On Certain New Subclass of Bi-Univalent Functions using Quasi Subordination and q-Derivative

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Candidate of Master of Science in Mathematics at the National University of Modern Languages do hereby declare that the thesis On Certain New Subclass of Bi-Univalent Functions using Quasi Subordination and q-Derivative submitted by me in partial fulfillment of MS degree, is my original work and has not been submitted or published earlier. I also solemnly declare that it shall not, in the future, be submitted by me for obtaining any other degree from this or any other university or institution. I also understand that if evidence of plagiarism is found in my thesis/dissertation at any stage, even after the award of a degree, the work may be canceled and the degree revoked.

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ABSTRACT

Title: On Certain New Subclass of Bi-Univalent Functions using Quasi Subordination and q-Derivative

This research work aims to establish and determine the new subclasses of bi-univalent functions. The ideas of quantum calculus will be applied to examine the q-extensions of previously defined classes of bi-univalent functions. We investigate the bounds of initial coefficients of subclasses of q-bi univalent functions by employing quasi-subordination. Also, we analyze the upper bounds of the initial coefficients, Fekete-Szego inequality, and upper bound of second order Hankel determinant of a subclass of bi-starlike functions by employing the q-Salagean operator. It will be indicated that the newly defined results of this research are refined and advanced compared to previous results proved by various researchers in this field. Corollaries of new estimated results will also be shown in this thesis which shows the relation between previously derived and newly estimated results.

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LIST OF SYMBOLS

V	-	Open unit disk
A	-	Class of Analytic functions
S	-	Class of Univalent functions
P	-	Class of Caratheodory functions
∇	-	class of bi-univalent functions
$\mathbb{S}^*_\nabla$	-	Class of bi-starlike functions
$\prec_q$	-	Quasi-subordination symbol
$\mathfrak{R}^{\partial,y}_{\nabla,\gamma,c}$	-	subclass of bi-univalent function using Quasi-subordination symbol
$\mathfrak{K}^{\partial,y}_{\nabla,\gamma,c}$	-	subclass of bi-univalent functions using Quasi-subordination symbol
O^u	-	Salagean operator symbol
O^u_q	-	q-Salagean operator symbol
$\mathbb{S}^*_\nabla(\gamma,\Theta,u,v)$	-	subclass of bi-starlike function using Salagean operator symbol
$\mathbb{S}^*_{\nabla,q}(\gamma,\Theta,u,v)$	-	subclass of bi-starlike function using q-Salagean operator symbol
D_q	-	q-Derivative operator symbol
H	-	Hankel Determinant symbol

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DEDICATION

To my father Ghulam Ali Malik and my mother Khalida Parveen, to my teachers, and to everyone who encourages me to make this success achievable.

CHAPTER 1

INTRODUCTION AND LITERATURE REVIEW

1.1 Overview

This chapter offers a thorough introduction and literature review of the framework of Geometric Function Theory. It covers the classes and their respective subclasses of analytic, univalent, and bi-univalent functions. It provides the basic details about Quasi-subordination and Hankel determinants. This chapter also provides a brief discussion of quantum calculus.

1.2 Riemann Mapping Theorem

Riemann provides essential support within the discipline of complex analysis. He built a geometric framework by using Cauchy-Riemann equations and conformal mappings. His quality of work opened the doors for a new field of study of the mathematics of complex-valued functions known as Geometric Function Theory. This field of mathematics relates the mathematical concepts of complex analysis with geometry and analyzes structural behavior in analytic functions. Various researchers have provided many important results about geometric functions. Still, this is one of the most active fields in the current research. One of the most important and basic results is the Riemann mapping theorem. Riemann gave this result in 1981 [1]. This theorem describes that on the complex plane, most simply connected domains (other than the complete complex plane) can be conformally mapped (analytic and one-to-one) upon the open unit disk.

1.3 Analytic Function and Univalent Function

In 1907, Koebe [2] brings the advancements in the work of Riemann. He discovered that analytic and univalent functions completely utilized Riemann's theorem in simply connected domains. A function is analytic in a domain if its derivative takes place at every point of the domain, and a function is analytic at a point if its derivative takes place at that point and also in the neighborhood of that point. These functions have a series representation form as given:

$$f(\mathtt{s}) = \mathtt{s} + \sum_{t=2}^{\infty} \alpha_t \mathtt{s}^t, \tag{1.1}$$

and if a function (1.1) takes a value zero at the origin, that is $f(0) = 0$ and if the derivative of the function has value 1 at the origin, that is $f'(0) = 1$. Then function 'f' is considered to be normalized. So, it can be stated that if a function having form (1.1) is analytic and normalized then function 'f' is considered to be in class A. And, if an analytic, and normalized function is also injective(one-to-one) in an open unit disk(V) then the function is considered to be in class S. Functions in class S are known as univalent or schlicht functions ([3, 4, 5]). A Function is univalent in a domain if different values in the domain provide different values of the range, and a function is univalent at a point if it is univalent at that point and at the neighborhood of that point. The condition $f'(\mathtt{s}_0) \neq 0$ shows that analytic function f is univalent at a point $\mathtt{s}_0$. Also, many researchers provide different conditions through which we can examine whether a function is univalent or not in a domain. The most important theorem given by Noshiro and Warchawski is that if the derivative of an analytic function is positive in V then the function is univalent. Class S has many subclasses. Most popular are starlike, convex, quasi-convex, and close-to-convex functions. In 1916, Bieberbach [6] provided an estimate for coefficients of functions of class S. That is, $|\alpha_t| \leq t, \ \forall\, t \geq 2$.

This conjecture became a challenge for many mathematicians for many years, and finally, in 1985, de-branges solved it (see [7]). The Koebe function $K(\mathtt{s})$ has a very essential role in the class S because of its extremal nature. These subclasses of S whose have the factor that real parts are positive provide the presence of the series form

$$p(\mathrm{s}) = 1 + n_1\mathrm{s} + n_2\mathrm{s}^2 + n_3\mathrm{s}^3 + \ldots \tag{1.2}$$

satisfies $p(0) = 1$ and $Re(p(\mathrm{s})) > 0$ (positive real parts). A function is considered in the class P if it has the property of positivity of real parts. Such functions are named as Caratheodory functions. Specifically, $f \in P$ if and only if $f \in A$, $f(0) = 1$ and $Ref(\mathrm{s})>0$ for $|\mathrm{s}| < 1$. The function 'p' may be expressed as a function subordinate to Mobius function $L_o(\mathrm{s}) = \frac{1+\mathrm{s}}{1-\mathrm{s}}$. Mobius function assumes the best possible property for such functions. The coefficient estimate for this class is $|n_t| \leq 2$. For $t = 1, 2, 3\ldots$,

1.4 Bi-Univalent Function

As stated by Koebe one- quarter theorem, f^{-1} is defined in some disk consisting of the disk $|\omega|<1/4$. In some cases, this inverse function is expanded to the entire unit disk. That shows f^{-1} is also univalent. If f and f^{-1} both are univalent in an open unit disk(V) then such functions are considered to be bi-univalent. Many recent investigations give a brief study of bi-univalent functions (see, [8, 9, 10]).

The study of bi-univalent functions is considered important in Geometric function theory because of its relation with the univalent functions. These functions are a subclass of univalent functions that consider both function and its inverse. As these functions are the extension of univalent functions, therefore they are very significant in the study of Geometric Function Theory. Applications of bi-univalent functions in geometric function theory are in modeling geometric transformations, coefficient estimation, generalization of classical results, and approximation theory, etc. Recently, the analysis of bi-univalent functions has captured the attention of researchers. Specifically, through the work of Srivastava (for example, [11, 12, 13]). Since then, many other researchers have given their ideas which have contributed to many other developments in this field of mathematics. Specifically, researchers have determined many findings about the estimates of the coefficients, especially the initial coefficients, for different categories and types of bi-univalent functions (for example, [14, 15, 16]).

In 1967, Lewin [17] gave the idea of bi-univalent functions and proved the second coefficient estimate for these functions, that is $|\alpha_2|<1.51$. In the same year, Brannan and Clunie [18] obtained estimate for bi-univalent functions is $|\alpha_2|<\sqrt{2}$. Later, Netanyahu [19] showed that $\max_{f\in\nabla}|\alpha_2|=\frac{4}{3}$.Kedzierawski [20] verified the Brannan and Clunie postulate for bi-starlike functions in 1985. In the same year, Tan [21] find coefficient bound $|\alpha_2|<1.485$, which is known as best result for functions in class ∇. In 1986, Brannan and Taha [22] presented the subclasses of bi-univalent functions known as "bi-starlike function of order β" and "bi-convex function of order β" and attained evaluation on the primary coefficients. Recently, Deniz [23] and Kumar [24] enhanced and expanded the work of Brannan and Taha by applying the technique of subordination among analytic functions. In 2010, Srivastava et al. [25] presented new subclasses of "bi-univalent functions" and find the results of initial coefficient estimates. The challenge of evaluating the coefficients of bi-univalent functions is still open.

1.5 Quasi-Subordination

In 1909, Lindelof [26] provide the concept of subordination. Afterward, advancements were made by Littlewood and Rogosinski [27]. The subordination technique is provided by employing the Schwarz function. This is known as unit bound function. In 1970, Robertson [28] unveiled the concept of quasi-subordination and majorization. The idea of quasi-subordination can be described as a function $f(\mathrm{s})$ is quasi-subordinate to another function $g(\mathrm{s})$, if there are two analytic functions $\mu(\mathrm{s})$ and $F(\mathrm{s})$ such that for $|\mathrm{s}|<1$, exists $|\mu(\mathrm{s})| \leq 1$ and $|F(\mathrm{s})| \leq 1$. $'f(\mathrm{s})'$ is quasi-subordinate to $'g(\mathrm{s})'$ can be documented as $f(\mathrm{s}) \prec_{\mathrm{q}} g(\mathrm{s})$, which is equivalent to $f(\mathrm{s}) = \mu(\mathrm{s})\, g(F(\mathrm{s}))$. And a function $'f(\mathrm{s})'$ is majorized by $'g(\mathrm{s})'$ if there exist a function $\mu(\mathrm{s}) \leq 1$, and $F(\mathrm{s}) = 1$. For $\mu(\mathrm{s}) = 1$, the quasi-subordination becomes ordinary subordination. Quasi-subordination is the advancement of subordination and majorization. Quasi-subordination gives more flexibility by introducing inequalities and can be more useful when ordinary subordination gives too strict results or does not satisfy. This concept is used for growth and distortion properties, and for generalizations of results obtained by ordinary subordination.

In 1994, Ma and Minda [29] introduced the classes of starlike functions $(S^*(\mathrm{h}))$ and convex functions$(C(\mathrm{h}))$ by using the concept of ordinary subordination. Later in 2012, Haji Mohd et al. [30] defined many classes using quasi-subordination, and obtained their initial coefficient estimates and Fekete-Szego inequality. Out of those classes they defined $S_q^*(\mathrm{h})$ and $C_q(\mathrm{h})$ known as Ma-Minda starlike class and Ma-Minda convex class respectively defined in the form of quasi-subordination. In the past few years, many scholars have been working on the classes of bi-univalent functions applying quasi-subordination. Most of them find the initial coefficients estimates of class A (e.g. [31, 32, 33]).

1.6 Hankel Determinant

The Hankel determinant is a unique determinant of a matrix in which the components are placed in a Hankel pattern, which suggests that the components of the matrix are arranged in a structure that the values of each diagonal are constant. To examine the Fibonacci sequence or the Locas sequence Hankel determinants are usually used. It can be implemented to investigate the generating function of the sequence and its characteristics as the sequence grows. Hankel determinants are effective for expressing that a function with bounded qualities in V, particularly, a function that can be shown as the ratio of two bounded analytic functions, with having Laurent series about the origin possessing integral coefficients is rational [34]. In 1966, the nth Hankel determinant for $n \geq 1$ and $t \geq 0$ is pointed out by Pommerenke [35] (see also, [36]) as

$$H_n(t) = \begin{vmatrix} \alpha_t & \alpha_{t+1} & \alpha_{t+2} & \cdots & \alpha_{t+n-1} \\ \alpha_{t+1} & \alpha_{t+2} & \alpha_{t+3} & \cdots & \alpha_{t+n} \\ \vdots & \vdots & \vdots & \cdots & \alpha_{t+n+1} \\ \alpha_{t+n-1} & \alpha_{t+n} & \alpha_{t+n+1} & \cdots & \alpha_{t+2n-2} \end{vmatrix}$$

For distinct values of 'n' and 't' in the general form of Hankel determinant there are many different orders of Hankel determinants and their values depend upon coefficients of required univalent function. Pommerenke [37] analyzed the growth rate of Hankel determinant $H_n(t)$ as $t \to \infty$ of p-valent functions, univalent functions, and starlike functions (see also, [35]). Hankel determinant is helpful and observed by several authors [38]. Many characteristics of these determinants can be seen in [39]. The first order Hankel determinant $H_2(1) = \alpha_3 - \alpha_2^2$ is the famous Fekete-Szego functional for values of $n = 2$ and $t = 1$. The second order Hankel determinant is $H_2(2) = \alpha_2\alpha_4 - \alpha_3^2$. For the different subclasses of univalent functions, the first two orders of determinants are broadly examined by many authors (For example, [40, 41, 42, 43, 44]). For more study and work on this determinant, see ([45, 46, 47, 48]).

1.7 Quantum Calculus

Leonhard Euler(1707-1783) and Carl Gustav Jacobi(1804-1851) set the roots of quantum calculus or named q-calculus. But later in 1905, q-calculus gained attention by the paper of Albert Einstein which shows its applicability in quantum mechanics. In the years 1908 to 1910, Jackson (see, [49, 50]) gives a deep understanding of q-calculus. Later, the exploration of quantum groups shows the geometrical analysis of q-calculus. Quantum calculus is widely used in many branches of physics and mathematics including combinatorics, fractional calculus, orthogonal polynomials, calculus of variations, relativity, basic hypergeometric functions q-difference and q-integral equations, optimal control problems, and q-transform analysis. In traditional calculus concept of limits is applied to obtain the results of the functions, while in q- calculus there is no need for limits. In simpler words, q-calculus is the calculus without the impression of limits. q –operators are like modified versions of traditional calculus tools(like derivatives and integrals). q-calculus is the concept of calculus where smoothness is not needed. They include a parameter "q" and are used to handle discrete situations. The parameter "q" is a mathematical variable that introduces a level of deformation or magnification to traditional calculus operations. The q-derivative act is just like the difference quotient and q-integral as a sum in q-calculus. Some basic q-functions are q-exponential functions, q-trigonometric functions, q-logarithmic functions etc. In [51], Gasper and Rahman provide detailed explanations and applications of q-calculus in different fields like physics, combinatorics, and number theory. In 1990, Ismail et al. [52] gave the q-extension of starlike functions. This exploration allows access to further study of q-calculus in Geometric Function Theory. Srivastava [53] provides the basic concepts and practical applications of q-derivative operators in association with geometric function theory. Aldweby et al. [54] created q-analogs for numerous operations linked with the convolution of analytic functions. In [55], they provide the more advanced subfamilies of q-starlike functions using Janowski functions. The q-calculus gained a lot of attention from scholars as can be seen in numerous articles (see, [56, 57, 58, 59, 60, 61, 62, 63]). The idea of q-bi univalent functions is useful for many properties of analytic functions. The q-bi univalent functions are important and useful in real-world mathematics like in robotics, computational geometry, computer-aided design(CAD), Quantum physics, Quantum mechanics, signal and image processing, cryptography, control theory, discrete dynamical systems, relativity, and many other fields of sciences. In our study, we focus on subclasses of bi-univalent functions linked with q-analog applying Quasi subordination and we also explore the subclass of q-bi starlike function applying q-Salagean operator.

1.8 Preface

The purpose of this thesis is to characterize some sub-classes of bi-univalent functions through the implementation of the concepts of quasi-subordination and the Salagean operator. A summary of each chapter of this thesis is presented as follows:

In **Chapter 2**, basic ideas and concepts of Geometric Function Theory are briefly explained. It explains the structures of different domains and their mappings. It provides a clear explanation of analytic and univalent functions and their related important subclasses. It also includes concepts of bi-univalent function and quasi-subordination and their fundamental subclasses. This chapter has all the different operators that we used in our research work. A summary of q-calculus is also discussed. At the end, important preliminary lemmas are provided. This chapter does not contain new discoveries, it has only well-established basic concepts of this area of study.

In **Chapter 3** the subclasses of bi-univalent functions are provided. These subclasses involve the quasi-subordination. The initial coefficients estimate of these classes are investigated. A key element to note is that review work is properly referenced.

In **Chapter 4** new subclasses in the field of q-calculus are designated as q-bi univalent functions using quasi-subordination. The initial bounds of coefficients are analyzed. This chapter shows corollaries which tells that our new advanced results are aligned with previous findings.

In **Chapter 5** the subclass of bi-starlike functions is defined and analyzed which involves the Salagean operator. For this class of bi-univalent functions, the upper bounds of initial coefficients, Fekete-Szego inequality, and upper bound for second-order Hankel determinant are examined. A key element to note is that review work is properly referenced.

In **Chapter 6** new subclass in the area of q-calculus are presented as q-bi starlike functions using the q-Salagean operator. For this class of q-bi starlike function, upper bounds of initial coefficients, Fekete-Szego inequality, and upper bound for second order Hankel determinant are analyzed. This chapter contains corollaries which show that our new advanced results are aligned with previous findings.

In **Chapter 7** conclusion of the thesis is presented.

CHAPTER 2

DEFINITIONS AND PRELIMINARY CONCEPTS

2.1 Overview

This chapter is dedicated to providing the thorough study of normalized analytic univalent functions and bi-univalent functions along with certain remarkable functions, linear operators, and related lemmas. The chapter also introduces the basics of q-calculus, creating a detailed setup for the later analysis.

2.2 Topology of Domains and Mapping Theorem

The analysis of shapes or domains of analytic functions and their related properties in a complex plane is considered as the geometry of geometric functions. Here we discuss the structures of different types of domains and theorem about their conformal mapping.

Definition 2.2.1. *[1] Let*

$$E(\mathrm{s}_0, r) = \{|\mathrm{s} - \mathrm{s}_0| < r,\ \mathrm{s} \in C\}$$

is the neighbourhood of point s_0*, having* $r > 0$ *and* $\mathrm{s}_0 \in C$.

In geometric function theory, a domain of any function can be described as a set that is non-empty, open, and connected set. This shows that any open connected set in geometric function theory is known as the domain of the geometric function. This implies that the domain *is a set that cannot be divided into two disjoint sets and every point in a set has a neighborhood that should lie entirely within the set.*

Definition 2.2.2. *[1, 3] The open unit disk is an open connected set that has an origin of 0 and a radius less than 1. Mathematically, the open unit disk can be stated as*

$$V = \{|\mathtt{s}| < 1,\ \mathtt{s} \in C\}$$

Here $'V'$ represents the open unit disk, $\mathtt{s}$ is the complex number and C is the set of complex numbers.

Any connected domain that is conformally equivalent to an open unit disk is referred to as a simply connected domain. Geometrically, it can be interpreted as a simply connected domain is a region that has no holes.

Definition 2.2.3. *[64, 65] A simply connected region(U) which is not a complex plane having a point $'w'$ such that for a unique analytic function $f(\mathtt{s})$ on U exists $f(w) = 0,\ f'(w) > 0$, and function $f(\mathtt{s})$ conformally mapped the simply connected region(U) to open unit disk$(\{|\mathtt{s}| < 1,\ \mathtt{s} \in C\})$ which have center 0 and radius 1. Like through Mobius transformation upper half plane $(\{Im(h) > 0,\ h \in C\})$ to open unit disk $(\{|\mathtt{s}| < 1,\ \mathtt{s} \in C\})$.*

$$h = \frac{i(1+\mathtt{s})}{1-\mathtt{s}} \quad , \qquad \mathtt{s} = \frac{h-i}{h+i}.$$

This theorem is the basic concept for understanding the field of Geometric Function Theory known as Riemann's mapping theorem.

2.3 Analytic Functions

Analytic functions are the base of this area of study. These functions have interesting properties because analytic functions show a strong relation between analysis and geometry.

Definition 2.3.1. *[66, 67] A function is described as differentiable in its domain if its derivative at each point of a domain exists. A complex derivative at $\mathtt{s}_0 \neq \mathtt{s} \in U$ (open set) is shown as*

$$f'(\mathtt{s}) = \lim_{\mathtt{s} \to \mathtt{s}_0} \frac{f(\mathtt{s}) - f(\mathtt{s}_0)}{\mathtt{s} - \mathtt{s}_0}$$

Definition 2.3.2. *[65] In geometric function theory, a convergent power series representation is as follows*

$$f(\mathrm{s}) = \sum_{t=0}^{\infty} \alpha_t (\mathrm{s} - \mathrm{s}_0)^t,$$

where s_0 *is the center,* α_t *is the complex coefficients and* s *is the complex variable.*

Definition 2.3.3. *[68, 69] An analytic function or holomorphic function is a function differentiable at every point in its domain and has a power series of the following form*

$$f(\mathrm{s}) = \mathrm{s} + \sum_{t=2}^{\infty} \alpha_t \mathrm{s}^t, \quad \mathrm{s} \in V.$$

Locally, analytic functions maintain angles and shapes except at singular points where the derivative of the function does not exist.

Definition 2.3.4. *[3, 69] If* $f(0) = 0$ *and* $f'(0) = 1$, *then function is referred as normalized. Functions that are analytic and normalized are considered to be in the class* A.

2.4 Univalent Functions

The class univalent functions is the subclass of analytic functions. Univalent functions bring new turns in the field of Geometric Function Theory. In the complex plane, univalent functions enable the researchers to study the mappings of domains. These functions are useful for different distortion, covering, and growth theorems. These functions are also helpful in fields like complex dynamics, fluid mechanics, and potential theory. Univalent functions are useful not only in theoretical concepts but also in practical areas like engineering, physics, and computer graphics.

Definition 2.4.1. *[69] If one value of the domain of a function maps to one value from the range then the function is considered to be injective or one-to-one. That can be expressed mathematically as*

For

$$\mathrm{s}_0 = \mathrm{s}$$

Implies that

$$f(\mathrm{s}_0) = f(\mathrm{s}).$$

So, it can also be expressed as the function that does not occupy the same value twice is one-to-one.

Definition 2.4.2. *[70, 71] Univalent functions are normalized analytic functions which are also injective. The class that contains these types of functions is called class S. The sharp coefficient bound for the functions of class S is* $|\alpha_t| \leq t, \quad (t = 2, 3, 4, \dots)$. *where equality holds for the Koebe function.*

Koebe function is one of the most popular example of the univalent functions. This function and its rotations are the only extremal functions in the class S, whose geometry shows the mapping of an open unit disk to a complex plane except for a portion from $-\frac{1}{4}$ to $-\infty$ on the negative x-axis. Mathematically, the Koebe function can be shown as

$$K(\mathrm{s}) = \frac{\mathrm{s}}{(1-\mathrm{s})^2} = \sum_{t=1}^{\infty} t\mathrm{s}^t, \quad \mathrm{s} \in V.$$

2.5 Functions with Positive Real Parts

The functions that have positive real parts are from the class P. Many subclasses from class S are introduced based on the concept of class P. Here we discuss the basic concepts of this class.

Definition 2.5.1. *[3, 69] The functions p with positive real parts are the subclass of normalized analytic functions, having normalization conditions of* $Re(p(\mathrm{s})) > 0$ *and* $p(0) = 1$. *Mathematically, this function can be represented as*

$$p(\mathrm{s}) = 1 + \sum_{t=1}^{\infty} n_t \mathrm{s}^t, \quad \mathrm{s} \in V.$$

These functions are also named as Caratheodory functions. The class in which these functions belong is considered to be class P.

The most popular example of class P is the Mobius function. This shows the extremal behavior, but in class P Mobius function is not the only function that shows extremal nature. Geometrically, the Mobius function maps the open unit disk to the positive half-plane. Mathematically, the Mobius function can be shown as

$$M(\mathrm{s}) = \frac{1+\mathrm{s}}{1-\mathrm{s}} = 1 + 2\sum_{t=1}^{\infty} \mathrm{s}^t.$$

Sharp coefficient bound for functions of class P is $|n_t| \leq 2, \quad (t = 1, 2, 3, \ldots)$, where equality holds for Mobius function. The functions of class P show the behavior of the functions of subclasses of class S.

2.6 Subclasses of Class S

Functions with bounded turning, starlike functions, convex functions, quasi-convex functions, close-to-convex functions, and bazilevic functions are some of the important subclasses of class S. Here the class of starlike functions is discussed as follows:

Definition 2.6.1. *[69] If we draw a line from any fixed point to any other point of the domain and that line lies entirely within that domain then such domains are known as starlike. Those functions that have starlike domains are called starlike functions. Mathematically, we can say that a function $f \in S$ is in S^*(starlike function) if and only if Re* $\left(\frac{\mathrm{s}f'(\mathrm{s})}{f(\mathrm{s})}\right) > 0, \quad (\mathrm{s} \in V)$.

2.7 Bi-Univalent Functions

The idea of bi-univalent functions originates from univalent functions. These functions are also an important subclass of analytic functions, which shows the function is not only univalent in the domain of the function but also in the range of that function. These functions have many important characteristics in Geometric Function Theory.

Definition 2.7.1. *[72] Let ∇ represent the class of bi-univalent functions in V(open unit disk). A function is considered to be bi-univalent in V if $V \subset f(V)$ and if both function and its inverse that is f and f^{-1} are univalent in V.*

This means that function $f(\mathrm{s})$ is analytic and injective and its inverse $f^{-1}(\mathrm{s})$ is also analytic and injective in $f(V)$.
For analysis of growth, distortion, and other geometrical properties it's very important to find the coefficient bounds of the bi-univalent functions.
Inverse of Koebe function does not satisfy the property of univalency which indicates that Koebe function is not bi-univalent.

Some examples of bi-univalent functions are

$$f(\mathrm{s}) = \frac{\mathrm{s}}{1-\mathrm{s}},$$
$$f(\mathrm{s}) = -\log(1-\mathrm{s}),$$
$$f(\mathrm{s}) = \frac{1}{2}\log\left(\frac{1+\mathrm{s}}{1-\mathrm{s}}\right),$$

etc.

2.8 Subclasses of Bi-Univalent Functions

Bi-starlike functions, biconvex functions, bi close-to-convex functions, bi quasi-convex functions, etc are some subclasses of bi-univalent functions. In which functions are univalent in open unit disk (V), and their inverses are also univalent in the image $f(V)$. Here a subclass of bi-univalent functions named as bi-starlike function of order β is discussed as given below:

Definition 2.8.1. *[22] A function is considered to be in the class $S^*_{\nabla}(\beta)$(bi-starlike function of order β), where $0 \le \beta < 1$. If the conditions listed below hold:*
For $f \in \nabla$,

$$Re\left\{\frac{\mathrm{s}f'(\mathrm{s})}{f(\mathrm{s})}\right\} > \beta, \qquad |\mathrm{s}| < 1,$$
$$Re\left\{\frac{\omega g'(\omega)}{g(\omega)}\right\} > \beta, \qquad |\omega| < 1.$$

Here g is the inverse of f.

2.9 Quasi-Subordination

Definition 2.9.1. *[31] For two analytic functions $f(\mathrm{s})$ and $g(\mathrm{s})$, this can be stated as $f(\mathrm{s})$ is quasi-subordinate to $g(\mathrm{s})$ in V and express as $f(\mathrm{s}) \prec_{\mathrm{q}} g(\mathrm{s})$, this can be possible if there exists analytic functions $\mu(\mathrm{s})$ and $F(\mathrm{s})$, with $|\mu(\mathrm{s})| < 1$, $F(0) = 0$ and $|F(\mathrm{s})| < 1$ such that*

$$f(\mathrm{s}) = \mu(\mathrm{s})\, g(F(\mathrm{s})), \mathrm{s} \in V$$

Definition 2.9.2. *[31] For $\mu(\mathrm{s}) = 1$, the function $f(\mathrm{s})$ is subordinate to function $g(\mathrm{s})$ in V and express as $f(\mathrm{s}) \prec g(\mathrm{s})$, if there exists schwarz function $F(\mathrm{s})$, with conditions $F(0) = 0$ and $|F(\mathrm{s})| < 1$ such that*

$$f(\mathrm{s}) = g(F(\mathrm{s})),$$

and represented as $f(\mathrm{s}) \prec g(\mathrm{s})$ in V.

Definition 2.9.3. *[31] For $F(\mathrm{s}) = \mathrm{s}$, then for analytic function $\mu(\mathrm{s})$, the following expression*

$$f(\mathrm{s}) = \mu(\mathrm{s})\, g(\mathrm{s}),$$

shows that $f(\mathrm{s})$ is majorized by $g(\mathrm{s})$ and represented as $f(\mathrm{s}) \ll g(\mathrm{s})$ in V.

In 2012, Haji Mohd et al. [30] defined many classes using quasi-subordination, and obtained their initial coefficient estimates and Fekete-Szego inequality. Out of those classes they defined $S_q^*(\mathrm{h})$ and $C_q(\mathrm{h})$ known as Ma-Minda starlike class and Ma-Minda convex class respectively. Ma-Minda starlike class is defined in the form of quasi-subordination as

Definition 2.9.4. *A function $f \in A$ is considered to be in the class $S_q^*(h)$ if the condition listed below hold:*

$$\frac{\mathrm{s}f'(\mathrm{s})}{f(\mathrm{s})} - 1 \prec_{\mathrm{q}} h(\mathrm{s}) - 1,$$

where $\mathrm{h}(\mathrm{s})$ is analytic function from class P.

2.10 Convolution of Two Functions

Definition 2.10.1. *[31] The convolution of the two analytic functions f* and *h is denoted by $f(\mathrm{s}) * h(\mathrm{s})$ and is express as*

$$f(\mathrm{s}) * h(\mathrm{s}) = \mathrm{s} + \sum_{t=2}^{\infty} \alpha_t b_t \mathrm{s}^t, \mathrm{s} \in V.$$

Another name for this expression is Hadamard product.

2.11 Quantum Calculus

Definition 2.11.1. *The given expression*

$$D_q f(\mathrm{s}) = \frac{d_q f(\mathrm{s})}{d_q \mathrm{s}} = \begin{cases} \frac{f(q\mathrm{s})-f(\mathrm{s})}{(q-1)\mathrm{s}}, & if\ \mathrm{s} \neq 0, \\ f'(0), & if\ \mathrm{s} = 0, \\ f'(\mathrm{s}), & if\ q \to 1^-, \quad \mathrm{s} \neq 0. \end{cases}$$

is said to be the q-derivative or q-difference operator of the function $f(\mathrm{s})$ for $0 < q < 1$.

where

$$\lim_{q \to 1^-} D_q f(\mathrm{s}) = \frac{df(\mathrm{s})}{d\mathrm{s}}.$$

q-derivative was derived by Jackson [49] in 1908. That's why it is named as Jackson derivative. q-derivative is the q- analogue of the ordinary derivative. The q-derivative of the function having form (1.1) is

$$D_q f(\mathrm{s}) = D_q\left(\mathrm{s} + \sum_{t=2}^{\infty} \alpha_t \mathrm{s}^t\right) = 1 + \sum_{t=2}^{\infty} [t]_q \alpha_t \mathrm{s}^{t-1}, \qquad \mathrm{s} \in V.$$

and q-derivative of its inverse function is

$$D_q\left(f^{-1}(\mathrm{s})\right) = 1 - [2]_q \alpha_2 \mathrm{s} + [3]_q \left(2\alpha_2^3 - \alpha_3\right)\mathrm{s}^2 + \ldots, \quad \mathrm{s} \in V.$$

For example, the q-derivative of a function is

$$D_q \mathrm{s}^t = [t]_q \mathrm{s}^{t-1},$$

This becomes the ordinary derivative of s^t when $q \to 1^-$, $[t]_q = t$.
while, the q-analogue $[t]_q$(q-integer or q-bracket) can be express as

$$[t]_q = \begin{cases} \frac{1-q^t}{1-q}, & if\ t \in C/\{0\} \\ 1 + q + \cdots + q^{t-2} + q^{t-1} = \sum_{k=0}^{t-1} q^k, & if\ t \in N \\ 1 & if\ q \to 0^+, t \in C/\{0\} \\ t & if\ q \to 1^-, t \in C/\{0\} \end{cases},$$

With the helpful formula $[t+1]_q - [t]_q = q^t$.

2.12 Some Linear Operators

Some fundamental operators are listed below:

Srivastava-Attiya Operator

Srivastava-Attiya [73] gives the operator $\mathscr{D}_{\partial,\gamma} : A \longrightarrow A$, as

$$\mathscr{D}_{\partial,\gamma} f(\mathrm{s}) = (1+\gamma)^{\partial}\left[\zeta(\mathrm{s},\partial,\gamma) - \gamma^{-\partial}\right],$$

which can be expressed as

$$\mathscr{D}_{\partial,\gamma} f(\mathrm{s}) = \mathrm{s} + \sum_{t=2}^{\infty}\left(\frac{1+\gamma}{t+\gamma}\right)^{\partial}\alpha_t \mathrm{s}^t.$$

$$\gamma \in C\backslash\{0,-1,-2,\ldots\}, \partial \in C,\ \ \mathrm{s} \in V,\ f \in A.$$

where $\zeta(\mathrm{s},\partial,\gamma)$ is called as Hurwitz-Lerch Zeta function and expressed as

$$\zeta(\mathrm{s},\partial,\gamma) = \sum_{t=0}^{\infty}\frac{\mathrm{s}^t}{(t+\gamma)^{\partial}}.$$

Carlson and Shaffer Operator

Carlson and Shaffer [74] gives the integral operator $\mathscr{T}_y f(\mathrm{s})$ as

$$\mathscr{T}_y f(\mathrm{s}) = \mathrm{s} + \sum_{t=2}^{\infty}\frac{(y)_{t-1}}{(c)_{t-1}}\alpha_t \mathrm{s}^t.$$

$$\left(\mathrm{s} \in V, c \notin Z_0{}^{-} = \{0,-1,-2...\}\right)$$

and $(c)_t$ is the Pochhammer symbol that can be defined by concerning the gamma function as given:

$$(c)_t = \frac{\Gamma(c+t)}{\Gamma(c)} = \begin{cases} 1 & (t=0; c \neq 0), \\ c(c+1)...(c+t-1) & (t \in N). \end{cases}$$

Convolution of Operators $\mathscr{D}_{\partial,\gamma} f(\mathrm{s})$ and $\mathscr{T}_y f(\mathrm{s})$

In [31], Convolution of the operators $\mathscr{D}_{\partial,\gamma} f(\mathrm{s})$ and $\mathscr{T}_y f(\mathrm{s})$ is given as

$$\mathbb{N}_{y,c}^{\partial,\gamma} f(\mathrm{s}) = \mathscr{D}_{\partial,\gamma} f(\mathrm{s}) * \mathscr{T}_y f(\mathrm{s}) = \mathrm{s} + \sum_{t=2}^{\infty}\left(\frac{1+\gamma}{t+\gamma}\right)^{\partial}\frac{(y)_{t-1}}{(c)_{t-1}}\alpha_t \mathrm{s}^t,$$

which can be written in simplified form as

$$\mathbb{N}_{y,c}^{\partial,\gamma} f(\mathrm{s}) = \mathrm{s} + \sum_{t=2}^{\infty}\varphi_{t,y}\alpha_t \mathrm{s}^t,$$

where

$$\varphi_{t,y} = \left(\frac{1+\gamma}{t+\gamma}\right)^{\partial} \frac{(y)_{t-1}}{(c)_{t-1}}.$$

Salagean Differential Operator

In [75], Salagean provided the differential operator for analytic functions as:

$$\begin{aligned} O^0 f(\mathrm{s}) =& f(\mathrm{s}), \\ O^1 f(\mathrm{s}) =& Of(\mathrm{s}) = \mathrm{s}f'(\mathrm{s}), \\ &\vdots \\ O^u f(\mathrm{s}) =& O\left(O^{u-1} f(\mathrm{s})\right), \quad (u \in N = 1,2,3,\dots). \end{aligned}$$

Consider that

$$O^u f(\mathrm{s}) = \mathrm{s} + \sum_{t=2}^{\infty} t^u \alpha_t \mathrm{s}^t, \quad (u \in N_0 = N \cup \{0\}).$$

Many classes are defined by applying the Salagean operator, for example, see [76, 77, 78]

q-Salagean Differential Operator

For $f \in A$, Govindaraj et al.[79] provided the Salagean q-differential operator as shown below:

$$\begin{aligned} O_q^0 f(\mathrm{s}) =& f(\mathrm{s}) \\ O_q^1 f(\mathrm{s}) =& \mathrm{s}D_q f(\mathrm{s}), \\ &\vdots \\ O_q^u f(\mathrm{s}) =& \mathrm{s}O_q^u\left(O_q^{u-1} f(\mathrm{s})\right), \\ O_q^u f(\mathrm{s}) =& \mathrm{s} + \sum_{t=2}^{\infty} [t]_q^u \alpha_t \mathrm{s}^t, \quad (u \in N_0 = N \cup \{0\}, \mathrm{s} \in V). \end{aligned}$$

Considered that $\lim_{q \to 1^-} O_q^u f(\mathrm{s})$

$$O^u f(\mathrm{s}) = \mathrm{s} + \sum_{t=2}^{\infty} t^u \alpha_t \mathrm{s}^t, \quad (u \in N_0 = N \cup \{0\}, \mathrm{s} \in V).$$

Many classes are defined by applying the Salagean operator, for example, see [80, 81]

2.13 Some Important Lemmas

Let class P containing analytic functions such that $P : V \to C$, obeying the conditions $p(0) = 1$ and $Re(p(\mathrm{s})) > 0$.

Lemma 2.13.1. *[82] Let the function $p \in P$ is shown by the following series:*

$$p(\mathtt{s}) = 1 + n_1\mathtt{s} + n_2\mathtt{s}^2 + \ldots,$$

then the sharp estimate provided by

$$|n_t| \leq 2, \quad (t = 1, 2, 3, \ldots),$$

holds.

Lemma 2.13.2. *[82] Let the function $p \in P$ is shown by the series* (1.2)*, then*

$$2n_2 = n_1^2 + x\left(4 - n_1^2\right),$$
$$4n_3 = n_1^3 + 2\left(4 - n_1^2\right)n_1 x - n_1\left(4 - n_1^2\right)x^2 + 2\left(4 - n_1^2\right)\left(1 - |x|^2\right)\mathtt{s},$$

for values x and $\mathtt{s}$ with conditions $|x| \leq 1$and $|\mathtt{s}| \leq 1$.

CHAPTER 3

ON A CLASS OF BI-UNIVALENT FUNCTIONS USING QUASI-SUBORDINATION

3.1 Overview

This chapter will determine the coefficient estimates of subclasses of bi-univalent functions involving quasi-subordination. The outcomes of these classes are the refinement of previous findings by Atshan et al. [31].

Definition 3.1.1. *The function having form* (1.1) *is considered to be in the class* $\mathfrak{R}^{\partial,y}_{\nabla,\gamma,c}(\xi,\sigma,\tau,\Psi)$, $0\leq\xi\leq1,\ 0\leq\sigma\leq1,$ *and* $\tau\in C\backslash\{0\}$, *if the conditions listed below are hold:*

$$\frac{1}{\tau}\left[\left\{\frac{\mathrm{s}\left(\mathbb{N}^{\partial,\gamma}_{y,c}f(\mathrm{s})\right)'+\mathrm{s}^2\left(\mathbb{N}^{\partial,\gamma}_{y,c}f(\mathrm{s})\right)''}{(1-\xi)\,\mathrm{s}+\xi\,\mathrm{s}\left(\mathbb{N}^{\partial,\gamma}_{y,c}f(\mathrm{s})\right)'}+\sigma\mathrm{s}\left(\mathbb{N}^{\partial,\gamma}_{y,c}f(\mathrm{s})\right)''\right\}-1\right]\prec_{\mathrm{q}}[\Psi(\mathrm{s})-1], \tag{3.1}$$

and

$$\frac{1}{\tau}\left[\left\{\frac{\omega\left(\mathbb{N}^{\partial,\gamma}_{y,c}g(\omega)\right)'+\omega^2\left(\mathbb{N}^{\partial,\gamma}_{y,c}g(\omega)\right)''}{(1-\xi)\,\omega+\xi\,\omega\left(\mathbb{N}^{\partial,\gamma}_{y,c}g(\omega)\right)'}+\sigma\omega\left(\mathbb{N}^{\partial,\gamma}_{y,c}g(\omega)\right)''\right\}-1\right]\prec_{\mathrm{q}}[\Psi(\omega)-1], \tag{3.2}$$

where $\mathrm{s},\omega\in V$ *and* g *is the inverse of* f.

Definition 3.1.2. *The function having form* (1.1) *is considered to be in the class* $\mathfrak{K}^{\partial,y}_{\nabla,\gamma,c}(\xi,\sigma,\kappa,\Psi)$, $0\leq\xi,\ \kappa\geq1,$ *and* $\sigma\ \in C\backslash\{0\}$, *if the conditions listed below are hold true*

$$\frac{1}{\sigma}\left[\left\{(1-\kappa)\frac{\mathrm{s}\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)'}{\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})}+\kappa\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)'+\xi\mathrm{s}\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)''\right\}-1\right]\prec_{\mathrm{q}}[\Psi(\mathrm{s})-1], \quad (3.3)$$

and

$$\frac{1}{\sigma}\left[\left\{(1-\kappa)\frac{\omega\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)'}{\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)}+\kappa\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)'+\xi\omega\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)''\right\}-1\right]\prec_{\mathrm{q}}[\Psi(\omega)-1], \quad (3.4)$$

where $\mathrm{s},\omega\in V$ *and* g *is the inverse of* f.

3.2 Main Result

Theorem 3.2.1. If f presented by (1.1) is part of the subclass $\mathfrak{R}_{\nabla,\gamma,c}^{\partial,y}(\xi,\sigma,\tau,\Psi)$, then

$$|\alpha_2|\leq\frac{\tau|h_0|\beta_1\sqrt{\beta_1}}{\sqrt{3\tau h_0\beta_1^2(3-\xi+2\sigma)\varphi_{3,y}-4\{\xi(2-\xi)\tau h_0\beta_1^2+(\beta_2-\beta_1)(2-\xi+\sigma)^2\}\varphi_{2,y}^2}}, \quad (3.5)$$

$$|\alpha_3|\leq\frac{\tau(|h_1|+|h_0|)|\beta_1}{3(3-\xi+2\sigma)\varphi_{3,y}}+\frac{\tau^2h_0^2\beta_1^2}{4(2-\xi+\sigma)^2\varphi_{2,y}^2},\beta_1>1, \quad (3.6)$$

where

$$\varphi_{2,y}=\left(\frac{1+\gamma}{2+\gamma}\right)^{\partial}\frac{(y)_{2-1}}{(c)_{2-1}},$$

and

$$\varphi_{3,y}=\left(\frac{1+\gamma}{3+\gamma}\right)^{\partial}\frac{(y)_{3-1}}{(c)_{3-1}}.$$

Proof. Let $f\in\mathfrak{R}_{\nabla,\gamma,c}^{\partial,y}(\xi,\sigma,\tau,\Psi)$, then there lie two analytic functions μ, F in V and $\mu,F:V\to V$, hold the given conditions:

$$\frac{1}{\tau}\left[\left\{\frac{\mathrm{s}\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)'+\mathrm{s}^2\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)''}{(1-\xi)\mathrm{s}+\xi\mathrm{s}\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)'}+\sigma\mathrm{s}\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)''\right\}-1\right]=$$

$$[\mu(\mathrm{s})(\Psi(F(\mathrm{s}))-1], \quad (3.7)$$

and

$$\frac{1}{\tau}\left[\left\{\frac{\omega\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)'+\omega^2\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)''}{(1-\xi)\omega+\xi\omega\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)'}+\sigma\omega\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)''\right\}-1\right]=$$

$$[\mu(\omega)\,(\Psi(F(\omega))-1], \tag{3.8}$$

where $\mathrm{s}, \omega \in V$ and g is the inverse of f.

There exist functions n and m as

$$n(\mathrm{s}) = \frac{1+F(\mathrm{s})}{1-F(\mathrm{s})} \;=\; 1+n_1\mathrm{s}+n_2\mathrm{s}^2+n_3\mathrm{s}^3+\ldots, \tag{3.9}$$

and

$$m(\omega) = \frac{1+F(\omega)}{1-F(\omega)} \;=\; 1+m_1\omega+m_2\omega^2+m_3\omega^3+\ldots. \tag{3.10}$$

This can also be shown as,

$$\begin{aligned}
n(\mathrm{s}) &= \frac{1+F(\mathrm{s})}{1-F(\mathrm{s})}\\
(1-F(\mathrm{s}))\,n(\mathrm{s}) &= 1+F(\mathrm{s}),\\
n(\mathrm{s})-n(\mathrm{s})F(\mathrm{s}) &= 1+F(\mathrm{s}),\\
n(\mathrm{s})-1 &= F(\mathrm{s})+n(\mathrm{s})F(\mathrm{s}),\\
F(\mathrm{s}) &= \frac{n(\mathrm{s})-1}{n(\mathrm{s})+1}.
\end{aligned}$$

Using series of $n(\mathrm{s})$ from (3.9), this leads to

$$\begin{aligned}
F(\mathrm{s}) &= \frac{1+n_1\mathrm{s}+n_2\mathrm{s}^2+n_3\mathrm{s}^3+\cdots-1}{1+n_1\mathrm{s}+n_2\mathrm{s}^2+n_3\mathrm{s}^3+\cdots+1}\\
F(\mathrm{s}) &= \frac{n_1\mathrm{s}+n_2\mathrm{s}^2+n_3\mathrm{s}^3+\ldots}{2+n_1\mathrm{s}+n_2\mathrm{s}^2+n_3\mathrm{s}^3+\ldots}
\end{aligned}$$

So,

$$F(\mathrm{s}) = \frac{n(\mathrm{s})-1}{n(\mathrm{s})+1} = \frac{1}{2}\left[n_1\mathrm{s}+\left(n_2-\frac{n_1^2}{2}\right)\mathrm{s}^2+\ldots,\right. \tag{3.11}$$

similarly from (3.10), gives us

$$F(\omega) = \frac{m(\omega)-1}{m(\omega)+1} = \frac{1}{2}\left[m_1\omega+\left(m_2-\frac{m_1^2}{2}\right)\omega^2+\ldots.\right. \tag{3.12}$$

Using (3.11) and (3.12) in (3.7) and (3.8), this provide

$$\frac{1}{\tau}\left[\left\{\frac{\mathrm{s}\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)'+\mathrm{s}^2\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)''}{(1-\xi)\,\mathrm{s}+\xi\,\mathrm{s}\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)'}+\sigma\mathrm{s}\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)''\right\}-1\right]=$$

$$\left[\mu(\mathrm{s})\;\left(\Psi\left(\frac{n(\mathrm{s})-1}{n(\mathrm{s})+1}\right)\right)-1\right], \tag{3.13}$$

and

$$\frac{1}{\tau}\left[\left\{\frac{\omega\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)'+\omega^2\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)''}{(1-\xi)\,\omega+\xi\,\omega\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)'}+\sigma\omega\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)''\right\}-1\right]=$$

$$\left[\mu(\omega)\ \left(\Psi\left(\frac{m(\omega)-1}{m(\omega)+1}\right)\right)-1\right]. \tag{3.14}$$

This can be shown as:

$$\mu(\mathrm{s})\ \left(\Psi\left(\frac{n(\mathrm{s})-1}{n(\mathrm{s})+1}\right)\right)-1\ =\mu(\mathrm{s})\Psi\left(\frac{1}{2}\left[n_1\mathrm{s}+\left(n_2-\frac{n_1^2}{2}\right)\mathrm{s}^2+\dots\right)-1.$$

Since,

$$\Psi(\mathrm{s})=1+\beta_1\mathrm{s}+\beta_2\mathrm{s}^2+\beta_3\mathrm{s}^3+\dots,$$

$$\mu(\mathrm{s})\ \left(\Psi\left(\frac{n(\mathrm{s})-1}{n(\mathrm{s})+1}\right)\right)-1=\mu(\mathrm{s})\left[1+\frac{1}{2}\beta_1n_1\mathrm{s}+\frac{1}{2}\beta_1\left(n_2-\frac{n_1^2}{2}\right)\mathrm{s}^2+\frac{1}{4}\beta_2n_1^2\mathrm{s}^2+\dots,-1\right].$$

Here,

$$\mu(\mathrm{s})=h_0+h_1\mathrm{s}+h_2\mathrm{s}^2+\dots,$$

$$\mu(\mathrm{s})\ \left(\Psi\left(\frac{n(\mathrm{s})-1}{n(\mathrm{s})+1}\right)\right)-1=$$

$$\left(h_0+h_1\mathrm{s}+h_2\mathrm{s}^2+\dots\right)\left(\frac{1}{2}\beta_1n_1\mathrm{s}+\frac{1}{2}\beta_1\left(n_2-\frac{n_1^2}{2}\right)\mathrm{s}^2+\frac{1}{4}\beta_2n_1^2\mathrm{s}^2+\dots,\right)$$

$$\mu(\mathrm{s})\ \left(\Psi\left(\frac{n(\mathrm{s})-1}{n(\mathrm{s})+1}\right)\right)-1=$$

$$\frac{1}{2}h_0\beta_1n_1\mathrm{s}+\left\{\frac{1}{2}h_1\beta_1n_1+\frac{1}{2}h_0\beta_1\left(n_2-\frac{n_1^2}{2}\right)+\frac{1}{4}h_0\beta_2n_1^2\right\}\mathrm{s}^2+\dots, \tag{3.15}$$

Similarly,

$$\mu(\omega)\ \left(\Psi\left(\frac{m(\omega)-1}{m(\omega)+1}\right)\right)-1=$$

$$\frac{1}{2}h_0\beta_1m_1\omega+\left\{\frac{1}{2}h_1\beta_1m_1+\frac{1}{2}h_0\beta_1\left(m_2-\frac{m_1^2}{2}\right)+\frac{1}{4}h_0\beta_2m_1^2\right\}\omega^2+\dots. \tag{3.16}$$

Since,

$$\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})=\mathrm{s}+\sum_{t=2}^{\infty}\varphi_{t,y}\alpha_t\mathrm{s}^t,$$

$$\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})=\mathrm{s}+\varphi_{2,y}\alpha_2\mathrm{s}^2+\varphi_{3,y}\alpha_3\mathrm{s}^3+\dots,$$

$$\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)'=1+2\varphi_{2,y}\alpha_2\mathrm{s}+3\varphi_{3,y}\alpha_3\mathrm{s}^2+\dots,$$

$$\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)''=2\varphi_{2,y}\alpha_2+6\varphi_{3,y}\alpha_3\mathrm{s}+\dots,$$

$$\frac{1}{\tau}\left[\left\{\frac{\mathrm{s}\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)'+\mathrm{s}^2\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)''}{(1-\xi)\,\mathrm{s}+\xi\,\mathrm{s}\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)'}+\sigma\mathrm{s}\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)''\right\}-1\right]=$$
$$\frac{1}{\tau}\left[\left\{\frac{\mathrm{s}+\ 2\varphi_{2,y}\alpha_2\mathrm{s}^2+3\varphi_{3,y}\alpha_3\mathrm{s}^3+2\varphi_{2,y}\alpha_2\mathrm{s}^2+6\varphi_{3,y}\alpha_3\mathrm{s}^3+\dots}{\mathrm{s}-\xi\mathrm{s}+\xi\mathrm{s}+2\xi\varphi_{2,y}\alpha_2\mathrm{s}^2+3\xi\varphi_{3,y}\alpha_3\mathrm{s}^3+\dots}+\right.\right.$$
$$\left.\left.\sigma\mathrm{s}(2\varphi_{2,y}\alpha_2+6\varphi_{3,y}\alpha_3\mathrm{s}+\dots)\right\}-1\right],$$

$$\frac{1}{\tau}\left[\left\{\frac{\mathrm{s}\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)'+\mathrm{s}^2\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)''}{(1-\xi)\,\mathrm{s}+\xi\,\mathrm{s}\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)'}+\sigma\mathrm{s}\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)''\right\}-1\right]=$$
$$\frac{1}{\tau}\left[\left\{\frac{\mathrm{s}+\ 4\varphi_{2,y}\alpha_2\mathrm{s}^2+9\varphi_{3,y}\alpha_3\mathrm{s}^3+\dots}{\mathrm{s}+\ 2\xi\varphi_{2,y}\alpha_2\mathrm{s}^2+3\xi\varphi_{3,y}\alpha_3\mathrm{s}^3+\dots}+\right.\right.$$
$$\left.\left.2\sigma\varphi_{2,y}\alpha_2\mathrm{s}+6\sigma\varphi_{3,y}\alpha_3\mathrm{s}^2+\dots\ \right\}-1\right],$$

$$\frac{1}{\tau}\left[\left\{\frac{\mathrm{s}\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)'+\mathrm{s}^2\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)''}{(1-\xi)\,\mathrm{s}+\xi\,\mathrm{s}\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)'}+\sigma\mathrm{s}\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)''\right\}-1\right]=$$
$$\frac{1}{\tau}\left[1+2\varphi_{2,y}\alpha_2\,(2-\xi)\,\mathrm{s}+\left(3\varphi_{3,y}\alpha_3\,(3-\xi\ )-\right.\right.$$
$$\left.\left.4\varphi_{2,y}^2\alpha_2^2\xi(2-\xi\ )\right)\mathrm{s}^2+2\sigma\varphi_{2,y}\alpha_2\mathrm{s}+6\sigma\varphi_{3,y}\alpha_3\mathrm{s}^2-1\right],$$

$$\frac{1}{\tau}\left[\left\{\frac{\mathrm{s}\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)'+\mathrm{s}^2\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)''}{(1-\xi)\,\mathrm{s}+\xi\,\mathrm{s}\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)'}+\sigma\mathrm{s}\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)''\right\}-1\right]=$$
$$\frac{1}{\tau}\left[2\,(2-\xi+\sigma)\,\varphi_{2,y}\alpha_2\mathrm{s}+\left(3\,(3-\xi+2\sigma)\,\varphi_{3,y}\alpha_3-\right.\right.$$
$$\left.\left.4\xi(2-\xi)\varphi_{2,y}^2\alpha_2^2\ \right)\mathrm{s}^2+\dots\right], \tag{3.17}$$

and

$$\mathbb{N}_{y,c}^{\partial,\gamma}\,g\,(\omega)=\omega-\varphi_{2,y}\alpha_2\omega^2+\varphi_{3,y}(2\alpha_2^2-\alpha_3)\omega^3-\dots,$$
$$\left(\mathbb{N}_{y,c}^{\partial,\gamma}g\,(\omega)\right)'=1-2\varphi_{2,y}\alpha_2\omega+3\left(2\alpha_2^2-\alpha_3\right)\varphi_{3,y}\omega^2-\dots,$$
$$\left(\mathbb{N}_{y,c}^{\partial,\gamma}g\,(\omega)\right)''=-2\varphi_{2,y}\alpha_2+6(2\alpha_2^2-\alpha_3)\varphi_{3,y}\omega-\dots,$$

$$\frac{1}{\tau}\left[\left\{\frac{\omega\left(\mathbb{N}_{y,c}^{\partial;\gamma}g(\omega)\right)'+\omega^2\left(\mathbb{N}_{y,c}^{\partial;\gamma}g(\omega)\right)''}{(1-\xi)\,\omega+\xi\,\omega\left(\mathbb{N}_{y,c}^{\partial;\gamma}g(\omega)\right)'}+\sigma\omega\left(\mathbb{N}_{y,c}^{\partial;\gamma}g(\omega)\right)''\right\}-1\right]=$$
$$\frac{1}{\tau}\left[\left\{\frac{\omega-2\varphi_{2,y}\alpha_2\omega^2+3\left(2\alpha_2^2-\alpha_3\right)\varphi_{3,y}\omega^3-\ 2\varphi_{2,y}\alpha_2\omega^2+6\left(2\alpha_2^2-\alpha_3\right)\varphi_{3,y}\omega^3}{\omega-\xi\,\omega+\xi\,\omega-2\xi\,\varphi_{2,y}\alpha_2\omega^2+3\xi\,\left(2\alpha_2^2-\alpha_3\right)\varphi_{3,y}\omega^3}\right.\right.$$
$$\left.\left.+(-\ 2\sigma\varphi_{2,y}\alpha_2\omega+6\sigma\left(2\alpha_2^2-\alpha_3\right)\varphi_{3,y}\omega^2-\dots)\right\}-1\right],$$

$$\frac{1}{\tau}\left[\left\{\frac{\omega\left(\mathbb{N}_{y,c}^{\partial;\gamma}g(\omega)\right)'+\omega^2\left(\mathbb{N}_{y,c}^{\partial;\gamma}g(\omega)\right)''}{(1-\xi)\,\omega+\xi\,\omega\left(\mathbb{N}_{y,c}^{\partial;\gamma}g(\omega)\right)'}+\sigma\omega\left(\mathbb{N}_{y,c}^{\partial;\gamma}g(\omega)\right)''\right\}-1\right]=$$
$$\frac{1}{\tau}\left[\left\{\frac{\omega-\ 4\varphi_{2,y}\alpha_2\omega^2+9\varphi_{3,y}(2\alpha_2^2-\alpha_3)\omega^3-\dots}{\omega-\ 2\xi\,\varphi_{2,y}\alpha_2\omega^2+3\xi\,\varphi_{3,y}(2\alpha_2^2-\alpha_3)\omega^3-\dots}\right.\right.$$
$$\left.\left.-2\sigma\varphi_{2,y}\alpha_2\omega+6\sigma\left(2\alpha_2^2-\alpha_3\right)\varphi_{3,y}\omega^2-\dots\right\}-1\right],$$

$$\frac{1}{\tau}\left[\left\{\frac{\omega\left(\mathbb{N}_{y,c}^{\partial;\gamma}g(\omega)\right)'+\omega^2\left(\mathbb{N}_{y,c}^{\partial;\gamma}g(\omega)\right)''}{(1-\xi)\,\omega+\xi\,\omega\left(\mathbb{N}_{y,c}^{\partial;\gamma}g(\omega)\right)'}+\sigma\omega\left(\mathbb{N}_{y,c}^{\partial;\gamma}g(\omega)\right)''\right\}-1\right]=$$
$$\frac{1}{\tau}\left[1-2\,(2-\xi)\,\varphi_{2,y}\alpha_2\omega+3\,(3-\xi)\,\varphi_{3,y}\left(2\alpha_2^2-\alpha_3\right)\omega^2\right.$$
$$\left.-4\xi\,(2-\xi)\,\varphi_{2,y}^2\alpha_2^2\omega^2-2\sigma\varphi_{2,y}\alpha_2\omega+6\sigma\left(2\alpha_2^2-\alpha_3\right)\varphi_{3,y}\omega^2-\dots-1\right].$$

Therefore,

$$\frac{1}{\tau}\left[\left\{\frac{\omega\left(\mathbb{N}_{y,c}^{\partial;\gamma}g(\omega)\right)'+\omega^2\left(\mathbb{N}_{y,c}^{\partial;\gamma}g(\omega)\right)''}{(1-\xi)\,\omega+\xi\,\omega\left(\mathbb{N}_{y,c}^{\partial;\gamma}g(\omega)\right)'}+\sigma\omega\left(\mathbb{N}_{y,c}^{\partial;\gamma}g(\omega)\right)''\right\}-1\right]=$$
$$\frac{1}{\tau}\left[-2\,(2-\xi+\sigma)\,\varphi_{2,y}\alpha_2\omega+(3\,(3-\xi+2\sigma)\,\varphi_{3,y}\left(2\alpha_2^2-\alpha_3\right)-\right.$$
$$\left.4\xi(2-\xi)\varphi_{2,y}^2\alpha_2^2)\omega^2-\dots\right]. \tag{3.18}$$

Putting (3.15) and (3.17) in (3.13), this leads to

$$\frac{1}{\tau}\left[2\varphi_{2,y}\alpha_2\,(2-\xi+\sigma)\,\mathrm{s}+\left(3\varphi_{3,y}\alpha_3\,(3-\xi+2\sigma)-4\varphi_{2,y}^2\alpha_2^2\xi\,(2-\xi)\ \right)\mathrm{s}^2+\dots\right]=$$
$$\frac{1}{2}h_0\beta_1n_1\,\mathrm{s}+\left\{\frac{1}{2}h_1\beta_1n_1+\frac{1}{2}h_0\beta_1\left(n_2-\frac{n_1^2}{2}\right)+\frac{1}{4}h_0\beta_2n_1^2\right\}\mathrm{s}^2+\dots\ .$$

Comparing coefficients on both sides, this gives us

$$\frac{1}{\tau}[2\varphi_{2,y}\alpha_2\,(2-\xi+\sigma)]=\frac{1}{2}h_0\beta_1n_1, \tag{3.19}$$

$$\frac{1}{\tau}\left[3\varphi_{3,y}\alpha_3\left(3-\xi+2\sigma\right)-4\varphi_{2,y}^2\alpha_2^2\xi\left(2-\xi\right)\right]=$$
$$\frac{1}{2}h_1\beta_1n_1+\frac{1}{2}h_0\beta_1\left(n_2-\frac{n_1^2}{2}\right)+\frac{1}{4}h_0\beta_2n_1^2. \tag{3.20}$$

Putting (3.16) and (3.18) in (3.14), this gives us

$$\frac{1}{\tau}\left[-2\left(2-\xi+\sigma\right)\varphi_{2,y}\alpha_2\omega+\left(3\left(3-\xi+2\sigma\right)\varphi_{3,y}\left(2\alpha_2^2-\alpha_3\right)-4\xi\left(2-\xi\right)\varphi_{2,y}^2\alpha_2^2\right)\omega^2-\ldots\right]=$$
$$\frac{1}{2}h_0\beta_1m_1\omega+\left\{\frac{1}{2}h_1\beta_1m_1+\frac{1}{2}h_0\beta_1\left(m_2-\frac{m_1^2}{2}\right)+\frac{1}{4}h_0\beta_2m_1^2\right\}\omega^2+\ldots .$$

Comparing coefficients on both sides, this provides

$$\frac{1}{\tau}[-2\left(2-\xi+\sigma\right)\varphi_{2,y}\alpha_2]=\frac{1}{2}h_0\beta_1m_1, \tag{3.21}$$

$$\frac{1}{\tau}[3\left(3-\xi+2\sigma\right)\varphi_{3,y}\left(2\alpha_2^2-\alpha_3\right)-4\xi\left(2-\xi\right)\varphi_{2,y}^2\alpha_2^2]=$$
$$\frac{1}{2}h_1\beta_1m_1+\frac{1}{2}h_0\beta_1\left(m_2-\frac{m_1^2}{2}\right)+\frac{1}{4}h_0\beta_2m_1^2. \tag{3.22}$$

From (3.19) and (3.21) , this gives

$$\alpha_2=\frac{\tau h_0\beta_1n_1}{4\left(2-\xi+\sigma\right)\varphi_{2,y}},$$

$$\alpha_2=\frac{-\tau h_0\beta_1m_1}{4\left(2-\xi+\sigma\right)\varphi_{2,y}},$$

therefore,

$$\alpha_2=\frac{\tau h_0\beta_1n_1}{4\left(2-\xi+\sigma\right)\varphi_{2,y}}=\frac{-\tau h_0\beta_1m_1}{4\left(2-\xi+\sigma\right)\varphi_{2,y}}, \tag{3.23}$$

it gives

$$n_1=-m_1. \tag{3.24}$$

By taking square of (3.23), this yields

$$\alpha_2^2=\frac{\tau^2h_0^2\beta_1^2n_1^2}{16(2-\xi+\sigma)^2\varphi_{2,y}^2},$$

$$\alpha_2^2=\frac{\tau^2h_0^2\beta_1^2m_1^2}{16(2-\xi+\sigma)^2\varphi_{2,y}^2}.$$

By adding both of them, this provides

$$2\alpha_2^2=\frac{\tau^2h_0^2\beta_1^2(n_1^2+m_1^2)}{16(2-\xi+\sigma)^2\varphi_{2,y}^2}$$

$$32\alpha_2^2\left(2-\xi+\sigma\right)^2\varphi_{2,y}^2=\tau^2h_0^2\beta_1^2(n_1^2+m_1^2). \tag{3.25}$$

Adding (3.20) and (3.22), this gives

$$\begin{aligned}&\tfrac{1}{\tau}\left[3\varphi_{3,y}\alpha_3\left(3-\xi+2\sigma\right)-4\varphi_{2,y}^2\alpha_2^2\xi\left(2-\xi\right)\right]+\\&\tfrac{1}{\tau}[3\left(3-\xi+2\sigma\right)\varphi_{3,y}\left(2\alpha_2^2-\alpha_3\right)-4\xi\left(2-\xi\right)\varphi_{2,y}^2\alpha_2^2]=\\&\frac{1}{2}h_1\beta_1n_1+\frac{1}{2}h_0\beta_1\left(n_2-\frac{n_1^2}{2}\right)+\frac{1}{4}h_0\beta_2n_1^2+\\&\frac{1}{2}h_1\beta_1m_1+\frac{1}{2}h_0\beta_1\left(m_2-\frac{m_1^2}{2}\right)+\frac{1}{4}h_0\beta_2m_1^2.\end{aligned}$$

By using (3.24), this leads to

$$\begin{aligned}&\frac{1}{\tau}\left[-8\xi\left(2-\xi\right)\varphi_{2,y}^2\alpha_2^2+6\left(3-\xi+2\sigma\right)\varphi_{3,y}\alpha_2^2\right]=\\&\frac{1}{2}h_0\beta_1\left(n_2+m_2\right)-\frac{1}{4}h_0\beta_1\left(n_1^2+m_1^2\right)+\frac{1}{4}h_0\beta_2\left(n_1^2+m_1^2\right),\end{aligned}$$

$$\begin{aligned}&\frac{1}{\tau}\left[-8\xi\left(2-\xi\right)\varphi_{2,y}^2\alpha_2^2+6\left(3-\xi+2\sigma\right)\varphi_{3,y}\alpha_2^2\right]=\\&\frac{1}{2}h_0\beta_1\left(n_2+m_2\right)+\frac{1}{4}h_0(\beta_2-\beta_1)(n_1^2+m_1^2).\end{aligned}$$

By using (3.25), gives us

$$\begin{aligned}&\frac{4}{\tau}\left[-8\xi\left(2-\xi\right)\varphi_{2,y}^2\alpha_2^2+6\left(3-\xi+2\sigma\right)\varphi_{3,y}\alpha_2^2\right]=\\&2h_0\beta_1\left(n_2+m_2\right)+\\&h_0(\beta_2-\beta_1)\left(\frac{32\alpha_2^2\left(2-\xi+\sigma\right)^2\varphi_{2,y}^2}{\tau^2h_0^2\beta_1^2}\right),\end{aligned}$$

$$\begin{aligned}&\frac{4\tau^2h_0\beta_1^2}{\tau}\left[-8\xi\left(2-\xi\right)\varphi_{2,y}^2\alpha_2^2+6\left(3-\xi+2\sigma\right)\varphi_{3,y}\alpha_2^2\right]=\\&2\tau^2h_0^2\beta_1^3\left(n_2+m_2\right)+32\left(\beta_2-\beta_1\right)\left(2-\xi+\sigma\right)^2\varphi_{2,y}^2\alpha_2^2,\end{aligned}$$

$$\begin{aligned}&8\left[3\tau h_0\beta_1^2\left(3-\xi+2\sigma\right)\varphi_{3,y}\alpha_2^2-4\xi\left(2-\xi\right)\tau h_0\beta_1^2\varphi_{2,y}^2\alpha_2^2\right]=\\&2\tau^2h_0^2\beta_1^3\left(n_2+m_2\right)+32\left(\beta_2-\beta_1\right)\left(2-\xi+\sigma\right)^2\varphi_{2,y}^2\alpha_2^2,\end{aligned} \tag{3.26}$$

which gives

$$\alpha_2^2=\frac{2\tau^2h_0^2\beta_1^3\left(n_2+m_2\right)}{8\left[z_1\varphi_{3,y}-z_2\varphi_{2,y}^2\right]}. \tag{3.27}$$

Using Lemma 2.13.1 in (3.27), provides us

$$\left|\alpha_2^2\right| \leq \frac{2\tau^2 h_0^2 \beta_1^3 \left(|n_2|+|m_2|\right)}{8\left[\mathrm{z}_1 \varphi_{3,y} - \mathrm{z}_2 \varphi_{2,y}^2\right]},$$

$$\left|\alpha_2^2\right| \leq \frac{8\tau^2 h_0^2 \beta_1^3}{8\left[\mathrm{z}_1 \varphi_{3,y} - \mathrm{z}_2 \varphi_{2,y}^2\right]}.$$

Taking the square root on both sides results in (3.5).

$$|\alpha_2| \leq \frac{\tau |h_0| \beta_1 \sqrt{\beta_1}}{\sqrt{\mathrm{z}_1 \varphi_{3,y} - \mathrm{z}_2 \varphi_{2,y}^2}},$$

where,

$$\mathrm{z}_1 = 3\tau h_0 \beta_1^2 \left(3-\xi+2\sigma\right),$$
$$\mathrm{z}_2 = 4\{\xi \left(2-\xi\right) \tau h_0 \beta_1^2 + (\beta_2-\beta_1)(2-\xi+\sigma)^2\}.$$

Now, for finding the bound of the coefficient $|\alpha_3|$, by subtracting (3.20) and (3.22), this yields us

$$\begin{aligned}
&\tfrac{1}{\tau}\left[3\varphi_{3,y}\alpha_3 \left(3-\xi+2\sigma\right) - 4\varphi_{2,y}^2 \alpha_2^2 \xi \left(2-\xi\right)\right] - \\
&\tfrac{1}{\tau}[3\left(3-\xi+2\sigma\right)\varphi_{3,y}\left(2\alpha_2^2-\alpha_3\right) - 4\xi\left(2-\xi\right)\varphi_{2,y}^2\alpha_2^2] = \\
&\frac{1}{2}h_1\beta_1 n_1 + \frac{1}{2}h_0\beta_1\left(n_2 - \frac{n_1^2}{2}\right) + \frac{1}{4}h_0\beta_2 n_1^2 - \\
&\frac{1}{2}h_1\beta_1 m_1 - \frac{1}{2}h_0\beta_1\left(m_2 - \frac{m_1^2}{2}\right) - \frac{1}{4}h_0\beta_2 m_1^2.
\end{aligned}$$

By using (3.24), gives us

$$\begin{aligned}
&\frac{1}{\tau}\left[6(3-\xi+2\sigma)\right]\varphi_{3,y}\alpha_3 - 6\left(3-\xi+2\sigma\right)\varphi_{3,y}\alpha_2^2] = \\
&\frac{1}{2}h_1\beta_1(2n_1) + \frac{1}{2}h_0\beta_1\left(n_2-m_2\right) + \frac{1}{4}h_0(\beta_2-\beta_1)(n_1^2-n_1^2),
\end{aligned}$$

therefore,

$$\begin{aligned}
&\frac{2}{\tau}\left[6(3-\xi+2\sigma)\right]\varphi_{3,y}\alpha_3 - 6\left(3-\xi+2\sigma\right)\varphi_{3,y}\alpha_2^2] = \\
&2h_1\beta_1 n_1 + h_0\beta_1\left(n_2-m_2\right).
\end{aligned} \tag{3.28}$$

Further evaluation using (3.25), provides us

$$12\left(3-\xi+2\sigma\right)\varphi_{3,y}\alpha_3 = 2\tau h_1\beta_1 n_1 + \tau h_0\beta_1\left(n_2-m_2\right) + 12\left(3-\xi+2\sigma\right)\varphi_{3,y}\alpha_2^2,$$

$$\alpha_3 = \frac{2\tau h_1 \beta_1 n_1}{12(3-\xi+2\sigma)\varphi_{3,y}} + \frac{\tau h_0 \beta_1 (n_2 - m_2)}{12(3-\xi+2\sigma)\varphi_{3,y}} + \frac{12(3-\xi+2\sigma)\varphi_{3,y}\alpha_2^2}{12(3-\xi+2\sigma)\varphi_{3,y}},$$
$$\alpha_3 = \frac{2\tau h_1 \beta_1 n_1}{12(3-\xi+2\sigma)\varphi_{3,y}} + \frac{\tau h_0 \beta_1 (n_2 - m_2)}{12(3-\xi+2\sigma)\varphi_{3,y}} + \frac{\tau^2 h_0^2 \beta_1^2 (n_1^2 + m_1^2)}{32(2-\xi+\sigma)^2 \varphi_{2,y}^2}. \quad (3.29)$$

Using Lemma 2.13.1 in (3.29), this leads to (3.6).

$$|\alpha_3| \le \frac{4\tau |h_1| \beta_1}{12(3-\xi+2\sigma)\varphi_{3,y}} + \frac{4\tau |h_0| \beta_1}{12(3-\xi+2\sigma)\varphi_{3,y}} + \frac{8\tau^2 h_0^2 \beta_1^2}{32(2-\xi+\sigma)^2 \varphi_{2,y}^2},$$

$$|\alpha_3| \le \frac{\tau(|h_1+h_0|)|\beta_1}{3(3-\xi+2\sigma)\varphi_{3,y}} + \frac{\tau^2 h_0^2 \beta_1^2}{4(2-\xi+\sigma)^2 \varphi_{2,y}^2}, \qquad \beta_1 > 1.$$

Theorem 3.2.1 is complete. □

Theorem 3.2.2 If f provided by (1.1) is the part of the subclass $\mathfrak{K}_{\nabla,\gamma,c}^{\partial,y}(\xi,\sigma,\kappa,\Psi)$, then

$$|\alpha_2| \le \min\left\{ \frac{\sigma |h_0| \beta_1}{(1+\kappa+2\xi)\varphi_{2,y}} \ , \ \sqrt{\frac{\sigma |h_0| (\beta_1 + (|\beta_2 - \beta_1|))}{\left[(2+\kappa+6\xi)\varphi_{3,y} - (1-\kappa)\varphi_{2,y}^2\right]}} \right\}, \quad (3.30)$$

and

$$|\alpha_3| \le \min\left\{ \frac{\sigma(h_1\beta_1 + h_0\beta_1)}{(2+\kappa+6\xi)\varphi_{3,y}} + \frac{\sigma |h_0| (\beta_1 + |\beta_2 - \beta_1|)}{\left[(2+\kappa+6\xi)\varphi_{3,y} - (1-\kappa)\varphi_{2,y}^2\right]}, \right.$$
$$\left. \frac{\sigma(h_1\beta_1 + h_0\beta_1)}{(2+\kappa+6\xi)\varphi_{3,y}} + \frac{\sigma^2 h_0^2 \beta_1^2}{(1+\kappa+2\xi)^2 \varphi_{2,y}^2} \right\}, \qquad \beta_1 > 1. \quad (3.31)$$

Proof. If $f \in \mathfrak{K}_{\nabla,\gamma,c}^{\partial,y}(\xi,\sigma,\kappa,\Psi)$ then exist two analytic functions μ, F in V and $\mu, F : V \to V$, hold the given conditions:

$$\frac{1}{\sigma}\left[\left\{(1-\kappa)\frac{\mathrm{s}\left(\mathbb{N}_{y,c}^{\partial,\gamma} f(\mathrm{s})\right)'}{\mathbb{N}_{y,c}^{\partial,\gamma} f(\mathrm{s})} + \kappa\left(\mathbb{N}_{y,c}^{\partial,\gamma} f(\mathrm{s})\right)' + \xi \mathrm{s}\left(\mathbb{N}_{y,c}^{\partial,\gamma} f(\mathrm{s})\right)''\right\} - 1\right] =$$
$$[\mu(\mathrm{s})(\Psi(F(\mathrm{s})) - 1], \quad (3.32)$$

and

$$\frac{1}{\sigma}\left[\left\{(1-\kappa)\frac{\omega\left(\mathbb{N}_{y,c}^{\partial,\gamma} g(\omega)\right)'}{\mathbb{N}_{y,c}^{\partial,\gamma} g(\omega)} + \kappa\left(\mathbb{N}_{y,c}^{\partial,\gamma} g(\omega)\right)' + \xi \omega\left(\mathbb{N}_{y,c}^{\partial,\gamma} g(\omega)\right)''\right\} - 1\right] =$$

$$[\mu(\omega)\,(\Psi(F(\omega))-1]\,. \tag{3.33}$$

The function $n(\mathrm{s})$ and $m(\omega)$ express by (3.9) and (3.10) respectively.

By taking process similarly as in Theorem 3.2.1, provides us

$$\frac{1}{\sigma}\left[\left\{(1-\kappa)\,\frac{\mathrm{s}\left(\mathbb{N}^{\partial;\gamma}_{y,c}f(\mathrm{s})\right)'}{\mathbb{N}^{\partial;\gamma}_{y,c}f(\mathrm{s})}+\kappa\left(\mathbb{N}^{\partial;\gamma}_{y,c}f(\mathrm{s})\right)'+\xi\,\mathrm{s}\left(\mathbb{N}^{\partial;\gamma}_{y,c}f(\mathrm{s})\right)''\right\}-1\right]=$$

$$\left[\mu(\mathrm{s})\;\left(\Psi\left(\frac{n(\mathrm{s})-1}{n(\mathrm{s})+1}\right)\right)-1\right], \tag{3.34}$$

and

$$\frac{1}{\sigma}\left[\left\{(1-\kappa)\,\frac{\omega\left(\mathbb{N}^{\partial;\gamma}_{y,c}g(\omega)\right)'}{\mathbb{N}^{\partial;\gamma}_{y,c}g(\omega)}+\kappa\left(\mathbb{N}^{\partial;\gamma}_{y,c}g(\omega)\right)'+\xi\,\omega\left(\mathbb{N}^{\partial;\gamma}_{y,c}g(\omega)\right)''\right\}-1\right]=$$

$$\left[\mu(\omega)\;\left(\Psi\left(\frac{m(\omega)-1}{m(\omega)+1}\right)\right)-1\right]. \tag{3.35}$$

The following are (3.15) and (3.16) respectively.

$$\mu(\mathrm{s})\;\left(\Psi\left(\frac{n(\mathrm{s})-1}{n(\mathrm{s})+1}\right)\right)-1=$$

$$\frac{1}{2}h_0\beta_1 n_1\mathrm{s}+\left\{\frac{1}{2}h_1\beta_1 n_1+\frac{1}{2}h_0\beta_1\left(n_2-\frac{n_1^2}{2}\right)+\frac{1}{4}h_0\beta_2 n_1^2\right\}\mathrm{s}^2+\ldots,$$

and

$$\mu(\omega)\;\left(\Psi\left(\frac{m(\omega)-1}{m(\omega)+1}\right)\right)-1=$$

$$\frac{1}{2}h_0\beta_1 m_1\omega+\left\{\frac{1}{2}h_1\beta_1 m_1+\frac{1}{2}h_0\beta_1\left(m_2-\frac{m_1^2}{2}\right)+\frac{1}{4}h_0\beta_2 m_1^2\right\}\omega^2+\ldots\,.$$

Since,

$$\begin{aligned}
\mathbb{N}^{\partial;\gamma}_{y,c}f(\mathrm{s}) &=\mathrm{s}+\varphi_{2,y}\alpha_2\mathrm{s}^2+\varphi_{3,y}\alpha_3\mathrm{s}^3+\ldots,\\
\left(\mathbb{N}^{\partial;\gamma}_{y,c}f(\mathrm{s})\right)' &=1+2\varphi_{2,y}\alpha_2\mathrm{s}+3\varphi_{3,y}\alpha_3\mathrm{s}^2+\ldots,\\
\left(\mathbb{N}^{\partial;\gamma}_{y,c}f(\mathrm{s})\right)'' &=2\varphi_{2,y}\alpha_2+6\varphi_{3,y}\alpha_3\mathrm{s}+\ldots,
\end{aligned}$$

(3.34) implies that,

$$\frac{1}{\sigma}\left[\left\{(1-\kappa)\,\frac{\mathrm{s}\left(\mathbb{N}^{\partial;\gamma}_{y,c}f(\mathrm{s})\right)'}{\mathbb{N}^{\partial;\gamma}_{y,c}f(\mathrm{s})}+\kappa\left(\mathbb{N}^{\partial;\gamma}_{y,c}f(\mathrm{s})\right)'+\xi\,\mathrm{s}\left(\mathbb{N}^{\partial;\gamma}_{y,c}f(\mathrm{s})\right)''\right\}-1\right]=$$

$$\frac{1}{\sigma}\left[\left\{(1-\kappa)\,\frac{\mathrm{s}+\;2\varphi_{2,y}\alpha_2\mathrm{s}^2+3\varphi_{3,y}\alpha_3\mathrm{s}^3+\ldots}{\mathrm{s}+\;\varphi_{2,y}\alpha_2\mathrm{s}^2+\varphi_{3,y}\alpha_3\mathrm{s}^3+\ldots}+\kappa+\;2\kappa\varphi_{2,y}\alpha_2\mathrm{s}+\right.\right.$$

$$\left.\left.3\kappa\varphi_{3,y}\alpha_3\mathrm{s}^2+\ldots\;\;+2\xi\varphi_{2,y}\alpha_2\mathrm{s}+6\xi\varphi_{3,y}\alpha_3\mathrm{s}^2+\ldots\;\right\}-1\right],$$

$$\frac{1}{\sigma}\left[\left\{(1-\kappa)\frac{\mathrm{s}\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)'}{\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})}+\kappa\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)'+\xi\,\mathrm{s}\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)''\right\}-1\right]=$$
$$\frac{1}{\sigma}\Big[\Big((1-\kappa)\Big(1+\ \varphi_{2,y}\alpha_2\mathrm{s}+2\varphi_{3,y}\alpha_3\mathrm{s}^2-\varphi_{2,y}^2\alpha_2^2\mathrm{s}^2+...\ \Big)+\kappa+$$
$$2\kappa\varphi_{2,y}\alpha_2\mathrm{s}+3\kappa\varphi_{3,y}\alpha_3\mathrm{s}^2+2\xi\,\varphi_{2,y}\alpha_2\mathrm{s}+6\xi\,\varphi_{3,y}\alpha_3\mathrm{s}^2+\ldots\Big)-1\Big],$$

$$\frac{1}{\sigma}\left[\left\{(1-\kappa)\frac{\mathrm{s}\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)'}{\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})}+\kappa\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)'+\xi\,\mathrm{s}\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)''\right\}-1\right]=$$
$$\frac{1}{\sigma}\Big[\Big(\Big(1+\ \varphi_{2,y}\alpha_2\mathrm{s}+2\varphi_{3,y}\alpha_3\mathrm{s}^2-\varphi_{2,y}^2\alpha_2^2\mathrm{s}^2-\kappa-\ \kappa\varphi_{2,y}\alpha_2\mathrm{s}-$$
$$2\kappa\varphi_{3,y}\alpha_3\mathrm{s}^2+\kappa\varphi_{2,y}^2\alpha_2^2\mathrm{s}^2\Big)+\kappa+\ 2\kappa\varphi_{2,y}\alpha_2\mathrm{s}+3\kappa\varphi_{3,y}\alpha_3\mathrm{s}^2+$$
$$2\xi\,\varphi_{2,y}\alpha_2\mathrm{s}+6\xi\,\varphi_{3,y}\alpha_3\mathrm{s}^2+\ldots\Big)-1\Big],$$

$$\frac{1}{\sigma}\left[\left\{(1-\kappa)\frac{\mathrm{s}\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)'}{\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})}+\kappa\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)'+\xi\,\mathrm{s}\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)''\right\}-1\right]=$$
$$\frac{1}{\sigma}\Big[\varphi_{2,y}\alpha_2\mathrm{s}+2\varphi_{3,y}\alpha_3\mathrm{s}^2-\varphi_{2,y}^2\alpha_2^2\mathrm{s}^2+\kappa\varphi_{2,y}\alpha_2\mathrm{s}+\kappa\varphi_{3,y}\alpha_3\mathrm{s}^2$$
$$+\kappa\varphi_{2,y}^2\alpha_2^2\mathrm{s}^2+2\xi\,\varphi_{2,y}\alpha_2\mathrm{s}+6\xi\,\varphi_{3,y}\alpha_3\mathrm{s}^2+\ldots\Big],$$

Therefore,

$$\frac{1}{\sigma}\left[\left\{(1-\kappa)\frac{\mathrm{s}\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)'}{\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})}+\kappa\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)'+\xi\,\mathrm{s}\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)''\right\}-1\right]=$$
$$\frac{1}{\sigma}\left[(1+\kappa+2\xi)\,\varphi_{2,y}\alpha_2\mathrm{s}+(2+\kappa+6\xi)\,\varphi_{3,y}\alpha_3\mathrm{s}^2-(1-\kappa)\,\varphi_{2,y}^2\alpha_2^2\mathrm{s}^2\right], \tag{3.36}$$

and

$$\begin{aligned}\mathbb{N}_{y,c}^{\partial,\gamma}\,g\,(\omega)=&\,\omega-\varphi_{2,y}\alpha_2\omega^2+\varphi_{3,y}(2\alpha_2^2-\alpha_3)\omega^3-\ldots,\\ \left(\mathbb{N}_{y,c}^{\partial,\gamma}g\,(\omega)\right)'=&\,1-2\varphi_{2,y}\alpha_2\omega+3\left(2\alpha_2^2-\alpha_3\right)\varphi_{3,y}\omega^2-\ldots,\\ \left(\mathbb{N}_{y,c}^{\partial,\gamma}g\,(\omega)\right)''=&-2\varphi_{2,y}\alpha_2+6(2\alpha_2^2-\alpha_3)\varphi_{3,y}\omega-\ldots.\end{aligned}$$

Hence, (3.35) become

$$\frac{1}{\sigma}\left[\left\{(1-\kappa)\frac{\omega\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)'}{\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)}+\kappa\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)'+\xi\,\omega\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)''\right\}-1\right]=$$

$$\frac{1}{\sigma}\left[\left\{(1-\kappa)\frac{\omega-2\varphi_{2,y}\alpha_2\omega^2+3\left(2\alpha_2^2-\alpha_3\right)\varphi_{3,y}\omega^3-\dots}{\omega-\ \varphi_{2,y}\alpha_2\omega^2+\varphi_{3,y}(2\alpha_2^2-\alpha_3)\omega^3-\dots}+\kappa-2\kappa\varphi_{2,y}\alpha_2\omega\right.\right.$$

$$\left.\left.+3\left(2\alpha_2^2-\alpha_3\right)\kappa\varphi_{3,y}\omega^2-\dots-2\xi\,\varphi_{2,y}\alpha_2\omega+6\left(2\alpha_2^2-\alpha_3\right)\xi\,\varphi_{3,y}\omega^2-\dots\right\}-1\right],$$

$$\frac{1}{\sigma}\left[\left\{(1-\kappa)\frac{\omega\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)'}{\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)}+\kappa\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)'+\xi\,\omega\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)''\right\}-1\right]=$$

$$\frac{1}{\sigma}\left[\left((1-\kappa)\left(1-\varphi_{2,y}\alpha_2\omega+2\left(2\alpha_2^2-\alpha_3\right)\varphi_{3,y}\omega^2-\alpha_2^2\varphi_{2,y}^2\omega^2\right)+\kappa-2\kappa\varphi_{2,y}\alpha_2\omega\right.\right.$$

$$\left.\left.+3\left(2\alpha_2^2-\alpha_3\right)\kappa\varphi_{3,y}\omega^2-\dots+-2\xi\,\varphi_{2,y}\alpha_2\omega+6\left(2\alpha_2^2-\alpha_3\right)\xi\,\varphi_{3,y}\omega^2-\dots\right)-1\right],$$

$$\frac{1}{\sigma}\left[\left\{(1-\kappa)\frac{\omega\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)'}{\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)}+\kappa\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)'+\xi\,\omega\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)''\right\}-1\right]=$$

$$\frac{1}{\sigma}\left[\left(\left(1-\varphi_{2,y}\alpha_2\omega+2\left(2\alpha_2^2-\alpha_3\right)\varphi_{3,y}\omega^2-\alpha_2^2\varphi_{2,y}^2\omega^2-\kappa+\kappa\varphi_{2,y}\alpha_2\omega\right.\right.\right.$$

$$\left.-2\left(2\alpha_2^2-\alpha_3\right)\kappa\varphi_{3,y}\omega^2+\kappa\alpha_2^2\varphi_{2,y}^2\omega^2\right)+\kappa-2\kappa\varphi_{2,y}\alpha_2\omega$$

$$\left.\left.+3\left(2\alpha_2^2-\alpha_3\right)\kappa\varphi_{3,y}\omega^2-2\xi\,\varphi_{2,y}\alpha_2\omega+6\left(2\alpha_2^2-\alpha_3\right)\xi\,\varphi_{3,y}\omega^2-\dots\right)-1\right],$$

$$\frac{1}{\sigma}\left[\left\{(1-\kappa)\frac{\omega\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)'}{\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)}+\kappa\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)'+\xi\,\omega\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)''\right\}-1\right]=$$

$$\frac{1}{\sigma}\left[-\varphi_{2,y}\alpha_2\omega+2\left(2\alpha_2^2-\alpha_3\right)\varphi_{3,y}\omega^2-\alpha_2^2\varphi_{2,y}^2\omega^2-\kappa\varphi_{2,y}\alpha_2\omega+\right.$$

$$\left.\left(2\alpha_2^2-\alpha_3\right)\kappa\varphi_{3,y}\omega^2+\kappa\alpha_2^2\varphi_{2,y}^2\omega^2-2\xi\,\varphi_{2,y}\alpha_2\omega+6\left(2\alpha_2^2-\alpha_3\right)\xi\,\varphi_{3,y}\omega^2\right],$$

$$\frac{1}{\sigma}\left[\left\{(1-\kappa)\frac{\omega\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)'}{\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)}+\kappa\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)'+\xi\,\omega\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)''\right\}-1\right]=$$

$$\frac{1}{\sigma}\left[-(1+\kappa+2\xi)\,\varphi_{2,y}\alpha_2\omega+(2+\kappa+6\xi)\left(2\alpha_2^2-\alpha_3\right)\varphi_{3,y}\omega^2-(1-\kappa)\,\alpha_2^2\varphi_{2,y}^2\omega^2\right]. \quad (3.37)$$

Using (3.36) with (3.15) in (3.34), this leads to

$$\frac{1}{\sigma}\left[(1+\kappa+2\xi)\,\varphi_{2,y}\alpha_2\mathrm{s}+(2+\kappa+6\xi)\,\varphi_{3,y}\alpha_3\mathrm{s}^2-(1-\kappa)\,\varphi_{2,y}^2\alpha_2^2\mathrm{s}^2\right]=$$

$$\frac{1}{2}h_0\beta_1n_1\mathrm{s}+\left\{\frac{1}{2}h_1\beta_1n_1+\frac{1}{2}h_0\beta_1\left(n_2-\frac{n_1^2}{2}\right)+\frac{1}{4}h_0\beta_2n_1^2\right\}\mathrm{s}^2+\dots.$$

Comparing coefficients on both sides, this yields us

$$\frac{1}{\sigma}\left[(1+\kappa+2\xi)\,\varphi_{2,y}\alpha_2\right]=\frac{1}{2}h_0\beta_1 n_1, \tag{3.38}$$

$$\begin{aligned}&\frac{1}{\sigma}\left[(2+\kappa+6\xi)\,\varphi_{3,y}\alpha_3-(1-\kappa)\,\varphi_{2,y}^2\alpha_2^2\right]=\\&\frac{1}{2}h_1\beta_1 n_1+\frac{1}{2}h_0\beta_1\left(n_2-\frac{n_1^2}{2}\right)+\frac{1}{4}h_0\beta_2 n_1^2.\end{aligned} \tag{3.39}$$

Using (3.37) and (3.16) in (3.35), this gives

$$\begin{aligned}&\frac{1}{\sigma}\left[-(1+\kappa+2\xi)\,\varphi_{2,y}\alpha_2\omega+(2+\kappa+6\xi)\left(2\alpha_2^2-\alpha_3\right)\varphi_{3,y}\omega^2-(1-\kappa)\,\alpha_2^2\varphi_{2,y}^2\omega^2\right]=\\&\frac{1}{2}h_0\beta_1 m_1\omega+\left\{\frac{1}{2}h_1\beta_1 m_1+\frac{1}{2}h_0\beta_1\left(m_2-\frac{m_1^2}{2}\right)+\frac{1}{4}h_0\beta_2 m_1^2\right\}\omega^2+\ldots.\end{aligned}$$

Comparing coefficients on both sides, this yields

$$\frac{1}{\sigma}\left[-(1+\kappa+2\xi)\,\varphi_{2,y}\alpha_2\right]=\frac{1}{2}h_0\beta_1 m_1, \tag{3.40}$$

$$\begin{aligned}&\frac{1}{\sigma}\left[(2+\kappa+6\xi)\left(2\alpha_2^2-\alpha_3\right)\varphi_{3,y}-(1-\kappa)\,\alpha_2^2\varphi_{2,y}^2\right]=\\&\frac{1}{2}h_1\beta_1 m_1+\frac{1}{2}h_0\beta_1\left(m_2-\frac{m_1^2}{2}\right)+\frac{1}{4}h_0\beta_2 m_1^2.\end{aligned} \tag{3.41}$$

From (3.38) and (3.40), this results in

$$n_1=-m_1. \tag{3.42}$$

Taking square of (3.38) and (3.40) and add them, this provides

$$\begin{aligned}2\alpha_2^2=&\frac{\sigma^2h_0^2\beta_1^2n_1^2}{4(1+\kappa+2\xi)^2\varphi_{2,y}^2}+\frac{\sigma^2h_0^2\beta_1^2m_1^2}{4(1+\kappa+2\xi)^2\varphi_{2,y}^2},\\2\alpha_2^2=&\frac{\sigma^2h_0^2\beta_1^2\left(n_1^2+m_1^2\right)}{4(1+\kappa+2\xi)^2\varphi_{2,y}^2},\end{aligned}$$

$$8(1+\kappa+2\xi)^2\varphi_{2,y}^2\alpha_2^2=\sigma^2h_0^2\beta_1^2\left(n_1^2+m_1^2\right). \tag{3.43}$$

Adding (3.39) and (3.41), this gives

$$\begin{aligned}&\frac{1}{\sigma}\left[(2+\kappa+6\xi)\,\varphi_{3,y}(\alpha_3+2\alpha_2^2-\alpha_3)-2\,(1-\kappa)\,\varphi_{2,y}^2\alpha_2^2\right]=\\&\frac{1}{2}h_1\beta_1(n_1+m_1)+\frac{1}{2}h_0\beta_1\left(n_2+m_2-\frac{n_1^2}{2}-\frac{m_1^2}{2}\right)+\frac{1}{4}h_0\beta_2(n_1^2+m_1^2),\end{aligned}$$

by using (3.42), gives

$$\frac{1}{\sigma}\left[(2+\kappa+6\xi)\,\varphi_{3,y}(2\alpha_2^2)-2\,(1-\kappa)\,\varphi_{2,y}^2\alpha_2^2\right]=$$
$$\frac{1}{2}h_1\beta_1(-m_1+m_1)+\frac{1}{2}h_0\beta_1\,(n_2+m_2)+\frac{1}{4}h_0(\beta_2-\beta_1)(n_1^2+m_1^2),$$

$$\frac{8}{\sigma}\left[(2+\kappa+6\xi)\,\varphi_{3,y}-(1-\kappa)\,\varphi_{2,y}^2\right]\alpha_2^2=$$
$$2h_0\beta_1\,(n_2+m_2)+h_0(\beta_2-\beta_1)(n_1^2+m_1^2), \tag{3.44}$$

which gives,

$$\alpha_2^2=\frac{2\sigma h_0\beta_1\,(n_2+m_2)+\sigma h_0(\beta_2-\beta_1)(n_1^2+m_1^2)}{8\left[(2+\kappa+6\xi)\,\varphi_{3,y}-(1-\kappa)\,\varphi_{2,y}^2\right]}. \tag{3.45}$$

Applying Lemma 2.13.1 for the coefficients $n_1, n_2, m_1 and\ m_2$, it follows from (3.43) and (3.45), this yields

$$\alpha_2^2=\frac{\sigma^2h_0^2\beta_1^2\left(n_1^2+m_1^2\right)}{8(1+\kappa+2\xi)^2\varphi_{2,y}^2}, \tag{3.46}$$

$$\left|\alpha_2^2\right|\leq\frac{8\sigma^2h_0^2\beta_1^2}{8(1+\kappa+2\xi)^2\varphi_{2,y}^2}.$$

By taking square root on both sides, this provides

$$|\alpha_2|\leq\frac{\sigma h_0\beta_1}{(1+\kappa+2\xi)\,\varphi_{2,y}},$$

and

$$\alpha_2^2\;=\frac{2\sigma h_0\beta_1\,(n_2+m_2)+\sigma h_0(\beta_2-\beta_1)(n_1^2+m_1^2)}{8\left[(2+\kappa+6\xi)\,\varphi_{3,y}-(1-\kappa)\,\varphi_{2,y}^2\right]}, \tag{3.47}$$

$$\left|\alpha_2^2\right|\leq\frac{8\sigma h_0\beta_1+8\sigma h_0(\beta_2-\beta_1)}{8\left[(2+\kappa+6\xi)\,\varphi_{3,y}-(1-\kappa)\,\varphi_{2,y}^2\right]}.$$

By taking square root on both sides, this provides

$$|\alpha_2|\leq\sqrt{\frac{\sigma\,|h_0|\,(\beta_1+|\beta_2-\beta_1|)}{\left[(2+\kappa+6\xi)\,\varphi_{3,y}-(1-\kappa)\,\varphi_{2,y}^2\right]}},$$

Hence,

$$|\alpha_2|\leq\frac{\sigma|h_0|\beta_1}{(1+\kappa+2\xi)\,\varphi_{2,y}},$$

and

$$|\alpha_2| \leq \sqrt{\frac{\sigma |h_0| (\beta_1 + (\beta_2 - \beta_1))}{\left[(2+\kappa+6\xi)\,\varphi_{3,y} - (1-\kappa)\,\varphi_{2,y}^2\right]}},$$

which provide the required estimate on $|\alpha_2|$ as shown in (3.30).

Next, to obtain the estimate of the coefficient $|\alpha_3|$, by subtracting (3.39) and (3.41), this leads to

$$\frac{1}{\sigma}\left[(2+\kappa+6\xi)\,\varphi_{3,y}(\alpha_3 - 2\alpha_2^2 + \alpha_3) - (1-\kappa)\,\varphi_{2,y}^2\alpha_2^2 + (1-\kappa)\,\varphi_{2,y}^2\alpha_2^2\right] =$$
$$\frac{1}{2}h_1\beta_1(n_1 - m_1) + \frac{1}{2}h_0\beta_1\left(n_2 - m_2 - \frac{n_1^2}{2} + \frac{m_1^2}{2}\right) + \frac{1}{4}h_0\beta_2(n_1^2 - m_1^2).$$

Using (3.42), gives

$$\frac{1}{\sigma}\left[(2+\kappa+6\xi)\,\varphi_{3,y}(2\alpha_3 - 2\alpha_2^2)\right] = \frac{1}{2}h_1\beta_1(2n_1) + \frac{1}{2}h_0\beta_1\,(n_2 - m_2),$$

$$\frac{4}{\sigma}\left[(2+\kappa+6\xi)\,\varphi_{3,y}(\alpha_3 - \alpha_2^2)\right] = h_1\beta_1(2n_1) + h_0\beta_1\,(n_2 - m_2). \tag{3.48}$$

By substituting the value of α_2^2 from (3.43) in (3.48), it gives

$$4\left[(2+\kappa+6\xi)\,\varphi_{3,y}\alpha_3 - (2+\kappa+6\xi)\,\varphi_{3,y}\alpha_2^2\right] = 2\sigma h_1\beta_1 n_1 + \sigma h_0\beta_1\,(n_2 - m_2),$$

$$4\,(2+\kappa+6\xi)\,\varphi_{3,y}\alpha_3 = 2\sigma h_1\beta_1 n_1 + \sigma h_0\beta_1\,(n_2 - m_2)$$
$$+\frac{4\sigma^2 h_0^2\beta_1^2\,(2+\kappa+6\xi)\,\varphi_{3,y}\left(n_1^2 + m_1^2\right)}{8(1+\kappa+2\xi)^2\varphi_{2,y}^2},$$

$$\alpha_3 = \frac{2\sigma h_1\beta_1 n_1}{4\,(2+\kappa+6\xi)\,\varphi_{3,y}} + \frac{\sigma h_0\beta_1\,(n_2 - m_2)}{4\,(2+\kappa+6\xi)\,\varphi_{3,y}} +$$
$$\frac{4\sigma^2 h_0^2\beta_1^2\,(2+\kappa+6\xi)\,\varphi_{3,y}\left(n_1^2 + m_1^2\right)}{32\,(2+\kappa+6\xi)\,\varphi_{3,y}(1+\kappa+2\xi)^2\varphi_{2,y}^2},$$

$$|\alpha_3| \leq \frac{2\sigma h_1\beta_1 n_1 + \sigma h_0\beta_1\,(n_2 - m_2)}{4\,(2+\kappa+6\xi)\,\varphi_{3,y}} + \frac{\sigma^2 h_0^2\beta_1^2\left(n_1^2 + m_1^2\right)}{8(1+\kappa+2\xi)^2\varphi_{2,y}^2}. \tag{3.49}$$

Now, by substituting the value of α_2^2 from (3.45) in (3.48), it provides

$$4\,(2+\kappa+6\xi)\,\varphi_{3,y}\alpha_3 = 2\sigma h_1\beta_1 n_1 + \sigma h_0\beta_1\,(n_2 - m_2)$$
$$+\frac{4\,(2+\kappa+6\xi)\,\varphi_{3,y}(2\sigma h_0\beta_1\,(n_2 + m_2) + \sigma h_0(\beta_2 - \beta_1)(n_1^2 + m_1^2))}{8\left[(2+\kappa+6\xi)\,\varphi_{3,y} - (1-\kappa)\,\varphi_{2,y}^2\right]},$$

$$\alpha_3 = \frac{2\sigma h_1 \beta_1 n_1 + \sigma h_0 \beta_1 (n_2 - m_2)}{4(2+\kappa+6\xi)\varphi_{3,y}} + \\ \frac{(2\sigma h_0 \beta_1 (n_2 + m_2) + \sigma h_0 (\beta_2 - \beta_1)(n_1^2 + m_1^2))}{8\left[(2+\kappa+6\xi)\varphi_{3,y} - (1-\kappa)\varphi_{2,y}^2\right]},$$

$$|\alpha_3| \leq \frac{2\sigma h_1 \beta_1 n_1 + \sigma h_0 \beta_1 (n_2 - m_2)}{4(2+\kappa+6\xi)\varphi_{3,y}} + \\ \frac{(2\sigma h_0 \beta_1 (n_2 + m_2) + \sigma h_0 (\beta_2 - \beta_1)(n_1^2 + m_1^2))}{8\left[(2+\kappa+6\xi)\varphi_{3,y} - (1-\kappa)\varphi_{2,y}^2\right]}. \tag{3.50}$$

Applying Lemma 2.13.1 in eq (3.49), (3.50) for the coefficients n_1, n_2, m_1 and m_2, this results in

$$|\alpha_3| \leq \frac{\sigma(h_1\beta_1 + h_0\beta_1)}{(2+\kappa+6\xi)\varphi_{3,y}} + \frac{\sigma^2 h_0^2 \beta_1^2}{(1+\kappa+2\xi)^2 \varphi_{2,y}^2}. \tag{3.51}$$

$$|\alpha_3| \leq \frac{\sigma(h_1\beta_1 + h_0\beta_1)}{(2+\kappa+6\xi)\varphi_{3,y}} + \frac{\sigma h_0(\beta_1 + |\beta_2 - \beta_1|)}{\left[(2+\kappa+6\xi)\varphi_{3,y} - (1-\kappa)\varphi_{2,y}^2\right]}, \tag{3.52}$$

which provide the required estimate on $|\alpha_3|$, as shown in (3.31).

Theorem 3.2.2 is complete. $\square$

CHAPTER 4

ON A NEW CLASS OF BI-UNIVALENT FUNCTIONS USING QUASI-SUBORDINATION

4.1 Overview

This chapter determines new subclasses of bi-univalent functions which involve quasi-subordination and q-derivatives. This chapter central objective is to find the initial coefficients estimate for these subclasses.

Definition 4.1.1. *A function having form* (1.1) *is considered to be in the class* $\mathfrak{R}_{\nabla,\gamma,c}^{\partial,y}(\xi,\sigma,\tau,\Psi,q)$, $0\leq\xi\leq1,\ 0\leq\sigma\leq1,$ *and* $\tau\in C\backslash\{0\}$, *for the conditions given below are satisfy:*

$$\frac{1}{\tau}\left[\left\{\frac{\mathrm{s}D_q\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)+\mathrm{s}^2D_q^2\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)}{(1-\xi)\,\mathrm{s}+\xi\,\mathrm{s}D_q\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)}+\sigma\mathrm{s}D_q^2\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)\right\}-1\right]$$

$$\prec_{\mathsf{q}}[\Psi(\mathrm{s})-1], \tag{4.1}$$

and

$$\frac{1}{\tau}\left[\left\{\frac{\omega D_q\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)+\omega^2D_q^2\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)}{(1-\xi)\,\omega+\xi\,\omega D_q\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)}+\sigma\omega D_q^2\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)\right\}-1\right]$$

$$\prec_{\mathsf{q}}[\Psi(\omega)-1], \tag{4.2}$$

where $\mathtt{s}, \omega \in V$ *and* g *is the inverse of* f.

Definition 4.1.2. *A function having form* (1.1) *is considered to be in the class* $\mathfrak{K}^{\partial,y}_{\nabla,\gamma,c}(\xi,\sigma,\kappa,\Psi,\mathrm{q})$, $0 \leq \xi$, $\kappa \geq 1$, *and* $\sigma \in C\backslash\{0\}$, *if the conditions listed below are hold true:*

$$\frac{1}{\sigma}\left[\left\{(1-\kappa)\frac{\mathtt{s}D_q\left(\mathbb{N}^{\partial,\gamma}_{y,c}f(\mathtt{s})\right)}{\mathbb{N}^{\partial,\gamma}_{y,c}f(\mathtt{s})}+\kappa D_q\left(\mathbb{N}^{\partial,\gamma}_{y,c}f(\mathtt{s})\right)+\xi\,\mathtt{s}D^2_q\left(\mathbb{N}^{\partial,\gamma}_{y,c}f(\mathtt{s})\right)\right\}-1\right]$$

$$\prec_{\mathsf{q}}[\Psi(\mathtt{s})-1], \tag{4.3}$$

and

$$\frac{1}{\sigma}\left[\left\{(1-\kappa)\frac{\omega D_q\left(\mathbb{N}^{\partial,\gamma}_{y,c}g(\omega)\right)}{\mathbb{N}^{\partial,\gamma}_{y,c}g(\omega)}+\kappa D_q\left(\mathbb{N}^{\partial,\gamma}_{y,c}g(\omega)\right)+\xi\,\omega D^2_q\left(\mathbb{N}^{\partial,\gamma}_{y,c}g(\omega)\right)\right\}-1\right]$$

$$\prec_{\mathsf{q}}[\Psi(\omega)-1], \tag{4.4}$$

where $\mathtt{s}, \omega \in V$ *and* g *is the inverse of* f.

4.2 Main Result

Theorem 4.2.1 If f provided by (1.1) is part of the subclass $\mathfrak{R}^{\partial,y}_{\nabla,\gamma,c}(\xi,\sigma,\tau,\Psi,q)$, then

$$|\alpha_2| \leq \frac{\tau|h_0|\beta_1\sqrt{\beta_1}}{\sqrt{\mathtt{r}_1\varphi_{3,y}-\mathtt{r}_2\varphi^2_{2,y}}}, \tag{4.5}$$

$$|\alpha_3| \leq \frac{\tau(|h_1|+|h_0|)|\beta_1}{\mathtt{r}_1\varphi_{3,y}}+\frac{\tau^2h_0^2\beta_1^2}{\mathtt{r}_3\varphi^2_{2,y}},\ \beta_1>1. \tag{4.6}$$

where

$$\begin{aligned}
\bar{q} &= 1+q,\\
\mathtt{r}_1 &= (1+q\bar{q})\,\tau h_0\beta_1^2\,(1+\bar{q}-\xi+\bar{q}\sigma),\\
\mathtt{r}_2 &= \bar{q}^2\{\xi\,(2-\xi)\,\tau h_0\beta_1^2+(\beta_2-\beta_1)(2-\xi+\sigma)^2\},\\
\mathtt{r}_3 &= \bar{q}^2(2-\xi+\sigma)^2,\\
\varphi_{2,y} &= \left(\frac{1+\gamma}{2+\gamma}\right)^{\partial}\frac{(y)_{2-1}}{(c)_{2-1}},
\end{aligned}$$

and

$$\varphi_{3,y}=\left(\frac{1+\gamma}{3+\gamma}\right)^{\partial}\frac{(y)_{3-1}}{(c)_{3-1}}.$$

Proof. If $f\in\Re^{\partial,y}_{\nabla,\gamma,c}(\xi,\sigma,\tau,\Psi,q)$, for the analytic functions μ,F in V such that $\mu,F:V\rightarrow V$, hold the given conditions:

$$\frac{1}{\tau}\left[\left\{\frac{\mathrm{s}D_q\left(\mathbb{N}^{\partial,\gamma}_{y,c}f(\mathrm{s})\right)+\mathrm{s}^2D_q^2\left(\mathbb{N}^{\partial,\gamma}_{y,c}f(\mathrm{s})\right)}{(1-\xi)\,\mathrm{s}+\xi\,\mathrm{s}D_q\left(\mathbb{N}^{\partial,\gamma}_{y,c}f(\mathrm{s})\right)}+\sigma\mathrm{s}D_q^2\left(\mathbb{N}^{\partial,\gamma}_{y,c}f(\mathrm{s})\right)\right\}-1\right]=$$

$$[\mu(\mathrm{s})\,(\Psi(F(\mathrm{s}))-1],\tag{4.7}$$

and

$$\frac{1}{\tau}\left[\left\{\frac{\omega D_q\left(\mathbb{N}^{\partial,\gamma}_{y,c}g(\omega)\right)+\omega^2D_q^2\left(\mathbb{N}^{\partial,\gamma}_{y,c}g(\omega)\right)}{(1-\xi)\,\omega+\xi\,\omega D_q\left(\mathbb{N}^{\partial,\gamma}_{y,c}g(\omega)\right)}+\sigma\omega D_q^2\left(\mathbb{N}^{\partial,\gamma}_{y,c}g(\omega)\right)\right\}-1\right]=$$

$$[\mu(\omega)\,(\Psi(F(\omega))-1],\tag{4.8}$$

where $\mathrm{s},\omega\in V$ and g is the inverse of f.

Express the caratheodory functions n and m by

$$n(\mathrm{s})=\frac{1+F(\mathrm{s})}{1-F(\mathrm{s})}\;=\;1+n_1\mathrm{s}+n_2\mathrm{s}^2+n_3\mathrm{s}^3+\ldots,\tag{4.9}$$

and

$$m(\omega)=\frac{1+F(\omega)}{1-F(\omega)}\;=\;1+m_1\omega+m_2\omega^2+m_3\omega^3+\ldots.\tag{4.10}$$

Also, it can be written as

$$F(\mathrm{s})=\;\frac{n(\mathrm{s})-1}{n(\mathrm{s})+1}=\frac{1}{2}\left[n_1\mathrm{s}+\left(n_2-\frac{n_1^2}{2}\right)\mathrm{s}^2+\ldots,\right.\tag{4.11}$$

and

$$F(\omega)=\frac{m(\omega)-1}{m(\omega)+1}=\;\frac{1}{2}\left[m_1\omega+\left(m_2-\frac{m_1^2}{2}\right)\omega^2+\ldots.\right.\tag{4.12}$$

Substituting (4.11), (4.12) in (4.7), (4.8), this gives us

$$\frac{1}{\tau}\left[\left\{\frac{\mathrm{s}D_q\left(\mathbb{N}^{\partial,\gamma}_{y,c}f(\mathrm{s})\right)+\mathrm{s}^2D_q^2\left(\mathbb{N}^{\partial,\gamma}_{y,c}f(\mathrm{s})\right)}{(1-\xi)\,\mathrm{s}+\xi\,\mathrm{s}D_q\left(\mathbb{N}^{\partial,\gamma}_{y,c}f(\mathrm{s})\right)}+\sigma\mathrm{s}D_q^2\left(\mathbb{N}^{\partial,\gamma}_{y,c}f(\mathrm{s})\right)\right\}-1\right]=$$

$$\left[\mu(\mathrm{s})\;\left(\Psi\left(\frac{n(\mathrm{s})-1}{n(\mathrm{s})+1}\right)\right)-1\right],\tag{4.13}$$

and

$$\frac{1}{\tau}\left[\left\{\frac{\omega D_q\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)+\omega^2 D_q^2\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)}{(1-\xi)\,\omega+\xi\,\omega D_q\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)}+\sigma\omega D_q^2\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)\right\}-1\right]=$$

$$\left[\mu(\omega)\ \left(\Psi\left(\frac{m(\omega)-1}{m(\omega)+1}\right)\right)-1\right]. \tag{4.14}$$

This can be written as

$$\mu(\mathrm{s})\ \left(\Psi\left(\frac{n(\mathrm{s})-1}{n(\mathrm{s})+1}\right)\right)-1=$$

$$\frac{1}{2}h_0\beta_1 n_1\mathrm{s}+\left\{\frac{1}{2}h_1\beta_1 n_1+\frac{1}{2}h_0\beta_1\left(n_2-\frac{n_1^2}{2}\right)+\frac{1}{4}h_0\beta_2 n_1^2\right\}\mathrm{s}^2+\ldots, \tag{4.15}$$

and

$$\mu(\omega)\ \left(\Psi\left(\frac{m(\omega)-1}{m(\omega)+1}\right)\right)-1=$$

$$\frac{1}{2}h_0\beta_1 m_1\omega+\left\{\frac{1}{2}h_1\beta_1 m_1+\frac{1}{2}h_0\beta_1\left(m_2-\frac{m_1^2}{2}\right)+\frac{1}{4}h_0\beta_2 m_1^2\right\}\omega^2+\ldots. \tag{4.16}$$

Since,

$$\begin{aligned}
\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s}) &=\mathrm{s}+\sum_{t=2}^{\infty}\varphi_{t,y}\alpha_t\mathrm{s}^t,\\
\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s}) &=\mathrm{s}+\varphi_{2,y}\alpha_2\mathrm{s}^2+\varphi_{3,y}\alpha_3\mathrm{s}^3+\ldots,\\
D_q\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right) &=1+[2]_q\varphi_{2,y}\alpha_2\mathrm{s}+[3]_q\varphi_{3,y}\alpha_3\mathrm{s}^2+\ldots,\\
D_q^2\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right) &=[2]_q\varphi_{2,y}\alpha_2+[2]_q[3]_q\varphi_{3,y}\alpha_3\mathrm{s}+\ldots,
\end{aligned}$$

(4.13) implies that

$$\frac{1}{\tau}\left[\left\{\frac{\mathrm{s}D_q\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)+\mathrm{s}^2 D_q^2\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)}{(1-\xi)\,\mathrm{s}+\xi\,\mathrm{s}D_q\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)}+\sigma\mathrm{s}D_q^2\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)\right\}-1\right]=$$

$$\frac{1}{\tau}\left[\left\{\frac{\mathrm{s}+[2]_q\varphi_{2,y}\alpha_2\mathrm{s}^2+[3]_q\varphi_{3,y}\alpha_3\mathrm{s}^3+[2]_q\varphi_{2,y}\alpha_2\mathrm{s}^2+[2]_q[3]_q\varphi_{3,y}\alpha_3\mathrm{s}^3+\ldots}{\mathrm{s}-\xi\mathrm{s}+\xi\mathrm{s}+[2]_q\xi\varphi_{2,y}\alpha_2\mathrm{s}^2+[3]_q\xi\varphi_{3,y}\alpha_3\mathrm{s}^3+\ldots}\right.\right.$$

$$\left.\left.+\left([2]_q\sigma\varphi_{2,y}\alpha_2\mathrm{s}+[2]_q[3]_q\sigma\varphi_{3,y}\alpha_3\mathrm{s}^2+\ldots\right)\right\}-1\right],$$

$$\frac{1}{\tau}\left[\left\{\frac{\mathrm{s}D_q\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)+\mathrm{s}^2 D_q^2\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)}{(1-\xi)\,\mathrm{s}+\xi\,\mathrm{s}D_q\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)}+\sigma\mathrm{s}D_q^2\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)\right\}-1\right]=$$

$$\frac{1}{\tau}\left[\left\{\frac{\mathrm{s}+2[2]_q\varphi_{2,y}\alpha_2\mathrm{s}^2+\left(1+[2]_q\right)[3]_q\varphi_{3,y}\alpha_3\mathrm{s}^3+\ldots}{\mathrm{s}+[2]_q\xi\varphi_{2,y}\alpha_2\mathrm{s}^2+[3]_q\xi\varphi_{3,y}\alpha_3\mathrm{s}^3+\ldots}+[2]_q\sigma\varphi_{2,y}\alpha_2\mathrm{s}+\right.\right.$$

$$\left.\left.[2]_q[3]_q\sigma\varphi_{3,y}\alpha_3\mathrm{s}^2+\ldots\right\}-1\right],$$

$$\frac{1}{\tau}\left[\left\{\frac{\mathrm{s}D_q\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)+\mathrm{s}^2D_q^2\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)}{(1-\xi)\,\mathrm{s}+\xi\,\mathrm{s}D_q\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)}+\sigma\mathrm{s}D_q^2\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)\right\}-1\right]=$$
$$\frac{1}{\tau}\Big[1+\ (2-\xi)\,[2]_q\varphi_{2,y}\alpha_2\mathrm{s}+\left(1+[2]_q-\xi\right)[3]_q\varphi_{3,y}\alpha_3\mathrm{s}^2-$$
$$\xi\,(2-\xi)\left([2]_q\right)^2\varphi_{2,y}^2\alpha_2^2\mathrm{s}^2+[2]_q\sigma\varphi_{2,y}\alpha_2\mathrm{s}+[2]_q[3]_q\sigma\varphi_{3,y}\alpha_3\mathrm{s}^2+\cdots-1\Big],$$

$$\frac{1}{\tau}\left[\left\{\frac{\mathrm{s}D_q\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)+\mathrm{s}^2D_q^2\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)}{(1-\xi)\,\mathrm{s}+\xi\,\mathrm{s}D_q\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)}+\sigma\mathrm{s}D_q^2\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)\right\}-1\right]=$$
$$\frac{1}{\tau}\Big[\ (2-\xi+\sigma)\,[2]_q\varphi_{2,y}\alpha_2\mathrm{s}+\left(1+[2]_q-\xi+[2]_q\sigma\right)[3]_q\varphi_{3,y}\alpha_3\mathrm{s}^2-$$
$$\xi\,(2-\xi)\left([2]_q\right)^2\varphi_{2,y}^2\alpha_2^2\mathrm{s}^2\Big]. \tag{4.17}$$

Now,

$$\begin{aligned}\mathbb{N}_{y,c}^{\partial,\gamma}\,g(\omega)=&\,\omega-\varphi_{2,y}\alpha_2\omega^2+\varphi_{3,y}(2\alpha_2^2-\alpha_3)\omega^3-\ldots,\\ D_q\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)=&\,1-[2]_q\varphi_{2,y}\alpha_2\omega+[3]_q\left(2\alpha_2^2-\alpha_3\right)\varphi_{3,y}\omega^2-\ldots,\\ D_q^2\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)=&-[2]_q\varphi_{2,y}\alpha_2+[2]_q[3]_q\left(2\alpha_2^2-\alpha_3\right)\varphi_{3,y}\omega-\ldots.\end{aligned}$$

Therefore,

$$\frac{1}{\tau}\left[\left\{\frac{\omega D_q\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)+\omega^2D_q^2\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)}{(1-\xi)\,\omega+\xi\,\omega D_q\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)}+\sigma\omega D_q^2\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)\right\}-1\right]=$$
$$\frac{1}{\tau}\left[\left\{\frac{\omega-2[2]_q\varphi_{2,y}\alpha_2\omega^2+\left(1+[2]_q\right)[3]_q\left(2\alpha_2^2-\alpha_3\right)\varphi_{3,y}\omega^3-\ldots}{\omega-[2]_q\xi\varphi_{2,y}\alpha_2\omega^2+[3]_q\xi\left(2\alpha_2^2-\alpha_3\right)\varphi_{3,y}\omega^3-\ldots}\right.\right.$$
$$\left.\left.-[2]_q\sigma\varphi_{2,y}\alpha_2\omega+[2]_q[3]_q\sigma\left(2\alpha_2^2-\alpha_3\right)\varphi_{3,y}\omega^2-\ldots)\ \right\}-1\right],$$

$$\frac{1}{\tau}\left[\left\{\frac{\omega D_q\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)+\omega^2D_q^2\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)}{(1-\xi)\,\omega+\xi\,\omega D_q\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)}+\sigma\omega D_q^2\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)\right\}-1\right]=$$
$$\frac{1}{\tau}\Big[1-(2-\xi)\,[2]_q\varphi_{2,y}\alpha_2\omega+\left(1+[2]_q-\xi\right)[3]_q\left(2\alpha_2^2-\alpha_3\right)\varphi_{3,y}\omega^2-$$
$$\xi\,(2-\xi)\,[2]_q^2\,\varphi_{2,y}^2\alpha_2^2\omega^2-[2]_q\sigma\varphi_{2,y}\alpha_2\omega+[2]_q[3]_q\sigma\left(2\alpha_2^2-\alpha_3\right)\varphi_{3,y}\omega^2-1\Big],$$

$$\frac{1}{\tau}\left[\left\{\frac{\omega D_q\left(\mathbb{N}_{y,c}^{\partial;\gamma}g(\omega)\right)+\omega^2 D_q^2\left(\mathbb{N}_{y,c}^{\partial;\gamma}g(\omega)\right)}{(1-\xi)\,\omega+\xi\,\omega D_q\left(\mathbb{N}_{y,c}^{\partial;\gamma}g(\omega)\right)}+\sigma\omega D_q^2\left(\mathbb{N}_{y,c}^{\partial;\gamma}g(\omega)\right)\right\}-1\right]=$$

$$\frac{1}{\tau}\Big[-(2-\xi+\sigma)\,[2]_q\varphi_{2,y}\alpha_2\omega+\left(1+[2]_q-\xi+\sigma[2]_q\right)[3]_q\left(2\alpha_2^2-\alpha_3\right)\varphi_{3,y}\omega^2$$

$$-\xi\,(2-\xi)\left([2]_q\right)^2\varphi_{2,y}^2\alpha_2^2\omega^2\Big].\tag{4.18}$$

Substituting (4.17) and (4.15) in (4.13) and comparing coefficients on both sides, it leads to

$$\frac{1}{\tau}\left[(2-\xi+\sigma)\,[2]_q\varphi_{2,y}\alpha_2\right]=\frac{1}{2}h_0\beta_1 n_1,\tag{4.19}$$

$$\frac{1}{\tau}\left[\left(1+[2]_q-\xi+[2]_q\sigma\right)[3]_q\varphi_{3,y}\alpha_3-\xi\,(2-\xi)\left([2]_q\right)^2\varphi_{2,y}^2\alpha_2^2\right]=$$

$$\frac{1}{2}h_1\beta_1 n_1+\frac{1}{2}h_0\beta_1\left(n_2-\frac{n_1^2}{2}\right)+\frac{1}{4}h_0\beta_2 n_1^2.\tag{4.20}$$

Now, by substituting (4.18) and (4.16) in (4.14) and comparing coefficients, it gives

$$\frac{1}{\tau}\left[-(2-\xi+\sigma)\,[2]_q\varphi_{2,y}\alpha_2\right]=\frac{1}{2}h_0\beta_1 m_1,\tag{4.21}$$

$$\frac{1}{\tau}\left[\left(1+[2]_q-\xi+\sigma[2]_q\right)[3]_q\left(2\alpha_2^2-\alpha_3\right)\varphi_{3,y}-\xi\,(2-\xi)\left([2]_q\right)^2\varphi_{2,y}^2\alpha_2^2\right]=$$

$$\frac{1}{2}h_1\beta_1 m_1+\frac{1}{2}h_0\beta_1\left(m_2-\frac{m_1^2}{2}\right)+\frac{1}{4}h_0\beta_2 m_1^2.\tag{4.22}$$

From (4.19) and (4.21), it results in

$$\alpha_2=\frac{\tau h_0\beta_1 n_1}{2[2]_q\,(2-\xi+\sigma)\,\varphi_{2,y}},\tag{4.23}$$

$$\alpha_2=\frac{-\tau h_0\beta_1 m_1}{2[2]_q\,(2-\xi+\sigma)\,\varphi_{2,y}},\tag{4.24}$$

$$\alpha_2=\frac{\tau h_0\beta_1 n_1}{2[2]_q\,(2-\xi+\sigma)\,\varphi_{2,y}}=\frac{-\tau h_0\beta_1 m_1}{2[2]_q\,(2-\xi+\sigma)\,\varphi_{2,y}}.\tag{4.25}$$

It provides

$$n_1=-m_1.\tag{4.26}$$

By taking square of (4.25) and adding them, this results in

$$2\alpha_2^2=\frac{\tau^2h_0^2\beta_1^2(n_1^2+m_1^2)}{4\left([2]_q\right)^2(2-\xi+\sigma)^2\varphi_{2,y}^2},$$

$$8\left([2]_q\right)^2(2-\xi+\sigma)^2\varphi_{2,y}^2\alpha_2^2=\tau^2h_0^2\beta_1^2\left(n_1^2+m_1^2\right). \tag{4.27}$$

Adding (4.20) and (4.22), this leads to

$$\frac{1}{\tau}[-2\left([2]_q\right)^2\xi\,(2-\xi)\,\varphi_{2,y}^2\alpha_2^2+2[3]_q\left(1+[2]_q-\xi+[2]_q\sigma\right)\varphi_{3,y}\alpha_2^2]=$$
$$\frac{1}{2}h_1\beta_1(n_1+m_1)+\frac{1}{2}h_0\beta_1\,(n_2+m_2)+\frac{1}{2}h_0\beta_1\left(\frac{-n_1^2}{2}-\frac{m_1^2}{2}\right)+\frac{1}{4}h_0\beta_2\left(n_1^2+m_1^2\right).$$

By using (4.26),gives us

$$\frac{1}{\tau}[-2\left([2]_q\right)^2\xi\,(2-\xi)\,\varphi_{2,y}^2\alpha_2^2+2[3]_q\left(1+[2]_q-\xi+[2]_q\sigma\right)\varphi_{3,y}\alpha_2^2]=$$
$$\frac{1}{2}h_0\beta_1\,(n_2+m_2)+\frac{1}{4}h_0\;(\beta_2-\beta_1)\left(n_1^2+m_1^2\right).$$

By using (4.27),gives us

$$\frac{4}{\tau}[-2\left([2]_q\right)^2\xi\,(2-\xi)\,\varphi_{2,y}^2\alpha_2^2+2[3]_q\left(1+[2]_q-\xi+[2]_q\sigma\right)\varphi_{3,y}\alpha_2^2]=$$
$$2h_0\beta_1\,(n_2+m_2)+h_0\;(\beta_2-\beta_1)\left(\frac{8\left([2]_q\right)^2(2-\xi+\sigma)^2\varphi_{2,y}^2\alpha_2^2}{\tau^2h_0^2\beta_1^2}\right)$$

$$\frac{4\tau^2h_0\beta_1^2}{\tau}[-2\left([2]_q\right)^2\xi\,(2-\xi)\,\varphi_{2,y}^2\alpha_2^2+2[3]_q\left(1+[2]_q-\xi+[2]_q\sigma\right)\varphi_{3,y}\alpha_2^2]=$$
$$2\tau^2h_0^2\beta_1^3\,(n_2+m_2)+\;(\beta_2-\beta_1)\left(8\left([2]_q\right)^2(2-\xi+\sigma)^2\varphi_{2,y}^2\alpha_2^2\right),$$

$$8\left[[3]_q\tau h_0\beta_1^2\left(1+[2]_q-\xi+[2]_q\sigma\right)\varphi_{3,y}-\left([2]_q\right)^2\tau h_0\beta_1^2\xi\,(2-\xi)\,\varphi_{2,y}^2\right]\alpha_2^2=$$
$$2\tau^2h_0^2\beta_1^3\,(n_2+m_2)+\;(\beta_2-\beta_1)\,8\left([2]_q\right)^2(2-\xi+\sigma)^2\varphi_{2,y}^2\alpha_2^2. \tag{4.28}$$

which implies

$$\alpha_2^2=\frac{2\tau^2h_0^2\beta_1^3\,(n_2+m_2)}{8\left[\mathtt{r}_1\varphi_{3,y}-\mathtt{r}_2\varphi_{2,y}^2\right]}. \tag{4.29}$$

Using Lemma 2.13.1 in (4.29), this yields (4.5)

$$|\alpha_2^2|=\left|\frac{2\tau^2h_0^2\beta_1^3\,(n_2+m_2)}{8\left[\mathtt{r}_1\varphi_{3,y}-\mathtt{r}_2\varphi_{2,y}^2\right]}\right|,$$

$$|\alpha_2^2|\leq\frac{8\tau^2h_0^2\beta_1^3}{8\left[\mathtt{r}_1\varphi_{3,y}-\mathtt{r}_2\varphi_{2,y}^2\right]}.$$

By taking square root on both sides, it gives

$$|\alpha_2| \leq \frac{\tau\,|h_0|\,\beta_1\sqrt{\beta_1}}{\sqrt{\mathtt{r}_1\varphi_{3,y}-\mathtt{r}_2\varphi_{2,y}^2}}.$$

Now, to determine the estimate of the coefficient $|\alpha_3|$, by subtracting (4.20) and (4.22), it provides

$$\frac{2}{\tau}\left[-\left(1+[2]_q-\xi+[2]_q\sigma\right)\alpha_2^2+\left(1+[2]_q-\xi+[2]_q\sigma\right)\alpha_3\right][3]_q\varphi_{3,y}=$$
$$\frac{1}{2}h_1\beta_1(n_1-m_1)+\frac{1}{2}h_0\beta_1(n_2-m_2)+\frac{1}{2}h_0\beta_1\left(\frac{-n_1^2}{2}+\frac{m_1^2}{2}\right)+\frac{1}{4}h_0\beta_2\left(n_1^2-m_1^2\right).$$

By using (4.26), it yields

$$\frac{4}{\tau}\left[-\left(1+[2]_q-\xi+[2]_q\sigma\right)\alpha_2^2+\left(1+[2]_q-\xi+[2]_q\sigma\right)\alpha_3\right][3]_q\varphi_{3,y}=$$
$$2h_1\beta_1n_1+h_0\beta_1(n_2-m_2), \tag{4.30}$$

$$-4\left(1+[2]_q-\xi+[2]_q\sigma\right)[3]_q\varphi_{3,y}\alpha_2^2+4\left(1+[2]_q-\xi+[2]_q\sigma\right)[3]_q\varphi_{3,y}\alpha_3=$$
$$2\tau h_1\beta_1n_1+\tau h_0\beta_1(n_2-m_2),$$

$$\alpha_3=\frac{2\tau h_1\beta_1n_1}{4\left(1+[2]_q-\xi+[2]_q\sigma\right)[3]_q\varphi_{3,y}}+\frac{\tau h_0\beta_1(n_2-m_2)}{4\left(1+[2]_q-\xi+[2]_q\sigma\right)[3]_q\varphi_{3,y}}+$$
$$\frac{4\left(1+[2]_q-\xi+[2]_q\sigma\right)[3]_q\varphi_{3,y}\alpha_2^2}{4\left(1+[2]_q-\xi+[2]_q\sigma\right)[3]_q\varphi_{3,y}}.$$

By using (4.27), this results in

$$\alpha_3=\frac{\tau h_1\beta_1n_1}{2\left(1+[2]_q-\xi+[2]_q\sigma\right)[3]_q\varphi_{3,y}}+\frac{\tau h_0\beta_1(n_2-m_2)}{4\left(1+[2]_q-\xi+[2]_q\sigma\right)[3]_q\varphi_{3,y}}+$$
$$\frac{\tau^2h_0^2\beta_1^2(n_1^2+m_1^2)}{8\left([2]_q\right)^2(2-\xi+\sigma)^2\varphi_{2,y}^2}. \tag{4.31}$$

$$|\alpha_3|=\left|\frac{\tau h_1\beta_1n_1}{2\left(1+[2]_q-\xi+[2]_q\sigma\right)[3]_q\varphi_{3,y}}+\frac{\tau h_0\beta_1(n_2-m_2)}{4\left(1+[2]_q-\xi+[2]_q\sigma\right)[3]_q\varphi_{3,y}}+\right.$$
$$\left.\frac{\tau^2h_0^2\beta_1^2(n_1^2+m_1^2)}{8\left([2]_q\right)^2(2-\xi+\sigma)^2\varphi_{2,y}^2}\right|.$$

Using Lemma 2.13.1, this gives

$$|\alpha_3| \leq \frac{2\tau h_1\beta_1}{2\left(1+[2]_q-\xi+[2]_q\sigma\right)[3]_q\varphi_{3,y}} + \frac{4\tau h_0\beta_1}{4\left(1+[2]_q-\xi+[2]_q\sigma\right)[3]_q\varphi_{3,y}} + \frac{8\tau^2h_0^2\beta_1^2}{8\left([2]_q\right)^2(2-\xi+\sigma)^2\varphi_{2,y}^2},$$

$$|\alpha_3| \leq \frac{\tau(|h_1|+|h_0|)|\beta_1}{(1+q\bar{q})\,\tau h_0\beta_1^2\,(1+\bar{q}-\xi+\bar{q}\sigma)\,\varphi_{3,y}} + \frac{\tau^2h_0^2\beta_1^2}{\bar{q}^2(2-\xi+\sigma)^2\varphi_{2,y}^2},\ \beta_1 > 1.$$

$$|\alpha_3| \leq \frac{\tau(|h_1|+|h_0|)|\beta_1}{\mathtt{r}_1\varphi_{3,y}} + \frac{\tau^2h_0^2\beta_1^2}{\mathtt{r}_3\varphi_{2,y}^2},\ \beta_1 > 1.$$

where,

$$\begin{aligned}
\bar{q} &= 1+q.\\
\mathtt{r}_1 &= (1+q\bar{q})\,\tau h_0\beta_1^2\,(1+\bar{q}-\xi+\bar{q}\sigma),\\
\mathtt{r}_2 &= \bar{q}^2\{\xi\,(2-\xi)\,\tau h_0\beta_1^2+(\beta_2-\beta_1)(2-\xi+\sigma)^2\},\\
\mathtt{r}_3 &= \bar{q}^2(2-\xi+\sigma)^2.
\end{aligned}$$

The Theorem 4.2.1 is complete. □

For $q \to 1^-$, using this value in the above result gives an advanced result that perfectly aligns with the previous findings by Atshan et al. [31], as shown in the given corollary.

Corollary 4.2.1.1. If f provided by (1.1) be a part of the subclass $\mathfrak{R}^{\partial,y}_{\nabla,\gamma,c}(\xi,\sigma,\tau,\Psi)$, then

$$|\alpha_2| \leq \frac{\tau\,|h_0|\,\beta_1\sqrt{\beta_1}}{\sqrt{3\tau h_0\beta_1^2\,(3-\xi+2\sigma)\,\varphi_{3,y}-4\{\xi\,(2-\xi)\,\tau h_0\beta_1^2+(\beta_2-\beta_1)(2-\xi+\sigma)^2\}\varphi_{2,y}^2}},$$

$$|\alpha_3| \leq \frac{\tau(|h_1|+|h_0|)|\beta_1}{3\,(3-\xi+2\sigma)\,\varphi_{3,y}} + \frac{\tau^2h_0^2\beta_1^2}{4(2-\xi+\sigma)^2\varphi_{2,y}^2}, \beta_1 > 1.$$

Theorem 4.2.2. If f provided by (1.1) be a part of the subclass $\mathfrak{K}^{\partial,y}_{\nabla,\gamma,c}(\xi,\sigma,\kappa,\Psi,q)$, then

$$|\alpha_2| \leq \min\left\{\frac{\sigma|h_0|\beta_1}{(q+\kappa+\bar{q}\xi)\,\varphi_{2,y}},\ \sqrt{\frac{\sigma\,|h_0|\,(\beta_1+|\beta_2-\beta_1|)}{\left[(\kappa+q\bar{q}+\bar{q}_o\xi)\,\varphi_{3,y}-q\,(1-\kappa)\,\varphi_{2,y}^2\right]}}\right\}, \tag{4.32}$$

and

$$|\alpha_3| \leq \min\left\{\begin{array}{c}\frac{\sigma(|h_1|\beta_1+|h_0|\beta_1)}{(\kappa+q\bar{q}+\bar{q}_o\xi)\varphi_{3,y}} + \frac{\sigma^2h_0^2\beta_1^2}{(q+\kappa+\bar{q}\xi)^2\varphi_{2,y}^2},\\ \frac{\sigma(|h_1|\beta_1+|h_0|\beta_1)}{(\kappa+q\bar{q}+\bar{q}_o\xi)\varphi_{3,y}} + \frac{\sigma|h_0|(\beta_1+|\beta_2-\beta_1|)}{\left[(\kappa+q\bar{q}+\bar{q}_o\xi)\varphi_{3,y}-q(1-\kappa)\varphi_{2,y}^2\right]}\end{array}\right\}, \qquad \beta_1 > 1. \tag{4.33}$$

where

$$\bar{q} = 1+q,$$
$$\bar{q}_o = 1+2q+2q^2+q^3.$$

Proof. For $f \in \Re^{\partial,y}_{\nabla,\gamma,c}(\xi,\sigma,\kappa,\Psi,q)$, for analytic functions μ, F in V exists such that $\mu, F : V \to V$, hold the given conditions:

$$\frac{1}{\sigma}\left[\left\{(1-\kappa)\frac{\mathrm{s}D_q\left(\mathbb{N}^{\partial,\gamma}_{y,c}f(\mathrm{s})\right)}{\mathbb{N}^{\partial,\gamma}_{y,c}f(\mathrm{s})}+\kappa D_q\left(\mathbb{N}^{\partial,\gamma}_{y,c}f(\mathrm{s})\right)+\xi\,\mathrm{s}D_q^2\left(\mathbb{N}^{\partial,\gamma}_{y,c}f(\mathrm{s})\right)\right\}-1\right]=$$
$$[\mu(\mathrm{s})(\Psi(F(\mathrm{s}))-1], \tag{4.34}$$

and

$$\frac{1}{\sigma}\left[\left\{(1-\kappa)\frac{\omega D_q\left(\mathbb{N}^{\partial,\gamma}_{y,c}g(\omega)\right)}{\mathbb{N}^{\partial,\gamma}_{y,c}g(\omega)}+\kappa D_q\left(\mathbb{N}^{\partial,\gamma}_{y,c}g(\omega)\right)+\xi\,\omega D_q^2\left(\mathbb{N}^{\partial,\gamma}_{y,c}g(\omega)\right)\right\}-1\right]=$$
$$[\mu(\omega)(\Psi(F(\omega))-1]. \tag{4.35}$$

Express the function $n(\mathrm{s})$ and $m(\mathrm{s})$ by (4.9) and (4.10) respectively, this yields

$$\frac{1}{\sigma}\left[\left\{(1-\kappa)\frac{\mathrm{s}D_q\left(\mathbb{N}^{\partial,\gamma}_{y,c}f(\mathrm{s})\right)}{\mathbb{N}^{\partial,\gamma}_{y,c}f(\mathrm{s})}+\kappa D_q\left(\mathbb{N}^{\partial,\gamma}_{y,c}f(\mathrm{s})\right)+\xi\,\mathrm{s}D_q^2\left(\mathbb{N}^{\partial,\gamma}_{y,c}f(\mathrm{s})\right)\right\}-1\right]=$$
$$\left[\mu(\mathrm{s})\left(\Psi\left(\frac{n(\mathrm{s})-1}{n(\mathrm{s})+1}\right)\right)-1\right], \tag{4.36}$$

and

$$\frac{1}{\sigma}\left[\left\{(1-\kappa)\frac{\omega D_q\left(\mathbb{N}^{\partial,\gamma}_{y,c}g(\omega)\right)}{\mathbb{N}^{\partial,\gamma}_{y,c}g(\omega)}+\kappa D_q\left(\mathbb{N}^{\partial,\gamma}_{y,c}g(\omega)\right)+\xi\,\omega D_q^2\left(\mathbb{N}^{\partial,\gamma}_{y,c}g(\omega)\right)\right\}-1\right]=$$
$$\left[\mu(\omega)\left(\Psi\left(\frac{m(\omega)-1}{m(\omega)+1}\right)\right)-1\right]. \tag{4.37}$$

Since (4.15) and (4.16) are

$$\mu(\mathrm{s})\ \left(\Psi\left(\frac{n(\mathrm{s})-1}{n(\mathrm{s})+1}\right)\right)-1=$$
$$\frac{1}{2}h_0\beta_1 n_1\mathrm{s}+\left\{\frac{1}{2}h_1\beta_1 n_1+\frac{1}{2}h_0\beta_1\left(n_2-\frac{n_1^2}{2}\right)+\frac{1}{4}h_0\beta_2 n_1^2\right\}\mathrm{s}^2+\ldots,$$

and

$$\mu(\omega)\ \left(\Psi\left(\frac{m(\omega)-1}{m(\omega)+1}\right)\right)-1=$$

$$\frac{1}{2}h_0\beta_1 m_1\omega+\left\{\frac{1}{2}h_1\beta_1 m_1+\frac{1}{2}h_0\beta_1\left(m_2-\frac{m_1^2}{2}\right)+\frac{1}{4}h_0\beta_2 m_1^2\right\}\omega^2+\dots .$$

Since,

$$\begin{aligned}\mathbb{N}_{y,c}^{\partial;\gamma}f(\mathrm{s}) &=\mathrm{s}+\varphi_{2,y}\alpha_2\mathrm{s}^2+\varphi_{3,y}\alpha_3\mathrm{s}^3+\dots,\\ D_q\left(\mathbb{N}_{y,c}^{\partial;\gamma}f(\mathrm{s})\right) &=1+[2]_q\varphi_{2,y}\alpha_2\mathrm{s}+[3]_q\varphi_{3,y}\alpha_3\mathrm{s}^2+\dots,\\ D_q^2\left(\mathbb{N}_{y,c}^{\partial;\gamma}f(\mathrm{s})\right) &=[2]_q\varphi_{2,y}\alpha_2+[2]_q[3]_q\varphi_{3,y}\alpha_3\mathrm{s}+\dots,\end{aligned}$$

(4.36) implies that

$$\frac{1}{\sigma}\left[\left\{(1-\kappa)\frac{\mathrm{s}D_q\left(\mathbb{N}_{y,c}^{\partial;\gamma}f(\mathrm{s})\right)}{\mathbb{N}_{y,c}^{\partial;\gamma}f(\mathrm{s})}+\kappa D_q\left(\mathbb{N}_{y,c}^{\partial;\gamma}f(\mathrm{s})\right)+\xi\mathrm{s}D_q^2\left(\mathbb{N}_{y,c}^{\partial;\gamma}f(\mathrm{s})\right)\right\}-1\right]=$$
$$\frac{1}{\sigma}\left[\left\{(1-\kappa)\frac{\mathrm{s}+[2]_q\varphi_{2,y}\alpha_2\mathrm{s}^2+[3]_q\varphi_{3,y}\alpha_3\mathrm{s}^3+\dots}{\mathrm{s}+\ \varphi_{2,y}\alpha_2\mathrm{s}^2+\varphi_{3,y}\alpha_3\mathrm{s}^3+\dots}+\right.\right.$$
$$\left(\kappa+[2]_q\kappa\varphi_{2,y}\alpha_2\mathrm{s}+[3]_q\kappa\varphi_{3,y}\alpha_3\mathrm{s}^2+\dots\right)$$
$$\left.\left.+\left([2]_q\xi\varphi_{2,y}\alpha_2\mathrm{s}+[2]_q[3]_q\xi\varphi_{3,y}\alpha_3\mathrm{s}^2+\dots\right)\right\}-1\right],$$

$$\frac{1}{\sigma}\left[\left\{(1-\kappa)\frac{\mathrm{s}D_q\left(\mathbb{N}_{y,c}^{\partial;\gamma}f(\mathrm{s})\right)}{\mathbb{N}_{y,c}^{\partial;\gamma}f(\mathrm{s})}+\kappa D_q\left(\mathbb{N}_{y,c}^{\partial;\gamma}f(\mathrm{s})\right)+\xi\mathrm{s}D_q^2\left(\mathbb{N}_{y,c}^{\partial;\gamma}f(\mathrm{s})\right)\right\}-1\right]=$$
$$\frac{1}{\sigma}\left[\left\{(1-\kappa)\left(1+\left([2]_q-1\right)\varphi_{2,y}\alpha_2\mathrm{s}+\left([3]_q-1\right)\varphi_{3,y}\alpha_3\mathrm{s}^2-\left([2]_q-1\right)\varphi_{2,y}^2\alpha_2^2\mathrm{s}^2+\dots\right)\right.\right.$$
$$\left.\left.+\kappa+[2]_q\kappa\varphi_{2,y}\alpha_2\mathrm{s}+[3]_q\kappa\varphi_{3,y}\alpha_3\mathrm{s}^2+[2]_q\xi\varphi_{2,y}\alpha_2\mathrm{s}+[2]_q[3]_q\xi\varphi_{3,y}\alpha_3\mathrm{s}^2+\dots\right\}-1\right],$$

$$\frac{1}{\sigma}\left[\left\{(1-\kappa)\frac{\mathrm{s}D_q\left(\mathbb{N}_{y,c}^{\partial;\gamma}f(\mathrm{s})\right)}{\mathbb{N}_{y,c}^{\partial;\gamma}f(\mathrm{s})}+\kappa D_q\left(\mathbb{N}_{y,c}^{\partial;\gamma}f(\mathrm{s})\right)+\xi\mathrm{s}D_q^2\left(\mathbb{N}_{y,c}^{\partial;\gamma}f(\mathrm{s})\right)\right\}-1\right]=$$
$$\frac{1}{\sigma}\left[1+\left([2]_q-1\right)\varphi_{2,y}\alpha_2\mathrm{s}+\left([3]_q-1\right)\varphi_{3,y}\alpha_3\mathrm{s}^2-\left([2]_q-1\right)\varphi_{2,y}^2\alpha_2^2\mathrm{s}^2-\kappa\right.$$
$$-\left([2]_q-1\right)\kappa\varphi_{2,y}\alpha_2\mathrm{s}-\left([3]_q-1\right)\kappa\varphi_{3,y}\alpha_3\mathrm{s}^2+\left([2]_q-1\right)\kappa\varphi_{2,y}^2\alpha_2^2\mathrm{s}^2+\kappa+$$
$$\left.[2]_q\kappa\varphi_{2,y}\alpha_2\mathrm{s}+[3]_q\kappa\varphi_{3,y}\alpha_3\mathrm{s}^2+[2]_q\xi\varphi_{2,y}\alpha_2\mathrm{s}+[2]_q[3]_q\xi\varphi_{3,y}\alpha_3\mathrm{s}^2-1\right],$$

$$\frac{1}{\sigma}\left[\left\{(1-\kappa)\frac{\mathrm{s}D_q\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)}{\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})}+\kappa D_q\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)+\xi\,\mathrm{s}D_q^2\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)\right\}-1\right]=$$
$$\frac{1}{\sigma}\left[\left(\left([2]_q-1\right)-\left([2]_q-1\right)\kappa+[2]_q\kappa+[2]_q\xi\right)\varphi_{2,y}\alpha_2\mathrm{s}+\right.$$
$$\left(\left([3]_q-1\right)-\left([3]_q-1\right)\kappa+[3]_q\kappa+[2]_q[3]_q\xi\right)\varphi_{3,y}\alpha_3\mathrm{s}^2+$$
$$\left.\left(-\left([2]_q-1\right)+\left([2]_q-1\right)\kappa\right)\varphi_{2,y}^2\alpha_2^2\mathrm{s}^2\right],$$
$$\frac{1}{\sigma}\left[\left\{(1-\kappa)\frac{\mathrm{s}D_q\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)}{\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})}+\kappa D_q\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)+\xi\,\mathrm{s}D_q^2\left(\mathbb{N}_{y,c}^{\partial,\gamma}f(\mathrm{s})\right)\right\}-1\right]=$$
$$\frac{1}{\sigma}\left[\left(\kappa+[2]_q\xi+[2]_q-1\right)\varphi_{2,y}\alpha_2\mathrm{s}+\left(\kappa+[3]_q+[2]_q[3]_q\xi-1\right)\varphi_{3,y}\alpha_3\mathrm{s}^2\right.$$
$$\left.+\left(-\left([2]_q-1\right)(1-\kappa)\right)\varphi_{2,y}^2\alpha_2^2\mathrm{s}^2\right]. \tag{4.38}$$

Since,

$$\mathbb{N}_{y,c}^{\partial,\gamma}\,g(\omega)=\omega-\varphi_{2,y}\alpha_2\omega^2+\varphi_{3,y}(2\alpha_2^2-\alpha_3)\omega^3-\ldots,$$
$$D_q\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)=1-[2]_q\varphi_{2,y}\alpha_2\omega+[3]_q\left(2\alpha_2^2-\alpha_3\right)\varphi_{3,y}\omega^2-\ldots,$$
$$D_q^2\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)=-[2]_q\varphi_{2,y}\alpha_2+[2]_q[3]_q\left(2\alpha_2^2-\alpha_3\right)\varphi_{3,y}\omega-\ldots,$$

(4.37) implies that

$$\frac{1}{\sigma}\left[\left\{(1-\kappa)\frac{\omega D_q\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)}{\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)}+\kappa D_q\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)+\xi\,\omega D_q^2\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)\right\}-1\right]=$$
$$\frac{1}{\sigma}\left[\left\{(1-\kappa)\frac{\left(\omega-[2]_q\varphi_{2,y}\alpha_2\omega^2+[3]_q\left(2\alpha_2^2-\alpha_3\right)\varphi_{3,y}\omega^3-\ldots\right)}{\omega-\ \varphi_{2,y}\alpha_2\omega^2+\varphi_{3,y}(2\alpha_2^2-\alpha_3)\omega^3-\ldots}\right.\right.$$
$$+\kappa\left(1-[2]_q\varphi_{2,y}\alpha_2\omega+[3]_q\left(2\alpha_2^2-\alpha_3\right)\varphi_{3,y}\omega^2-\ldots\right)+$$
$$\left.\left.\xi\,\omega\left(-[2]_q\varphi_{2,y}\alpha_2+[2]_q[3]_q\left(2\alpha_2^2-\alpha_3\right)\varphi_{3,y}\omega\right)\right\}-1\right],$$

$$\frac{1}{\sigma}\left[\left\{(1-\kappa)\frac{\omega D_q\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)}{\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)}+\kappa D_q\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)+\xi\,\omega D_q^2\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)\right\}-1\right]=$$
$$\frac{1}{\sigma}\left[\left\{(1-\kappa)\left(1-\left([2]_q-1\right)\varphi_{2,y}\alpha_2\omega+\left([3]_q-1\right)\varphi_{3,y}\left(2\alpha_2^2-\alpha_3\right)\omega^2\right.\right.\right.$$
$$\left.-\left([2]_q-1\right)\varphi_{2,y}^2\alpha_2^2\omega^2\right)+\kappa-[2]_q\kappa\varphi_{2,y}\alpha_2\omega+[3]_q\kappa\left(2\alpha_2^2-\alpha_3\right)\varphi_{3,y}\omega^2$$
$$\left.\left.-[2]_q\xi\varphi_{2,y}\alpha_2\omega+[2]_q[3]_q\xi\left(2\alpha_2^2-\alpha_3\right)\varphi_{3,y}\omega^2\right\}-1\right],$$

$$\frac{1}{\sigma}\left[\left\{(1-\kappa)\frac{\omega D_q\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)}{\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)}+\kappa D_q\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)+\xi\omega D_q^2\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)\right\}-1\right]=$$
$$\frac{1}{\sigma}\left[\left\{1-\left([2]_q-1\right)\varphi_{2,y}\alpha_2\omega+\left([3]_q-1\right)\varphi_{3,y}\left(2\alpha_2^2-\alpha_3\right)\omega^2\right.\right.$$
$$-\left([2]_q-1\right)\varphi_{2,y}^2\alpha_2^2\omega^2-\kappa+\kappa\left([2]_q-1\right)\varphi_{2,y}\alpha_2\omega-$$
$$\kappa\left([3]_q-1\right)\varphi_{3,y}\left(2\alpha_2^2-\alpha_3\right)\omega^2+\kappa\left([2]_q-1\right)\varphi_{2,y}^2\alpha_2^2\omega^2+$$
$$\kappa[2]_q\kappa\varphi_{2,y}\alpha_2\omega+[3]_q\kappa\left(2\alpha_2^2-\alpha_3\right)\varphi_{3,y}\omega^2$$
$$\left.\left.-[2]_q\xi\varphi_{2,y}\alpha_2\omega+[2]_q[3]_q\xi\left(2\alpha_2^2-\alpha_3\right)\varphi_{3,y}\omega^2\right\}-1\right],$$

$$\frac{1}{\sigma}\left[\left\{(1-\kappa)\frac{\omega D_q\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)}{\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)}+\kappa D_q\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)+\xi\omega D_q^2\left(\mathbb{N}_{y,c}^{\partial,\gamma}g(\omega)\right)\right\}-1\right]=$$
$$\frac{1}{\sigma}\left[-\left([2]_q-1+\kappa+[2]_q\xi\right)\varphi_{2,y}\alpha_2\omega+\left(-1+[3]_q+\kappa+[2]_q[3]_q\xi\right)\left(2\alpha_2^2-\alpha_3\right)\varphi_{3,y}\omega^2\right.$$
$$\left.-\left([2]_q-1\right)(1-\kappa)\alpha^2{}_2\varphi_{2,y}^2\omega^2\right]. \tag{4.39}$$

Equating the coefficients of (4.38) with (4.15), this leads to

$$\frac{1}{\sigma}\left[\left(\kappa+[2]_q\xi+[2]_q-1\right)\varphi_{2,y}\alpha_2\right]=\frac{1}{2}h_0\beta_1n_1, \tag{4.40}$$

$$\frac{1}{\sigma}\left[+\left(\kappa+[3]_q+[2]_q[3]_q\xi-1\right)\varphi_{3,y}\alpha_3+\left(-\left([2]_q-1\right)(1-\kappa)\right)\varphi_{2,y}^2\alpha_2^2\right]=$$
$$\frac{1}{2}h_1\beta_1n_1+\frac{1}{2}h_0\beta_1\left(n_2-\frac{n_1^2}{2}\right)+\frac{1}{4}h_0\beta_2n_1^2. \tag{4.41}$$

Now, equating the coefficients of (4.39) with (4.16), this provides

$$\frac{1}{\sigma}\left[-\left(\kappa+[2]_q\xi+[2]_q-1\right)\varphi_{2,y}\alpha_2\right]=\frac{1}{2}h_0\beta_1m_1, \tag{4.42}$$

$$\frac{1}{\sigma}\left[\left(-1+[3]_q+\kappa+[2]_q[3]_q\xi\right)\left(2\alpha_2^2-\alpha_3\right)\varphi_{3,y}-\left([2]_q-1\right)(1-\kappa)\alpha^2{}_2\varphi_{2,y}^2\right]=$$
$$\frac{1}{2}h_1\beta_1m_1+\frac{1}{2}h_0\beta_1\left(m_2-\frac{m_1^2}{2}\right)+\frac{1}{4}h_0\beta_2m_1^2. \tag{4.43}$$

From (4.40) and (4.42) this results in

$$n_1=-m_1. \tag{4.44}$$

Taking square of (4.40) and (4.42) and add them, this gives

$$2\alpha_2^2=\frac{\sigma^2h_0^2\beta_1^2n_1^2}{4\left([2]_q-1+\kappa+[2]_q\xi\right)^2\varphi_{2,y}^2}+\frac{\sigma^2h_0^2\beta_1^2m_1^2}{4\left([2]_q-1+\kappa+[2]_q\xi\right)^2\varphi_{2,y}^2},$$
$$2\alpha_2^2=\frac{\sigma^2h_0^2\beta_1^2(n_1^2+m_1^2)}{4\left([2]_q-1+\kappa+[2]_q\xi\right)^2\varphi_{2,y}^2},$$

$$8\Big([2]_q - 1 + \kappa + [2]_q \xi\Big)^2 \varphi_{2,y}^2 \alpha^2{}_2 = \sigma^2 h_0^2 \beta_1^2 \left(n_1^2 + m_1^2\right). \tag{4.45}$$

Adding (4.41) and (4.43), this gives

$$\frac{1}{\sigma}\Big[2\Big(\kappa + [3]_q + [2]_q[3]_q\xi - 1\Big)\varphi_{3,y}\alpha_2^2 + \Big(-2\Big([2]_q - 1\Big)(1-\kappa)\Big)\varphi_{2,y}^2\alpha_2^2\Big] =$$
$$\frac{1}{2}h_1\beta_1(n_1 + m_1) + \frac{1}{2}h_0\beta_1\left(n_2 + m_2 - \frac{n_1^2}{2} - \frac{m_1^2}{2}\right) + \frac{1}{4}h_0\beta_2(n_1^2 + m_1^2).$$

By using (4.44), this yields

$$\frac{2}{\sigma}\Big[\Big(\kappa + [3]_q + [2]_q[3]_q\xi - 1\Big)\varphi_{3,y}\alpha_2^2 + \Big(-\Big([2]_q - 1\Big)(1-\kappa)\Big)\varphi_{2,y}^2\alpha_2^2\Big] =$$
$$\frac{1}{2}h_1\beta_1(-m_1 + m_1) + \frac{1}{2}h_0\beta_1(n_2 + m_2) + \frac{1}{4}h_0(\beta_2 - \beta_1)(n_1^2 + m_1^2),$$

$$\frac{8}{\sigma}\Big[\Big(\kappa + [3]_q + [2]_q[3]_q\xi - 1\Big)\varphi_{3,y}\alpha_2^2 + \Big(-\Big([2]_q - 1\Big)(1-\kappa)\Big)\varphi_{2,y}^2\alpha_2^2\Big] =$$
$$2h_0\beta_1(n_2 + m_2) + h_0(\beta_2 - \beta_1)(n_1^2 + m_1^2). \tag{4.46}$$

which gives

$$\alpha_2^2 = \frac{2\sigma h_0\beta_1(n_2 + m_2) + \sigma h_0(\beta_2 - \beta_1)(n_1^2 + m_1^2)}{8\Big[\Big(\kappa + [3]_q + [2]_q[3]_q\xi - 1\Big)\varphi_{3,y} - \Big([2]_q - 1\Big)(1-\kappa)\varphi_{2,y}^2\Big]}. \tag{4.47}$$

$$\alpha_2^2 = \frac{\sigma^2 h_0^2\beta_1^2(n_1^2 + m_1^2)}{8\Big([2]_q - 1 + \kappa + [2]_q\xi\Big)^2\varphi_{2,y}^2}. \tag{4.48}$$

Using Lemma 2.13.1 for the coefficients, this provides

$$|\alpha_2^2| \le \frac{8\sigma^2 h_0^2\beta_1^2}{8\Big([2]_q - 1 + \kappa + [2]_q\xi\Big)^2\varphi_{2,y}^2},$$

Taking square root on both sides, this gives

$$|\alpha_2| \le \frac{\sigma h_0\beta_1}{\Big([2]_q - 1 + \kappa + [2]_q\xi\Big)\varphi_{2,y}},$$

and

$$|\alpha_2^2| \le \frac{8\sigma h_0\beta_1 + 8\sigma h_0(\beta_2 - \beta_1)}{8\Big[\Big(\kappa + [3]_q + [2]_q[3]_q\xi - 1\Big)\varphi_{3,y} - \Big([2]_q - 1\Big)(1-\kappa)\varphi_{2,y}^2\Big]}. \tag{4.49}$$

By taking square root on both sides, this provides

$$|\alpha_2| \le \sqrt{\frac{\sigma|h_0|(\beta_1 + |\beta_2 - \beta_1|)}{\Big[\Big(\kappa + [3]_q + [2]_q[3]_q\xi - 1\Big)\varphi_{3,y} - \Big([2]_q - 1\Big)(1-\kappa)\varphi_{2,y}^2\Big]}},$$

which provide the required estimate on $|\alpha_2|$.

Now, to determine the coefficient estimate of α_3, by subtracting (4.41) and (4.43), this yields

$$\frac{1}{\sigma}\left[\left(-1+[3]_q+\kappa+[2]_q[3]_q\xi\right)\left(-2\alpha_2^2+2\alpha_3\right)\varphi_{3,y}\right]=$$
$$\frac{1}{2}h_1\beta_1(n_1-m_1)+\frac{1}{2}h_0\beta_1\left(n_2-m_2-\frac{n_1^2}{2}+\frac{m_1^2}{2}\right)+\frac{1}{4}h_0\beta_2(n_1^2-m_1^2).$$

By using (4.44), this leads us to

$$\frac{2}{\sigma}\left[\left(-1+[3]_q+\kappa+[2]_q[3]_q\xi\right)\left(-\alpha_2^2+\alpha_3\right)\varphi_{3,y}\right]=\frac{1}{2}h_1\beta_1(2n_1)+\frac{1}{2}h_0\beta_1\left(n_2-m_2\right),$$

$$\frac{4}{\sigma}\left[\left(-1+[3]_q+\kappa+[2]_q[3]_q\xi\right)\left(-\alpha_2^2+\alpha_3\right)\varphi_{3,y}\right]=h_1\beta_1(2n_1)+h_0\beta_1\left(n_2-m_2\right). \qquad (4.50)$$

By substituting the value of α_2^2 from (4.48) in (4.50), this yields us

$$4\left[\left(-1+[3]_q+\kappa+[2]_q[3]_q\xi\right)\varphi_{3,y}\alpha_3-\left(-1+[3]_q+\kappa+[2]_q[3]_q\xi\right)\varphi_{3,y}\alpha_2^2\right]=$$
$$2\sigma h_1\beta_1n_1+\sigma h_0\beta_1\left(n_2-m_2\right),$$

$$\left[4\left(-1+[3]_q+\kappa+[2]_q[3]_q\xi\right)\varphi_{3,y}\alpha_3\right]=$$
$$2\sigma h_1\beta_1n_1+\sigma h_0\beta_1\left(n_2-m_2\right)+$$
$$\left(-1+[3]_q+\kappa+[2]_q[3]_q\xi\right)\varphi_{3,y}\frac{4\sigma^2h_0^2\beta_1^2(n_1^2+m_1^2)}{8\left([2]_q-1+\kappa+[2]_q\xi\right)^2\varphi_{2,y}^2},$$

$$\alpha_3=\frac{2\sigma h_1\beta_1n_1}{4\left(-1+[3]_q+\kappa+[2]_q[3]_q\xi\right)\varphi_{3,y}}+$$
$$\frac{\sigma h_0\beta_1\left(n_2-m_2\right)}{4\left(-1+[3]_q+\kappa+[2]_q[3]_q\xi\right)\varphi_{3,y}}+$$
$$\frac{4\sigma^2h_0^2\beta_1^2\varphi_{3,y}\left(-1+[3]_q+\kappa+[2]_q[3]_q\xi\right)\left(n_1^2+m_1^2\right)}{32\left(-1+[3]_q+\kappa+[2]_q[3]_q\xi\right)\varphi_{3,y}\left([2]_q-1+\kappa+[2]_q\xi\right)^2\varphi_{2,y}^2},$$

$$|\alpha_3|\leq\frac{2\sigma h_1\beta_1n_1+\sigma h_0\beta_1\left(n_2-m_2\right)}{4\left(-1+[3]_q+\kappa+[2]_q[3]_q\xi\right)\varphi_{3,y}}+\frac{\sigma^2h_0^2\beta_1^2\left(n_1^2+m_1^2\right)}{8\left([2]_q-1+\kappa+[2]_q\xi\right)^2\varphi_{2,y}^2}. \qquad (4.51)$$

By substituting the value of α_2^2 from (4.47) in (4.50), this gives us

$$\alpha_3=\frac{2\sigma h_1\beta_1n_1+\sigma h_0\beta_1\left(n_2-m_2\right)}{4\left(-1+[3]_q+\kappa+[2]_q[3]_q\xi\right)\varphi_{3,y}}+$$
$$\frac{\left(2\sigma h_0\beta_1\left(n_2+m_2\right)+\sigma h_0(\beta_2-\beta_1)(n_1^2+m_1^2)\right)}{8\left[\left(-1+[3]_q+\kappa+[2]_q[3]_q\xi\right)\varphi_{3,y}-\left([2]_q-1\right)(1-\kappa)\varphi_{2,y}^2\right]},$$

$$|\alpha_3| \leq \frac{2\sigma h_1\beta_1 n_1+\sigma h_0\beta_1\left(n_2-m_2\right)}{4\left(-1+[3]_q+\kappa+[2]_q[3]_q\xi\right)\varphi_{3,y}}+$$
$$\frac{\left(2\sigma h_0\beta_1\left(n_2+m_2\right)+\sigma h_0(\beta_2-\beta_1)(n_1^2+m_1^2)\right)}{8\left[\left(-1+[3]_q+\kappa+[2]_q[3]_q\xi\right)\varphi_{3,y}-\left([2]_q-1\right)(1-\kappa)\varphi_{2,y}^2\right]}. \tag{4.52}$$

By Using Lemma 2.13.1 in (4.51) and (4.52), this leads to

$$|\alpha_3| \leq \frac{\sigma(|h_1|\beta_1+|h_0|\beta_1)}{\left(-1+[3]_q+\kappa+[2]_q[3]_q\xi\right)\varphi_{3,y}}+\frac{\sigma^2 h_0^2\beta_1^2}{\left([2]_q-1+\kappa+[2]_q\xi\right)^2\varphi_{2,y}^2}, \tag{4.53}$$

$$|\alpha_3| \leq \frac{\sigma(|h_1|\beta_1+|h_0|\beta_1)}{\left(-1+[3]_q+\kappa+[2]_q[3]_q\xi\right)\varphi_{3,y}}+$$
$$\frac{\sigma h_0(\beta_1+|\beta_2-\beta_1|)}{\left[\left(-1+[3]_q+\kappa+[2]_q[3]_q\xi\right)\varphi_{3,y}-\left([2]_q-1\right)(1-\kappa)\varphi_{2,y}^2\right]}. \tag{4.54}$$

which gives the required estimate on $|\alpha_3|$, as given in (4.33).

The Theorem 4.2.2 is complete. □

For $q \to 1^-$, using this value in the above result gives an advanced result that perfectly aligns with the previous findings by Atshan et al. [31], as given in the corollary below:

Corollary 4.2.2.1. If f provided by (1.1) be a part of the subclass $\mathfrak{K}_{\nabla,\gamma,c}^{\partial,y}(\xi,\sigma,\kappa,\Psi)$, then

$$|\alpha_2| \leq \min\left\{\frac{\sigma|h_0|\beta_1}{(1+\kappa+2\xi)\varphi_{2,y}}\ ,\ \sqrt{\frac{\sigma|h_0|(\beta_1+(|\beta_2-\beta_1|))}{\left[(2+\kappa+6\xi)\varphi_{3,y}-(1-\kappa)\varphi_{2,y}^2\right]}}\right\},$$

and

$$|\alpha_3| \leq \min\left\{\frac{\sigma(h_1\beta_1+h_0\beta_1)}{(2+\kappa+6\xi)\varphi_{3,y}}+\frac{\sigma|h_0|(\beta_1+|\beta_2-\beta_1|)}{\left[(2+\kappa+6\xi)\varphi_{3,y}-(1-\kappa)\varphi_{2,y}^2\right]},\right.$$
$$\left.\frac{\sigma(h_1\beta_1+h_0\beta_1)}{(2+\kappa+6\xi)\varphi_{3,y}}+\frac{\sigma^2h_0^2\beta_1^2}{(1+\kappa+2\xi)^2\varphi_{2,y}^2}\right\},\ \ \beta_1>1.$$

CHAPTER 5

ON A CERTAIN SUBCLASS OF BI-STARLIKE FUNCTION DEFINE BY DIFFERENTIAL OPERATOR

5.1 Overview

This chapter is about to examine the subclass of bi-starlike functions that involve the Salagean operator. The aim is to find the upper bound of initial coefficients, the Fekete-Szego Inequality, and the second Hankel determinant. The results of this class are the refinement of previous findings of Orhan et al. [82].

Definition 5.1.1. *A function* $f(\mathtt{s})$ *having form* (1.1) *is considered to be in the class* $f \in \mathbb{S}^{*}_{\nabla}(\gamma, \Theta, u, v)$*, if the conditions given below hold:*
For $f \in \nabla$

$$Re\left\{e^{i\Theta}\left[\frac{D^{u}f(\mathtt{s})}{D^{v}f(\mathtt{s})}\right]\right\} > \gamma,$$

$$(\mathtt{s} \in V; u > v,\ \ 0 \le \gamma < 1,\ \ |\Theta| < \pi\ \ and\ \ \ cos\Theta > \gamma),$$

$$Re\left\{e^{i\Theta}\left[\frac{D^{u}g(\omega)}{D^{v}g(\omega)}\right]\right\} > \gamma,$$

$$(\omega \in V; u > v,\ \ 0 \le \gamma < 1,\ \ |\Theta| < \pi\ \ and\ \ \ cos\Theta > \gamma),$$

where the function g has a series form of

$$g(\omega) = \omega - \alpha_2\omega^2 + \left(2\alpha_2^2 - \alpha_3\right)\omega^3 - \left(5\alpha_2^3 - 5\alpha_2\alpha_3 + \alpha_4\right)\omega^4 + \ldots$$

5.2 Main Result

Theorem 5.2.1. If $f(\mathrm{s})$ having form (1.1) be a part of the class $f \in \mathbb{S}^*_\nabla(\gamma, \Theta, u, v)$ for $u > v+1,\ 0 \leq \gamma < 1,\ |\Theta| < \pi$ and $cos\Theta > \gamma$. Then

$$|\alpha_2| \leq \frac{2(cos\Theta - \gamma)}{2^u - 2^v}, \tag{5.1}$$

$$|\alpha_3| \leq \frac{4(cos\Theta - \gamma)^2}{(2^u - 2^v)^2} + \frac{2\,(cos\Theta - \gamma)}{3^u - 3^v}, \tag{5.2}$$

$$\begin{aligned} |\alpha_4| \leq & \frac{8(cos\Theta - \gamma)^3 \left[(2^u - 2^v)\left(3^v - 2^{2v}\right) + 2^v\,(3^u - 3^v)\right]}{(2^u + 2^v)\,(2^u - 2^v)^4} + \\ & \frac{10(cos\Theta - \gamma)^2}{(2^u - 2^v)\,(3^u - 3^v)} + \frac{2\,(cos\Theta - \gamma)}{4^u - 4^v}. \end{aligned} \tag{5.3}$$

For $\lambda \in C$, $\quad \left|\alpha_3 - \lambda \alpha_2^2\right| \leq \begin{cases} \frac{cos\Theta - \gamma}{3^u - 3^v}, & 0 \leq |T(\lambda, u, v)| \leq \frac{1}{2(3^u - 3^v)}, \\ 2\,|T(\lambda, u, v)|\,[cos\Theta - \gamma], & |T(\lambda, u, v)| \geq \frac{1}{2(3^u - 3^v)}, \end{cases}$

where

$$T\,(\lambda, u, v) = \frac{1 - \lambda}{2\left[(3^u - 3^v) - (2^{u+v} - 2^{2v})\right]}.$$

Proof. If $f \in \mathbb{S}^*_\nabla(\gamma, \Theta, u, v)$, Then

$$\frac{O^u f(\mathrm{s})}{O^v f(\mathrm{s})} = 1 + \sum_{t=1}^{\infty} h_t \mathrm{s}^t,$$

and

$$\frac{O^u g(\omega)}{O^v g(\omega)} = 1 + \sum_{t=1}^{\infty} h_t \omega^t.$$

Hence

$$e^{i\Theta}\left[\frac{O^u f(\mathrm{s})}{O^v f(\mathrm{s})}\right] - \gamma = e^{i\Theta}\left(1 + \sum_{t=1}^{\infty} h_t \mathrm{s}^t\right) - \gamma,$$

and

$$e^{i\Theta}\left[\frac{O^u g(\omega)}{O^v g(\omega)}\right] - \gamma = e^{i\Theta}\left(1 + \sum_{t=1}^{\infty} h_t \omega^t\right) - \gamma.$$

Now, by simplifying the equation, gives

$$e^{i\Theta}\left[\frac{O^u f(\mathrm{s})}{O^v f(\mathrm{s})}\right] - \gamma = e^{i\Theta} + e^{i\Theta}\left(\sum_{t=1}^{\infty} h_t \mathrm{s}^t\right) - \gamma,$$

and

$$e^{i\Theta}\left[\frac{O^u g(\omega)}{O^v g(\omega)}\right] - \gamma = e^{i\Theta} + e^{i\Theta}\left(\sum_{t=1}^{\infty} h_t \omega^t\right) - \gamma.$$

$$\left(\because e^{i\Theta} = cos\Theta + isin\Theta\right)$$

$$e^{i\Theta}\left[\frac{O^u f(\mathrm{s})}{O^v f(\mathrm{s})}\right] - \gamma = cos\Theta + isin\Theta + e^{i\Theta}\left(\sum_{t=1}^{\infty} h_t \mathrm{s}^t\right) - \gamma,$$

and

$$e^{i\Theta}\left[\frac{O^u g(\omega)}{O^v g(\omega)}\right] - \gamma = cos\Theta + isin\Theta + e^{i\Theta}\left(\sum_{t=1}^{\infty} h_t \omega^t\right) - \gamma.$$

$$e^{i\Theta}\left[\frac{O^u f(\mathrm{s})}{O^v f(\mathrm{s})}\right] - \gamma - isin\Theta = cos\Theta - \gamma + e^{i\Theta}\left(\sum_{t=1}^{\infty} h_t \mathrm{s}^t\right),$$

and

$$e^{i\Theta}\left[\frac{O^u g(\omega)}{O^v g(\omega)}\right] - \gamma - isin\Theta = cos\Theta - \gamma + e^{i\Theta}\left(\sum_{t=1}^{\infty} h_t \omega^t\right),$$

which gives

$$e^{i\Theta}\left[\frac{O^u f(\mathrm{s})}{O^v f(\mathrm{s})}\right] - \gamma - isin\Theta = cos\Theta - \gamma\left[1 + \frac{e^{i\Theta}\left(\sum_{t=1}^{\infty} h_t \mathrm{s}^t\right)}{cos\Theta - \gamma}\right],$$

and

$$e^{i\Theta}\left[\frac{O^u g(\omega)}{O^v g(\omega)}\right] - \gamma - isin\Theta = cos\Theta - \gamma\left[1 + \frac{e^{i\Theta}\left(\sum_{t=1}^{\infty} h_t \omega^t\right)}{cos\Theta - \gamma}\right].$$

$$\begin{aligned} \frac{e^{i\Theta}\left[\frac{O^u f(\mathrm{s})}{O^v f(\mathrm{s})}\right] - \gamma - isin\Theta}{cos\Theta - \gamma} &= 1 + \frac{e^{i\Theta}\left(\sum_{t=1}^{\infty} h_t \mathrm{s}^t\right)}{cos\Theta - \gamma}, \\ \frac{e^{i\Theta}\left[\frac{O^u g(\omega)}{O^v g(\omega)}\right] - \gamma - isin\Theta}{cos\Theta - \gamma} &= 1 + \frac{e^{i\Theta}\left(\sum_{t=1}^{\infty} h_t \omega^t\right)}{cos\Theta - \gamma}. \end{aligned} \tag{5.4}$$

From (5.4) and Lemma 2.13.2, this leads to

$$\begin{aligned} \frac{e^{i\Theta}\left[\frac{O^u f(\mathrm{s})}{O^v f(\mathrm{s})}\right] - \gamma - isin\Theta}{cos\Theta - \gamma} &= p(\mathrm{s}), \\ \frac{e^{i\Theta}\left[\frac{O^u g(\omega)}{O^v g(\omega)}\right] - \gamma - isin\Theta}{cos\Theta - \gamma} &= q(\omega). \end{aligned} \tag{5.5}$$

$$\begin{aligned} O^u f(\mathrm{s}) &= \mathrm{s} + \sum_{t=2}^{\infty} t^u \alpha_t \mathrm{s}^t, \\ O^u f(\mathrm{s}) &= \mathrm{s} + 2^u \alpha_2 \mathrm{s}^2 + 3^u \alpha_3 \mathrm{s}^3 + 4^u \alpha_4 \mathrm{s}^4 + \ldots, \end{aligned}$$

$$\begin{aligned} O^v f(\mathrm{s}) &= \mathrm{s} + \sum_{t=2}^{\infty} t^v \alpha_t \mathrm{s}^t, \\ O^v f(\mathrm{s}) &= \mathrm{s} + 2^v \alpha_2 \mathrm{s}^2 + 3^v \alpha_3 \mathrm{s}^3 + 4^v \alpha_4 \mathrm{s}^4 + \ldots, \end{aligned}$$

$$\frac{O^u f(\mathrm{s})}{O^v f(\mathrm{s})} = \frac{\mathrm{s}+2^u\alpha_2\mathrm{s}^2+3^u\alpha_3\mathrm{s}^3+4^u\alpha_4\mathrm{s}^4+\dots}{\mathrm{s}+2^v\alpha_2\mathrm{s}^2+3^v\alpha_3\mathrm{s}^3+4^v\alpha_4\mathrm{s}^4+\dots},$$

$$\begin{aligned}\frac{O^u f(\mathrm{s})}{O^v f(\mathrm{s})} =& 1+(2^u-2^v)\alpha_2\mathrm{s}+\left[(3^u-3^v)\alpha_3-\left(2^{u+v}-2^{2v}\right)\alpha_2^2\right]\mathrm{s}^2+\\ &\left[(4^u-4^v)\alpha_4+\left(2^{u+2v}-2^{3v}\right)\alpha_2^3-3^v(2^u-2^v)-2^v(3^u-3^v)\alpha_2\alpha_3\right]\mathrm{s}^3+\dots .\end{aligned}$$

Thus,

$$\frac{e^{i\Theta}\left\{\begin{array}{c}1+(2^u-2^v)\alpha_2\mathrm{s}+\left[(3^u-3^v)\alpha_3-\left(2^{u+v}-2^{2v}\right)\alpha_2^2\right]\mathrm{s}^2+\\ \left[\begin{array}{c}(4^u-4^v)\alpha_4+\left(2^{u+2v}-2^{3v}\right)\alpha_2^3-\\ 3^v(2^u-2^v)-2^v(3^u-3^v)\alpha_2\alpha_3\end{array}\right]\mathrm{s}^3+\dots\end{array}\right\}-\gamma-isin\Theta}{cos\Theta-\gamma}=p(\mathrm{s}),$$

and

$$O^u g(\omega)=\omega-2^u\alpha_2\omega^2+3^u\left(2\alpha_2^2-\alpha_3\right)\omega^3-4^u\left(5\alpha_2^3-5\alpha_2\alpha_3+\alpha_4\right)\omega^4+\dots,$$

$$O^v g(\omega)=\omega-2^v\alpha_2\omega^2+3^v\left(2\alpha_2^2-\alpha_3\right)\omega^3-4^v\left(5\alpha_2^3-5\alpha_2\alpha_3+\alpha_4\right)\omega^4+\dots,$$

$$\begin{aligned}\frac{O^u g(\omega)}{O^v g(\omega)} =& 1+(2^v-2^u)\alpha_2\omega+\left[(3^u-3^v)\left(2\alpha_2^2-\alpha_3\right)+\left(2^{2v}-2^{v+u}\right)\alpha_2^2\right]\omega^2\\ &+\left[(4^v-4^u)\left(5\alpha_2^3-5\alpha_2\alpha_3+\alpha_4\right)-3^v(2^v-2^u)\alpha_2\left(2\alpha_2^2-\alpha_3\right)+\right.\\ &\left.2^v(3^u-3^v)\alpha_2\left(2\alpha_2^2-\alpha_3\right)+\left(2^{2v}-2^{2v+u}\right)\alpha_2^3\right]\omega^3+\dots,\end{aligned}$$

and

$$\frac{e^{i\Theta}\left\{\begin{array}{c}1-(2^u-2^v)\alpha_2\omega+\left[(3^u-3^v)\left(2\alpha_2^2-\alpha_3\right)-\left(2^{v+u}-2^{2v}\right)\alpha_2^2\right]\omega^2\\ -\left[\begin{array}{c}(4^u-4^v)\left(5\alpha_2^3-5\alpha_2\alpha_3+\alpha_4\right)+\\ 3^v(2^u-2^v)\alpha_2\left(\alpha_3-2\alpha_2^2\right)+\\ 2^v(3^u-3^v)\alpha_2\left(\alpha_3-2\alpha_2^2\right)+2^{2v}(2^u-2^v)\alpha_2^3\end{array}\right]\omega^3+\dots\end{array}\right\}-\gamma-isin\Theta}{cos\Theta-\gamma}=q(\omega).$$

Here, $p(\mathrm{s})$ and $q(\omega)$ provided in the following series form

$$p(\mathrm{s})=1+n_1\mathrm{s}+n_2\mathrm{s}^2+\dots,$$

and

$$q(\omega)=1+m_1\omega+m_2\omega^2+\dots .$$

By comparing coefficients in (5.5), gives

$$\frac{e^{i\Theta}\left(2^{u}-2^{v}\right)\alpha_{2}}{cos\Theta-\gamma}=n_{1},$$

$$\alpha_{2}=\frac{n_{1}\left[cos\Theta-\gamma\right]}{e^{i\Theta}\left(2^{u}-2^{v}\right)}. \tag{5.6}$$

$$\frac{e^{i\Theta}\left[\left(3^{u}-3^{v}\right)\alpha_{3}-\left(2^{v+u}-2^{2v}\right)\alpha_{2}^{2}\right]}{cos\Theta-\gamma}=n_{2},$$

$$\left[\left(3^{u}-3^{v}\right)\alpha_{3}-\left(2^{v+u}-2^{2v}\right)\alpha_{2}^{2}\right]=\frac{n_{2}\left(cos\Theta-\gamma\right)}{e^{i\Theta}}. \tag{5.7}$$

$$\frac{e^{i\Theta}\left(4^{u}-4^{v}\right)\alpha_{4}+e^{i\Theta}\left(2^{u+2v}-2^{3v}\right)\alpha_{2}^{3}-e^{i\Theta}3^{v}\left(2^{u}-2^{v}\right)\alpha_{2}\alpha_{3}-e^{i\Theta}2^{v}\left(3^{u}-3^{v}\right)\alpha_{2}\alpha_{3}}{cos\Theta-\gamma}=n_{3},$$

$$\begin{array}{c}\left(4^{u}-4^{v}\right)\alpha_{4}+\left(2^{u+2v}-2^{3v}\right)\alpha_{2}^{3}-\\ 3^{v}\left(2^{u}-2^{v}\right)\alpha_{2}\alpha_{3}-2^{v}\left(3^{u}-3^{v}\right)\alpha_{2}\alpha_{3}\end{array}=\frac{n_{3}\left(cos\Theta-\gamma\right)}{e^{i\Theta}}. \tag{5.8}$$

$$\frac{-e^{i\Theta}\left(2^{u}-2^{v}\right)\alpha_{2}}{cos\Theta-\gamma}=m_{1},$$

$$-\alpha_{2}=\frac{m_{1}\left(cos\Theta-\gamma\right)}{e^{i\Theta}\left(2^{u}-2^{v}\right)}. \tag{5.9}$$

$$\left(3^{u}-3^{v}\right)\left(2\alpha_{2}^{2}-\alpha_{3}\right)-\left(2^{v+u}-2^{2v}\right)\alpha_{2}^{2}=\frac{m_{2}\left(cos\Theta-\gamma\right)}{e^{i\Theta}},$$

$$\left[2\left(3^{u}-3^{v}\right)\alpha_{2}^{2}-\left(2^{u+v}-2^{2v}\right)\alpha_{2}^{2}-\left(3^{u}-3^{v}\right)\alpha_{3}\right]=\frac{m_{2}\left(cos\Theta-\gamma\right)}{e^{i\Theta}}. \tag{5.10}$$

$$\begin{array}{c}-\left(4^{u}-4^{v}\right)\left(5\alpha_{2}^{3}-5\alpha_{2}\alpha_{3}+\alpha_{4}\right)-3^{v}\left(2^{u}-2^{v}\right)\alpha_{2}\left(\alpha_{3}-2\alpha_{2}^{2}\right)\\ -2^{v}\left(3^{u}-3^{v}\right)\alpha_{2}\left(\alpha_{3}-2\alpha_{2}^{2}\right)-2^{2v}\left(2^{u}-2^{v}\right)\alpha_{2}^{3}\end{array}=\frac{m_{3}\left(cos\Theta-\gamma\right)}{e^{i\Theta}}. \tag{5.11}$$

From (5.6) and (5.9), this yields

$$n_{1}=-m_{1}, \tag{5.12}$$

and

$$\alpha_{2}=\frac{n_{1}\left[cos\Theta-\gamma\right]}{e^{i\Theta}\left(2^{u}-2^{v}\right)}. \tag{5.13}$$

Now, from (5.7), (5.10) and (5.13), Subtract (5.7) and (5.10), this results in

$$\begin{aligned}(3^u-3^v)\,\alpha_3-\left(2^{v+u}-2^{2v}\right)\alpha_2^2-2\,(3^u-3^v)\,\alpha_2^2+\\ \left(2^{u+v}-2^{2v}\right)\alpha_2^2+(3^u-3^v)\,\alpha_3\end{aligned}=\frac{n_2\,(cos\Theta-\gamma)}{e^{i\Theta}}-\frac{m_2\,(cos\Theta-\gamma)}{e^{i\Theta}},$$

$$2\,(3^u-3^v)\,\alpha_3-2\,(3^u-3^v)\,\alpha_2^2=\frac{(n_2-m_2)\,(cos\Theta-\gamma)}{e^{i\Theta}},$$

$$2\,(3^u-3^v)\,\alpha_3-2\,(3^u-3^v)\left(\frac{n_1^2(cos\Theta-\gamma)^2}{e^{2i\Theta}(2^u-2^v)^2}\right)=\frac{(n_2-m_2)\,(cos\Theta-\gamma)}{e^{i\Theta}},$$

$$\alpha_3-\left(\frac{n_1^2(cos\Theta-\gamma)^2}{e^{2i\Theta}(2^u-2^v)^2}\right)=\frac{(n_2-m_2)\,(cos\Theta-\gamma)}{2\,(3^u-3^v)\,e^{i\Theta}},$$

$$\alpha_3=\frac{n_1^2(cos\Theta-\gamma)^2}{e^{2i\Theta}(2^u-2^v)^2}+\frac{(n_2-m_2)\,(cos\Theta-\gamma)}{2\,(3^u-3^v)\,e^{i\Theta}}. \tag{5.14}$$

Also, by subtracting (5.8) and (5.11), and using (5.13) and (5.14),this gives

$$\begin{aligned}2\,(4^u-4^v)\,\alpha_4+2\times2^{2v}\,(2^u-2^v)\,\alpha_2^3+5\,(4^u-4^v)\,\alpha_2^3-\\ 5\,(4^u-4^v)\,\alpha_2\alpha_3-2\times3^v\,(2^u-2^v)\,\alpha_2^3-2\times2^v\,(3^u-3^v)\,\alpha_2^3\end{aligned}=\frac{(n_3-m_3)\,(cos\Theta-\gamma)}{e^{i\Theta}},$$

$$\begin{aligned}2\,(4^u-4^v)\,\alpha_4+\big[2\times2^{2v}\,(2^u-2^v)+5\,(4^u-4^v)-\\ 2\times3^v\,(2^u-2^v)-2\times2^v\,(3^u-3^v)\big]\left(\tfrac{n_1^3(cos\Theta-\gamma)^3}{e^{3i\Theta}(2^u-2^v)^3}\right)\\ -5\,(4^u-4^v)\left(\tfrac{n_1[cos\Theta-\gamma]}{e^{i\Theta}(2^u-2^v)}\right)\left(\tfrac{(n_2-m_2)(cos\Theta-\gamma)}{2(3^u-3^v)e^{i\Theta}}+\tfrac{n_1^2(cos\Theta-\gamma)^2}{e^{2i\Theta}(2^u-2^v)^2}\right)\end{aligned}=\frac{(n_3-m_3)\,(cos\Theta-\gamma)}{e^{i\Theta}},$$

$$\begin{aligned}2\,(4^u-4^v)\,\alpha_4+\big[2\times2^{2v}\,(2^u-2^v)-2\times3^v\,(2^u-2^v)-\\ 2\times2^v\,(3^u-3^v)\big]\left(\tfrac{n_1^3(cos\Theta-\gamma)^3}{e^{3i\Theta}(2^u-2^v)^3}\right)+5\,(4^u-4^v)\left(\tfrac{n_1^3(cos\Theta-\gamma)^3}{e^{3i\Theta}(2^u-2^v)^3}\right)\\ -\tfrac{5(4^u-4^v)n_1(n_2-m_2)(cos\Theta-\gamma)^2}{2e^{2i\Theta}(2^u-2^v)(3^u-3^v)}-\tfrac{5(4^u-4^v)n_1^3(cos\Theta-\gamma)^3}{e^{3i\Theta}(2^u-2^v)^3}\end{aligned}=\frac{(n_3-m_3)\,(cos\Theta-\gamma)}{e^{i\Theta}},$$

$$\begin{aligned}2\,(4^u-4^v)\,\alpha_4+\big[2\times2^{2v}\,(2^u-2^v)-2\times3^v\,(2^u-2^v)\\ -2\times2^v\,(3^u-3^v)\big]\left(\tfrac{n_1^3(cos\Theta-\gamma)^3}{e^{3i\Theta}(2^u-2^v)^3}\right)-\tfrac{5(4^u-4^v)n_1(n_2-m_2)(cos\Theta-\gamma)^2}{2e^{2i\Theta}(2^u-2^v)(3^u-3^v)}\end{aligned}=\frac{(n_3-m_3)\,(cos\Theta-\gamma)}{e^{i\Theta}},$$

$$\begin{aligned}2\,(4^u-4^v)\,\alpha_4=&-2\left[2^{2v}\,(2^u-2^v)-3^v\,(2^u-2^v)-2^v\,(3^u-3^v)\right]\left(\frac{n_1^3(cos\Theta-\gamma)^3}{e^{3i\Theta}(2^u-2^v)^3}\right)\\ &+\frac{5\,(4^u-4^v)\,n_1\,(n_2-m_2)\,(cos\Theta-\gamma)^2}{2e^{2i\Theta}\,(2^u-2^v)\,(3^u-3^v)}+\frac{(n_3-m_3)\,(cos\Theta-\gamma)}{e^{i\Theta}},\end{aligned}$$

$$2\,(4^u-4^v)\,\alpha_4 = 2\left[\,(3^v-2^{2v})\,(2^u-2^v)+2^v\,(3^u-3^v)\right]\left(\frac{n_1^3(cos\Theta-\gamma)^3}{e^{3i\Theta}(2^u-2^v)^3}\right) + \frac{5\,(4^u-4^v)\,n_1\,(n_2-m_2)\,(cos\Theta-\gamma)^2}{2e^{2i\Theta}\,(2^u-2^v)\,(3^u-3^v)} + \frac{(n_3-m_3)\,(cos\Theta-\gamma)}{e^{i\Theta}},$$

$$\alpha_4 = \frac{n_1^3\left[\,(3^v-2^{2v})\,(2^u-2^v)+2^v\,(3^u-3^v)\right](cos\Theta-\gamma)^3}{e^{3i\Theta}(4^u-4^v)\,(2^u-2^v)^3} + \frac{5n_1\,(n_2-m_2)\,(cos\Theta-\gamma)^2}{4e^{2i\Theta}\,(2^u-2^v)\,(3^u-3^v)} + \frac{(n_3-m_3)\,(cos\Theta-\gamma)}{2\,(4^u-4^v)\,e^{i\Theta}}. \tag{5.15}$$

For using Lemma 2.13.1 to (5.13), this yields (5.1)

$$|\alpha_2| = \left|\frac{n_1\,(cos\Theta-\gamma)}{e^{i\Theta}\,(2^u-2^v)}\right|,$$

$$\because \left|e^{i\Theta}\right| = 1$$

$$|\alpha_2| \le \frac{2(cos\Theta-\gamma)}{2^u-2^v},$$

Using Lemma 2.13.1 to (5.14), yields (5.2)

$$|\alpha_3| = \left|\frac{n_1^2(cos\Theta-\gamma)^2}{e^{2i\Theta}(2^u-2^v)^2} + \frac{(n_2-m_2)\,(cos\Theta-\gamma)}{2\,(3^u-3^v)\,e^{i\Theta}}\right|,$$

$$|\alpha_3| \le \frac{4(cos\Theta-\gamma)^2}{(2^u-2^v)^2} + \frac{2\,(cos\Theta-\gamma)}{3^u-3^v}.$$

Apply Lemma 2.13.1 to (5.15), yields (5.3)

$$|\alpha_4| = \left|\frac{n_1^3\left[\,(3^v-2^{2v})\,(2^u-2^v)+2^v\,(3^u-3^v)\right](cos\Theta-\gamma)^3}{e^{3i\Theta}(4^u-4^v)\,(2^u-2^v)^3} + \frac{5n_1\,(n_2-m_2)\,(cos\Theta-\gamma)^2}{4e^{2i\Theta}\,(2^u-2^v)\,(3^u-3^v)} + \frac{(n_3-m_3)\,(cos\Theta-\gamma)}{2\,(4^u-4^v)\,e^{i\Theta}}\right|,$$

$$|\alpha_4| \le \frac{8(cos\Theta-\gamma)^3\left[(2^u-2^v)\left(3^v-2^{2v}\right)+2^v\,(3^u-3^v)\right]}{(2^u+2^v)\,(2^u-2^v)^4} + \frac{10(cos\Theta-\gamma)^2}{(2^u-2^v)\,(3^u-3^v)} + \frac{2\,(cos\Theta-\gamma)}{4^u-4^v}.$$

By adding (5.7) and (5.10), this leads to

$$\begin{array}{c}(3^u-3^v)\,\alpha_3-\left(2^{v+u}-2^{2v}\right)\alpha_2^2+\\ 2\,(3^u-3^v)\,\alpha_2^2-\left(2^{u+v}-2^{2v}\right)\alpha_2^2-\\ (3^u-3^v)\,\alpha_3\end{array} = \frac{(n_2+m_2)\,(cos\Theta-\gamma)}{e^{i\Theta}},$$

$$\left[2\left(3^{u}-3^{v}\right)-2\left(2^{u+v}-2^{2v}\right)\right]\alpha_2^2=\frac{(n_2+m_2)\,(cos\Theta-\gamma)}{e^{i\Theta}},$$
$$2\left[\left(3^{u}-3^{v}\right)-\left(2^{u+v}-2^{2v}\right)\right]\alpha_2^2=\frac{(n_2+m_2)\,(cos\Theta-\gamma)}{e^{i\Theta}}. \tag{5.16}$$

Also by subtracting (5.10) from (5.7), gives

$$\alpha_3=\frac{(n_2-m_2)\,(cos\Theta-\gamma)}{2\,(3^{u}-3^{v})\,e^{i\Theta}}+\frac{n_1^2(cos\Theta-\gamma)^2}{e^{2i\Theta}(2^{u}-2^{v})^2},$$
$$\alpha_3=\frac{(n_2-m_2)\,(cos\Theta-\gamma)}{2\,(3^{u}-3^{v})\,e^{i\Theta}}+\alpha_2^2, \tag{5.17}$$

From equations (5.16) and (5.17), gives

$$\alpha_3-\lambda\alpha_2^2=\frac{(n_2-m_2)\,(cos\Theta-\gamma)}{2\,(3^{u}-3^{v})\,e^{i\Theta}}+\alpha_2^2-\lambda\alpha_2^2,$$

$$\alpha_3-\lambda\alpha_2^2=\frac{(n_2-m_2)\,(cos\Theta-\gamma)}{2\,(3^{u}-3^{v})\,e^{i\Theta}}+(1-\lambda)\alpha_2^2\ ,$$

$$\alpha_3-\lambda\alpha_2^2\ =\frac{(n_2-m_2)\,(cos\Theta-\gamma)}{2\,(3^{u}-3^{v})\,e^{i\Theta}}+(1-\lambda)\frac{(n_2+m_2)\,(cos\Theta-\gamma)}{2[(3^{u}-3^{v})-(2^{u+v}-2^{2v})]\,e^{i\Theta}},$$

$$\alpha_3-\lambda\alpha_2^2\ =\frac{(cos\Theta-\gamma)}{e^{i\Theta}}\left\{\frac{(n_2-m_2)}{2\,(3^{u}-3^{v})}+\frac{(1-\lambda)(n_2+m_2)}{2\,[(3^{u}-3^{v})-(2^{u+v}-2^{2v})]}\right\},$$

$$\begin{aligned}\alpha_3-\lambda\alpha_2^2\ =&\frac{(cos\Theta-\gamma)}{e^{i\Theta}}\left\{\frac{n_2}{2\,(3^{u}-3^{v})}-\frac{m_2}{2\,(3^{u}-3^{v})}\right.\\&\left.+\frac{(1-\lambda)\,n_2}{2\,[(3^{u}-3^{v})-(2^{u+v}-2^{2v})]}+\frac{(1-\lambda)\ m_2}{2\,[(3^{u}-3^{v})-(2^{u+v}-2^{2v})]}\right\},\end{aligned}$$

$$\begin{aligned}\alpha_3-\lambda\alpha_2^2\ =&\frac{(cos\Theta-\gamma)}{e^{i\Theta}}\left\{\left(\frac{(1-\lambda)}{2\,[(3^{u}-3^{v})-(2^{u+v}-2^{2v})]}+\frac{1}{2\,(3^{u}-3^{v})}\right)n_2\right.\\&\left.+\left(\frac{(1-\lambda)}{2\,[(3^{u}-3^{v})-(2^{u+v}-2^{2v})]}-\frac{1}{2\,(3^{u}-3^{v})}\right)m_2\right\},\end{aligned}$$

$$\begin{aligned}\alpha_3-\lambda\alpha_2^2=&\frac{(cos\Theta-\gamma)}{e^{i\Theta}}\left\{\left(T\,(\lambda,u,v)+\frac{1}{2\,(3^{u}-3^{v})}\right)n_2\right.\\&\left.+\left(T\,(\lambda,u,v)-\frac{1}{2\,(3^{u}-3^{v})}\right)m_2\right\},\end{aligned}$$

where,

$$T\,(\lambda,u,v)=\frac{1-\lambda}{2\,[(3^{u}-3^{v})-(2^{u+v}-2^{2v})]}\ .$$

The Theorem 5.2.1 is complete. □

Theorem 5.2.2. Let $f(\mathrm{s})$ provided by (1.1) be a part of the class $f \in \mathbb{S}^*_\nabla(\gamma,\Theta,u,v)$ and if $X = cos\Theta - \gamma$ for $u > v+1$, $\left[\left(3^v - 2^{2v}\right)(2^u - 2^v) + 2^v(3^u - 3^v) - (4^u - 4^v)\right] \neq 0$, $|\Theta| < \pi$ and $cos\Theta > \gamma$. Then

$$\left|\alpha_2\alpha_4 - \alpha_3^2\right| \leq \begin{cases} \frac{4X^2}{(4^u-4^v)}\left\{\frac{4X^2\left[\left(3^v-2^{2v}\right)(2^u-2^v)+2^v(3^u-3^v)-(4^u-4^v)\right]}{(2^u-2^v)^4} + \frac{1}{(2^u-2^v)}\right\}, & X \in \left[0, \phi_{(u,v)}\right], \\ \frac{X^2}{16(4^u-4^v)}\left\{\frac{64(4^u-4^v)}{(3^u-3^v)^2} - \frac{4\left(\Pi+\frac{2X(2^u+2^v)}{(2^u-2^v)(3^u-3^v)}\right)^2}{(\Omega+\Sigma)}\right\}, & X \in \left[\phi_{(u,v)}, 1\right], \end{cases}$$

where

$$\phi_{(u,v)} = \frac{(2^u-2^v)^2(4^u-4^v)}{8(3^u-3^v)\left[\left(3^v-2^{2v}\right)(2^u-2^v)+2^v(3^u-3^v)-(4^u-4^v)\right]} \times$$

$$\left(1+\sqrt{1-\frac{16\left[\left(3^v-2^{2v}\right)(2^u-2^v)+2^v(3^u-3^v)-(4^u-4^v)\right]\times\left[(4^u-4^v)(2^u-2^v)-2(3^u-3^v)^2\right]}{(4^u-4^v)^2(2^u-2^v)}}\right),$$

$$\begin{aligned} \Pi =& \frac{6(3^u-3^v)^2 - 4(4^u-4^v)(2^u-2^v)}{(2^u-2^v)(3^u-3^v)^2}, \\ \Sigma =& -\frac{X(4^u-4^v)}{(2^u-2^v)^2(3^u-3^v)} + \frac{(4^u-4^v)(2^u-2^v)-2(3^u-3^v)^2}{(2^u-2^v)(3^u-3^v)^2}, \\ \Omega =& \frac{4\left[\left(3^v-2^{2v}\right)(2^u-2^v)+2^v(3^u-3^v)-(4^u-4^v)\right]X^2}{(2^u-2^v)^4}. \end{aligned}$$

Proof. From (5.13), (5.14) and (5.15) and letting $X = cos\Theta - \gamma$, this results in

$$\begin{aligned} \alpha_2 =& \frac{n_1 X}{e^{i\Theta}(2^u-2^v)}, \\ \alpha_3 =& \frac{(n_2-m_2)X}{2(3^u-3^v)e^{i\Theta}} + \frac{n_1^2X^2}{e^{2i\Theta}(2^u-2^v)^2}, \end{aligned}$$

and

$$\alpha_4 = \frac{n_1^3\left[(3^v-2^{2v})(2^u-2^v)+2^v(3^u-3^v)\right]X^3}{e^{3i\Theta}(4^u-4^v)(2^u-2^v)^3} + \frac{5n_1(n_2-m_2)X^2}{4e^{2i\Theta}(2^u-2^v)(3^u-3^v)} + \frac{(n_3-m_3)X}{2(4^u-4^v)e^{i\Theta}}.$$

Therefore, the functional $\alpha_2\alpha_4 - \alpha_3^2$ will become

$$\begin{aligned} \alpha_2\alpha_4 - \alpha_3^2 =& \left(\frac{n_1X}{e^{i\Theta}(2^u-2^v)}\right)\left\{\frac{n_1^3\left[(3^v-2^{2v})(2^u-2^v)+2^v(3^u-3^v)\right]X^3}{e^{3i\Theta}(4^u-4^v)(2^u-2^v)^3}\right. \\ &\left. + \frac{5n_1(n_2-m_2)X^2}{4e^{2i\Theta}(2^u-2^v)(3^u-3^v)} + \frac{(n_3-m_3)X}{2(4^u-4^v)e^{i\Theta}}\right\} - \\ &\left(\frac{(n_2-m_2)X}{2(3^u-3^v)e^{i\Theta}} + \frac{n_1^2X^2}{e^{2i\Theta}(2^u-2^v)^2}\right)^2, \end{aligned}$$

$$\alpha_2\alpha_4-\alpha_3^2=\left\{\frac{n_1^4\left[\,(3^v-2^{2v})(2^u-2^v)+2^v(3^u-3^v)\right]X^4}{e^{4i\Theta}(4^u-4^v)(2^u-2^v)^4}+\frac{5n_1^2(n_2-m_2)X^3}{4e^{3i\Theta}(2^u-2^v)^2(3^u-3^v)}\right.$$
$$\left.+\frac{n_1(n_3-m_3)X^2}{2(2^u-2^v)(4^u-4^v)e^{2i\Theta}}\right\}-\left\{\frac{n_1^4X^4}{e^{4i\Theta}(2^u-2^v)^4}+\frac{|n_2-m_2|^2X^2}{4e^{2i\Theta}(3^u-3^v)^2}+\right.$$
$$\left.\frac{2n_1^2X^3|n_2-m_2|}{2e^{3i\Theta}(2^u-2^v)^2(3^u-3^v)}\right\},$$

$$\alpha_2\alpha_4-\alpha_3^2=\left(\frac{n_1^4\left[\,\left(3^v-2^{2v}\right)(2^u-2^v)+2^v(3^u-3^v)\right]X^4}{e^{4i\Theta}(4^u-4^v)(2^u-2^v)^4}-\frac{n_1^4X^4}{e^{4i\Theta}(2^u-2^v)^4}\right)+$$
$$\left(\frac{5n_1^2(n_2-m_2)X^3}{4e^{3i\Theta}(2^u-2^v)^2(3^u-3^v)}-\frac{2n_1^2X^3|n_2-m_2|}{2e^{3i\Theta}(2^u-2^v)^2(3^u-3^v)}\right)+$$
$$\left(\frac{n_1(n_3-m_3)X^2}{2(2^u-2^v)(4^u-4^v)e^{2i\Theta}}-\frac{|n_2-m_2|^2X^2}{4e^{2i\Theta}(3^u-3^v)^2}\right),$$

$$\alpha_2\alpha_4-\alpha_3^2=\frac{n_1^4\left[\left(3^v-2^{2v}\right)(2^u-2^v)+2^v(3^u-3^v)-(4^u-4^v)\right]X^4}{e^{4i\Theta}(4^u-4^v)(2^u-2^v)^4}+$$
$$\left(\frac{5}{4}-1\right)\frac{n_1^2X^3|n_2-m_2|}{e^{3i\Theta}(2^u-2^v)^2(3^u-3^v)}+\frac{n_1(n_3-m_3)X^2}{2(2^u-2^v)(4^u-4^v)e^{2i\Theta}}-$$
$$\frac{|n_2-m_2|^2X^2}{4e^{2i\Theta}(3^u-3^v)^2},$$

$$\alpha_2\alpha_4-\alpha_3^2=\frac{n_1^4\left[\,\left(3^v-2^{2v}\right)(2^u-2^v)+2^v(3^u-3^v)-(4^u-4^v)\right]X^4}{e^{4i\Theta}(4^u-4^v)(2^u-2^v)^4}+$$
$$\frac{n_1^2X^3|n_2-m_2|}{4e^{3i\Theta}(2^u-2^v)^2(3^u-3^v)}+\frac{n_1(n_3-m_3)X^2}{2(2^u-2^v)(4^u-4^v)e^{2i\Theta}}- \tag{5.18}$$
$$\frac{|n_2-m_2|^2X^2}{4e^{2i\Theta}(3^u-3^v)^2}.$$

According to Lemma 2.13.2 , yields us

$$2n_2=n_1^2+x\left(4-n_1^2\right),$$
$$2m_2=m_1^2+y\left(4-m_1^2\right).$$

By solving, it gives us

$$2n_2-2m_2=n_1^2-m_1^2+x\left(4-n_1^2\right)-y\left(4-m_1^2\right).$$

Using (5.12), this results in

$$2(n_2-m_2)=n_1^2-n_1^2+x\left(4-n_1^2\right)-y\left(4-n_1^2\right),$$

$$
\begin{aligned}
2(n_2-m_2) &= (x-y)\left(4-n_1^2\right),\\
n_2-m_2 &= \frac{4-n_1^2}{2}\,(x-y),
\end{aligned}
\tag{5.19}
$$

and

$$
\begin{aligned}
4n_3 &= n_1^3+2\left(4-n_1^2\right)n_1x-n_1\left(4-n_1^2\right)x^2+2\left(4-n_1^2\right)\left(1-|x|^2\right)\mathrm{s},\\
4m_3 &= m_1^3+2\left(4-m_1^2\right)m_1y-m_1\left(4-m_1^2\right)y^2+2\left(4-m_1^2\right)\left(1-|y|^2\right)\omega.
\end{aligned}
\tag{5.20}
$$

By solving (5.20), gives

$$
\begin{aligned}
4n_3-4m_3 =& n_1^3-m_1^3+2\left(4-n_1^2\right)n_1x-2\left(4-m_1^2\right)m_1y-n_1\left(4-n_1^2\right)x^2+\\
& m_1\left(4-m_1^2\right)y^2+2\left(4-n_1^2\right)\left(1-|x|^2\right)\mathrm{s}-2\left(4-m_1^2\right)\left(1-|y|^2\right)\omega.
\end{aligned}
$$

$$
\begin{aligned}
4\left(n_3-m_3\right) =& n_1^3+n_1^3+2\left(4-n_1^2\right)n_1x+2\left(4-n_1^2\right)n_1y-n_1\left(4-n_1^2\right)x^2-\\
& n_1\left(4-n_1^2\right)y^2+2\left(4-n_1^2\right)\left(1-|x|^2\right)\mathrm{s}-2\left(4-n_1^2\right)\left(1-|y|^2\right)\omega.
\end{aligned}
$$

$$
\begin{aligned}
n_3-m_3 =& \frac{1}{4}\Big[2n_1^3+2\left(4-n_1^2\right)n_1(x+y)-n_1\left(4-n_1^2\right)(x^2+y^2)\\
& +2\left(4-n_1^2\right)(\left(1-|x|^2\right)\mathrm{s}-\left(1-|y|^2\right)\omega)\Big],
\end{aligned}
$$

$$
\begin{aligned}
n_3-m_3 =& \frac{n_1^3}{2}+\frac{\left(4-n_1^2\right)n_1}{2}(x+y)-\frac{n_1\left(4-n_1^2\right)}{4}(x^2+y^2)+\\
& \frac{\left(4-n_1^2\right)}{2}(\left(1-|x|^2\right)\mathrm{s}-\left(1-|y|^2\right)\omega).
\end{aligned}
\tag{5.21}
$$

From (5.19), gives

$$
2n_2+2m_2 = n_1^2+m_1^2+x\left(4-n_1^2\right)+y\left(4-m_1^2\right).
$$

Using (5.12), this leads to

$$
2(n_2+m_2) = n_1^2+n_1^2+x\left(4-n_1^2\right)+y\left(4-n_1^2\right),
$$

$$
\begin{aligned}
2(n_2+m_2) &= 2n_1^2+(x+y)\left(4-n_1^2\right),\\
n_2+m_2 &= n_1^2+\frac{4-n_1^2}{2}\,(x+y).
\end{aligned}
\tag{5.22}
$$

The terms x, y and s, ω are provided with conditions $|x|\leq 1, |y|\leq 1, |\mathrm{s}|\leq 1, |\omega|\leq 1$ and $\left|e^{i\Theta}\right|=1$. Using triangle inequality on (5.18), this leads us

$$
\begin{aligned}
\left|\alpha_2\alpha_4-\alpha_3^2\right| =& \left|\frac{n_1^4\left[\left(3^v-2^{2v}\right)\left(2^u-2^v\right)+2^v\left(3^u-3^v\right)-\left(4^u-4^v\right)\right]X^4}{e^{4i\Theta}(4^u-4^v)\left(2^u-2^v\right)^4}+\right.\\
& \left.\frac{n_1^2X^3\,|n_2-m_2|}{4e^{3i\Theta}(2^u-2^v)^2\left(3^u-3^v\right)}+\frac{n_1\left(n_3-m_3\right)X^2}{2\left(2^u-2^v\right)\left(4^u-4^v\right)e^{2i\Theta}}-\frac{|n_2-m_2|^2X^2}{4e^{2i\Theta}(3^u-3^v)^2}\right|.
\end{aligned}
$$

Using (5.19) and (5.21), this yields

$$\begin{aligned}\left|\alpha_2\alpha_4-\alpha_3^2\right| = &\left| \frac{n_1^4\left[\,\left(3^v-2^{2v}\right)\left(2^u-2^v\right)+2^v\left(3^u-3^v\right)-\left(4^u-4^v\right)\right]X^4}{\left(4^u-4^v\right)\left(2^u-2^v\right)^4}+\right.\\ &\frac{n_1^2(4-n_1^2)X^3}{8(2^u-2^v)^2\left(3^u-3^v\right)}\left(x-y\right)+\frac{n_1X^2}{2\left(2^u-2^v\right)\left(4^u-4^v\right)}\left(\frac{n_1^3}{2}+\right.\\ &\frac{\left(4-n_1^2\right)n_1}{2}\left(x+y\right)-\frac{n_1\left(4-n_1^2\right)}{4}(x^2+y^2)+\\ &\left.\left.\frac{\left(4-n_1^2\right)}{2}(\left(1-|x|^2\right)\mathrm{s}-\left(1-|y|^2\right)\omega)\right)-\frac{\left(4-n_1^2\right)^2X^2}{16(3^u-3^v)^2}(x-y)^2\right|,\end{aligned}$$

$$\begin{aligned}\left|\alpha_2\alpha_4-\alpha_3^2\right| \leq &\left| \frac{n_1^4\left[\,\left(3^v-2^{2v}\right)\left(2^u-2^v\right)+2^v\left(3^u-3^v\right)-\left(4^u-4^v\right)\right]X^4}{\left(4^u-4^v\right)\left(2^u-2^v\right)^4}\right|+\\ &\left|\frac{n_1^2(4-n_1^2)X^3}{8(2^u-2^v)^2\left(3^u-3^v\right)}\left(x-y\right)\right|+\left|\frac{n_1X^2}{2\left(2^u-2^v\right)\left(4^u-4^v\right)}\left(\frac{n_1^3}{2}+\right.\right.\\ &\frac{\left(4-n_1^2\right)n_1}{2}\left(x+y\right)-\frac{n_1\left(4-n_1^2\right)}{4}(x^2+y^2)+\\ &\left.\left.\frac{\left(4-n_1^2\right)}{2}(\left(1-|x|^2\right)\mathrm{s}-\left(1-|y|^2\right)\omega)\right)\right|+\left|\frac{\left(4-n_1^2\right)^2X^2}{16(3^u-3^v)^2}(x-y)^2\right|,\end{aligned}$$

$$\begin{aligned}\left|\alpha_2\alpha_4-\alpha_3^2\right| \leq &\frac{n_1^4\left[\,\left(3^v-2^{2v}\right)\left(2^u-2^v\right)+2^v\left(3^u-3^v\right)-\left(4^u-4^v\right)\right]X^4}{\left(4^u-4^v\right)\left(2^u-2^v\right)^4}+\\ &\frac{n_1^2(4-n_1^2)X^3}{8(2^u-2^v)^2\left(3^u-3^v\right)}\left(|x|+|y|\right)+\frac{n_1^4X^2}{4\left(2^u-2^v\right)\left(4^u-4^v\right)}+\\ &\frac{n_1^2\left(4-n_1^2\right)X^2}{4\left(2^u-2^v\right)\left(4^u-4^v\right)}\left(|x|+|y|\right)+\frac{n_1^2\left(4-n_1^2\right)X^2}{8\left(2^u-2^v\right)\left(4^u-4^v\right)}\left(|x|^2+|y|^2\right)+\\ &\frac{n_1X^2\left(4-n_1^2\right)}{4\left(2^u-2^v\right)\left(4^u-4^v\right)}\left(|\left(1-|x|^2\right)-\left(1-|y|^2\right)|\right)+\\ &\frac{\left(4-n_1^2\right)^2X^2}{16(3^u-3^v)^2}(|x|+|y|)^2,\end{aligned}$$

$$\left|\alpha_2\alpha_4-\alpha_3^2\right| \leq \frac{n_1^4\left[\left(3^v-2^{2v}\right)\left(2^u-2^v\right)+2^v\left(3^u-3^v\right)-\left(4^u-4^v\right)\right]X^4}{\left(4^u-4^v\right)\left(2^u-2^v\right)^4}+ \frac{n_1^4X^2}{4\left(2^u-2^v\right)\left(4^u-4^v\right)}+\frac{n_1X^2\left(4-n_1^2\right)}{2\left(2^u-2^v\right)\left(4^u-4^v\right)}+ \left[\frac{n_1^2(4-n_1^2)X^3}{8(2^u-2^v)^2\left(3^u-3^v\right)}+\frac{n_1^2\left(4-n_1^2\right)X^2}{4\left(2^u-2^v\right)\left(4^u-4^v\right)}\right](|x|+|y|)+ \left[\frac{n_1^2\left(4-n_1^2\right)X^2}{8\left(2^u-2^v\right)\left(4^u-4^v\right)}-\frac{n_1X^2\left(4-n_1^2\right)}{4\left(2^u-2^v\right)\left(4^u-4^v\right)}\right]\left(|x|^2+|y|^2\right)+ \frac{\left(4-n_1^2\right)^2X^2}{16(3^u-3^v)^2}(|x|+|y|)^2.$$

Since, the function $p\left(e^{i\Theta}\mathrm{s}\right)$ for $\Theta\in R$ is from class P, therefore this can suppose without loss of generality that $n_1=n\in[0,2]$. Thus, for $\tau = |x|\leq 1$ and $\delta = |y|\leq 1$, given by

$$\left|\alpha_2\alpha_4-\alpha_3^2\right|\leq W_1+W_2\left(\tau+\delta\right)+W_3\left(\tau^2+\delta^2\right)+W_4(\tau+\delta)^2=F\left(\tau,\delta\right),$$

where

$$W_1=\frac{n_1^4\left[\left(3^v-2^{2v}\right)\left(2^u-2^v\right)+2^v\left(3^u-3^v\right)-\left(4^u-4^v\right)\right]X^4}{\left(4^u-4^v\right)\left(2^u-2^v\right)^4}+ \frac{n_1^4X^2}{4\left(2^u-2^v\right)\left(4^u-4^v\right)}+\frac{n_1X^2\left(4-n_1^2\right)}{2\left(2^u-2^v\right)\left(4^u-4^v\right)},$$

$$W_1\left(n\right)=\frac{n^4\left[\left(3^v-2^{2v}\right)\left(2^u-2^v\right)+2^v\left(3^u-3^v\right)-\left(4^u-4^v\right)\right]X^4}{\left(4^u-4^v\right)\left(2^u-2^v\right)^4}+ \frac{n^4X^2}{4\left(2^u-2^v\right)\left(4^u-4^v\right)}+\frac{nX^2\left(4-n^2\right)}{2\left(2^u-2^v\right)\left(4^u-4^v\right)},$$

$$W_1\left(n\right)=\frac{X^2}{4\left(4^u-4^v\right)}\left[\frac{4n^4\left[\left(3^v-2^{2v}\right)\left(2^u-2^v\right)+2^v\left(3^u-3^v\right)-\left(4^u-4^v\right)\right]X^2}{\left(4^u-4^v\right)\left(2^u-2^v\right)^4} +\frac{n^4}{\left(2^u-2^v\right)}+\frac{2n\left(4-n^2\right)}{\left(2^u-2^v\right)}\right],$$

$$W_1\left(n\right)=\frac{X^2}{4\left(4^u-4^v\right)}\left[\frac{4n^4\left[\left(3^v-2^{2v}\right)\left(2^u-2^v\right)+2^v\left(3^u-3^v\right)-\left(4^u-4^v\right)\right]X^2}{\left(4^u-4^v\right)\left(2^u-2^v\right)^4}+ \frac{n^4}{\left(2^u-2^v\right)}+\frac{8n-2n^3}{\left(2^u-2^v\right)}\right],$$

$$
\begin{aligned}
W_2 =& \frac{n_1^2(4-n_1^2)X^3}{8(2^u-2^v)^2(3^u-3^v)} + \frac{n_1^2\left(4-n_1^2\right)X^2}{4\left(2^u-2^v\right)\left(4^u-4^v\right)},\\
W_2 =& W_2\left(n\right) = \frac{n^2(4-n^2)X^3}{8(2^u-2^v)^2(3^u-3^v)} + \frac{n^2\left(4-n^2\right)X^2}{4\left(2^u-2^v\right)\left(4^u-4^v\right)},\\
W_2 =& W_2\left(n\right) = \frac{X^2}{24\left(3^u-3^v\right)}\left\{\frac{3X}{(2^u-2^v)^2} + \frac{6\left(3^u-3^v\right)}{\left(2^u-2^v\right)\left(4^u-4^v\right)}\right\}n^2\left(4-n^2\right),\\
W_2 =& W_2\left(n\right) = \frac{X^2}{24\left(3^u-3^v\right)}\left[n^2\left(4-n^2\right)\left\{\frac{3\left(4^u-4^v\right)X+6\left(3^u-3^v\right)\left(2^u-2^v\right)}{(2^u-2^v)^2\left(4^u-4^v\right)}\right\}\right] \geq 0,
\end{aligned}
$$

$$
\begin{aligned}
W_3 =& \frac{n_1^2\left(4-n_1^2\right)X^2}{8\left(2^u-2^v\right)\left(4^u-4^v\right)} - \frac{n_1X^2\left(4-n_1^2\right)}{4\left(2^u-2^v\right)\left(4^u-4^v\right)},\\
W_3 =& W_3\left(n\right) = \frac{n^2\left(4-n^2\right)X^2}{8\left(2^u-2^v\right)\left(4^u-4^v\right)} - \frac{nX^2\left(4-n^2\right)}{4\left(2^u-2^v\right)\left(4^u-4^v\right)},\\
W_3 =& W_3\left(n\right) = \frac{X^2}{8\left(2^u-2^v\right)\left(4^u-4^v\right)}n\left(4-n^2\right)\left(n-2\right) \leq 0,
\end{aligned}
$$

$$
\begin{aligned}
W_4 =& \frac{\left(4-n_1^2\right)^2X^2}{16(3^u-3^v)^2},\\
W_4 =& \frac{X^2}{16(3^u-3^v)^2}\left(4-n_1^2\right)^2 \geq 0.
\end{aligned}
$$

Now, to maximize $F(\tau,\delta)$ in the closed square

$$S = \{(\tau,\delta) : 0 \leq \tau \leq 1,\ 0 \leq \delta \leq 1\}, \text{for } n \in [0,2].$$

By differentiate partially the function $F\left(\tau,\delta\right)$, this approach leads to

$$\frac{\partial F}{\partial \tau} = W_2 + 2W_3\tau + 2W_4\left(\tau+\delta\right) = 0, \tag{5.23}$$

and

$$\frac{\partial F}{\partial \delta} = W_2 + 2W_3\delta + 2W_4\left(\tau+\delta\right) = 0. \tag{5.24}$$

By comparing (5.23) and (5.24), this results in

$$\tau = \delta\ .$$

Using in (5.23), provides

$$W_2 + 2W_3\tau + 2W_4\left(\tau+\delta\right) = 0,$$

$$W_2 + 2W_3\tau + 2W_4\left(\tau+\tau\right) = 0,$$

$$W_2+2W_3\tau+4W_4\tau=0,$$

$$(2W_3+4W_4)\,\tau=-W_2,$$

$$\tau=\frac{-W_2}{2(W_3+2W_4)},$$

$$\delta=\frac{-W_2}{2(W_3+2W_4)}.$$

So

$$(\tau,\delta)=\left(\frac{-W_2}{2(W_3+2W_4)},\frac{-W_2}{2(W_3+2W_4)}\right).$$

This is the critical point.

As the function $F(\tau,\delta)$ doesn't show a relative maximum, we examine the maximum of $F(\tau,\delta)$ on the boundary. For $\tau=0$ and $0\le\tau\ \le 1$ (Similar to $\delta=0$ and $0\ \le\delta\ \le 1$), results in

$$\begin{aligned}
F\,(\tau,\delta)=&W_1+W_2\,(\tau+\delta)+W_3\left(\tau^2+\delta^2\right)+W_4(\tau+\delta)^2,\\
F\,(0,\delta)=&W_1+W_2\,(\delta)+(W_3+W_4)\left(\delta^2\right)=G\,(\delta)\,,\\
G'\,(\delta)=&W_2+2\,(W_3+W_4)\,\delta,
\end{aligned}$$

we need to check the boundaries,

At $\delta=0$,

$$G\,(0)=W_1,$$

At $\delta=1$,

$$G\,(1)=W_1+W_2+W_3+W_4,$$

Since

$$G'\,(\delta)=W_2+2\,(W_3+W_4)\,\delta.$$

The sign of W_3+W_4 affects the behavior. If $W_3+W_4\ge 0,\ \ G'(\delta)\ge 0$, $G(\delta)$ is increasing. The interior point of $0\le n\le 2$ for $0\le\delta\le 1$ is achieved, when $W_3+W_4\ge 0$. The function $G'\,(\delta)>0$ for δ>0 shows the positive slope of function 'F'. Hence, the upper bound $\left|\alpha_2\alpha_4-\alpha_3^2\right|$ leads to $\delta=1$ and $n=0$, which can be simplified into $G'\,(\delta)=2\,(W_3+W_4)\,\delta+W_2\ge 0$. Therefore, the maximum of $G(\delta)$ happens at $\delta=1$ and

$$\max\left\{G(\delta)\right\}=G\,(1)=W_1+W_2+W_3+W_4.$$

For the case when $W_3+W_4<0$, we note that $2\,(W_3+W_4)\,\delta+W_2\ge 0$ for $0\le\delta\le 1$ and any fixed n with $0\le n\le 2$. It shows that $2\,(W_3+W_4)+W_2<\ 2\,(W_3+W_4)\,\delta+W_2<\ W_2$, and so

$G'(\delta) > 0.$

Therefore, for $n = 2$, results in

$$F(\tau,\delta) = \frac{n_1^4\left[\left(3^v - 2^{2v}\right)(2^u - 2^v) + 2^v(3^u - 3^v) - (4^u - 4^v)\right]X^4}{(4^u - 4^v)(2^u - 2^v)^4} + \frac{n_1^4X^2}{4(2^u - 2^v)(4^u - 4^v)} + \frac{n_1X^2\left(4 - n_1^2\right)}{2(2^u - 2^v)(4^u - 4^v)} + \left[\frac{n_1^2(4 - n_1^2)X^3}{8(2^u - 2^v)^2(3^u - 3^v)} + \frac{n_1^2\left(4 - n_1^2\right)X^2}{4(2^u - 2^v)(4^u - 4^v)}\right](\tau + \delta) + \left[\frac{n_1^2\left(4 - n_1^2\right)X^2}{8(2^u - 2^v)(4^u - 4^v)} - \frac{n_1X^2\left(4 - n_1^2\right)}{4(2^u - 2^v)(4^u - 4^v)}\right]\left(\tau^2 + \delta^2\right) + \frac{\left(4 - n_1^2\right)^2X^2}{16(3^u - 3^v)^2}(\tau + \delta)^2,$$

$$F(\tau,\delta) = \frac{16\left[\left(3^v - 2^{2v}\right)(2^u - 2^v) + 2^v(3^u - 3^v) - (4^u - 4^v)\right]X^4}{(4^u - 4^v)(2^u - 2^v)^4} + \frac{16X^2}{4(2^u - 2^v)(4^u - 4^v)},$$

$$\begin{aligned} F(\tau,\delta) =& \frac{4X^2}{(4^u - 4^v)}\left\{\frac{4X^2\left[\left(3^v - 2^{2v}\right)(2^u - 2^v) + 2^v(3^u - 3^v) - (4^u - 4^v)\right]}{(2^u - 2^v)^4} + \frac{4}{4(2^u - 2^v)}\right\}, \\ \left|\alpha_2\alpha_4 - \alpha_3^2\right| =& \frac{4X^2}{(4^u - 4^v)}\left\{\frac{4X^2\left[\left(3^v - 2^{2v}\right)(2^u - 2^v) + 2^v(3^u - 3^v) - (4^u - 4^v)\right]}{(2^u - 2^v)^4} + \frac{1}{(2^u - 2^v)}\right\}. \end{aligned}$$

Next for $\tau = 1$ and $0 \le \tau \le 1$ (similar to $\delta = 1$ and $0 \le \delta \le 1$), this gives

$$\begin{aligned} F(1,\delta) = H(\delta) =& W_1 + W_2 + W_3 + W_4 + (W_2 + 2W_4)\,\delta + (W_3 + W_4)\,\delta^2, \\ H'(\delta) =& W_2 + 2W_4 + 2(W_3 + W_4)\,\delta. \end{aligned}$$

Similarly, to the above cases of $W_3 + W_4$ where $\delta = 1$, this provides

$$\max\{H(\delta)\} = H(1) = W_1 + 2W_2 + 2W_3 + 4W_4.$$

Since $G(1) \le H(1)$, the interior point of $n \in [0,2]$ is obtain, where maximum of F happens at $\tau = 1$ *and* $\delta = 1$. Therefore,

$$F(\tau,\delta) = F(1,1) = W_1 + 2W_2 + 2W_3 + 4W_4 = L(n).$$

By putting the value of $W_1 + W_2 + W_3 + W_4$ in the function L, we get

$$F(1,1) = L(n) = W_1 + 2W_2 + 2W_3 + 4W_4,$$

$$
\begin{aligned}
L(n) = & \frac{n^4\left[\left(3^v-2^{2v}\right)\left(2^u-2^v\right)+2^v\left(3^u-3^v\right)-\left(4^u-4^v\right)\right]X^4}{\left(4^u-4^v\right)\left(2^u-2^v\right)^4}+\\
& \frac{n^4X^2}{4\left(2^u-2^v\right)\left(4^u-4^v\right)}+\frac{nX^2\left(4-n^2\right)}{2\left(2^u-2^v\right)\left(4^u-4^v\right)}+\\
& 2\left[\frac{n^2(4-n^2)X^3}{8(2^u-2^v)^2\left(3^u-3^v\right)}+\frac{n^2(4-n^2)X^2}{4\left(2^u-2^v\right)\left(4^u-4^v\right)}\right]+\\
& 2\left[\frac{n^2(4-n^2)X^2}{8\left(2^u-2^v\right)\left(4^u-4^v\right)}-\frac{nX^2(4-n^2)}{4\left(2^u-2^v\right)\left(4^u-4^v\right)}\right]+\\
& 4\left[\frac{\left(4-n^2\right)^2X^2}{16(3^u-3^v)^2}\right],
\end{aligned}
$$

$$
\begin{aligned}
L(n) = & \frac{n^4\left[\left(3^v-2^{2v}\right)\left(2^u-2^v\right)+2^v\left(3^u-3^v\right)-\left(4^u-4^v\right)\right]X^4}{\left(4^u-4^v\right)\left(2^u-2^v\right)^4}+\\
& \frac{n^4X^2}{4\left(2^u-2^v\right)\left(4^u-4^v\right)}+\frac{4nX^2}{2\left(2^u-2^v\right)\left(4^u-4^v\right)}-\\
& \frac{n^3X^2}{2\left(2^u-2^v\right)\left(4^u-4^v\right)}+\frac{8n^2X^3}{8(2^u-2^v)^2\left(3^u-3^v\right)}-\\
& \frac{2n^4X^3}{8(2^u-2^v)^2\left(3^u-3^v\right)}+\frac{8n^2X^2}{4\left(2^u-2^v\right)\left(4^u-4^v\right)}-\\
& \frac{2n^4X^2}{4\left(2^u-2^v\right)\left(4^u-4^v\right)}+\frac{8n^2X^2}{8\left(2^u-2^v\right)\left(4^u-4^v\right)}-\\
& \frac{2n^4X^2}{8\left(2^u-2^v\right)\left(4^u-4^v\right)}-\frac{8nX^2}{4\left(2^u-2^v\right)\left(4^u-4^v\right)}+\\
& \frac{2n^3X^2}{4\left(2^u-2^v\right)\left(4^u-4^v\right)}+\frac{64X^2}{16(3^u-3^v)^2}+\\
& \frac{4n^4X^2}{16(3^u-3^v)^2}-\frac{32n^4X^2}{16(3^u-3^v)^2},
\end{aligned}
$$

$$
\begin{aligned}
L(n) =& \frac{n^4\left[\left(3^v-2^{2v}\right)\left(2^u-2^v\right)+2^v\left(3^u-3^v\right)-\left(4^u-4^v\right)\right]X^4}{\left(4^u-4^v\right)\left(2^u-2^v\right)^4}+\\
& \frac{n^4X^2}{4\left(2^u-2^v\right)\left(4^u-4^v\right)}+\frac{2nX^2}{\left(2^u-2^v\right)\left(4^u-4^v\right)}-\\
& \frac{n^3X^2}{2\left(2^u-2^v\right)\left(4^u-4^v\right)}+\frac{n^2X^3}{\left(2^u-2^v\right)^2\left(3^u-3^v\right)}-\\
& \frac{n^4X^3}{4(2^u-2^v)^2\left(3^u-3^v\right)}+\frac{2n^2X^2}{\left(2^u-2^v\right)\left(4^u-4^v\right)}-\\
& \frac{n^4X^2}{2\left(2^u-2^v\right)\left(4^u-4^v\right)}+\frac{n^2X^2}{\left(2^u-2^v\right)\left(4^u-4^v\right)}-\\
& \frac{n^4X^2}{4\left(2^u-2^v\right)\left(4^u-4^v\right)}-\frac{2nX^2}{\left(2^u-2^v\right)\left(4^u-4^v\right)}+\\
& \frac{n^3X^2}{2\left(2^u-2^v\right)\left(4^u-4^v\right)}+\frac{4X^2}{\left(3^u-3^v\right)^2}+\\
& \frac{n^4X^2}{4(3^u-3^v)^2}-\frac{2n^4X^2}{\left(3^u-3^v\right)^2},
\end{aligned}
$$

$$
\begin{aligned}
L(n) =& n^4\left[\frac{\left[\left(3^v-2^{2v}\right)\left(2^u-2^v\right)+2^v\left(3^u-3^v\right)-\left(4^u-4^v\right)\right]X^4}{\left(4^u-4^v\right)\left(2^u-2^v\right)^4}-\right.\\
& \left.\frac{X^3}{4(2^u-2^v)^2\left(3^u-3^v\right)}-\frac{X^2}{2\left(2^u-2^v\right)\left(4^u-4^v\right)}+\frac{X^2}{4(3^u-3^v)^2}\right]+\\
& n^2\left[\frac{X^3}{\left(2^u-2^v\right)^2\left(3^u-3^v\right)}+\frac{X^2}{\left(2^u-2^v\right)\left(4^u-4^v\right)}-\frac{2X^2}{\left(3^u-3^v\right)^2}\right]+\\
& \frac{4X^2}{\left(3^u-3^v\right)^2},
\end{aligned}
$$

$$
\begin{aligned}
L(n) =& \frac{X^2}{16\left(4^u-4^v\right)}\left\{n^4\left[\frac{16\left[\left(3^v-2^{2v}\right)\left(2^u-2^v\right)+2^v\left(3^u-3^v\right)-\left(4^u-4^v\right)\right]X^2}{\left(2^u-2^v\right)^4}-\right.\right.\\
& \left.\frac{4X\left(4^u-4^v\right)}{\left(2^u-2^v\right)^2\left(3^u-3^v\right)}-\frac{8}{\left(2^u-2^v\right)}+\frac{4\left(4^u-4^v\right)}{\left(3^u-3^v\right)^2}\right]+n^2\left[\frac{16\left(4^u-4^v\right)X}{\left(2^u-2^v\right)^2\left(3^u-3^v\right)}+\right.\\
& \left.\left.\frac{48}{\left(2^u-2^v\right)}-\frac{32\left(4^u-4^v\right)}{\left(3^u-3^v\right)^2}\right]+\frac{64\left(4^u-4^v\right)}{\left(3^u-3^v\right)^2}\right\},
\end{aligned}
$$

$$L(n) = \frac{X^2}{16(4^u-4^v)}\left\{n^4\left[\frac{16\left[\left(3^v-2^{2v}\right)(2^u-2^v)+2^v(3^u-3^v)-(4^u-4^v)\right]X^2}{(2^u-2^v)^4} - \frac{4X(4^u-4^v)}{(2^u-2^v)^2(3^u-3^v)} + \frac{4(4^u-4^v)(2^u-2^v)-8(3^u-3^v)^2}{(2^u-2^v)(3^u-3^v)^2}\right] + n^2\left[\frac{16(4^u-4^v)X}{(2^u-2^v)^2(3^u-3^v)} + \frac{48(3^u-3^v)^2-32(4^u-4^v)(2^u-2^v)}{(2^u-2^v)(3^u-3^v)^2}\right] + \frac{64(4^u-4^v)}{(3^u-3^v)^2}\right\}.$$

Suppose a maximum of $L(n)$ exists in an interior point n of [0,2]. By taking derivative of the function $L(n)$ with respect to n, this results in

$$L'(n) = \frac{X^2}{16(4^u-4^v)}\left\{4n^3\left[\frac{16\left[\left(3^v-2^{2v}\right)(2^u-2^v)+2^v(3^u-3^v)-(4^u-4^v)\right]X^2}{(2^u-2^v)^4} - \frac{4X(4^u-4^v)}{(2^u-2^v)^2(3^u-3^v)} + \frac{4(4^u-4^v)(2^u-2^v)-8(3^u-3^v)^2}{(2^u-2^v)(3^u-3^v)^2}\right] + 2n\left[\frac{16(4^u-4^v)X}{(2^u-2^v)^2(3^u-3^v)} + \frac{48(3^u-3^v)^2-32(4^u-4^v)(2^u-2^v)}{(2^u-2^v)(3^u-3^v)^2}\right]\right\},$$

By taking

$$\left[\frac{16\left[\left(3^v-2^{2v}\right)(2^u-2^v)+2^v(3^u-3^v)-(4^u-4^v)\right]X^2}{(2^u-2^v)^4} - \frac{4X(4^u-4^v)}{(2^u-2^v)^2(3^u-3^v)} + \frac{4(4^u-4^v)(2^u-2^v)-8(3^u-3^v)^2}{(2^u-2^v)(3^u-3^v)^2}\right] \geq 0.$$

Solving for the value of X leads to

$$X = \frac{\frac{4(4^u-4^v)}{(2^u-2^v)^2(3^u-3^v)} + \sqrt{\left(\frac{-4(4^u-4^v)}{(2^u-2^v)^2(3^u-3^v)}\right)^2 - 4\left(\frac{16\left[\left(3^v-2^{2v}\right)(2^u-2^v)+2^v(3^u-3^v)-(4^u-4^v)\right]}{(2^u-2^v)^4}\right)\left(\frac{4(4^u-4^v)(2^u-2^v)-8(3^u-3^v)^2}{(2^u-2^v)(3^u-3^v)^2}\right)}}{2\left(\frac{16\left[\left(3^v-2^{2v}\right)(2^u-2^v)+2^v(3^u-3^v)-(4^u-4^v)\right]}{(2^u-2^v)^4}\right)},$$

$$X = \frac{\frac{4(4^u-4^v)}{(2^u-2^v)^2(3^u-3^v)} + \frac{4(4^u-4^v)}{(2^u-2^v)^2(3^u-3^v)}\sqrt{1 - \frac{16\left[\left(3^v-2^{2v}\right)(2^u-2^v)+2^v(3^u-3^v)-(4^u-4^v)\right]\times\left[(4^u-4^v)(2^u-2^v)-2(3^u-3^v)^2\right]}{(4^u-4^v)^2(2^u-2^v)}}}{\left(\frac{32\left[\left(3^v-2^{2v}\right)(2^u-2^v)+2^v(3^u-3^v)-(4^u-4^v)\right]}{(2^u-2^v)^4}\right)},$$

$$X = \frac{(2^u-2^v)^2(4^u-4^v)}{8(3^u-3^v)\left[(3^v-2^{2v})(2^u-2^v)+2^v(3^u-3^v)-(4^u-4^v)\right]}\times$$

$$\left(1+\sqrt{1-\frac{16\left[\left(3^{v}-2^{2v}\right)(2^{u}-2^{v})+2^{v}(3^{u}-3^{v})-(4^{u}-4^{v})\right]\times\left[(4^{u}-4^{v})(2^{u}-2^{v})-2(3^{u}-3^{v})^{2}\right]}{(4^{u}-4^{v})^{2}(2^{u}-2^{v})}}\right)=\phi_{(u,v)},$$

therefore

$$X\in\left[0,\phi_{(u,v)}\right],$$

Therefore, $L'(n)>0$ for $n\in[0,2]$. As L is an increasing function in the interval [0,2] so the maximum point of L is on the boundary for $n=2$. Thus,

$$\max L(n)=L(2).$$

Thus,

$$L(2)=\frac{X^{2}}{16(4^{u}-4^{v})}\left\{16\left[\frac{16\left[\left(3^{v}-2^{2v}\right)(2^{u}-2^{v})+2^{v}(3^{u}-3^{v})-(4^{u}-4^{v})\right]X^{2}}{(2^{u}-2^{v})^{4}}-\frac{4X(4^{u}-4^{v})}{(2^{u}-2^{v})^{2}(3^{u}-3^{v})}-\frac{8}{(2^{u}-2^{v})}+\frac{4(4^{u}-4^{v})}{(3^{u}-3^{v})^{2}}\right]+4\left[\frac{16(4^{u}-4^{v})X}{(2^{u}-2^{v})^{2}(3^{u}-3^{v})}+\frac{48}{(2^{u}-2^{v})}-\frac{32(4^{u}-4^{v})}{(3^{u}-3^{v})^{2}}\right]+\frac{64(4^{u}-4^{v})}{(3^{u}-3^{v})^{2}}\right\},$$

$$L(2)=\frac{X^{2}}{(4^{u}-4^{v})}\left[\frac{16\left[\left(3^{v}-2^{2v}\right)(2^{u}-2^{v})+2^{v}(3^{u}-3^{v})-(4^{u}-4^{v})\right]X^{2}}{(2^{u}-2^{v})^{4}}-\frac{4X(4^{u}-4^{v})}{(2^{u}-2^{v})^{2}(3^{u}-3^{v})}-\frac{8}{(2^{u}-2^{v})}+\frac{4(4^{u}-4^{v})}{(3^{u}-3^{v})^{2}}+\frac{4(4^{u}-4^{v})X}{(2^{u}-2^{v})^{2}(3^{u}-3^{v})}+\frac{12}{(2^{u}-2^{v})}-\frac{8(4^{u}-4^{v})}{(3^{u}-3^{v})^{2}}+\frac{4(4^{u}-4^{v})}{(3^{u}-3^{v})^{2}}\right],$$

$$L(2)=\frac{4X^{2}}{(4^{u}-4^{v})}\left[\frac{4\left[\left(3^{v}-2^{2v}\right)(2^{u}-2^{v})+2^{v}(3^{u}-3^{v})-(4^{u}-4^{v})\right]X^{2}}{(2^{u}-2^{v})^{4}}+\frac{(3^{u}-3^{v})^{2}-(4^{u}-4^{v})(2^{u}-2^{v})}{(2^{u}-2^{v})(3^{u}-3^{v})^{2}}+\frac{(4^{u}-4^{v})}{(3^{u}-3^{v})^{2}}\right].$$

Also, by letting

$$\left[\frac{16\left[\left(3^{v}-2^{2v}\right)(2^{u}-2^{v})+2^{v}(3^{u}-3^{v})-(4^{u}-4^{v})\right]X^{2}}{(2^{u}-2^{v})^{4}}-\frac{4X(4^{u}-4^{v})}{(2^{u}-2^{v})^{2}(3^{u}-3^{v})}+\frac{4(4^{u}-4^{v})(2^{u}-2^{v})-8(3^{u}-3^{v})^{2}}{(2^{u}-2^{v})(3^{u}-3^{v})^{2}}\right]\geq 0.$$

That is , $X\in\left[\phi_{(u,v)},1\right]$. we note that $n_0<2$, that is n_0 is in the interval $[0,2]$. As $L'(n_0)\leq 0$, the maximum of $L(n)$ exist at $n=n_0$.

Therefore,

$$L'(n) = \frac{X^2}{16(4^u-4^v)}\left\{4n^3\left[\frac{16\left[\left(3^v-2^{2v}\right)(2^u-2^v)+2^v(3^u-3^v)-(4^u-4^v)\right]X^2}{(2^u-2^v)^4} - \frac{4X(4^u-4^v)}{(2^u-2^v)^2(3^u-3^v)} + \frac{4(4^u-4^v)(2^u-2^v)-8(3^u-3^v)^2}{(2^u-2^v)(3^u-3^v)^2}\right] + 2n\left[\frac{16(4^u-4^v)X}{(2^u-2^v)^2(3^u-3^v)} + \frac{48(3^u-3^v)^2-32(4^u-4^v)(2^u-2^v)}{(2^u-2^v)(3^u-3^v)^2}\right]\right\} = 0,$$

$$2n^2\left[\frac{16\left[\left(3^v-2^{2v}\right)(2^u-2^v)+2^v(3^u-3^v)-(4^u-4^v)\right]X^2}{(2^u-2^v)^4} - \frac{4X(4^u-4^v)}{(2^u-2^v)^2(3^u-3^v)} + \frac{4(4^u-4^v)(2^u-2^v)-8(3^u-3^v)^2}{(2^u-2^v)(3^u-3^v)^2}\right] = -\left[\frac{16(4^u-4^v)X}{(2^u-2^v)^2(3^u-3^v)} + \frac{48(3^u-3^v)^2-32(4^u-4^v)(2^u-2^v)}{(2^u-2^v)(3^u-3^v)^2}\right],$$

$$8n^2\left[\frac{4\left[\left(3^v-2^{2v}\right)(2^u-2^v)+2^v(3^u-3^v)-(4^u-4^v)\right]X^2}{(2^u-2^v)^4} - \frac{X(4^u-4^v)}{(2^u-2^v)^2(3^u-3^v)} + \frac{(4^u-4^v)(2^u-2^v)-2(3^u-3^v)^2}{(2^u-2^v)(3^u-3^v)^2}\right] = -8\left[\frac{2(4^u-4^v)X}{(2^u-2^v)^2(3^u-3^v)} + \frac{6(3^u-3^v)^2-4(4^u-4^v)(2^u-2^v)}{(2^u-2^v)(3^u-3^v)^2}\right],$$

$$n^2 = \frac{-\left[\frac{2(4^u-4^v)X}{(2^u-2^v)^2(3^u-3^v)} + \frac{6(3^u-3^v)^2-4(4^u-4^v)(2^u-2^v)}{(2^u-2^v)(3^u-3^v)^2}\right]}{\left[\frac{4\left[\left(3^v-2^{2v}\right)(2^u-2^v)+2^v(3^u-3^v)-(4^u-4^v)\right]X^2}{(2^u-2^v)^4} - \frac{X(4^u-4^v)}{(2^u-2^v)^2(3^u-3^v)} + \frac{(4^u-4^v)(2^u-2^v)-2(3^u-3^v)^2}{(2^u-2^v)(3^u-3^v)^2}\right]},$$

$$n_0 = \sqrt{\frac{-\left[\frac{2(4^u-4^v)X}{(2^u-2^v)^2(3^u-3^v)} + \frac{6(3^u-3^v)^2-4(4^u-4^v)(2^u-2^v)}{(2^u-2^v)(3^u-3^v)^2}\right]}{\left[\frac{4\left[\left(3^v-2^{2v}\right)(2^u-2^v)+2^v(3^u-3^v)-(4^u-4^v)\right]X^2}{(2^u-2^v)^4} - \frac{X(4^u-4^v)}{(2^u-2^v)^2(3^u-3^v)} + \frac{(4^u-4^v)(2^u-2^v)-2(3^u-3^v)^2}{(2^u-2^v)(3^u-3^v)^2}\right]}},$$

$$\max(L(n_0))$$

$$= L\left(\sqrt{\frac{-\left[\frac{2(4^u-4^v)X}{(2^u-2^v)^2(3^u-3^v)} + \frac{6(3^u-3^v)^2-4(4^u-4^v)(2^u-2^v)}{(2^u-2^v)(3^u-3^v)^2}\right]}{\left[\frac{4\left[\left(3^v-2^{2v}\right)(2^u-2^v)+2^v(3^u-3^v)-(4^u-4^v)\right]X^2}{(2^u-2^v)^4} - \frac{X(4^u-4^v)}{(2^u-2^v)^2(3^u-3^v)} + \frac{(4^u-4^v)(2^u-2^v)-2(3^u-3^v)^2}{(2^u-2^v)(3^u-3^v)^2}\right]}}\right),$$

$$
\begin{aligned}
&\max\left(L\left(n_0\right)\right)\\
&=\frac{X^2}{16\left(4^u-4^v\right)}\left\{\begin{array}{c}
\left(\frac{-\left[\frac{2(4^u-4^v)X}{(2^u-2^v)^2(3^u-3^v)}+\frac{6(3^u-3^v)^2-4(4^u-4^v)(2^u-2^v)}{(2^u-2^v)(3^u-3^v)^2}\right]}{\left[\frac{4\left[\left(3^v-2^{2v}\right)(2^u-2^v)+2^v(3^u-3^v)-(4^u-4^v)\right]X^2}{(2^u-2^v)^4}-\frac{X(4^u-4^v)}{(2^u-2^v)^2(3^u-3^v)}+\frac{(4^u-4^v)(2^u-2^v)-2(3^u-3^v)^2}{(2^u-2^v)(3^u-3^v)^2}\right]}\right)^2\\
\times\left[\begin{array}{c}\frac{16\left[\left(3^v-2^{2v}\right)(2^u-2^v)+2^v(3^u-3^v)-(4^u-4^v)\right]X^2}{(2^u-2^v)^4}\\
-\frac{4X(4^u-4^v)}{(2^u-2^v)^2(3^u-3^v)}+\frac{4(4^u-4^v)(2^u-2^v)-8(3^u-3^v)^2}{(2^u-2^v)(3^u-3^v)^2}\end{array}\right]\\
+\left(\frac{-\left[\frac{2(4^u-4^v)X}{(2^u-2^v)^2(3^u-3^v)}+\frac{6(3^u-3^v)^2-4(4^u-4^v)(2^u-2^v)}{(2^u-2^v)(3^u-3^v)^2}\right]}{\left[\frac{4\left[\left(3^v-2^{2v}\right)(2^u-2^v)+2^v(3^u-3^v)-(4^u-4^v)\right]X^2}{(2^u-2^v)^4}-\frac{X(4^u-4^v)}{(2^u-2^v)^2(3^u-3^v)}+\frac{(4^u-4^v)(2^u-2^v)-2(3^u-3^v)^2}{(2^u-2^v)(3^u-3^v)^2}\right]}\right)\\
\times\left[\frac{16(4^u-4^v)X}{(2^u-2^v)^2(3^u-3^v)}+\frac{48(3^u-3^v)^2-32(4^u-4^v)(2^u-2^v)}{(2^u-2^v)(3^u-3^v)^2}\right]\\
+\frac{64(4^u-4^v)}{(3^u-3^v)^2}
\end{array}\right\},
\end{aligned}
$$

$$
=\frac{X^2}{16\left(4^u-4^v\right)}\left\{\begin{array}{c}\frac{\left(\Pi+\frac{2X(2^u+2^v)}{(2^u-2^v)(3^u-3^v)}\right)^2}{(\Omega+\Sigma)^2}4(\Omega+\Sigma)-\frac{\left(\Pi+\frac{2X(2^u+2^v)}{(2^u-2^v)(3^u-3^v)}\right)}{(\Omega+\Sigma)}8\left(\Pi+\frac{2X(2^u+2^v)}{(2^u-2^v)(3^u-3^v)}\right)\\
+\frac{64(4^u-4^v)}{(3^u-3^v)^2}\end{array}\right\},
$$

$$
=\frac{X^2}{16\left(4^u-4^v\right)}\left\{\frac{64\left(4^u-4^v\right)}{\left(3^u-3^v\right)^2}+\frac{4\left(\Pi+\frac{2X(2^u+2^v)}{(2^u-2^v)(3^u-3^v)}\right)^2}{(\Omega+\Sigma)}-\frac{8\left(\Pi+\frac{2X(2^u+2^v)}{(2^u-2^v)(3^u-3^v)}\right)^2}{(\Omega+\Sigma)}\right\},
$$

$$
\left|\alpha_2\alpha_4-\alpha_3^2\right|\leq\frac{X^2}{16\left(4^u-4^v\right)}\left\{\frac{64\left(4^u-4^v\right)}{\left(3^u-3^v\right)^2}-\frac{4\left(\Pi+\frac{2X(2^u+2^v)}{(2^u-2^v)(3^u-3^v)}\right)^2}{(\Omega+\Sigma)}\right\},
$$

where

$$
\begin{aligned}
\Pi=&\frac{6(3^u-3^v)^2-4\left(4^u-4^v\right)\left(2^u-2^v\right)}{\left(2^u-2^v\right)\left(3^u-3^v\right)^2},\\
\Sigma=&-\frac{X\left(4^u-4^v\right)}{\left(2^u-2^v\right)^2\left(3^u-3^v\right)}+\frac{\left(4^u-4^v\right)\left(2^u-2^v\right)-2(3^u-3^v)^2}{\left(2^u-2^v\right)\left(3^u-3^v\right)^2},\\
\Omega=&\frac{4\left[\left(3^v-2^{2v}\right)\left(2^u-2^v\right)+2^v\left(3^u-3^v\right)-\left(4^u-4^v\right)\right]X^2}{\left(2^u-2^v\right)^4}.
\end{aligned}
$$

The Theorem 5.2.2 is complete. □

CHAPTER 6

ON A NEW CLASS OF q-BI STARLIKE FUNCTION USING q-SALAGEAN DIFFERENTIAL OPERATOR

6.1 Overview

This chapter will examine the new class of bi-stalike functions in q-calculus using the q-Salagean operator. The aim is to find the upper bounds of initial coefficients, the Fekete-Szego inequality, and the second Hankel determinant of the new class.

Definition 6.1.1. *If function* $f(\mathtt{s})$ *provided by* (1.1) *is considered to be in the class* $f \in \mathbb{S}^*_{\nabla,q}(\gamma,\Theta,u,v)$*, if the conditions listed below are hold:*
$f \in \nabla$ *and*

$$Re\left\{e^{i\Theta}\left[\frac{O^u_q f(\mathtt{s})}{O^v_q f(\mathtt{s})}\right]\right\} > \gamma,$$

$$(\mathtt{s} \in V; u > v,\ 0 \leq \gamma < 1,\ |\Theta| < \pi \ \ and \ \ cos\Theta > \gamma),$$

$$Re\left\{e^{i\Theta}\left[\frac{O^u_q g(\omega)}{O^v_q g(\omega)}\right]\right\} > \gamma,$$

$$(\omega \in V; u > v,\ 0 \leq \gamma < 1,\ |\Theta| < \pi \ \ and \ \ cos\Theta > \gamma),$$

where the function g has a series form of

$$g(\omega) = \omega - \alpha_2\omega^2 + \left(2\alpha_2^2 - \alpha_3\right)\omega^3 - \left(5\alpha_2^3 - 5\alpha_2\alpha_3 + \alpha_4\right)\omega^4 + \dots$$

6.2 Main Result

Theorem 6.2.1. If $f(\mathrm{s})$ provided by (1.1) be a part of the class $f \in \mathbb{S}^{*}_{\nabla,q}(\gamma,\Theta,u,v)$ for $u > v+1,\ 0 \leq \gamma < 1,\ |\Theta| < \pi$ and $cos\Theta > \gamma$. Then

$$|\alpha_2| \leq \frac{2X}{i}, \tag{6.1}$$

$$|\alpha_3| \leq \frac{4X^2}{i^2} + \frac{2X}{j}, \tag{6.2}$$

$$|\alpha_4| \leq \frac{8\,(l+k)\,X^3}{yi^4} + \frac{10X^2}{ij} + \frac{2X}{k}. \tag{6.3}$$

So for $\lambda \in C$,

$$\left|\alpha_3 - \lambda\alpha_2^2\right| \leq \begin{cases} \frac{X}{j} & 0 \leq |T(\lambda,u,v)| \leq \frac{1}{2j}, \\ 2\,|T(\lambda,u,v)|\,X & |T(\lambda,u,v)| \geq \frac{1}{2j}, \end{cases}$$

where

$$\begin{aligned} T\,(\lambda,u,v) =& \frac{1-\lambda}{2(j-z)}, \\ X =& cos\Theta - \gamma, \\ i =& (1+q)^u - (1+q)^v, \\ j =& (1+q+q^2)^u - (1+q+q^2)^v, \\ k =& (1+q+q^2+q^3)^u - (1+q+q^2+q^3)^v, \\ y =& (1+q)^u + (1+q)^v, \\ z =& (1+q)^{u+v} - (1+q)^{2v}, \\ l =& ((1+q+q^2)^v - (1+q)^{2v})((1+q)^u - (1+q)^v)(1+q)^v((1+q+q^2)^u - \\ & (1+q+q^2)^v) - ((1+q+q^2+q^3)^u - (1+q+q^2+q^3)^v). \end{aligned}$$

Proof. If $f \in \mathbb{S}^{*}_{\nabla,q}(\gamma,\Theta,u,v)$ and $g = f^{-1}$.
Then

$$\frac{O_q^u f(\mathrm{s})}{O_q^v f(\mathrm{s})} = 1 + \sum_{t=1}^{\infty} h_t \mathrm{s}^t,$$

and

$$\frac{O_q^u g(\omega)}{O_q^v g(\omega)} = 1 + \sum_{t=1}^{\infty} h_t \omega^t.$$

Therefore,

$$e^{i\Theta}\left[\frac{O_q^u f(\mathrm{s})}{O_q^v f(\mathrm{s})}\right]-\gamma=e^{i\Theta}\left(1+\sum_{t=1}^{\infty}h_t\mathrm{s}^t\right)-\gamma,$$

and

$$e^{i\Theta}\left[\frac{O_q^u g(\omega)}{O_q^v g(\omega)}\right]-\gamma=e^{i\Theta}\left(1+\sum_{t=1}^{\infty}h_t\omega^t\right)-\gamma.$$

Now by simplifying the equation, provides

$$e^{i\Theta}\left[\frac{O_q^u f(\mathrm{s})}{O_q^v f(\mathrm{s})}\right]-\gamma=e^{i\Theta}+e^{i\Theta}\left(\sum_{t=1}^{\infty}h_t\mathrm{s}^t\right)-\gamma,$$

and

$$e^{i\Theta}\left[\frac{O_q^u g(\omega)}{O_q^v g(\omega)}\right]-\gamma=e^{i\Theta}+e^{i\Theta}\left(\sum_{t=1}^{\infty}h_t\omega^t\right)-\gamma.$$

$$\left(\because e^{i\Theta}=cos\Theta+isin\Theta\right)$$

$$e^{i\Theta}\left[\frac{O_q^u f(\mathrm{s})}{O_q^v f(\mathrm{s})}\right]-\gamma=cos\Theta+isin\Theta+e^{i\Theta}\left(\sum_{t=1}^{\infty}h_t\mathrm{s}^t\right)-\gamma,$$

and

$$e^{i\Theta}\left[\frac{O_q^u g(\omega)}{O_q^v g(\omega)}\right]-\gamma=cos\Theta+isin\Theta+e^{i\Theta}\left(\sum_{t=1}^{\infty}h_t\omega^t\right)-\gamma.$$

$$e^{i\Theta}\left[\frac{O_q^u f(\mathrm{s})}{O_q^v f(\mathrm{s})}\right]-\gamma-isin\Theta=cos\Theta-\gamma+e^{i\Theta}\left(\sum_{t=1}^{\infty}h_t\mathrm{s}^t\right),$$

and

$$e^{i\Theta}\left[\frac{O_q^u g(\omega)}{O_q^v g(\omega)}\right]-\gamma-isin\Theta=cos\Theta-\gamma+e^{i\Theta}\left(\sum_{t=1}^{\infty}h_t\omega^t\right),$$

which gives

$$e^{i\Theta}\left[\frac{O_q^u f(\mathrm{s})}{O_q^v f(\mathrm{s})}\right]-\gamma-isin\Theta=cos\Theta-\gamma\left[1+\frac{e^{i\Theta}\left(\sum_{t=1}^{\infty}h_t\mathrm{s}^t\right)}{cos\Theta-\gamma}\right],$$

and

$$e^{i\Theta}\left[\frac{O_q^u g(\omega)}{O_q^v g(\omega)}\right]-\gamma-isin\Theta=cos\Theta-\gamma\left[1+\frac{e^{i\Theta}\left(\sum_{t=1}^{\infty}h_t\omega^t\right)}{cos\Theta-\gamma}\right].$$

$$\begin{aligned}\frac{e^{i\Theta}\left[\frac{O_q^u f(\mathrm{s})}{O_q^v f(\mathrm{s})}\right]-\gamma-isin\Theta}{cos\Theta-\gamma}&=1+\frac{e^{i\Theta}\left(\sum_{t=1}^{\infty}h_t\mathrm{s}^t\right)}{cos\Theta-\gamma},\\ \frac{e^{i\Theta}\left[\frac{O_q^u g(\omega)}{O_q^v g(\omega)}\right]-\gamma-isin\Theta}{cos\Theta-\gamma}&=1+\frac{e^{i\Theta}\left(\sum_{t=1}^{\infty}h_t\omega^t\right)}{cos\Theta-\gamma}.\end{aligned}\tag{6.4}$$

Hence, from (6.4) and lemma 2.13.1, this results in

$$\begin{aligned} \frac{e^{i\Theta}\left[\frac{O_q^u f(\mathrm{s})}{O_q^v f(\mathrm{s})}\right]-\gamma-i sin\Theta}{cos\Theta-\gamma} &= p(\mathrm{s}), \\ \frac{e^{i\Theta}\left[\frac{O_q^u g(\omega)}{O_q^v g(\omega)}\right]-\gamma-i sin\Theta}{cos\Theta-\gamma} &= q(\omega). \end{aligned} \tag{6.5}$$

$$\begin{aligned} O_q^u\, f(\mathrm{s}) &= \mathrm{s}+\sum_{t=2}^{\infty}[t]_q^u\alpha_t\mathrm{s}^t, \\ O_q^u\, f(\mathrm{s}) &= \mathrm{s}+[2]_q^u\alpha_2\mathrm{s}^2+[3]_q^u\alpha_3\mathrm{s}^3+[4]_q^u\alpha_4\mathrm{s}^4+\ldots, \end{aligned}$$

and

$$\begin{aligned} O_q^v\, f(\mathrm{s}) &= \mathrm{s}+\sum_{t=2}^{\infty}[t]_q^v\alpha_t\mathrm{s}^t, \\ O_q^v\, f(\mathrm{s}) &= \mathrm{s}+[2]_q^v\alpha_2\mathrm{s}^2+[3]_q^v\alpha_3\mathrm{s}^3+[4]_q^v\alpha_4\mathrm{s}^4+\ldots. \end{aligned}$$

Therefore,

$$\frac{O_q^u f(\mathrm{s})}{O_q^v f(\mathrm{s})}=\frac{\mathrm{s}+[2]_q^u\alpha_2\mathrm{s}^2+[3]_q^u\alpha_3\mathrm{s}^3+[4]_q^u\alpha_4\mathrm{s}^4+\ldots}{\mathrm{s}+[2]_q^v\alpha_2\mathrm{s}^2+[3]_q^v\alpha_3\mathrm{s}^3+[4]_q^v\alpha_4\mathrm{s}^4+\ldots},$$

$$\begin{aligned} \frac{O_q^u f(\mathrm{s})}{O_q^v f(\mathrm{s})} =& 1+\left([2]_q^u-[2]_q^v\right)\alpha_2\mathrm{s}+\left[\left([3]_q^u-[3]_q^v\right)\alpha_3-[2]_q^v\left([2]_q^u-[2]_q^v\right)\alpha_2^2\right]\mathrm{s}^2+ \\ &\left[\left([4]_q^u-[4]_q^v\right)\alpha_4+[2]_q^{2v}\left([2]_q^u-[2]_q^v\right)\alpha_2^3-[3]_q^v\left([2]_q^u-[2]_q^v\right)\alpha_2\alpha_3-\right. \\ &\left.[2]_q^v\left([3]_q^u-[3]_q^v\right)\alpha_2\alpha_3\right]\mathrm{s}^3+\ldots. \end{aligned}$$

Thus,

$$\frac{e^{i\Theta}\left\{\begin{array}{c} 1+\left([2]_q^u-[2]_q^v\right)\alpha_2\mathrm{s}+\left[\left([3]_q^u-[3]_q^v\right)\alpha_3-[2]_q^v\left([2]_q^u-[2]_q^v\right)\alpha_2^2\right]\mathrm{s}^2 \\ +\left[\begin{array}{c} \left([4]_q^u-[4]_q^v\right)\alpha_4+[2]_q^{2v}\left([2]_q^u-[2]_q^v\right)\alpha_2^3 \\ -[3]_q^v\left([2]_q^u-[2]_q^v\right)\alpha_2\alpha_3-[2]_q^v\left([3]_q^u-[3]_q^v\right)\alpha_2\alpha_3 \end{array}\right]\mathrm{s}^3+\ldots \end{array}\right\}-\gamma-i sin\Theta}{cos\Theta-\gamma}=p(\mathrm{s}),$$

and

$$\begin{aligned} O_q^u\, g(\omega) &= \omega-[2]_q^u\alpha_2\omega^2+[3]_q^u\left(2\alpha_2^2-\alpha_3\right)\omega^3-[4]_q^u\left(5\alpha_2^3-5\alpha_2\alpha_3+\alpha_4\right)\omega^4+\ldots, \\ O_q^v\, g(\omega) &= \omega-[2]_q^v\alpha_2\omega^2+[3]_q^v\left(2\alpha_2^2-\alpha_3\right)\omega^3-[4]_q^v\left(5\alpha_2^3-5\alpha_2\alpha_3+\alpha_4\right)\omega^4+\ldots, \end{aligned}$$

$$\begin{aligned} \frac{O_q^u g(\omega)}{O_q^v g(\omega)} =& 1-\left([2]_q^u-[2]_q^v\right)\alpha_2\omega+\left[\left([3]_q^u-[3]_q^v\right)\left(2\alpha_2^2-\alpha_3\right)-[2]_q^v\left([2]_q^u-[2]_q^v\right)\alpha_2^2\right]\omega^2 \\ &+\left[-\left([4]_q^u-[4]_q^v\right)\left(5\alpha_2^3-5\alpha_2\alpha_3+\alpha_4\right)-[3]_q^v\left([2]_q^u-[2]_q^v\right)\alpha_2\left(\alpha_3-2\alpha_2^2\right)-\right. \\ &\left.[2]_q^v\left([3]_q^u-[3]_q^v\right)\alpha_2\left(\alpha_3-2\alpha_2^2\right)-[2]_q^{2v}\left([2]_q^u-[2]_q^v\right)\alpha_2^3\right]\omega^3+\ldots, \end{aligned}$$

and

$$\frac{e^{i\Theta}\left\{\begin{array}{c} 1-\left([2]_q^u-[2]_q^v\right)\alpha_2\omega+\left[\left([3]_q^u-[3]_q^v\right)\left(2\alpha_2^2-\alpha_3\right)-[2]_q^v\left([2]_q^u-[2]_q^v\right)\alpha_2^2\right]\omega^2+ \\ \left[\begin{array}{c} -\left([4]_q^u-[4]_q^v\right)\left(5\alpha_2^3-5\alpha_2\alpha_3+\alpha_4\right)- \\ [3]_q^v\left([2]_q^u-[2]_q^v\right)\alpha_2\left(\alpha_3-2\alpha_2^2\right) \\ -[2]_q^v\left([3]_q^u-[3]_q^v\right)\alpha_2\left(\alpha_3-2\alpha_2^2\right)-[2]_q^{2v}\left([2]_q^u-[2]_q^v\right)\alpha_2^3 \end{array}\right]\omega^3+\ldots \end{array}\right\} -\gamma-isin\Theta}{cos\Theta-\gamma}=q(\omega),$$

where the functions $p(\mathrm{s})$ and $q(\omega)$ provided in the series form by

$$p(\mathrm{s})=1+n_1\mathrm{s}+n_2\mathrm{s}^2+\ldots,$$

and

$$q(\omega)=1+m_1\omega+m_2\omega^2+\ldots.$$

Comparing coefficients in (6.5), this gives

$$\mathrm{s}:\frac{e^{i\Theta}\left([2]_q^u-[2]_q^v\right)\alpha_2}{cos\Theta-\gamma}=n_1,$$

$$\alpha_2=\frac{n_1\left[cos\Theta-\gamma\right]}{e^{i\Theta}\left([2]_q^u-[2]_q^v\right)}, \tag{6.6}$$

$$\mathrm{s}^2:\frac{e^{i\Theta}\left[\left([3]_q^u-[3]_q^v\right)\alpha_3-[2]_q^v\left([2]_q^u-[2]_q^v\right)\alpha_2^2\right]}{cos\Theta-\gamma}=n_2,$$

$$\left[\left([3]_q^u-[3]_q^v\right)\alpha_3-[2]_q^v\left([2]_q^u-[2]_q^v\right)\alpha_2^2\right]=\frac{n_2\left(cos\Theta-\gamma\right)}{e^{i\Theta}}, \tag{6.7}$$

$$\mathrm{s}^3:\frac{e^{i\Theta}\left[\begin{array}{c}\left([4]_q^u-[4]_q^v\right)\alpha_4+[2]_q^{2v}\left([2]_q^u-[2]_q^v\right)\alpha_2^3- \\ [3]_q^v\left([2]_q^u-[2]_q^v\right)\alpha_2\alpha_3-[2]_q^v\left([3]_q^u-[3]_q^v\right)\alpha_2\alpha_3\end{array}\right]}{cos\Theta-\gamma}=n_3,$$

$$\left[\begin{array}{c}\left([4]_q^u-[4]_q^v\right)\alpha_4+[2]_q^{2v}\left([2]_q^u-[2]_q^v\right)\alpha_2^3- \\ [3]_q^v\left([2]_q^u-[2]_q^v\right)\alpha_2\alpha_3-[2]_q^v\left([3]_q^u-[3]_q^v\right)\alpha_2\alpha_3\end{array}\right]=\frac{n_3\left(cos\Theta-\gamma\right)}{e^{i\Theta}}. \tag{6.8}$$

$$\omega:\frac{-e^{i\Theta}\left([2]_q^u-[2]_q^v\right)\alpha_2}{cos\Theta-\gamma}=m_1,$$

$$-\alpha_2 = \frac{m_1\left(cos\Theta - \gamma\right)}{e^{i\Theta}\left([2]_q^u - [2]_q^v\right)}, \tag{6.9}$$

$$\omega^2 : \left([3]_q^u - [3]_q^v\right)\left(2\alpha_2^2 - \alpha_3\right) - [2]_q^v\left([2]_q^u - [2]_q^v\right)\alpha_2^2 = \frac{m_2\left(cos\Theta - \gamma\right)}{e^{i\Theta}},$$

$$[2\left([3]_q^u - [3]_q^v\right)\alpha_2^2 - \left([3]_q^u - [3]_q^v\right)\alpha_3 - [2]_q^v\left([2]_q^u - [2]_q^v\right)\alpha_2^2] = \frac{m_2\left(cos\Theta - \gamma\right)}{e^{i\Theta}}, \tag{6.10}$$

$$\omega^3 : \begin{array}{c} -\left([4]_q^u - [4]_q^v\right)\left(5\alpha_2^3 - 5\alpha_2\alpha_3 + \alpha_4\right) - \\ [3]_q^v\left([2]_q^u - [2]_q^v\right)\alpha_2\left(\alpha_3 - 2\alpha_2^2\right) \\ -[2]_q^v\left([3]_q^u - [3]_q^v\right)\alpha_2\left(\alpha_3 - 2\alpha_2^2\right) - [2]_q^{2v}\left([2]_q^u - [2]_q^v\right)\alpha_2^3 \end{array} = \frac{m_3\left(cos\Theta - \gamma\right)}{e^{i\Theta}}. \tag{6.11}$$

From (6.6) and (6.9), this results in

$$n_1 = -m_1, \tag{6.12}$$

and

$$\alpha_2 = \frac{n_1\left[cos\Theta - \gamma\right]}{e^{i\Theta}\left([2]_q^u - [2]_q^v\right)}. \tag{6.13}$$

By subtracting (6.7) and (6.10), this provides

$$\left[\begin{array}{c} \left([3]_q^u - [3]_q^v\right)\alpha_3 - [2]_q^v\left([2]_q^u - [2]_q^v\right)\alpha_2^2 \\ -2\left([3]_q^u - [3]_q^v\right)\alpha_2^2 + \\ \left([3]_q^u - [3]_q^v\right)\alpha_3 + [2]_q^v\left([2]_q^u - [2]_q^v\right)\alpha_2^2 \end{array}\right] = \frac{n_2\left(cos\Theta - \gamma\right)}{e^{i\Theta}} - \frac{m_2\left(cos\Theta - \gamma\right)}{e^{i\Theta}},$$

$$2\left([3]_q^u - [3]_q^v\right)\alpha_3 - 2\left([3]_q^u - [3]_q^v\right)\alpha_2^2 = \frac{\left(n_2 - m_2\right)\left(cos\Theta - \gamma\right)}{e^{i\Theta}},$$

By using equation (6.13), this gives

$$2\left([3]_q^u - [3]_q^v\right)\alpha_3 - 2\left([3]_q^u - [3]_q^v\right)\left(\frac{n_1^2(cos\Theta - \gamma)^2}{e^{2i\Theta}\left([2]_q^u - [2]_q^v\right)^2}\right) = \frac{\left(n_2 - m_2\right)\left(cos\Theta - \gamma\right)}{e^{i\Theta}},$$

$$\alpha_3 - \frac{n_1^2(cos\Theta - \gamma)^2}{e^{2i\Theta}\left([2]_q^u - [2]_q^v\right)^2} = \frac{\left(n_2 - m_2\right)\left(cos\Theta - \gamma\right)}{2\left([3]_q^u - [3]_q^v\right)e^{i\Theta}},$$

$$\alpha_3 = \frac{n_1^2(cos\Theta - \gamma)^2}{e^{2i\Theta}\left([2]_q^u - [2]_q^v\right)^2} + \frac{\left(n_2 - m_2\right)\left(cos\Theta - \gamma\right)}{2\left([3]_q^u - [3]_q^v\right)e^{i\Theta}}, \tag{6.14}$$

$$\left[\begin{array}{c}\left([4]_q^u-[4]_q^v\right)\alpha_4+[2]_q^{2v}\left([2]_q^u-[2]_q^v\right)\alpha_2^3-\\ [3]_q^v\left([2]_q^u-[2]_q^v\right)\alpha_2\alpha_3-[2]_q^v\left([3]_q^u-[3]_q^v\right)\alpha_2\alpha_3\end{array}\right]=\frac{n_3\left(cos\Theta-\gamma\right)}{e^{i\Theta}},$$

$$\begin{array}{c}-5\left([4]_q^u-[4]_q^v\right)\ \alpha_2^3+5\left([4]_q^u-[4]_q^v\right)\alpha_2\alpha_3-\\ \left([4]_q^u-[4]_q^v\right)\alpha_4-[3]_q^v\left([2]_q^u-[2]_q^v\right)\left(\alpha_2\alpha_3-2\alpha_2^3\right)-\\ [2]_q^{2v}\left([2]_q^u-[2]_q^v\right)\alpha_2^3-[2]_q^v\left([3]_q^u-[3]_q^v\right)\left(\alpha_2\alpha_3-2\alpha_2^3\right)\end{array}=\frac{m_3\left(cos\Theta-\gamma\right)}{e^{i\Theta}}.$$

Subtracting (6.8) and (6.11), this provides

$$\left([4]_q^u-[4]_q^v\right)\alpha_4+[2]_q^{2v}\left([2]_q^u-[2]_q^v\right)\alpha_2^3-[3]_q^v\left([2]_q^u-[2]_q^v\right)\alpha_2\alpha_3-[2]_q^v\left([3]_q^u-[3]_q^v\right)\alpha_2\alpha_3$$

$$+5\left([4]_q^u-[4]_q^v\right)\ \alpha_2^3-5\left([4]_q^u-[4]_q^v\right)\alpha_2\alpha_3+\left([4]_q^u-[4]_q^v\right)\alpha_4+[3]_q^v\left([2]_q^u-[2]_q^v\right)\left(\alpha_2\alpha_3-2\alpha_2^3\right)+$$

$$[2]_q^{2v}\left([2]_q^u-[2]_q^v\right)\alpha_2^3+[2]_q^v\left([3]_q^u-[3]_q^v\right)\left(\alpha_2\alpha_3-2\alpha_2^3\right)$$

$$=\frac{\left(n_3-m_3\right)\left(cos\Theta-\gamma\right)}{e^{i\Theta}},$$

$$\left([4]_q^u-[4]_q^v\right)\alpha_4+[2]_q^{2v}\left([2]_q^u-[2]_q^v\right)\alpha_2^3-[3]_q^v\left([2]_q^u-[2]_q^v\right)\alpha_2\alpha_3-[2]_q^v\left([3]_q^u-[3]_q^v\right)\alpha_2\alpha_3$$

$$+5\left([4]_q^u-[4]_q^v\right)\ \alpha_2^3-5\left([4]_q^u-[4]_q^v\right)\alpha_2\alpha_3+\left([4]_q^u-[4]_q^v\right)\alpha_4+[3]_q^v\left([2]_q^u-[2]_q^v\right)\alpha_2\alpha_3$$

$$-2\,[3]_q^v\left([2]_q^u-[2]_q^v\right)\alpha_2^3+[2]_q^{2v}\left([2]_q^u-[2]_q^v\right)\alpha_2^3+[2]_q^v\left([3]_q^u-[3]_q^v\right)\alpha_2\alpha_3-2[2]_q^v\left([3]_q^u-[3]_q^v\right)\alpha_2^3$$

$$=\frac{\left(n_3-m_3\right)\left(cos\Theta-\gamma\right)}{e^{i\Theta}},$$

$$\begin{array}{l}2\left([4]_q^u-[4]_q^v\right)\alpha_4+\Big[[2]_q^{2v}\left([2]_q^u-[2]_q^v\right)+5\ \left([4]_q^u-[4]_q^v\right)\\ -2\,[3]_q^v\left([2]_q^u-[2]_q^v\right)+[2]_q^{2v}\left([2]_q^u-[2]_q^v\right)-2[2]_q^v\left([3]_q^u-[3]_q^v\right)\Big]\,\alpha_2^3\\ -5\left([4]_q^u-[4]_q^v\right)\alpha_2\alpha_3=\dfrac{\left(n_3-m_3\right)\left(cos\Theta-\gamma\right)}{e^{i\Theta}},\end{array}$$

$$\begin{array}{l}2\left([4]_q^u-[4]_q^v\right)\alpha_4+\Big[[2]_q^{2v}\left([2]_q^u-[2]_q^v\right)+5\ \left([4]_q^u-[4]_q^v\right)-2\,[3]_q^v\left([2]_q^u-[2]_q^v\right)+\\ [2]_q^{2v}\left([2]_q^u-[2]_q^v\right)-2[2]_q^v\left([3]_q^u-[3]_q^v\right)\Big]\left(\dfrac{n_1^3(cos\Theta-\gamma)^3}{e^{3i\Theta}\left([2]_q^u-[2]_q^v\right)^3}\right)-\\ 5\left([4]_q^u-[4]_q^v\right)\left(\dfrac{n_1\left[cos\Theta-\gamma\right]}{e^{i\Theta}\left([2]_q^u-[2]_q^v\right)}\right)\left(\dfrac{n_1^2(cos\Theta-\gamma)^2}{e^{2i\Theta}\left([2]_q^u-[2]_q^v\right)^2}+\dfrac{\left(n_2-m_2\right)\left(cos\Theta-\gamma\right)}{2\left([3]_q^u-[3]_q^v\right)e^{i\Theta}}\right)\\ =\dfrac{\left(n_3-m_3\right)\left(cos\Theta-\gamma\right)}{e^{i\Theta}},\end{array}$$

$$2\left([4]_q^u-[4]_q^v\right)\alpha_4+\Big[2\times[2]_q^{2v}\left([2]_q^u-[2]_q^v\right)$$
$$-2\times[3]_q^v\left([2]_q^u-[2]_q^v\right)-2\times[2]_q^v\left([3]_q^u-[3]_q^v\right)\Big]\left(\frac{n_1^3(cos\Theta-\gamma)^3}{e^{3i\Theta}\left([2]_q^u-[2]_q^v\right)^3}\right)$$
$$-5\left([4]_q^u-[4]_q^v\right)\left(\frac{n_1(n_2-m_2)\,(cos\Theta-\gamma)^2}{2\left([3]_q^u-[3]_q^v\right)e^{2i\Theta}\left([2]_q^u-[2]_q^v\right)}\right)=\frac{(n_3-m_3)\,(cos\Theta-\gamma)}{e^{i\Theta}},$$

$$\alpha_4=\Big[-2\times[2]_q^{2v}\left([2]_q^u-[2]_q^v\right)+2\times[3]_q^v\left([2]_q^u-[2]_q^v\right)+$$
$$2\times[2]_q^v\left([3]_q^u-[3]_q^v\right)\Big]\left(\frac{n_1^3(cos\Theta-\gamma)^3}{2\left([4]_q^u-[4]_q^v\right)e^{3i\Theta}\left([2]_q^u-[2]_q^v\right)^3}\right)+$$
$$\frac{5n_1(n_2-m_2)\,(cos\Theta-\gamma)^2}{4\left([3]_q^u-[3]_q^v\right)e^{2i\Theta}\left([2]_q^u-[2]_q^v\right)}+\frac{(n_3-m_3)\,(cos\Theta-\gamma)}{2\left([4]_q^u-[4]_q^v\right)e^{i\Theta}},$$

$$\alpha_4=\frac{n_1^3\Big[([3]_q^v-[2]_q^{2v})\left([2]_q^u-[2]_q^v\right)+[2]_q^v\left([3]_q^u-[3]_q^v\right)\Big](cos\Theta-\gamma)^3}{\left([4]_q^u-[4]_q^v\right)e^{3i\Theta}\left([2]_q^u-[2]_q^v\right)^3}+$$
$$\frac{5n_1(n_2-m_2)\,(cos\Theta-\gamma)^2}{4\left([3]_q^u-[3]_q^v\right)e^{2i\Theta}\left([2]_q^u-[2]_q^v\right)}+\frac{(n_3-m_3)\,(cos\Theta-\gamma)}{2\left([4]_q^u-[4]_q^v\right)e^{i\Theta}}. \tag{6.15}$$

For using Lemma 2.13.1 to (6.13), this gives (6.1)

$$|\alpha_2|=\left|\frac{n_1\,[cos\Theta-\gamma]}{e^{i\Theta}\left([2]_q^u-[2]_q^v\right)}\right|,$$

$$|\alpha_2|\leq\frac{2(cos\Theta-\gamma)}{\left([2]_q^u-[2]_q^v\right)},$$

$$\left(\because\left|e^{i\Theta}\right|=\sqrt{cos^2\Theta+sin^2\Theta}=1\right).$$

Using Lemma 2.13.1 to (6.14), this gives (6.2)

$$|\alpha_3|=\left|\frac{n_1^2(cos\Theta-\gamma)^2}{e^{2i\Theta}\left([2]_q^u-[2]_q^v\right)^2}+\frac{(n_2-m_2)\,(cos\Theta-\gamma)}{2\left([3]_q^u-[3]_q^v\right)e^{i\Theta}}\right|,$$
$$|\alpha_3|\leq\frac{4(cos\Theta-\gamma)^2}{\left([2]_q^u-[2]_q^v\right)^2}+\frac{2\,(cos\Theta-\gamma)}{\left([3]_q^u-[3]_q^v\right)}.$$

Using Lemma 2.13.1 to (6.15), this gives (6.3)

$$|\alpha_4| = \left| \frac{n_1^3\left[([3]_q^v - [2]_q^{2v})\left([2]_q^u - [2]_q^v\right) + [2]_q^v\left([3]_q^u - [3]_q^v\right)\right](cos\Theta-\gamma)^3}{\left([4]_q^u - [4]_q^v\right)e^{3i\Theta}\left([2]_q^u - [2]_q^v\right)^3} + \frac{5n_1(n_2-m_2)\,(cos\Theta-\gamma)^2}{2\left([3]_q^u - [3]_q^v\right)e^{2i\Theta}\left([2]_q^u - [2]_q^v\right)} + \frac{(n_3-m_3)\,(cos\Theta-\gamma)}{2\left([4]_q^u - [4]_q^v\right)e^{i\Theta}} \right|,$$

$$|\alpha_4| \leq \frac{8\left[([3]_q^v - [2]_q^{2v})\left([2]_q^u - [2]_q^v\right) + [2]_q^v\left([3]_q^u - [3]_q^v\right)\right](cos\Theta-\gamma)^3}{\left([2]_q^u + [2]_q^v\right)\left([2]_q^u - [2]_q^v\right)^4} + \frac{10(cos\Theta-\gamma)^2}{\left([3]_q^u - [3]_q^v\right)\left([2]_q^u - [2]_q^v\right)} + \frac{2\,(cos\Theta-\gamma)}{\left([4]_q^u - [4]_q^v\right)}.$$

By adding (6.7) and (6.10), this leads to

$$\left[\begin{array}{c} \left([3]_q^u - [3]_q^v\right)\alpha_3 - \\ [2]_q^v\left([2]_q^u - [2]_q^v\right)\alpha_2^2 + 2\left([3]_q^u - [3]_q^v\right)\alpha_2^2 - \\ \left([3]_q^u - [3]_q^v\right)\alpha_3 - [2]_q^v\left([2]_q^u - [2]_q^v\right)\alpha_2^2 \end{array}\right] = \frac{(n_2+m_2)\,(cos\Theta-\gamma)}{e^{i\Theta}},$$

$$\left[-2\times[2]_q^v\left([2]_q^u - [2]_q^v\right)\alpha_2^2 + 2\times\left([3]_q^u - [3]_q^v\right)\alpha_2^2\right] = \frac{(n_2+m_2)\,(cos\Theta-\gamma)}{e^{i\Theta}},$$

$$2\times\left[\left([3]_q^u - [3]_q^v\right) - [2]_q^v\left([2]_q^u - [2]_q^v\right)\right]\alpha_2^2 = \frac{(n_2+m_2)\,(cos\Theta-\gamma)}{e^{i\Theta}}. \tag{6.16}$$

By subtracting (6.7) and (6.10), this leads to

$$\alpha_3 = \frac{n_1^2(cos\Theta-\gamma)^2}{e^{2i\Theta}\left([2]_q^u - [2]_q^v\right)^2} + \frac{(n_2-m_2)\,(cos\Theta-\gamma)}{2\left([3]_q^u - [3]_q^v\right)e^{i\Theta}},$$

$$\alpha_3 = \frac{(n_2-m_2)\,(cos\Theta-\gamma)}{2\left([3]_q^u - [3]_q^v\right)e^{i\Theta}} + \alpha_2^2. \tag{6.17}$$

From equations (6.16) and (6.17), gives us

$$\alpha_3 - \lambda\alpha_2^2 = \frac{(n_2-m_2)\,(cos\Theta-\gamma)}{2\left([3]_q^u - [3]_q^v\right)e^{i\Theta}} + \alpha_2^2 - \lambda\alpha_2^2,$$

$$\alpha_3 - \lambda\alpha_2^2 = \frac{(n_2-m_2)\,(cos\Theta-\gamma)}{2\left([3]_q^u - [3]_q^v\right)e^{i\Theta}} + (1-\lambda)\alpha_2^2,$$

$$\alpha_3 - \lambda\alpha_2^2 = \frac{(n_2-m_2)\,(cos\Theta-\gamma)}{2\left([3]_q^u - [3]_q^v\right)e^{i\Theta}} + (1-\lambda)\frac{(n_2+m_2)\,(cos\Theta-\gamma)}{2\left[\left([3]_q^u - [3]_q^v\right) - [2]_q^v\left([2]_q^u - [2]_q^v\right)\right]e^{i\Theta}},$$

$$\alpha_3-\lambda\alpha_2^2=\frac{(cos\Theta-\gamma)}{e^{i\Theta}}\left\{\frac{(n_2-m_2)}{2\left([3]_q^u-[3]_q^v\right)}+(1-\lambda)\frac{(n_2+m_2)}{2\left[\left([3]_q^u-[3]_q^v\right)-[2]_q^v\left([2]_q^u-[2]_q^v\right)\right]}\right\},$$

$$\alpha_3-\lambda\alpha_2^2=\frac{(cos\Theta-\gamma)}{e^{i\Theta}}\left\{\left(\frac{(1-\lambda)}{2\left[\left([3]_q^u-[3]_q^v\right)-[2]_q^v\left([2]_q^u-[2]_q^v\right)\right]}+\frac{1}{2\left([3]_q^u-[3]_q^v\right)}\right)n_2\right.$$
$$\left.+\left(\frac{(1-\lambda)}{2\left[\left([3]_q^u-[3]_q^v\right)-[2]_q^v\left([2]_q^u-[2]_q^v\right)\right]}-\frac{1}{2\left([3]_q^u-[3]_q^v\right)}\right)m_2\right\},$$

$$\alpha_3-\lambda\alpha_2^2=\frac{(cos\Theta-\gamma)}{e^{i\Theta}}\left\{\left(T(\lambda,u,v)+\frac{1}{2\left([3]_q^u-[3]_q^v\right)}\right)n_2+\right.$$
$$\left.\left(T(\lambda,u,v)-\frac{1}{2\left([3]_q^u-[3]_q^v\right)}\right)m_2\right\},$$

where,

$$T(\lambda,u,v)=\frac{(1-\lambda)}{2\left[\left([3]_q^u-[3]_q^v\right)-[2]_q^v\left([2]_q^u-[2]_q^v\right)\right]}.$$

The Theorem 6.2.1 is complete. □

For $q\to 1^-$, using this value in the above result gives an advanced result that perfectly aligns with the previous findings by Orhan et al. [82], as shown in the given corollary.

Corollary 6.2.1.1. If $f(\mathrm{s})$ provided by (1.1) be a part of the class $f\in\mathbb{S}_{\nabla}^{*}(\gamma,\Theta,u,v)$ for $u>v+1,\ 0\le\gamma<1,\ |\Theta|<\pi$ and $cos\Theta>\gamma$. Then

$$|\alpha_2|\le\frac{2(cos\Theta-\gamma)}{2^u-2^v},$$
$$|\alpha_3|\le\frac{4(cos\Theta-\gamma)^2}{(2^u-2^v)^2}+\frac{2(cos\Theta-\gamma)}{3^u-3^v},$$
$$|\alpha_4|\le\frac{8(cos\Theta-\gamma)^3\left[(2^u-2^v)\left(3^v-2^{2v}\right)+2^v(3^u-3^v)\right]}{(2^u+2^v)(2^u-2^v)^4}+\frac{10(cos\Theta-\gamma)^2}{(2^u-2^v)(3^u-3^v)}+\frac{2(cos\Theta-\gamma)}{4^u-4^v}.$$

$$For\ \ \lambda\in\ \ C,\quad \left|\alpha_3-\lambda\alpha_2^2\right|\le\begin{cases}\frac{cos\Theta-\gamma}{3^u-3^v}, & 0\le|T(\lambda,u,v)|\le\frac{1}{2(3^u-3^v)},\\ 2|T(\lambda,u,v)|\left[cos\Theta-\gamma\right], & |T(\lambda,u,v)|\ge\frac{1}{2(3^u-3^v)},\end{cases}$$

where

$$T(\lambda,u,v)=\frac{1-\lambda}{2\left[(3^u-3^v)-(2^{u+v}-2^{2v})\right]}.$$

Theorem 6.2.2. If $f(\mathrm{s})$ provided by (1.1) be a part of the class $f \in \mathbb{S}^{*}_{\nabla,q}(\gamma,\Theta,u,v)$ and if $X = cos\Theta - \gamma$ for $u > v+1, l \neq 0,\ |\Theta| < \pi$ and $cos\Theta > \gamma$. Then

$$\left|\alpha_2\alpha_4 - \alpha_3^2\right| \leq \begin{cases} \frac{4X^2}{k}\left\{\frac{4lX^2}{i^4} + \frac{1}{i}\right\}, & X \in \left[0, \phi_{(u,v)}\right], \\ \frac{X^2}{16k}\left\{\frac{64k}{j^2} - \frac{4\left(\Pi + \frac{2yX}{ij}\right)^2}{(\Omega+\Sigma)}\right\}, & X \in \left[\phi_{(u,v)}, 1\right], \end{cases}$$

where

$$\begin{aligned}
\phi_{(u,v)} =& \frac{ki^2}{8jl}\left(1 + \sqrt{1 - \frac{16l(ki - 8j^2)}{ik^2}}\right), \\
\Pi =& \frac{6j^2 - 4ik}{ij^2}, \\
\Sigma =& \frac{Xk}{i^2 j} + \frac{ki - 2j^2}{ij^2}, \\
\Omega =& \frac{4lX^2}{i^4}, \\
i =& (1+q)^u - (1+q)^v, \\
j =& (1+q+q^2)^u - (1+q+q^2)^v, \\
k =& (1+q+q^2+q^3)^u - (1+q+q^2+q^3)^v, \\
y =& (1+q)^u + (1+q)^v, \\
l =& ((1+q+q^2)^v - (1+q)^{2v})((1+q)^u - (1+q)^v)(1+q)^v((1+q+q^2)^u - \\
& (1+q+q^2)^v) - ((1+q+q^2+q^3)^u - (1+q+q^2+q^3)^v).
\end{aligned}$$

Proof. From (6.13), (6.14) and (6.15) and taking $X = cos\Theta - \gamma$, this leads to

$$\alpha_2 = \frac{n_1\left[cos\Theta - \gamma\right]}{e^{i\Theta}\left([2]_q^u - [2]_q^v\right)},$$

$$\alpha_2 = \frac{n_1 X}{e^{i\Theta}\left([2]_q^u - [2]_q^v\right)},$$

$$\alpha_3 = \frac{n_1^2(cos\Theta - \gamma)^2}{e^{2i\Theta}\left([2]_q^u - [2]_q^v\right)^2} + \frac{(n_2 - m_2)(cos\Theta - \gamma)}{2\left([3]_q^u - [3]_q^v\right)e^{i\Theta}},$$

$$\alpha_3 = \frac{n_1^2 X^2}{e^{2i\Theta}\left([2]_q^u - [2]_q^v\right)^2} + \frac{(n_2 - m_2)X}{2\left([3]_q^u - [3]_q^v\right)e^{i\Theta}},$$

$$\alpha_4 = \frac{n_1^3\left[([3]_q^v - [2]_q^{2v})\left([2]_q^u - [2]_q^v\right) + [2]_q^v\left([3]_q^u - [3]_q^v\right)\right](cos\Theta - \gamma)^3}{\left([4]_q^u - [4]_q^v\right)e^{3i\Theta}\left([2]_q^u - [2]_q^v\right)^3} + \frac{5n_1(n_2 - m_2)\,(cos\Theta - \gamma)^2}{4\left([3]_q^u - [3]_q^v\right)e^{2i\Theta}\left([2]_q^u - [2]_q^v\right)} + \frac{(n_3 - m_3)\,(cos\Theta - \gamma)}{2\left([4]_q^u - [4]_q^v\right)e^{i\Theta}},$$

$$\alpha_4 = \frac{n_1^3\left[([3]_q^v - [2]_q^{2v})\left([2]_q^u - [2]_q^v\right) + [2]_q^v\left([3]_q^u - [3]_q^v\right)\right]X^3}{\left([4]_q^u - [4]_q^v\right)e^{3i\Theta}\left([2]_q^u - [2]_q^v\right)^3} + \frac{5n_1(n_2 - m_2)\,X^2}{4\left([3]_q^u - [3]_q^v\right)e^{2i\Theta}\left([2]_q^u - [2]_q^v\right)} + \frac{(n_3 - m_3)X}{2\left([4]_q^u - [4]_q^v\right)e^{i\Theta}}.$$

Therefore, the functional $\alpha_2\alpha_4 - \alpha_3^2$ will become

$$\alpha_2\alpha_4 - \alpha_3^2 = \left(\frac{n_1X}{e^{i\Theta}\left([2]_q^u - [2]_q^v\right)}\right)\left\{\frac{n_1^3\left[([3]_q^v - [2]_q^{2v})\left([2]_q^u - [2]_q^v\right) + [2]_q^v\left([3]_q^u - [3]_q^v\right)\right]X^3}{\left([4]_q^u - [4]_q^v\right)e^{3i\Theta}\left([2]_q^u - [2]_q^v\right)^3} + \frac{5n_1(n_2 - m_2)\,X^2}{4\left([3]_q^u - [3]_q^v\right)e^{2i\Theta}\left([2]_q^u - [2]_q^v\right)} + \frac{(n_3 - m_3)X}{2\left([4]_q^u - [4]_q^v\right)e^{i\Theta}}\right\} - \left(\frac{n_1^2X^2}{e^{2i\Theta}\left([2]_q^u - [2]_q^v\right)^2} + \frac{(n_2 - m_2)X}{2\left([3]_q^u - [3]_q^v\right)e^{i\Theta}}\right)^2,$$

$$\alpha_2\alpha_4 - \alpha_3^2 = \left\{\frac{n_1^4\left[([3]_q^v - [2]_q^{2v})\left([2]_q^u - [2]_q^v\right) + [2]_q^v\left([3]_q^u - [3]_q^v\right)\right]X^4}{\left([4]_q^u - [4]_q^v\right)e^{4i\Theta}\left([2]_q^u - [2]_q^v\right)^4} + \frac{5n_1^2(n_2 - m_2)\,X^3}{4\left([2]_q^u - [2]_q^v\right)^2\left([3]_q^u - [3]_q^v\right)e^{3i\Theta}} + \frac{n_1\,(n_3 - m_3)X^2}{2\left([4]_q^u - [4]_q^v\right)\left([2]_q^u - [2]_q^v\right)e^{2i\Theta}}\right\} - \left(\frac{n_1^2X^2}{e^{2i\Theta}\left([2]_q^u - [2]_q^v\right)^2} + \frac{(n_2 - m_2)X}{2\left([3]_q^u - [3]_q^v\right)e^{i\Theta}}\right)^2,$$

$$\alpha_2\alpha_4-\alpha_3^2=\frac{n_1^4\left[([3]_q^v-[2]_q^{2v})\left([2]_q^u-[2]_q^v\right)+[2]_q^v\left([3]_q^u-[3]_q^v\right)\right]X^4}{\left([4]_q^u-[4]_q^v\right)e^{4i\Theta}\left([2]_q^u-[2]_q^v\right)^4}$$
$$+\frac{5n_1^2(n_2-m_2)\,X^3}{4\left([2]_q^u-[2]_q^v\right)^2\left([3]_q^u-[3]_q^v\right)e^{3i\Theta}}+\frac{n_1\,(n_3-m_3)\,X^2}{2\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)e^{2i\Theta}}-$$
$$\frac{n_1^4X^4}{e^{4i\Theta}\left([2]_q^u-[2]_q^v\right)^4}-\frac{X^2(n_2-m_2)^2}{4\left([3]_q^u-[3]_q^v\right)^2e^{2i\Theta}}-\frac{n_1^2(n_2-m_2)X^3}{\left([2]_q^u-[2]_q^v\right)^2\left([3]_q^u-[3]_q^v\right)e^{3i\Theta}},$$

$$\alpha_2\alpha_4-\alpha_3^2=\frac{n_1^4\left[([3]_q^v-[2]_q^{2v})\left([2]_q^u-[2]_q^v\right)+[2]_q^v\left([3]_q^u-[3]_q^v\right)-\left([4]_q^u-[4]_q^v\right)\right]X^4}{\left([4]_q^u-[4]_q^v\right)e^{4i\Theta}\left([2]_q^u-[2]_q^v\right)^4}+$$
$$\left(\frac{5}{4}-1\right)\frac{n_1^2(n_2-m_2)\,X^3}{\left([2]_q^u-[2]_q^v\right)^2\left([3]_q^u-[3]_q^v\right)e^{3i\Theta}}+$$
$$\frac{n_1\,(n_3-m_3)\,X^2}{2\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)e^{2i\Theta}}-\frac{X^2(n_2-m_2)^2}{4\left([3]_q^u-[3]_q^v\right)^2e^{2i\Theta}},$$

$$\alpha_2\alpha_4-\alpha_3^2=\frac{n_1^4\left[([3]_q^v-[2]_q^{2v})\left([2]_q^u-[2]_q^v\right)+[2]_q^v\left([3]_q^u-[3]_q^v\right)-\left([4]_q^u-[4]_q^v\right)\right]X^4}{\left([4]_q^u-[4]_q^v\right)e^{4i\Theta}\left([2]_q^u-[2]_q^v\right)^4}+$$
$$\frac{n_1^2(n_2-m_2)\,X^3}{4\left([2]_q^u-[2]_q^v\right)^2\left([3]_q^u-[3]_q^v\right)e^{3i\Theta}}+\frac{n_1\,(n_3-m_3)\,X^2}{2\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)e^{2i\Theta}}- \tag{6.18}$$
$$\frac{X^2(n_2-m_2)^2}{4\left([3]_q^u-[3]_q^v\right)^2e^{2i\Theta}}.$$

According to Lemma 2.13.2, provides

$$2n_2=n_1^2+x\left(4-n_1^2\right),$$
$$2m_2=m_1^2+y\left(4-m_1^2\right).$$

By solving, it leads to

$$2n_2-2m_2=n_1^2-m_1^2+x\left(4-n_1^2\right)-y\left(4-m_1^2\right),$$

using (6.12), provides

$$2(n_2-m_2)=(x-y)\left(4-n_1^2\right),$$
$$n_2-m_2=\frac{4-n_1^2}{2}\,(x-y), \tag{6.19}$$

and

$$4n_3 = n_1^3 + 2\left(4-n_1^2\right)n_1x - n_1\left(4-n_1^2\right)x^2 + 2\left(4-n_1^2\right)\left(1-|x|^2\right)\mathrm{s},$$
$$4m_3 = m_1^3 + 2\left(4-m_1^2\right)m_1y - m_1\left(4-m_1^2\right)y^2 + 2\left(4-m_1^2\right)\left(1-|y|^2\right)\omega. \tag{6.20}$$

By solving (6.20), gives us

$$4n_3 - 4m_3 = n_1^3 - m_1^3 + 2\left(4-n_1^2\right)n_1x - 2\left(4-m_1^2\right)m_1y - n_1\left(4-n_1^2\right)x^2 + m_1\left(4-m_1^2\right)y^2 + 2\left(4-n_1^2\right)\left(1-|x|^2\right)\mathrm{s} - 2\left(4-m_1^2\right)\left(1-|y|^2\right)\omega.$$

$$4\left(n_3-m_3\right) = n_1^3 + n_1^3 + 2\left(4-n_1^2\right)n_1x + 2\left(4-n_1^2\right)n_1y - n_1\left(4-n_1^2\right)x^2 - n_1\left(4-n_1^2\right)y^2 + 2\left(4-n_1^2\right)\left(1-|x|^2\right)\mathrm{s} - 2\left(4-n_1^2\right)\left(1-|y|^2\right)\omega.$$

$$n_3 - m_3 = \frac{1}{4}\left[2n_1^3 + 2\left(4-n_1^2\right)n_1\left(x+y\right) - n_1\left(4-n_1^2\right)(x^2+y^2) + 2\left(4-n_1^2\right)(\left(1-|x|^2\right)\mathrm{s} - \left(1-|y|^2\right)\omega)\right],$$
$$n_3 - m_3 = \frac{n_1^3}{2} + \frac{\left(4-n_1^2\right)n_1}{2}\left(x+y\right) - \frac{n_1\left(4-n_1^2\right)}{4}(x^2+y^2) + \frac{\left(4-n_1^2\right)}{2}(\left(1-|x|^2\right)\mathrm{s} - \left(1-|y|^2\right)\omega). \tag{6.21}$$

By solving (6.19), also gives us

$$2n_2 + 2m_2 = n_1^2 + m_1^2 + x\left(4-n_1^2\right) + y\left(4-m_1^2\right),$$

using (6.12), leads to

$$2(n_2+m_2) = n_1^2 + n_1^2 + x\left(4-n_1^2\right) + y\left(4-n_1^2\right),$$

$$2(n_2+m_2) = 2n_1^2 + (x+y)\left(4-n_1^2\right),$$
$$n_2 + m_2 = n_1^2 + \frac{4-n_1^2}{2}\left(x+y\right). \tag{6.22}$$

For terms x, y and s, ω having conditions of $|x| \le 1, |y| \le 1, |\mathrm{s}| \le 1, |\omega| \le 1$ and $\left|e^{i\Theta}\right| = 1$.

Using triangle inequality on (6.18), gives

$$\left|\alpha_2\alpha_3-\alpha_3^2\right| = \left|\frac{n_1^4\left[([3]_q^v-[2]_q^{2v})\left([2]_q^u-[2]_q^v\right)+[2]_q^v\left([3]_q^u-[3]_q^v\right)-\left([4]_q^u-[4]_q^v\right)\right]X^4}{\left([4]_q^u-[4]_q^v\right)e^{4i\Theta}\left([2]_q^u-[2]_q^v\right)^4}+\right.$$
$$\frac{n_1^2(n_2-m_2)\,X^3}{4\left([2]_q^u-[2]_q^v\right)^2\left([3]_q^u-[3]_q^v\right)e^{3i\Theta}}+\frac{n_1\,(n_3-m_3)\,X^2}{2\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)e^{2i\Theta}}-$$
$$\left.\frac{X^2(n_2-m_2)^2}{4\left([3]_q^u-[3]_q^v\right)^2e^{2i\Theta}}\right|,$$

$$\left|\alpha_2\alpha_3-\alpha_3^2\right| = \left|\frac{n_1^4\left[([3]_q^v-[2]_q^{2v})\left([2]_q^u-[2]_q^v\right)+[2]_q^v\left([3]_q^u-[3]_q^v\right)-\left([4]_q^u-[4]_q^v\right)\right]X^4}{\left([4]_q^u-[4]_q^v\right)e^{4i\Theta}\left([2]_q^u-[2]_q^v\right)^4}+\right.$$
$$\frac{n_1^2X^3}{4\left([2]_q^u-[2]_q^v\right)^2\left([3]_q^u-[3]_q^v\right)e^{3i\Theta}}\left(\frac{4-n_1^2}{2}(x-y)\right)+$$
$$\frac{n_1X^2}{2\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)e^{2i\Theta}}\left(\frac{n_1^3}{2}+\frac{(4-n_1^2)\,n_1}{2}(x+y)-\right.$$
$$\left.\frac{n_1\,(4-n_1^2)}{4}(x^2+y^2)+\frac{(4-n_1^2)}{2}(\left(1-|x|^2\right)\mathrm{s}-\left(1-|y|^2\right)\omega).\right)-$$
$$\left.\frac{X^2}{4\left([3]_q^u-[3]_q^v\right)^2e^{2i\Theta}}\frac{(4-n_1^2)^2}{4}(x-y)^2\right|,$$

$$\left|\alpha_2\alpha_3-\alpha_3^2\right| = \left|\frac{n_1^4\left[([3]_q^v-[2]_q^{2v})\left([2]_q^u-[2]_q^v\right)+[2]_q^v\left([3]_q^u-[3]_q^v\right)-\left([4]_q^u-[4]_q^v\right)\right]X^4}{\left([4]_q^u-[4]_q^v\right)e^{4i\Theta}\left([2]_q^u-[2]_q^v\right)^4}\right.$$
$$+\frac{n_1^2(4-n_1^2)\,X^3}{8\left([2]_q^u-[2]_q^v\right)^2\left([3]_q^u-[3]_q^v\right)e^{3i\Theta}}(x-y)+$$
$$\frac{n_1X^2}{2\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)e^{2i\Theta}}\left(\frac{n_1^3}{2}+\frac{(4-n_1^2)\,n_1}{2}(x+y)-\right.$$
$$\left.\frac{n_1\,(4-n_1^2)}{4}(x^2+y^2)+\frac{(4-n_1^2)}{2}(\left(1-|x|^2\right)\mathrm{s}-\left(1-|y|^2\right)\omega)\right)-$$
$$\left.\frac{(4-n_1^2)^2X^2}{16\left([3]_q^u-[3]_q^v\right)^2e^{2i\Theta}}(x-y)^2\right|,$$

$$\left|\alpha_2\alpha_3-\alpha_3^2\right| \leq \left|\frac{n_1^4\left[([3]_q^v-[2]_q^{2v})\left([2]_q^u-[2]_q^v\right)+[2]_q^v\left([3]_q^u-[3]_q^v\right)-\left([4]_q^u-[4]_q^v\right)\right]X^4}{\left([4]_q^u-[4]_q^v\right)e^{4i\Theta}\left([2]_q^u-[2]_q^v\right)^4}\right|+$$

$$\left|\frac{n_1^2\left(4-n_1^2\right)X^3}{8\left([2]_q^u-[2]_q^v\right)^2\left([3]_q^u-[3]_q^v\right)e^{3i\Theta}}(x-y)\right|+$$

$$\left|\frac{n_1X^2}{2\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)e^{2i\Theta}}\left(\frac{n_1^3}{2}+\frac{\left(4-n_1^2\right)n_1}{2}(x+y)-\right.\right.$$

$$\left.\left.\frac{n_1\left(4-n_1^2\right)}{4}(x^2+y^2)+\frac{\left(4-n_1^2\right)}{2}(\left(1-|x|^2\right)\mathrm{s}-\left(1-|y|^2\right)\omega)\right)\right|+$$

$$\left|\frac{\left(4-n_1^2\right)^2X^2}{16\left([3]_q^u-[3]_q^v\right)^2e^{2i\Theta}}(x-y)^2\right|,$$

$$\left|\alpha_2\alpha_3-\alpha_3^2\right| \leq \frac{n_1^4\left[([3]_q^v-[2]_q^{2v})\left([2]_q^u-[2]_q^v\right)+[2]_q^v\left([3]_q^u-[3]_q^v\right)-\left([4]_q^u-[4]_q^v\right)\right]X^4}{\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)^4}+$$

$$\frac{n_1^2\left(4-n_1^2\right)X^3}{8\left([2]_q^u-[2]_q^v\right)^2\left([3]_q^u-[3]_q^v\right)}|(x-y)|+\frac{n_1^4X^2}{4\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)}+$$

$$\frac{n_1^2\left(4-n_1^2\right)X^2}{4\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)}|(x+y)|+\frac{n_1^2\left(4-n_1^2\right)X^2}{8\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)}|(x^2+y^2)|+$$

$$\frac{n_1\left(4-n_1^2\right)X^2}{4\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)}|\left(1-|x|^2\right)\mathrm{s}-\left(1-|y|^2\right)\omega|+$$

$$\frac{\left(4-n_1^2\right)^2X^2}{16\left([3]_q^u-[3]_q^v\right)^2}|(x-y)^2|.$$

$$\left|\alpha_2\alpha_3-\alpha_3^2\right| \leq \frac{n_1^4\left[([3]_q^v-[2]_q^{2v})\left([2]_q^u-[2]_q^v\right)+[2]_q^v\left([3]_q^u-[3]_q^v\right)-\left([4]_q^u-[4]_q^v\right)\right]X^4}{\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)^4}+$$
$$\frac{n_1^2(4-n_1^2)X^3}{8\left([2]_q^u-[2]_q^v\right)^2\left([3]_q^u-[3]_q^v\right)}(|x|+|y|)+\frac{n_1^4X^2}{4\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)}+$$
$$\frac{n_1^2(4-n_1^2)X^2}{4\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)}(|x|+|y|)+\frac{n_1^2(4-n_1^2)X^2}{8\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)}\left(|x|^2+|y|^2\right)+$$
$$\frac{n_1(4-n_1^2)X^2}{4\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)}(\left|\left(1-|x|^2\right)\mathrm{s}\right|+\left|\left(1-|y|^2\right)\omega\right|)+$$
$$\frac{(4-n_1^2)^2X^2}{16\left([3]_q^u-[3]_q^v\right)^2}(|x|+|y|)^2.$$

$$\left|\alpha_2\alpha_3-\alpha_3^2\right| \leq \frac{n_1^4\left[([3]_q^v-[2]_q^{2v})\left([2]_q^u-[2]_q^v\right)+[2]_q^v\left([3]_q^u-[3]_q^v\right)-\left([4]_q^u-[4]_q^v\right)\right]X^4}{\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)^4}+$$
$$\frac{n_1^4X^2}{4\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)}+\frac{n_1(4-n_1^2)X^2}{2\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)}+$$
$$\left[\frac{n_1^2(4-n_1^2)X^3}{8\left([2]_q^u-[2]_q^v\right)^2\left([3]_q^u-[3]_q^v\right)}+\frac{n_1^2(4-n_1^2)X^2}{4\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)}\right](|x|+|y|)+$$
$$\left[\frac{n_1^2(4-n_1^2)X^2}{8\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)}-\frac{n_1(4-n_1^2)X^2}{4\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)}\right]\left(|x|^2+|y|^2\right)+$$
$$\frac{(4-n_1^2)^2X^2}{16\left([3]_q^u-[3]_q^v\right)^2}(|x|+|y|)^2.$$

The function $p\left(e^{i\Theta}\mathrm{s}\right)$, for $\Theta \in R$ is in the class P therefore we can suppose for convenience and generality that $n_1 = n \in [0,2]$.Thus, for $\tau = |x| \leq 1$ and $\delta = |y| \leq 1$, this leads to

$$\left|\alpha_2\alpha_4-\alpha_3^2\right| \leq W_1+W_2(\tau+\delta)+W_3\left(\tau^2+\delta^2\right)+W_4(\tau+\delta)^2 = F(\tau,\delta),$$

where

$$W_1 = \frac{n_1^4\left[([3]_q^v-[2]_q^{2v})\left([2]_q^u-[2]_q^v\right)+[2]_q^v\left([3]_q^u-[3]_q^v\right)-\left([4]_q^u-[4]_q^v\right)\right]X^4}{\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)^4}+$$
$$\frac{n_1^4X^2}{4\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)}+\frac{n_1(4-n_1^2)X^2}{2\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)},$$

$$W_1(n) = \frac{n^4\left[([3]_q^v - [2]_q^{2v})\left([2]_q^u - [2]_q^v\right) + [2]_q^v\left([3]_q^u - [3]_q^v\right) - \left([4]_q^u - [4]_q^v\right)\right]X^4}{\left([4]_q^u - [4]_q^v\right)\left([2]_q^u - [2]_q^v\right)^4} + \frac{n^4X^2}{4\left([4]_q^u - [4]_q^v\right)\left([2]_q^u - [2]_q^v\right)} + \frac{n\left(4-n^2\right)X^2}{2\left([4]_q^u - [4]_q^v\right)\left([2]_q^u - [2]_q^v\right)},$$

$$W_1 = W_1(n) = \frac{X^2}{4\left([4]_q^u - [4]_q^v\right)}\left[\frac{4n^4\left[([3]_q^v - [2]_q^{2v})\left([2]_q^u - [2]_q^v\right) + [2]_q^v\left([3]_q^u - [3]_q^v\right) - \left([4]_q^u - [4]_q^v\right)\right]X^2}{\left([2]_q^u - [2]_q^v\right)^4} + \frac{n^4}{\left([2]_q^u - [2]_q^v\right)} + \frac{2n\left(4-n^2\right)}{\left([2]_q^u - [2]_q^v\right)}\right],$$

$$W_1 = W_1(n) = \frac{X^2}{4\left([4]_q^u - [4]_q^v\right)}\left[\left(\frac{1}{\left([2]_q^u - [2]_q^v\right)} + \frac{4\left[([3]_q^v - [2]_q^{2v})\left([2]_q^u - [2]_q^v\right) + [2]_q^v\left([3]_q^u - [3]_q^v\right) - \left([4]_q^u - [4]_q^v\right)\right]X^2}{\left([2]_q^u - [2]_q^v\right)^4}\right)n^4 + \frac{2n\left(4-n^2\right)}{\left([2]_q^u - [2]_q^v\right)}\right] \geq 0,$$

$$W_2 = \frac{n_1^2\left(4-n_1^2\right)X^3}{8\left([2]_q^u - [2]_q^v\right)^2\left([3]_q^u - [3]_q^v\right)} + \frac{n_1^2\left(4-n_1^2\right)X^2}{4\left([4]_q^u - [4]_q^v\right)\left([2]_q^u - [2]_q^v\right)},$$

$$W_2 = W_2(n) = \frac{n^2\left(4-n^2\right)X^3}{8\left([2]_q^u - [2]_q^v\right)^2\left([3]_q^u - [3]_q^v\right)} + \frac{n^2\left(4-n^2\right)X^2}{4\left([4]_q^u - [4]_q^v\right)\left([2]_q^u - [2]_q^v\right)},$$

$$W_2 = W_2(n) = \frac{n^2\left(4-n^2\right)X^2}{4}\left[\frac{X}{2\left([2]_q^u - [2]_q^v\right)^2\left([3]_q^u - [3]_q^v\right)} + \frac{1}{\left([4]_q^u - [4]_q^v\right)\left([2]_q^u - [2]_q^v\right)}\right],$$

$$W_2 = W_2(n) = \frac{n^2\left(4-n^2\right)X^2}{4}\left[\frac{\left([4]_q^u - [4]_q^v\right)X + 2\left([3]_q^u - [3]_q^v\right)\left([2]_q^u - [2]_q^v\right)}{2\left([2]_q^u - [2]_q^v\right)^2\left([3]_q^u - [3]_q^v\right)\left([4]_q^u - [4]_q^v\right)}\right],$$

$$W_2 = W_2(n) = \frac{X^2}{8\left([3]_q^u - [3]_q^v\right)}\left[\frac{n^2\left(4-n^2\right)\left\{\left([4]_q^u - [4]_q^v\right)X + 2\left([3]_q^u - [3]_q^v\right)\left([2]_q^u - [2]_q^v\right)\right\}}{\left([2]_q^u - [2]_q^v\right)^2\left([4]_q^u - [4]_q^v\right)}\right],$$

$$W_2 = W_2(n) = \frac{X^2}{24\left([3]_q^u - [3]_q^v\right)}\left[\frac{n^2\left(4-n^2\right)\left\{3\left([4]_q^u - [4]_q^v\right)X + 6\left([3]_q^u - [3]_q^v\right)\left([2]_q^u - [2]_q^v\right)\right\}}{\left([2]_q^u - [2]_q^v\right)^2\left([4]_q^u - [4]_q^v\right)}\right] \geq 0,$$

$$W_3 = \frac{n_1^2\left(4-n_1^2\right)X^2}{8\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)} - \frac{n_1\left(4-n_1^2\right)X^2}{4\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)},$$

$$W_3 = W_3\left(n\right) = \frac{n^2\left(4-n^2\right)X^2}{8\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)} - \frac{n\left(4-n^2\right)X^2}{4\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)},$$

$$W_3 = W_3\left(n\right) = \frac{X^2}{8\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)}\left\{n\left(4-n^2\right)\left(n-2\right)\right\} \leq 0,$$

$$W_4 = \frac{\left(4-n_1^2\right)^2 X^2}{16\left([3]_q^u-[3]_q^v\right)^2}$$

$$W_4 = W_4\left(n\right) = \frac{X^2}{16\left([3]_q^u-[3]_q^v\right)^2}\left(4-n^2\right)^2 \geq 0.$$

Next, there is need to maximize function $F(\tau,\delta)$ in the closed square

$$S = \{(\tau,\delta) : 0 \leq \tau \leq 1,\ 0 \leq \delta \leq 1\}, \text{for} \quad n \in [0,2].$$

By taking partial derivative of the function $F\left(\tau,\delta\right)$, results in

$$\frac{\partial F}{\partial \tau} = W_2 + 2W_3\tau + 2W_4\left(\tau+\delta\right) = 0, \tag{6.23}$$

and

$$\frac{\partial F}{\partial \delta} = W_2 + 2W_3\delta + 2W_4\left(\tau+\delta\right) = 0. \tag{6.24}$$

By comparing (6.23) and (6.24), this gives

$$\tau = \delta\ .$$

By using this result in (6.23), this leads to

$$W_2 + 2W_3\tau + 2W_4\left(\tau+\delta\right) = 0,$$

$$W_2 + 2W_3\tau + 2W_4\left(\tau+\tau\right) = 0,$$

$$W_2 + 2W_3\tau + 4W_4\tau = 0,$$

$$\left(2W_3 + 4W_4\right)\tau = -W_2,$$

$$\tau = \frac{-W_2}{2(W_3+2W_4)},$$

$$\delta = \frac{-W_2}{2(W_3+2W_4)}\ ,$$

So

$$(\tau,\delta)=\left(\frac{-W_2}{2(W_3+2W_4)},\frac{-W_2}{2(W_3+2W_4)}\right).$$

This is the critical point.

By first derivative test

$$\frac{\partial^2 F}{\partial\tau^2}=2W_3+2W_4>0,\qquad \frac{\partial^2 F}{\partial\delta^2}=2W_3+2W_4>0,\qquad \frac{\partial^2 F}{\partial\tau\partial\delta}=2W_4>0,$$

$$\begin{aligned}\frac{\partial^2 F}{\partial\tau^2}\times\frac{\partial^2 F}{\partial\delta^2}-\left(\frac{\partial^2 F}{\partial\tau\partial\delta}\right)^2&=(2W_3+2W_4)(2W_3+2W_4)-4W_4^2,\\ &=4W_3^2+4W_4^2+8W_3W_4-4W_4^2,\\ &=4W_3^2+8W_3W_4<0.\end{aligned}$$

Since the function $F(\tau,\delta)$ does not show a local maximum, we examine the maximum of $F(\tau,\delta)$ on the boundary.

For $\tau=0$ and $0\le\tau\ \le 1$ (Similar to $\delta=0$ and $0\ \le\delta\ \le 1$). this gives

$$\begin{aligned}F(\tau,\delta)=&W_1+W_2(\tau+\delta)+W_3\left(\tau^2+\delta^2\right)+W_4(\tau+\delta)^2,\\ F(0,\delta)=&W_1+W_2(\delta)+(W_3+W_4)\left(\delta^2\right)=G(\delta),\\ G'(\delta)=&W_2+2(W_3+W_4)\delta.\end{aligned}$$

To check at the boundaries

At $\delta=0$,

$$G(0)=W_1,$$

At $\delta=1$,

$$G(1)=W_1+W_2+W_3+W_4\ ,$$

Since

$$G'(\delta)=W_2+2(W_3+W_4)\delta,$$

The sign of W_3+W_4 affects the behavior. If

$$W_3+W_4\ge 0,\qquad G'(\delta)\ge 0,$$

$G(\delta)$ is increasing.

The interior point of $0\le n\le 2$ for $0\le\delta\le 1$ can be obtained, when $W_3+W_4\ge 0$.

The function $G'(\delta)>0$ for $\delta>0$ shows that function 'F' having a positive slope.

Hence, the upper bound for functional $\left|\alpha_2\alpha_4-\alpha_3^2\right|$ corresponds to $\delta=1$ and $n=0$, which can be evaluated into $G'(\delta)=2(W_3+W_4)\delta+W_2\ge 0$.

Hence the maximum of $G(\delta)$ exist at $\delta = 1$ and

$$\max\{G(\delta)\} = G(1) = W_1 + W_2 + W_3 + W_4.$$

For the case when $W_3 + W_4 < 0$, it can be noted that $2(W_3 + W_4)\delta + W_2 \geq 0$ for $0 \leq \delta \leq 1$ and any fixed n with $0 \leq n \leq 2$.
It is obvious that
$2(W_3 + W_4) + W_2 < 2(W_3 + W_4)\delta + W_2 < W_2$ and thus, $G'(\delta) > 0$.
Therefore, for $n = 2$, this results in

$$F(\tau,\delta) = \frac{4X^2}{\left([4]_q^u - [4]_q^v\right)} \left\{ \frac{4X^2\left[\left([3]_q^v - [2]_q^{2v}\right)\left([2]_q^u - [2]_q^v\right) + [2]_q^v\left([3]_q^u - [3]_q^v\right) - \left([4]_q^u - [4]_q^v\right)\right]}{\left([2]_q^u - [2]_q^v\right)^4} + \frac{1}{\left([2]_q^u - [2]_q^v\right)} \right\}.$$

Next, checking for $\tau = 1$ and $0 \leq \tau \leq 1$ (similar to $\delta = 1$ and $0 \leq \delta \leq 1$), gives us

$$F(1,\delta) = H(\delta) = W_1 + W_2 + W_3 + W_4 + (W_2 + 2W_4)\delta + (W_3 + W_4)\delta^2,$$
$$H'(\delta) = W_2 + 2W_4 + 2(W_3 + W_4)\delta.$$

Similarly, for the above cases of $W_3 + W_4$ where $\delta = 1$, provides

$$\max\{H(\delta)\} = H(1) = W_1 + 2W_2 + 2W_3 + 4W_4 .$$

Since $G(1) \leq H(1)$, we obtained the interior point of $n \in [0,2]$ where maximum of F exists at $\tau = 1$ and $\delta = 1$. Therefore,

$$F(\tau,\delta) = F(1,1) = W_1 + 2W_2 + 2W_3 + 4W_4 = L(n).$$

Substituting the value of $W_1 + W_2 + W_3 + W_4$ in the function L, provides

$$F(1,1) = L(n) = W_1 + 2W_2 + 2W_3 + 4W_4 ,$$

$$L(n)=\frac{n^4\left[([3]_q^v-[2]_q^{2v})\left([2]_q^u-[2]_q^v\right)+[2]_q^v\left([3]_q^u-[3]_q^v\right)-\left([4]_q^u-[4]_q^v\right)\right]X^4}{\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)^4}+$$
$$\frac{n^4X^2}{4\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)}+\frac{n\left(4-n^2\right)X^2}{2\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)}+$$
$$2\left[\frac{n^2\left(4-n^2\right)X^3}{8\left([2]_q^u-[2]_q^v\right)^2\left([3]_q^u-[3]_q^v\right)}+\frac{n^2\left(4-n^2\right)X^2}{4\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)}\right]+$$
$$2\left[\frac{n^2\left(4-n^2\right)X^2}{8\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)}-\frac{n\left(4-n^2\right)X^2}{4\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)}\right]+$$
$$\frac{4\left(4-n^2\right)^2X^2}{16\left([3]_q^u-[3]_q^v\right)^2},$$

$$L(n)=\frac{n^4\left[([3]_q^v-[2]_q^{2v})\left([2]_q^u-[2]_q^v\right)+[2]_q^v\left([3]_q^u-[3]_q^v\right)-\left([4]_q^u-[4]_q^v\right)\right]X^4}{\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)^4}+$$
$$\frac{n^4X^2}{4\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)}+\frac{4nX^2}{2\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)}-$$
$$\frac{n^3X^2}{2\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)}+\frac{4n^2X^3}{4\left([2]_q^u-[2]_q^v\right)^2\left([3]_q^u-[3]_q^v\right)}-$$
$$\frac{n^4X^3}{4\left([2]_q^u-[2]_q^v\right)^2\left([3]_q^u-[3]_q^v\right)}+\frac{4n^2X^2}{2\left([4]^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)}-$$
$$\frac{n^4X^2}{2\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)}+\frac{4n^2X^2}{4\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)}-$$
$$\frac{n^4X^2}{4\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)}-\frac{4nX^2}{2\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)}+$$
$$\frac{n^3X^2}{2\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)}+\frac{16X^2}{4\left([3]_q^u-[3]_q^v\right)^2}+$$
$$\frac{n^4X^2}{4\left([3]_q^u-[3]_q^v\right)^2}-\frac{8n^2X^2}{4\left([3]_q^u-[3]_q^v\right)^2},$$

$$L(n)=\frac{n^4\left[([3]_q^v-[2]_q^{2v})\left([2]_q^u-[2]_q^v\right)+[2]_q^v\left([3]_q^u-[3]_q^v\right)-\left([4]_q^u-[4]_q^v\right)\right]X^4}{\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)^4}+$$
$$\frac{n^2X^3}{\left([2]_q^u-[2]_q^v\right)^2\left([3]_q^u-[3]_q^v\right)}-\frac{n^4X^3}{4\left([2]_q^u-[2]_q^v\right)^2\left([3]_q^u-[3]_q^v\right)}+$$
$$\frac{2n^2X^2}{\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)}-\frac{n^4X^2}{2\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)}+$$
$$\frac{n^2X^2}{\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)}+\frac{4X^2}{\left([3]_q^u-[3]_q^v\right)^2}+\frac{n^4X^2}{4\left([3]_q^u-[3]_q^v\right)^2}-$$
$$\frac{2n^2X^2}{\left([3]_q^u-[3]_q^v\right)^2},$$

$$L(n)=\left[\frac{\left[([3]_q^v-[2]_q^{2v})\left([2]_q^u-[2]_q^v\right)+[2]_q^v\left([3]_q^u-[3]_q^v\right)-\left([4]_q^u-[4]_q^v\right)\right]X^4}{\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)^4}-\right.$$
$$\frac{X^3}{4\left([2]_q^u-[2]_q^v\right)^2\left([3]_q^u-[3]_q^v\right)}-\frac{X^2}{2\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)}+$$
$$\left.\frac{X^2}{4\left([3]_q^u-[3]_q^v\right)^2}\right]n^4+\left[\frac{X^3}{\left([2]_q^u-[2]_q^v\right)^2\left([3]_q^u-[3]_q^v\right)}+\frac{2X^2}{\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)}+\right.$$
$$\left.\frac{X^2}{\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)}-\frac{2X^2}{\left([3]_q^u-[3]_q^v\right)^2}\right]n^2+\frac{4X^2}{\left([3]_q^u-[3]_q^v\right)^2},$$

$$L(n)=\frac{X^2}{16\left([4]_q^u-[4]_q^v\right)}\left\{n^4\left[\frac{\left[([3]_q^v-[2]_q^{2v})\left([2]_q^u-[2]_q^v\right)+[2]_q^v\left([3]_q^u-[3]_q^v\right)-\left([4]_q^u-[4]_q^v\right)\right]16X^4}{\left([2]_q^u-[2]_q^v\right)^4}-\right.\right.$$
$$\left.\frac{4X\left([4]_q^u-[4]_q^v\right)}{\left([2]_q^u-[2]_q^v\right)^2\left([3]_q^u-[3]_q^v\right)}+\frac{4\left([2]_q^u-[2]_q^v\right)\left([4]_q^u-[4]_q^v\right)-8\left([3]_q^u-[3]_q^v\right)^2}{\left([2]_q^u-[2]_q^v\right)\left([3]_q^u-[3]_q^v\right)^2}\right]+$$
$$n^2\left[\frac{16X\left([4]_q^u-[4]_q^v\right)}{\left([2]_q^u-[2]_q^v\right)^2\left([3]_q^u-[3]_q^v\right)}+\frac{48\left([3]_q^u-[3]_q^v\right)^2-32\left([2]_q^u-[2]_q^v\right)\left([4]_q^u-[4]_q^v\right)}{\left([2]_q^u-[2]_q^v\right)\left([3]_q^u-[3]_q^v\right)^2}\right]+$$
$$\left.\frac{64\left([4]_q^u-[4]_q^v\right)}{\left([3]_q^u-[3]_q^v\right)^2}\right\}.$$

Suppose that $L(n)$ has a maximum in an interior point n of [0,2]. By taking derivative of $L(n)$ with respect to n, leads to

$$L'(n) = \frac{X^2}{16\left([4]_q^u - [4]_q^v\right)} \left\{ 4n^3 \left[\frac{16\left[([3]_q^v - [2]_q^{2v})\left([2]_q^u - [2]_q^v\right) + [2]_q^v\left([3]_q^u - [3]_q^v\right) - \left([4]_q^u - [4]_q^v\right)\right] X^2}{\left([2]_q^u - [2]_q^v\right)^4} - \frac{4X\left([4]_q^u - [4]_q^v\right)}{\left([2]_q^u - [2]_q^v\right)^2\left([3]_q^u - [3]_q^v\right)} + \frac{4\left([4]_q^u - [4]_q^v\right)\left([2]_q^u - [2]_q^v\right) - 8\left([3]_q^u - [3]_q^v\right)^2}{\left([2]_q^u - [2]_q^v\right)\left([3]_q^u - [3]_q^v\right)^2} \right] + 2n \left[\frac{16\left([4]_q^u - [4]_q^v\right) X}{\left([2]_q^u - [2]_q^v\right)^2\left([3]_q^u - [3]_q^v\right)} + \frac{48\left([3]_q^u - [3]_q^v\right)^2 - 32\left([2]_q^u - [2]_q^v\right)\left([4]_q^u - [4]_q^v\right)}{\left([2]_q^u - [2]_q^v\right)\left([3]_q^u - [3]_q^v\right)^2} \right] \right\}.$$

By taking

$$\left[\frac{16\left[([3]_q^v - [2]_q^{2v})\left([2]_q^u - [2]_q^v\right) + [2]_q^v\left([3]_q^u - [3]_q^v\right) - \left([4]_q^u - [4]_q^v\right)\right] X^2}{\left([2]_q^u - [2]_q^v\right)^4} - \frac{4X\left([4]_q^u - [4]_q^v\right)}{\left([2]_q^u - [2]_q^v\right)^2\left([3]_q^u - [3]_q^v\right)} + \frac{4\left([4]_q^u - [4]_q^v\right)\left([2]_q^u - [2]_q^v\right) - 8\left([3]_q^u - [3]_q^v\right)^2}{\left([2]_q^u - [2]_q^v\right)\left([3]_q^u - [3]_q^v\right)^2} \right] \geq 0.$$

For solving the value of $'X'$, this leads to

$$X = \frac{\frac{4\left([4]_q^u - [4]_q^v\right)}{\left([2]_q^u - [2]_q^v\right)^2\left([3]_q^u - [3]_q^v\right)} + \sqrt{\left(-\frac{4\left([4]_q^u - [4]_q^v\right)}{\left([2]_q^u - [2]_q^v\right)^2\left([3]_q^u - [3]_q^v\right)}\right)^2 - 4\left(\frac{16\left[([3]_q^v - [2]_q^{2v})\left([2]_q^u - [2]_q^v\right) + [2]_q^v\left([3]_q^u - [3]_q^v\right) - \left([4]_q^u - [4]_q^v\right)\right]}{\left([2]_q^u - [2]_q^v\right)^4}\right) \times \left(\frac{4\left([4]_q^u - [4]_q^v\right)\left([2]_q^u - [2]_q^v\right) - 8\left([3]_q^u - [3]_q^v\right)^2}{\left([2]_q^u - [2]_q^v\right)\left([3]_q^u - [3]_q^v\right)^2}\right)}}{2\left(\frac{16\left[([3]_q^v - [2]_q^{2v})\left([2]_q^u - [2]_q^v\right) + [2]_q^v\left([3]_q^u - [3]_q^v\right) - \left([4]_q^u - [4]_q^v\right)\right]}{\left([2]_q^u - [2]_q^v\right)^4}\right)},$$

$$X = \frac{\left([4]_q^u - [4]_q^v\right)\left([2]_q^u - [2]_q^v\right)^2}{8\left([3]_q^u - [3]_q^v\right)\left[([3]_q^v - [2]_q^{2v})\left([2]_q^u - [2]_q^v\right) + [2]_q^v\left([3]_q^u - [3]_q^v\right) - \left([4]_q^u - [4]_q^v\right)\right]} \times \left(1 + \sqrt{1 - \frac{16\left(\left([4]_q^u - [4]_q^v\right)\left([2]_q^u - [2]_q^v\right) - 8\left([3]_q^u - [3]_q^v\right)^2\right)\left[([3]_q^v - [2]_q^{2v})\left([2]_q^u - [2]_q^v\right) + [2]_q^v\left([3]_q^u - [3]_q^v\right) - \left([4]_q^u - [4]_q^v\right)\right]}{\left([2]_q^u - [2]_q^v\right)\left([4]_q^u - [4]_q^v\right)^2}} \right) = \phi_{(u,v)}.$$

So

$$X \in \left[0, \phi_{(u,v)}\right] .$$

Therefore, $L'(n) > 0$ for $n \in [0,2]$. Since L is an increasing function in interval [0,2], the maximum point of L exists on the boundary for $n = 2$. Thus,

$$\max L(n) = L(2),$$

$$L(2) = \frac{X^2}{16\left([4]_q^u - [4]_q^v\right)} \left\{ 16 \left[\frac{16\left[([3]_q^v - [2]_q^{2v})\left([2]_q^u - [2]_q^v\right) + [2]_q^v\left([3]_q^u - [3]_q^v\right) - \left([4]_q^u - [4]_q^v\right)\right] X^2}{\left([2]_q^u - [2]_q^v\right)^4} - \frac{4X\left([4]_q^u - [4]_q^v\right)}{\left([2]_q^u - [2]_q^v\right)^2\left([3]_q^u - [3]_q^v\right)} + \frac{4\left([4]_q^u - [4]_q^v\right)\left([2]_q^u - [2]_q^v\right) - 8\left([3]_q^u - [3]_q^v\right)^2}{\left([2]_q^u - [2]_q^v\right)\left([3]_q^u - [3]_q^v\right)^2} \right] + 4\left[\frac{16\left([4]_q^u - [4]_q^v\right)X}{\left([2]_q^u - [2]_q^v\right)^2\left([3]_q^u - [3]_q^v\right)} + \frac{48\left([3]_q^u - [3]_q^v\right)^2 - 32\left([2]_q^u - [2]_q^v\right)\left([4]_q^u - [4]_q^v\right)}{\left([2]_q^u - [2]_q^v\right)\left([3]_q^u - [3]_q^v\right)^2} \right] + \frac{64\left([4]_q^u - [4]_q^v\right)}{\left([3]_q^u - [3]_q^v\right)^2} \right\},$$

$$L(2) = \frac{4X^2}{\left([4]_q^u - [4]_q^v\right)} \left\{ \left[\frac{4\left[([3]_q^v - [2]_q^{2v})\left([2]_q^u - [2]_q^v\right) + [2]_q^v\left([3]_q^u - [3]_q^v\right) - \left([4]_q^u - [4]_q^v\right)\right] X^2}{\left([2]_q^u - [2]_q^v\right)^4} - \frac{X\left([4]_q^u - [4]_q^v\right)}{\left([2]_q^u - [2]_q^v\right)^2\left([3]_q^u - [3]_q^v\right)} - \frac{2}{\left([2]_q^u - [2]_q^v\right)} + \frac{\left([4]_q^u - [4]_q^v\right)}{\left([3]_q^u - [3]_q^v\right)^2} \right] + \left[\frac{\left([4]_q^u - [4]_q^v\right)X}{\left([2]_q^u - [2]_q^v\right)^2\left([3]_q^u - [3]_q^v\right)} + \frac{3}{\left([2]_q^u - [2]_q^v\right)} - \frac{2\left([4]_q^u - [4]_q^v\right)}{\left([3]_q^u - [3]_q^v\right)^2} \right] + \frac{\left([4]_q^u - [4]_q^v\right)}{\left([3]_q^u - [3]_q^v\right)^2} \right\}.$$

Hence,

$$\left|\alpha_2\alpha_4-\alpha_3^2\right| \leq \frac{4X^2}{\left([4]_q^u-[4]_q^v\right)}\left\{\frac{4\left[([3]_q^v-[2]_q^{2v})\left([2]_q^u-[2]_q^v\right)+[2]_q^v\left([3]_q^u-[3]_q^v\right)-\left([4]_q^u-[4]_q^v\right)\right]X^2}{\left([2]_q^u-[2]_q^v\right)^4}+\frac{1}{\left([2]_q^u-[2]_q^v\right)}\right\}.$$

Also, by taking

$$\left[\frac{16\left[([3]_q^v-[2]_q^{2v})\left([2]_q^u-[2]_q^v\right)+[2]_q^v\left([3]_q^u-[3]_q^v\right)-\left([4]_q^u-[4]_q^v\right)\right]X^2}{\left([2]_q^u-[2]_q^v\right)^4}-\frac{4X\left([4]_q^u-[4]_q^v\right)}{\left([2]_q^u-[2]_q^v\right)^2\left([3]_q^u-[3]_q^v\right)}+\frac{4\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)-8\left([3]_q^u-[3]_q^v\right)^2}{\left([2]_q^u-[2]_q^v\right)\left([3]_q^u-[3]_q^v\right)^2}\right] \geq 0.$$

This provide that

$$X \in \left[\phi_{(u,v)},1\right].$$

For a value $n_0 < 2$, that is n_0 is from the interval $[0,2]$. As $L'(n_0) \leq 0$, the maximum of $L(n)$ exists at $n = n_0$.

Hence, for $L'(n) = 0$, this leads to

$$2n\left\{2n^2\left[\frac{16\left[([3]_q^v-[2]_q^{2v})\left([2]_q^u-[2]_q^v\right)+[2]_q^v\left([3]_q^u-[3]_q^v\right)-\left([4]_q^u-[4]_q^v\right)\right]X^2}{\left([2]_q^u-[2]_q^v\right)^4}-\frac{4X\left([4]_q^u-[4]_q^v\right)}{\left([2]_q^u-[2]_q^v\right)^2\left([3]_q^u-[3]_q^v\right)}+\frac{4\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)-8\left([3]_q^u-[3]_q^v\right)^2}{\left([2]_q^u-[2]_q^v\right)\left([3]_q^u-[3]_q^v\right)^2}\right]+\left[\frac{16\left([4]_q^u-[4]_q^v\right)X}{\left([2]_q^u-[2]_q^v\right)^2\left([3]_q^u-[3]_q^v\right)}+\frac{48\left([3]_q^u-[3]_q^v\right)^2-32\left([2]_q^u-[2]_q^v\right)\left([4]_q^u-[4]_q^v\right)}{\left([2]_q^u-[2]_q^v\right)\left([3]_q^u-[3]_q^v\right)^2}\right]\right\}=0,$$

$$n^2 = \frac{-8\left[\frac{2\left([4]_q^u-[4]_q^v\right)X}{\left([2]_q^u-[2]_q^v\right)^2\left([3]_q^u-[3]_q^v\right)}+\frac{6\left([3]_q^u-[3]_q^v\right)^2-4\left([2]_q^u-[2]_q^v\right)\left([4]_q^u-[4]_q^v\right)}{\left([2]_q^u-[2]_q^v\right)\left([3]_q^u-[3]_q^v\right)^2}\right]}{8\left[\frac{4\left[([3]_q^v-[2]_q^{2v})\left([2]_q^u-[2]_q^v\right)+[2]_q^v\left([3]_q^u-[3]_q^v\right)-\left([4]_q^u-[4]_q^v\right)\right]X^2}{\left([2]_q^u-[2]_q^v\right)^4}-\frac{X\left([4]_q^u-[4]_q^v\right)}{\left([2]_q^u-[2]_q^v\right)^2\left([3]_q^u-[3]_q^v\right)}+\frac{\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)-2\left([3]_q^u-[3]_q^v\right)^2}{\left([2]_q^u-[2]_q^v\right)\left([3]_q^u-[3]_q^v\right)^2}\right]},$$

$$n=\sqrt{\frac{-\left[\frac{2\left([4]_q^u-[4]_q^v\right)X}{\left([2]_q^u-[2]_q^v\right)^2\left([3]_q^u-[3]_q^v\right)}+\frac{6\left([3]_q^u-[3]_q^v\right)^2-4\left([2]_q^u-[2]_q^v\right)\left([4]_q^u-[4]_q^v\right)}{\left([2]_q^u-[2]_q^v\right)\left([3]_q^u-[3]_q^v\right)^2}\right]}{\left[\frac{4\left[\left([3]_q^v-[2]_q^{2v}\right)\left([2]_q^u-[2]_q^v\right)+[2]_q^v\left([3]_q^u-[3]_q^v\right)-\left([4]_q^u-[4]_q^v\right)\right]X^2}{\left([2]_q^u-[2]_q^v\right)^4}-\frac{X\left([4]_q^u-[4]_q^v\right)}{\left([2]_q^u-[2]_q^v\right)^2\left([3]_q^u-[3]_q^v\right)}+\frac{\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)-2\left([3]_q^u-[3]_q^v\right)^2}{\left([2]_q^u-[2]_q^v\right)\left([3]_q^u-[3]_q^v\right)^2}\right]}},$$

$$\max\{L(n_0)\}=L\left(\sqrt{\frac{-\left[\frac{2\left([4]_q^u-[4]_q^v\right)X}{\left([2]_q^u-[2]_q^v\right)^2\left([3]_q^u-[3]_q^v\right)}+\frac{6\left([3]_q^u-[3]_q^v\right)^2-4\left([2]_q^u-[2]_q^v\right)\left([4]_q^u-[4]_q^v\right)}{\left([2]_q^u-[2]_q^v\right)\left([3]_q^u-[3]_q^v\right)^2}\right]}{\left[\frac{4\left[\left([3]_q^v-[2]_q^{2v}\right)\left([2]_q^u-[2]_q^v\right)+[2]_q^v\left([3]_q^u-[3]_q^v\right)-\left([4]_q^u-[4]_q^v\right)\right]X^2}{\left([2]_q^u-[2]_q^v\right)^4}-\frac{X\left([4]_q^u-[4]_q^v\right)}{\left([2]_q^u-[2]_q^v\right)^2\left([3]_q^u-[3]_q^v\right)}+\frac{\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)-2\left([3]_q^u-[3]_q^v\right)^2}{\left([2]_q^u-[2]_q^v\right)\left([3]_q^u-[3]_q^v\right)^2}\right]}}\right),$$

$$=\frac{X^2}{16\left([4]_q^u-[4]_q^v\right)}\left\{\begin{array}{c}\frac{\left(-\left[\frac{2\left([4]_q^u-[4]_q^v\right)X}{\left([2]_q^u-[2]_q^v\right)^2\left([3]_q^u-[3]_q^v\right)}+\frac{6\left([3]_q^u-[3]_q^v\right)^2-4\left([2]_q^u-[2]_q^v\right)\left([4]_q^u-[4]_q^v\right)}{\left([2]_q^u-[2]_q^v\right)\left([3]_q^u-[3]_q^v\right)^2}\right]\right)^2}{\left(\left[\frac{4\left[\left([3]_q^v-[2]_q^{2v}\right)\left([2]_q^u-[2]_q^v\right)+[2]_q^v\left([3]_q^u-[3]_q^v\right)-\left([4]_q^u-[4]_q^v\right)\right]X^2}{\left([2]_q^u-[2]_q^v\right)^4}-\frac{X\left([4]_q^u-[4]_q^v\right)}{\left([2]_q^u-[2]_q^v\right)^2\left([3]_q^u-[3]_q^v\right)}+\frac{\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)-2\left([3]_q^u-[3]_q^v\right)^2}{\left([2]_q^u-[2]_q^v\right)\left([3]_q^u-[3]_q^v\right)^2}\right]\right)^2}\times\\ \left[\frac{\left[\left([3]_q^v-[2]_q^{2v}\right)\left([2]_q^u-[2]_q^v\right)+[2]_q^v\left([3]_q^u-[3]_q^v\right)-\left([4]_q^u-[4]_q^v\right)\right]16X^4}{\left([2]_q^u-[2]_q^v\right)^4}-\frac{4X\left([4]_q^u-[4]_q^v\right)}{\left([2]_q^u-[2]_q^v\right)^2\left([3]_q^u-[3]_q^v\right)}+\frac{4\left([2]_q^u-[2]_q^v\right)\left([4]_q^u-[4]_q^v\right)-8\left([3]_q^u-[3]_q^v\right)^2}{\left([2]_q^u-[2]_q^v\right)\left([3]_q^u-[3]_q^v\right)^2}\right]\\ +\frac{\left(-\left[\frac{2\left([4]_q^u-[4]_q^v\right)X}{\left([2]_q^u-[2]_q^v\right)^2\left([3]_q^u-[3]_q^v\right)}+\frac{6\left([3]_q^u-[3]_q^v\right)^2-4\left([2]_q^u-[2]_q^v\right)\left([4]_q^u-[4]_q^v\right)}{\left([2]_q^u-[2]_q^v\right)\left([3]_q^u-[3]_q^v\right)^2}\right]\right)}{\left(\left[\frac{4\left[\left([3]_q^v-[2]_q^{2v}\right)\left([2]_q^u-[2]_q^v\right)+[2]_q^v\left([3]_q^u-[3]_q^v\right)-\left([4]_q^u-[4]_q^v\right)\right]X^2}{\left([2]_q^u-[2]_q^v\right)^4}-\frac{X\left([4]_q^u-[4]_q^v\right)}{\left([2]_q^u-[2]_q^v\right)^2\left([3]_q^u-[3]_q^v\right)}+\frac{\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)-2\left([3]_q^u-[3]_q^v\right)^2}{\left([2]_q^u-[2]_q^v\right)\left([3]_q^u-[3]_q^v\right)^2}\right]\right)}\times\\ \left[\frac{16X\left([4]_q^u-[4]_q^v\right)}{\left([2]_q^u-[2]_q^v\right)^2\left([3]_q^u-[3]_q^v\right)}+\frac{48\left([3]_q^u-[3]_q^v\right)^2-32\left([2]_q^u-[2]_q^v\right)\left([4]_q^u-[4]_q^v\right)}{\left([2]_q^u-[2]_q^v\right)\left([3]_q^u-[3]_q^v\right)^2}\right]+\frac{64\left([4]_q^u-[4]_q^v\right)}{\left([3]_q^u-[3]_q^v\right)^2}\end{array}\right\},$$

$$\max\{L(n_0)\} = \frac{X^2}{16\left([4]_q^u-[4]_q^v\right)}\left\{\frac{\left(\frac{2\left([4]_q^u-[4]_q^v\right)X}{\left([2]_q^u-[2]_q^v\right)^2\left([3]_q^u-[3]_q^v\right)}+\Pi\right)^2}{(\Omega+\Sigma)^2}4(\Omega+\Sigma)+\right.$$

$$-\frac{\left(\frac{2\left([4]_q^u-[4]_q^v\right)X}{\left([2]_q^u-[2]_q^v\right)^2\left([3]_q^u-[3]_q^v\right)}+\Pi\right)}{(\Omega+\Sigma)}8\left(\frac{2\left([4]_q^u-[4]_q^v\right)X}{\left([2]_q^u-[2]_q^v\right)^2\left([3]_q^u-[3]_q^v\right)}+\Pi\right)+$$

$$\left.\frac{64\left([4]_q^u-[4]_q^v\right)}{\left([3]_q^u-[3]_q^v\right)^2}\right\},$$

$$\max\{L(n_0)\} = \frac{X^2}{16\left([4]_q^u-[4]_q^v\right)}\left\{\frac{64\left([4]_q^u-[4]_q^v\right)}{\left([3]_q^u-[3]_q^v\right)^2}+4\frac{\left(\frac{2\left([4]_q^u-[4]_q^v\right)X}{\left([2]_q^u-[2]_q^v\right)^2\left([3]_q^u-[3]_q^v\right)}+\Pi\right)^2}{(\Omega+\Sigma)}-\right.$$

$$\left.8\frac{\left(\frac{2\left([4]_q^u-[4]_q^v\right)X}{\left([2]_q^u-[2]_q^v\right)^2\left([3]_q^u-[3]_q^v\right)}+\Pi\right)^2}{(\Omega+\Sigma)}\right\},$$

$$\max\{L(n_0)\} = \frac{X^2}{16\left([4]_q^u-[4]_q^v\right)}\left\{\frac{64\left([4]_q^u-[4]_q^v\right)}{\left([3]_q^u-[3]_q^v\right)^2}-\frac{4\left(\frac{2\left([2]_q^u+[2]_q^v\right)X}{\left([2]_q^u-[2]_q^v\right)\left([3]_q^u-[3]_q^v\right)}+\Pi\right)^2}{(\Omega+\Sigma)}\right\},$$

$$\left|\alpha_2\alpha_4-\alpha_3^2\right| \le \frac{X^2}{16\left([4]_q^u-[4]_q^v\right)}\left\{\frac{64\left([4]_q^u-[4]_q^v\right)}{\left([3]_q^u-[3]_q^v\right)^2}-\frac{4\left(\frac{2\left([2]_q^u+[2]_q^v\right)X}{\left([2]_q^u-[2]_q^v\right)\left([3]_q^u-[3]_q^v\right)}+\Pi\right)^2}{(\Omega+\Sigma)}\right\},$$

where

$$\Pi = \frac{6\left([3]_q^u-[3]_q^v\right)^2-4\left([2]_q^u-[2]_q^v\right)\left([4]_q^u-[4]_q^v\right)}{\left([2]_q^u-[2]_q^v\right)\left([3]_q^u-[3]_q^v\right)^2},$$

$$\Sigma = \frac{X\left([4]_q^u-[4]_q^v\right)}{\left([2]_q^u-[2]_q^v\right)^2\left([3]_q^u-[3]_q^v\right)}+\frac{\left([4]_q^u-[4]_q^v\right)\left([2]_q^u-[2]_q^v\right)-2\left([3]_q^u-[3]_q^v\right)^2}{\left([2]_q^u-[2]_q^v\right)\left([3]_q^u-[3]_q^v\right)^2},$$

$$\Omega = \frac{4\left[\left([3]_q^v-[2]_q^{2v}\right)\left([2]_q^u-[2]_q^v\right)+[2]_q^v\left([3]_q^u-[3]_q^v\right)-\left([4]_q^u-[4]_q^v\right)\right]X^2}{\left([2]_q^u-[2]_q^v\right)^4}.$$

The Theorem 6.2.2 is complete. □

For $q \to 1^-$, using this value in the above result gives an advanced result that perfectly aligns with the previous findings by Orhan et al. [82], as shown in the given corollary.

Corollary 6.2.2.1. If $f(s)$ provided by (1.1) be a part of the class $f \in \mathbb{S}^*_{\nabla}(\gamma, \Theta, u, v)$ and if $X = cos\Theta - \gamma$ for $u > v+1$, $\left[\left(3^v - 2^{2v}\right)(2^u - 2^v) + 2^v(3^u - 3^v) - (4^u - 4^v)\right] \neq 0$, $|\Theta| < \pi$ and $cos\Theta > \gamma$. Then

$$\left|\alpha_2\alpha_4 - \alpha_3^2\right| \leq \begin{cases} \frac{4X^2}{(4^u-4^v)}\left\{\frac{4X^2\left[\left(3^v-2^{2v}\right)(2^u-2^v)+2^v(3^u-3^v)-(4^u-4^v)\right]}{(2^u-2^v)^4} + \frac{1}{(2^u-2^v)}\right\}, & X \in \left[0, \phi_{(u,v)}\right], \\ \frac{X^2}{16(4^u-4^v)}\left\{\frac{64(4^u-4^v)}{(3^u-3^v)^2} - \frac{4\left(\Pi + \frac{2X(2^u+2^v)}{(2^u-2^v)(3^u-3^v)}\right)^2}{(\Omega+\Sigma)}\right\}, & X \in \left[\phi_{(u,v)}, 1\right], \end{cases}$$

where

$$\phi_{(u,v)} = \frac{(2^u-2^v)^2(4^u-4^v)}{8(3^u-3^v)\left[\left(3^v-2^{2v}\right)(2^u-2^v)+2^v(3^u-3^v)-(4^u-4^v)\right]} \times$$

$$\left(1 + \sqrt{1 - \frac{16\left[\left(3^v-2^{2v}\right)(2^u-2^v)+2^v(3^u-3^v)-(4^u-4^v)\right]\times\left[(4^u-4^v)(2^u-2^v)-2(3^u-3^v)^2\right]}{(4^u-4^v)^2(2^u-2^v)}}\right),$$

$$\Pi = \frac{6(3^u-3^v)^2 - 4(4^u-4^v)(2^u-2^v)}{(2^u-2^v)(3^u-3^v)^2},$$

$$\Sigma = -\frac{X(4^u-4^v)}{(2^u-2^v)^2(3^u-3^v)} + \frac{(4^u-4^v)(2^u-2^v)-2(3^u-3^v)^2}{(2^u-2^v)(3^u-3^v)^2},$$

$$\Omega = \frac{4\left[\left(3^v-2^{2v}\right)(2^u-2^v)+2^v(3^u-3^v)-(4^u-4^v)\right]X^2}{(2^u-2^v)^4}.$$

CHAPTER 7

CONCLUSION

This chapter concludes what we have done in this thesis and throws some light on future work related to our research work which other researchers can do.

This thesis analyzes bi-univalent functions in open unit disk V. It applies the basic concepts of Geometric Function Theory and q-calculus to examine how q-calculus ideas can be implemented to find the results for bi-univalent functions. This thesis examined three new subclasses of q-bi univalent functions.

Applying the quasi-subordination technique presented two new subclasses of q-bi univalent functions by implementing the q-difference operator. The analysis shows the q-version of subclasses of bi-univalent functions by applying quasi-subordination provided by Atshan et al. [31]. In q-calculus these subclasses are provided as $\mathfrak{R}^{\partial,y}_{\nabla,\gamma,c}(\xi,\sigma,\tau,\Psi,q)$, and $\mathfrak{K}^{\partial,y}_{\nabla,\gamma,c}(\xi,\sigma,\kappa,\Psi,q)$. The bounds of initial coefficients $|\alpha_2|$ and $|\alpha_3|$ of these classes were investigated.

Also, by applying the q-Salagean operator, a new subclass of q-bi starlike function is introduced using the q-difference operator. The analysis shows the q-version of a subclass of bi-starlike function investigated by the Salagean operator provided by Orhan et al. [82]. In q-calculus this subclass is provided as $\mathbb{S}^*_\nabla(\gamma,\theta,u,v,q)$. The upper bounds of initial coefficients $|\alpha_2|$, $|\alpha_3|$ and $|\alpha_4|$, Fekete-Szego inequality$\left(\left|\alpha_3-\lambda\alpha_2^2\right|\right)$ and second order Hankel determinant$\left(\left|\alpha_2\alpha_4-\alpha_3^2\right|\right)$ of this class were determined.

The results of the new classes are more advanced and refined than the previous findings by [31, 82], as can be verified by substituting the limit $q\to 1^-$. Hopefully, our contribution through this thesis will greatly strengthen the field of Geometric Function Theory and facilitate the new researchers for future work in this field.

7.1 Future Work

The central focus of this thesis is to examine the q-bi univalent functions by applying quasi-subordination and q-bi starlike functions applying the q-Salagean operator. Changing different parameters can generate well-known and new subclasses from these functions. Also, the Fekete-Szego inequality and Hankel determinant can be derived for subclasses of q-bi univalent functions implementing quasi-subordination. Further, Our results for the q-bi starlike function can also be derived for q-bi convex functions.

Appendices

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