1^2 + [192]_q \tau_1^2 \tau_2 \\ \\ -[192]_q \tau_1^4 \tau_2 + [96]_q (4\tau_1 - 4\tau_1^3)(1 - |\tau_2|^2)\tau_3) \end{array} \right]$$

$$= \frac{1}{[2304]_q} \left[\begin{array}{c} \frac{-[48]_q \tau_1^4}{[16]_q} - \frac{[256]_q \tau_2^2}{[16]_q} + \frac{[128]_q \tau_2^2 \tau_1^4}{[16]_q} + \frac{[128]_q \tau_2^2 \tau_1^2}{[16]_q} + \frac{[192]_q \tau_1^2 \tau_2}{[16]_q} \\ \\ -\frac{[192]_q \tau_1^4 \tau_2}{[16]_q} + \frac{[96]_q (4\tau_1 - 4\tau_1^3)(1 - |\tau_2|^2)\tau_3)}{[16]_q} \end{array} \right]$$

$$= \frac{1}{[2304]_q} \left[\begin{array}{c} -[3]_q \tau_1^4 - [16]_q \tau_2^2 + [8]_q \tau_2^2 \tau_1^4 + [8]_q \tau_2^2 \tau_1^2 + [12]_q \tau_1^2 \tau_2 \\ \\ -[12]_q \tau_1^4 \tau_2 + [6]_q (4\tau_1 - 4\tau_1^3)(1 - |\tau_2|^2)\tau_3) \end{array} \right]$$

As we know that

$$-[16]_q\tau_2^2+[8]_q\tau_2^2\tau_1^4+[8]_q\tau_2^2\tau_1^2 = -[8]_q(1-\tau_1^2)(2+\tau_1^2)\tau_2^2$$
$$[12]_q\tau_1^2\tau_2 - [12]_q\tau_1^4\tau_2 = [12]_q(1-\tau_1^2)\tau_1^2\tau_2^2,$$

then above term has become

$$\mathscr{H}_{2,1}(F_{f,q}/2) = \frac{1}{[2304]_q}\left[\begin{array}{c} -[3]_q\tau_1^4-[8]_q(1-\tau_1^2)(2+\tau_1^2)\tau_2^2 \\ \\ +[12]_q(1-\tau_1^2)\tau_1^2\tau_2^2 + [24]_q\tau_1\tau_3(1-\tau_1^2)(1-|\tau_2|^2) \end{array}\right]. \quad (4.23)$$

Examine the subsequent possible cases on τ_1:

Case 1

if $\tau_1 = 1$. Then (4.23) is

$$\mathscr{H}_{2,1}(F_{f,q}/2)| = \frac{1}{[2304]_q}[-[3]_q+0]$$

If $q \to 1^-$ then we have

$$|\mathscr{H}_{2,1}(F_f/2)| = \frac{1}{768}$$

Case 2

if $\tau_1 = 0$.Then (4.14) is

$$|\mathscr{H}_{2,1}(F_{f,q}/2)| = \frac{1}{[2304]_q}[-[8]_q(2)\tau_2^2 + 0]$$

$$|\mathscr{H}_{2,1}(F_{f,q}/2)| = \frac{[16]_q}{[2304]_q}|\tau_2|^2 \leq \frac{[16]_q}{[2304]_q}$$

If $q \to 1^-$ the we have

$$|\mathscr{H}_{2,1}(F_f/2)| = \frac{1}{144}|\tau_2|^2 \leq \frac{1}{144}$$

Case 3

Suppose $\tau \in (0,1)$. Utilizing the triangle inequality in (4.23) Additionally, we use the knowledge that $|\tau_3| \leq 1$, to get

$$|\mathscr{H}_{2,1}(F_{f,q}/2)| = \frac{1}{[2304]_q}\left[\begin{array}{c} -[3]_q\tau_1^4+[12]_q(1-\tau_1^2)\tau_1^2\tau_2^2-[8]_q(1-\tau_1^2)(2+\tau_1^2)\tau_2^2 \\ +[24]_q\tau_1\tau_3(1-\tau_1^2)(1-|\tau_2|^2) \end{array}\right]$$

taking $\frac{1}{[96]_q}\tau_1(1-\tau_1^2)$ from inside we have

$$= \frac{1}{[96]_q}\tau_1(1-\tau_1^2)\left\{\frac{-[3]_q\tau_1^3}{[24]_q(1-\tau_1^2)}+\frac{[12]_q\tau_1\tau_2}{[24]_q}-\frac{[8]_q(2-\tau_1^2)\tau_2^2}{[24]_q\tau_1}+(1-|\tau_2^2|)\right\}$$

If we take

$$A = \frac{-\tau_1^3}{[8]_q\left(1- \ \tau_1^2\right)}, \qquad B = \frac{\tau_1}{[2]_q} \qquad C = \frac{-\ (2+\tau_1^2)}{[3]_q\tau_1}.$$

Then we obtain

$$|\mathscr{H}_{2,1}(F_{f,q}/2)| = \frac{1}{[96]_q}\tau_1\ \left(1- \ \tau_1^2\right)(|A+B\tau_2+C\tau_2^2+1- \ |\tau_2^2|). \tag{4.24}$$

Since we can see that $AC > 0$, we can use Lemma (3.1.2) case (i).Currently, we examine every condition of case (i).

3(a) The inequality is seen.

$$|B|-2(1-|C|) = \frac{\tau_1}{[2]_q}\ -2\left(1-\frac{-\ (2+\tau_1^2)}{[3]_q\tau_1}\right)$$

$$= \frac{\tau_1}{[2]_q}\ -2+\frac{2\ (2+\tau_1^2)}{[3]_q\tau_1}$$

$$= \frac{\tau_1}{[2]_q}\ -2+\frac{4}{[3]_q\tau_1}+\frac{2\tau_1^2}{[3]_q\tau_1}$$

$$= \frac{\tau_1-[4]_q}{[2]_q}\ +\frac{4+2\tau_1^2}{[3]_q\tau_1}$$

$$= \frac{[3]_q\tau_1^2-[12]_q\tau_1+[8]_q+[4]_q\tau_1^2)}{[6]_q\tau_1}$$

$$= \frac{[7]_q\tau_1^2-[12]_q\tau_1+[8]_q}{[6]_q\tau_1}\ >0,$$

If $q \to 1^-$ then we have

$$|B|-2(1-|C|) = \frac{(7\tau_1^2-12\tau_1+8)}{(6\tau_1)} > 0.$$

Which is true for all $\tau_1 \in (0,1)$. Thus, Lemma (3.1.2) implies it. Not to mention the inequality

(4.24) that

$$|\mathscr{H}_{2,1}(F_{f,q}/2)| \leq \frac{1}{[96]_q}\tau_1\ \left(1- \ \tau_1^2\right)\left(|A|+|B|+|C|\right)$$

$$= \frac{1}{[96]_q}\tau_1\ \left(1- \ \tau_1^2\right)\left(\left|\frac{-\tau_1^3}{[8]_q\left(1- \ \tau_1^2\right)}\right|+\left|\frac{\tau_1}{[2]_q}\right|+\left|\frac{-\ (2+\tau_1^2)}{[3]_q\tau_1}\right|\right)$$

$$= \frac{\tau_1^4}{[768]_q}+\frac{\tau_1^2\left(1- \ \tau_1^2\right)}{[192]_q}+\frac{\tau_1\ \left(1- \ \tau_1^2\right)\left(2+\tau_1^2\right)}{[288]_q\tau_1}$$

$$= \frac{\tau_1^4}{[768]_q}+\frac{\tau_1^2- \ \tau_1^4}{[192]_q}+\frac{-2-2\tau_1^2+\tau_1^2-\tau_1^4}{[288]_q}$$

$$= \frac{[3]_q\tau_1^4+[12]_q\tau_1^2-[12]_q\tau_1^4-[16]_q-[16]_q\tau_1^2+[8]_q\tau_1^4-[8]_q\tau_1^4}{[2304]_q}$$

$$= \frac{[4]_q\tau_1^2+[16]_q-[17]_q\tau_1^4}{2304}$$

$$|\mathscr{H}_{2,1}(F_{f,q}/2)| = \frac{1}{[2304]_q}([16]_q\ +[4]_q\tau_1^2-[17]_q\tau_1^4),$$

if $\tau_1 = 0$ then

$$|\mathscr{H}_{2,1}(F_{f,q}/2)| \leq \left|\frac{[16]_q}{[2304]_q}\right| \leq \ \frac{1}{[144]_q}$$

If $q \to 1^-$ then we have

$$|\mathscr{H}_{2,1}(F_{f,q}/2)| \leq \frac{1}{144}$$

3(b) Next, it is easy to check that

$$|B|-2\left(1-|C|\right) = \frac{\tau_1}{[2]_q}\ -2\left(1-\frac{(2+\tau_1^2)}{[4]_q\tau_1}\right)$$

$$= \frac{([7]_q\tau_1^2-[12]_q\tau_1+[8]_q)}{[6]_q\tau_1} < 0,$$

If $q \to 1^-$ then we have

$$|B|-2\left(1-|C|\right) = \frac{(7\tau_1^2-12\tau_1+8)}{(6\tau_1)} < 0.$$

which is not true for all $\tau_1 \in (0,1)$.

After a summary of cases 1,2 and 3, the inequality (4.14) is proven.It is only to show that the boundary is sharp in order to finish the proof. To demonstrate that we consider the function $h \in C_{☾}(q)$, like this

$$h_0(\hat{y}) = \hat{y}\, exp_q\left(\frac{\sqrt{69}}{\sqrt{68}}\int_0^{\hat{y}} \frac{x^2 + \sqrt{1+x^4}\ -1}{x}\ d_qx\right) \tag{4.25}$$

$$= \hat{y} + \frac{\sqrt{69}}{6[2]_q\sqrt{17}}\hat{y}^3 + \frac{\sqrt{69}}{10[4]_q\sqrt{17}}\hat{y}^5 + \ldots,$$

let

$$h(\hat{y}) = \int_0^{\hat{y}} \frac{h_0(x)}{x} d_qx = \hat{y} + \frac{\sqrt{69}}{6[2]_q\sqrt{17}}\hat{y}^3 + \frac{\sqrt{69}}{10[4]_q\sqrt{17}}\hat{y}^5 + \ldots,$$

with $a_2 = a_4 = 0$ and $a_3 = \frac{\sqrt{69}}{6[2]_q\sqrt{17}}$ in (1.12). Then we have

$$|\mathscr{H}_{2,1}(F_h/2)| = \frac{1}{\bar{q}_1}.$$

This complete the proof. □

4.4 Summary

This chapter defines two subclasses of univalent functions: convex and starlike functions. Fekete–Szegő inequality, Hankel Determinants, and coefficient estimates were established for the newly defined classes. A few corollaries are also defined in this study, indicating that the resulting results are identical to those demonstrated by researchers when the limit $q \to 1^-$ is substituted.

CHAPTER 5

CONCLUSION

In this thesis, the initial coefficient bounds of analytic, univalent, normalised functions inside the open unit disc are the main emphasis. The first things we discussed were some fundamental terms and preliminary results from the Geometric Function Theory. We examined more recent concepts that were introduced in Quantum Calculus, but these fundamental concepts serve as the foundation for our innovative discoveries. A variety of distinctive classes of analytic functions related with symmetric points were defined using q-Calculus, and the applications of the operator for q-derivative in the Theory of Geometric Functions were thoroughly examined.

Our study focusses on two basic categories of univalent functions: convex functions connected with the second Hankel Determinant of logrithmic coefficients associated with lune and starlike functions. Building on earlier research by S. Mandal and Ahamed [51] on the $S^*_{☾}$ class of starlike functions with Second Hankel Determinant of logrithmic coefficients associated with lune,we investigated the extension of these classes using q-calculus lune-associated logrithmic coefficients belonging to the $C_{☾}$ class of convex functions with the Second Hankel Determinant. An extension of the original $S^*_{☾}$ class, we presented the $S^*_{☾}(\hat{q})$ class and $C_{☾}(\hat{q})$, which represent starlike functions and convex functions with logrithmic coefficients q-lune function. We demonstrated the q-extension of these classes by computing the second Hankel Determinant for q-starlike functions and q-convex functions, both of which are subordinate to the q-lune function we used the subordination technique to study the features of these classes, which were introduced by the q-derivative operator.

Within our recently established classes, we have investigated some important features of functions, such as the well-known Fekete–Szegő inequality and coefficient bounds. We have also looked into Hankel determinants of second order for functions that fall under our recently established classes. These new classes have been shown to be an improvement over the ones that already exist, and the resulting results indicate improvements over the theorems that many Geometric Function Theory scholars have already established. Our results were validated by considering the limit as $q \to 1^-$ which produced known results. The results of this work should greatly enhance the field of geometric function theory.

5.1 Future work

In univalent function theory, this thesis focusses on two main categories: starlike functions with the second Hankel Logrithmic coefficient and convex function determinant using Hankel In the fields of spherical forms, geometry, and spherical trigonometry, it is essential to determine the second order of logrithmic coefficients associated with lune, which are subservient to a particular lune function. These classes can be proficient by using the concept of close-to-convexity. Outcomes for the proficient class of q-quasi convex functions can be found, and an analytical and geometrical association in between these categories and the classes within this thesis can be shown.

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