On a New Subclasses of Starlike Functions Associated with Symmetric Points

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Candidate of Master of Science in Mathematics at the National University of Modern Languages University do hereby declare that the thesis On a New Subclasses of Starlike Functions Associated with Symmetric Points submitted by me in partial fulfillment of MS-MATH degree, is my original work and has not been submitted or published earlier. I also solemnly declare that it shall not, in the future, be submitted by me for obtaining any other degree from this or any other university or institution. I also understand that if evidence of plagiarism is found in my thesis/dissertation at any stage, even after the award of a degree, the work may be canceled and the degree revoked.

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ABSTRACT

Title: On a New Subclasses of Starlike Functions Associated with Symmetric Points

This research aims to introduce and examine new subclasses of analytic functions within the open unit disc. I will use q-calculus to develop the q-extension of starlike functions related to symmetric points. Additionally, i will explore significant properties such as coefficient bounds for analytic functions,the Fekete-Szego inequality and the Zalcman functional. I will also investigate upper bounds on Hankel Determinants for functions within these new class. The findings will be demonstrated to advance beyond previous results obtained by many researchers in Geometric Function Theory. Special cases of these new results will be presented as corollaries.

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$\mathbb{D}$	-	Open unit disc
A	-	The class of Analytic functions
S	-	Class of Univalent functions
C	-	Class of Convex functions
$\mathfrak{P}$	-	Class of Caratheodory functions
S^*	-	Class of Starlike functions
$\prec$	-	Subordination symbol
S_s^*	-	Class of Starlike functions with respect to Symmetric points
$S_s^*(1+tanh(ł))$	-	Class of Starlike functions Subordinated with Tangent hyperbolic funtion
$S_{s,\mathrm{q}}^*(1+tanh(\mathrm{q}ł))$	-	Class of q-Starlike function Subordinated with q-Tangent hyperbolic function
$\mathfrak{H}$	-	Henkel Determinant symbol
D_{q}	-	q-Derivative operator symbol

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DEDICATION

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CHAPTER 1

INTRODUCTION AND LITERATURE REVIEW

1.1 Overview

An extensive introduction and survey of the literature are provided in this chapter, with a focus on key ideas in geometric function theory. It covers various subclasses and the broader categories of analytic and univalent functions. Additionally, it briefly addresses the Fekete-Szegö inequality, Hankel determinants, the Zalcman Functional, and coefficient bounds within these classes. The chapter reviews some of the fundamental ideas in quantum calculus.

1.2 Riemann Mapping Theorem

Geometric function theory has its roots in Riemann's influential mapping theorem. Introduced by Bernard Riemann in 1851, the Riemann Mapping Theorem addresses the transformation of complex domains [1]. The first hard evidence emerged in the 20th century. Caratheodory providing a proved in 1912 using regular families and Riemann surfaces. As a result, the open disk, $\mathbb{D} = \{ł : |ł| < 1\}$ can now be regarded as a domain. The theorem is fundamental to Geometric Function Theory as it forms its foundation. The major contribution in the 19th century by Cauchy, Riemann, and Weierstrass built the basis of contemporary function theory [2].

1.3 Analytic and Univalent Functions

In 1907, Koebe investigated univalent and analytic functions within the open unit disk $\mathbb{D}$ [3]. His work contributed significantly to the understanding of these functions in this domain.Geometric Function theory was developed based on the concept of analytic functions, which were initally defined by Duren [3] in 1983. Duren defined the class A of analytic functions with the normalization conditions $\zeta(0) = 0$ and $\zeta'(0) = 1$, where functions are expressed as $\zeta(ł) = ł + \sum_{n=2}^{\infty} a_n ł^n$, with ł being a complex number such that $|ł| < 1$. Various classes categorize analytic functions, with further subclassifications based on geometric properties and the structure of their image domains. The geometric shape of these domains has been a subject of significant research and debate among scholars. Koebe [4] was the first who explored the theory of univalent functions in 1907, after that Robertson [5] in 1936 and Macgregor [6] in 1964 worked in the theory of univalent functions.

1.4 Subclasses of Analytic and Univalent Function

In 1907, Koebe made significant contributions to the study of univalent functions [4]. He introduced the concept of univalent functions, a key topic in complex analysis, specifically focusing on their behavior within the unit disk $\mathbb{D}$. Koebe's work laid the foundation for the class S of univalent functions, which includes functions that are analytic, normalized, and univalent within $\mathbb{D}$. This class S is central to much of geometric function theory. There are numerous significant subclasses that comprise the class S: The class of starlike functions is S^*. C: Convex functions are a class. The class of starlike functions with regard to symmetric points is denoted by S_s^*. The subclasses of starlike and convex functions were expanded by Robertson [5] and Sakaguchi [7] in 1959, introducing the class of starlike functions with regets to symmetric points in particularly. This classification started as part of the attempts to gather evidence for the Bieberbach conjecture [8]. In 1915, Alexander [5] formulated which is now called the Alexander relation, that links two categories of convex and starlike functions. In 1921, Nevanlinna [9] introduced the concept of starlike functions within $\mathbb{D}$. Subsequently, in 1975, Silverman [10] examined starlike of order α_o and convex of order α_o univalent functions with negative coefficients, providing insights into coefficient inequalities, covering features, and coefficient distortion.

1.5 Coefficient Bounds

There are many classes and subclasses in geometric function theory, but determining coefficient bounds is one of the main focuses. Usually, functions in this discipline are investigated within different classes of normalized analytic functions from A. The Bieberbach theorem, first proved by Ludwig Bieberbach in 1916, is a key finding in this field. This theorem pertains to the class S of univalent functions and specifies the bounds for the second coefficient α_2 in these functions, leading to the formulation of Bieberbach's conjecture. Numerous studies and efforts at proof have focused on this conjecture [8]. The coefficient conjecture asserts that for a function ł in the class S, the coefficients of ł satisfy $|c_n| \leq n$ for $n \in \{2,3,4,\ldots\}$. Specifically, it was proven that $|c_2| \leq 2$, with equality holding only if ł is the Koebe function or its rotations.Mathematicians have faced several difficulties in solving the Bieberbach conjecture, even though it is a simple concept. Despite numerous fruitless attempts to demonstrate it, substitute techniques have been devised. In 1923, Karl Loewner [11] proved that $|c_3| \leq 3$, a result that set the stage for further generalizations. In 1955, Gangadharan *et al.* [12] demonstrated that $|c_4| \leq 4$, addressing the conjecture for $m = 4$. The conjecture was ultimately resolved by Louis de Branges [13], in 1985. He provided a comprehensive proof of the general form . In more recent developments, Darus [14] derived estimates for the second and third coefficients in the classes of q-starlike and q-convex functions in 2016. Seoudy *et al.* [15] provided estimates for these coefficients in q-starlike and q-convex functions of complex order, in 2016.

1.6 Fekete-Szego Inequality

By focusing on the coefficients of specific polynomials, the Fekete-Szego inequality-which is strongly associated with the Bieberbach conjecture-contributes significantly to the field of complex analysis. This inequality was first introduced by Fekete and Szego in 1933 [16]. It has many important applications and implications in complex analysis. In 2020, Hari *et al.* [17] investigated the Fekete-Szego problem for analytic functions in subclasses of the class S.

1.7 Henkel Determinants

The determinant of the Hankel matrix connected to a function is referred to as the Hankel determinant. Pommerenke [18] first addressed the Hankel determinant of univalent and analytic functions in 1967 .After that, Hayman [19] investigated the univalent functions' second Hankel determinant. In 1976, Noonan and Thomas [20] expanded this research by introducing the q-th Hankel determinant. Noor [21] later examined the growth rate of $\mathfrak{H}_q(k)$ as $k \to \infty$ for analytic univalent functions with bounded boundaries. In 2007, Janteng *et al.* [22] provided a precise upper bound for the functional $|c_2c_4 - c_3^2|$ in the context of convex and starlike functions.

Babalola [23] introduced the Hankel determinant of order 3 for starlike and convex functions within the unit disk $\mathbb{D}$ in 2009. He found that for convex functions, the Hankel determinant is bounded by $\mathfrak{H}_3(1) \leq \frac{15}{24}$, while for starlike functions, it is bounded by $\mathfrak{H}_3(1) \leq 16$. Krishna *et al.* [24] determined that the Hankel determinant of order 3 for starlike functions with respect to symmetric points is $\frac{5}{2}$, and for convex functions, it is $\frac{19}{135}$, even still, they pointed out that these bounds weren't precise. Lecko *et al.* [25] refined these results in 2019 by providing precise bounds for the third-order Hankel determinant of starlike functions of order $\frac{1}{2}$. Bounds for the third-order Hankel determinant were established by Shi *et al.* [26] in certain classes of convex and starlike univalent functions associated with exponential functions in the $\mathbb{D}$. The third-order Hankel determinant for starlike functions associated with exponential functions within $\mathbb{D}$ was found by Joshi *et al.* [27]. They derived a new formula for the fourth coefficient of Carathéodory functions and established exact bounds for the third-order Hankel determinant. Arif *et al.* [28] examined the q-th Hankel determinant for certain subclasses of analytic functions and estimated its growth.

The exploration of the Hankel determinant has been addressed by various researchers, including Noor [21] and Pommerenke [18], as well as numerous others [29, 30, 31, 32, 33, 34].

1.8 Zalcman Functional

In Geometric Function Theory, a well-known conjecture proposed by Lawrence Zalcman in 1960 [35] postulate that the coefficients of functions within class S should satisfy the inequality:

$$|a_n^2 - a_{2n-1}| \leq (n-1)^2.$$

Equality in this form is only possible by the Koebe function $k(\text{ł}) = \frac{\text{ł}}{(1-\text{ł})^2}$ and its rotations. This well-known conjecture includes the bieberbach conjector, which has been proven true by many researchers but is still an extremely difficult open problem for all $n > 3$. For $n = 2$, equality in this form corresponds to the well-known Fekete-Szegö inequality. The Zalcman functional has been extensively studied by various researchers [36, 37, 38], with further details available in [39].

1.9 q-Calculus

Quantum calculus, or q-Calculus, is an extension of traditional calculus that focuses on q-analogous results without relying on limits. This field of mathematics is not only fasinating but also essential for various contexts, such as cosmic strings and black holes [40]. Quantum calculus includes both q-calculus and h-calculus, where q stands for quantum and h for Plancks constant. Euler laid the groundwork for q-calculus in the 17th century, and Jackson was among the early researchers to formalize q-derivatives and q-integrals in 1909 [41]. q-calculus can be seen as a version of classical calculus that does not use limits, and it has seen rapid development due to its wide applications in mathematics, mechanics, and physics.

Jackson [42] is credited with the systematic introduction of q-calculus, while his later work [43] presented the concepts of q-derivatives and q-integrals. The field gained further traction with the introduction of q-starlike functions by Ismail *et al.* [44] in 1990, introducing the integration of q-calculus into Geometric Function Theory. This was achieved through the use of the difference operator, and the class was initially referred to as the "class of q-starlike functions." Subsequent work includes Srivastava's exploration of generalizations and q-extensions of Bernoulli functions [45], and Darus's introduction of q-starlike and q-convex functions in 2016, utilizing the q-derivative operator [46]. Ramachandran *et al.* [47] further defined q-starlike and q-convex functions in relations to symmetric points. Recent research has focused on exploring practical properties of new classifications of meromorphic multivalent starlike functions using modified q-linear differential operators [48, 49].

1.10 Starlike Function with respect to a Symmetric Point

In 1959, Robertson [5] and Sakaguchi [7] expanded the concept of starlike and convex functions by introducing a new class of starlike functions defined with respect to symmetric points. This development introduced a broader framework for analyzing these functions. In 2019, Cho *et al.* [50] examined a specific class of starlike functions related to trigonometric functions, such as the sine function. Zaprawa [51] later explored coefficient inequalities for starlike functions with respect to symmetric points, addressing various problems within this framework. This approach has contributed to advancing the study of coefficient inequalities in function theory.

1.11 Preface

The goal of this thesis is to use the subordination idea to review and define certain analytic function sub-classes. It is divided into six chapters, each of having the following quick introduction:

In **Chapter 1**, an extensive examination of the literature arrives with a concentration on important ideas explored in the Geometric Function Theory classes. The classes of analytic functions, Carathéodory functions, and univalent functions are all covered in this investigation, along with certain significant subclasses. These ideas serve as the thesis's core.

In **Chapter 2**, concentrates on the basic principles in Geometric Function Theory, providing an important framework for the chapters to come. It begins by defining several basic subclasses of univalent functions and then researching into the concepts of analytic functions and normalized univalent functions under the $\mathbb{D}$. Basic lemmas are presented at the end of the chapter and will be used in later chapters. It is important that this chapter thoroughly cites and acknowledges well-established ideas in the subject rather than introducing any new results.

In **Chapter 3**, involves looking at the class of star-like functions that are associated with symmetric points. A few of the most significant findings are also looked at. Emphasizing the need of appropriately referencing the review work is essential.

In **Chapter 4**, after analyzing the review work results, we extend the research to include new subclass of starlike functions connected with symmetric points. Various important results are examined into with these defined classes.

In **Chapter 5**, futher advanced our research work extands the trigonometric q-tanh(qz) function to new subclasses, among which is q-starlike with regard to symmetric points. Various important results are examined into with these defined classes. Corollaries are provided to show how the recently acquired data compare to past discoveries obtained by other investigators.

In **Chapter 6**, our research has been concluded.

CHAPTER 2

PERLIMINARY CONCEPTS

2.1 Introduction

Examining fundamental ideas and findings that lay the groundwork for further research is the goal of this chapter. We will go into great detail on the normalized analytic univalent functions Carathéodory function. We'll take note of specific functions, a significant expressed proprietor, and several initial lemmas. The interaction of geometry and analysis is one of the most fascinating features of complex function theory. Futhermore, some brief overwiew of the basis of q-calculus is provided, followed by an examination of several recent categories of analytic functions.

2.2 Analytic and Univalent Function

The link between geometric functions and analytic structure is at the foundation of univalent function theory. In this framework, we propose categories for both analytic and univalent functions.

Definition 2.2.1. [52] If ζ is a complex valued function and is also analytic at point $ł_0$, it must be a single value function and its value occurs not only at $ł_0$ but also at all other points in the neighborhood $ł_0$. In this case, function ζ is considered analytic in its domain and analytic at all other points.

Definition 2.2.2. [3] Let a function ζ be analytic and normalized these conditions $\zeta(0) = 0$, $\zeta'(0) = 1$ and represent the form:

$$\zeta(\text{ł}) = \text{ł} + \sum_{n=2}^{\infty} a_n \text{ł}^n \quad \text{ł} \in \mathbb{D}. \tag{2.1}$$

then say that ζ is belong to Class A.

Definition 2.2.3. [53] Univalent, or one-to-one, functions are analytic functions that preserve injectivity. In particular, a function ζ is said to be univalent if it maps distinct complex numbers to distinct values. In other words, ζ is univalent within its domain if $(\text{ł}_1) \neq (\text{ł}_2)$ for any distinct complex numbers ł_1 and ł_2 in that domain. With applications in conformal mapping, complex dynamics, and Riemann surface theory, uniform functions play a significant role in both complex analysis and geometric function theory. They are valued for their geometric properties and for their role in understanding complex mappings and transformations.

2.3 Class S of Univalent Function

Class S plays very important role in Geometric Function Theory.

Definition 2.3.1. [53] Let a ζ function be analytic, normalized and univalent then $\zeta \in S$. The class of univalent function is also denoted by class S.

One well-known example from the class S of functions is the Koebe function, which has the following definition:

$$\zeta(\text{ł}) = \frac{1}{4}\left(\frac{1+\text{ł}}{1-\text{ł}}\right)^2 - \frac{1}{4} = \frac{\text{ł}}{(1-\text{ł})^2} = \sum_{c=1}^{\infty} c\text{ł}^c, \text{ where } \text{ł} \in \mathbb{D}. \tag{2.2}$$

2.4 Class $\mathfrak{P}$ of Caratheodory Functions

The class $\mathfrak{P}$ contains carathéodory functions, or functions with a positive real component. Numerous subclasses of univalent functions stem from this class. We will go over the core concepts from $\mathfrak{P}$ that are relevant to our task.

Definition 2.4.1. [54] Suppose that $\mathfrak{P}$ is an analytic function in the $\mathbb{D}$, with $Re[\mathfrak{P}(\text{ł})] > 0$ and

$\mathfrak{P}(0) = 1$. The Taylor series expansion of $\mathfrak{P}$ is as follows:

$$\mathfrak{P}(\mathfrak{k}) = 1 + \sum_{n=1}^{\infty} c_n \mathfrak{k}^n, \ \ \mathfrak{k} \in \mathbb{D}. \tag{2.3}$$

The Möbius function, which is known to be a function in class $\mathfrak{P}$, is defined as follows:

$$M_0(\mathfrak{k}) = \frac{1+\mathfrak{k}}{1-\mathfrak{k}} = 1 + 2\sum_{n=1}^{\infty} \mathfrak{k}_n, \ \ \mathfrak{k} \in \mathbb{D}. \tag{2.4}$$

2.5 Certain Subclasses of the Class S

The study of univalent functions is a long-established field that continues to evolve dynamically. Significant progress has been made over the past ten to fifteen years, leading to the introduction of various subclasses of univalent functions. The geometric characteristics of these subclasses' image domains essentially define them. Notable among them are the classes of Convex and Starlike functions.

Definition 2.5.1. [52, 55] Consider any point $\mathfrak{k}_0$ in the domain $\mathbb{D}$. $\mathbb{D}$ is said to be star-shaped with respect to $\mathfrak{k}_0$ if the line segment that joins $\mathfrak{k}_0$ to any other point $\mathfrak{k} \in \mathbb{D}$ stays completely inside $\mathbb{D}$. Let ζ be an analytical, univalent function with a star-shaped domain as its image in $\zeta(\mathbb{E})$. Then ζ is starlike domain and it can be defined as:

$$[56] \ Re(\frac{\mathfrak{k}\zeta'(\mathfrak{k})}{\zeta(\mathfrak{k})}) > 0 \iff f \in S^* and \, \mathfrak{k} \in \mathbb{D}. \tag{2.5}$$

Definition 2.5.2. [7] Sakaguchi introduced the class of starlike functions with regard to symmetric point, which is called S_s^*. It is defined as follows:

$$Re(\frac{2\mathfrak{k}\zeta'(\mathfrak{k})}{\zeta(\mathfrak{k}) - \zeta(-\mathfrak{k})}) > 0 \; for \, \mathfrak{k} = \mathfrak{k}_0, |\mathfrak{k}_0| = r \tag{2.6}$$

2.6 Subordination

In 1909, Lindelöf [57] proposed the idea of subordination for the first time. Subsequently, Littlewood [58] and Rogosinski [59] made further contributions to the field. In Geometric Function Theory, subordination serves as a valuable technique for relating two functions defined on different domains.

Definition 2.6.1. [60] Let class A is analytic in open disk $\mathbb{D}$ and let S be a subclass of A so it is univalent in $\mathbb{D}$. let any two functions (say ζ and j) be analytics in open disk $\mathbb{D}$, we say that ζ is subordination to j, written in mathematically as $\zeta \prec j$. If it exists a schwarz function w, which is analytic in open disk with $h(0) = 0$, $|h(ł)| < 1, ł \in \mathbb{D}$, such that $\zeta(ł) = j(h(ł)), ł \in \mathbb{D}$

2.7 q-Calculus

American mathematician Jackson invented quantum calculus at the beginning of the 20th century by presenting the q-analog of integral and derivative operators.

Definition 2.7.1. [61] A key element in q-calculus is the q-derivative operator, commonly represented as D_{q}, which plays a central role in this generalization of traditional calculus.

Definition 2.6.1. [42] Jackson introduced q-derivative and it is defined as following:

$$D_{\mathrm{q}}\zeta(ł) = (\frac{d}{dz})_{\mathrm{q}}\zeta(ł) = \frac{\zeta(\mathrm{q}ł) - \zeta(ł)}{\bar{\mathrm{q}}ł - ł}, \; ł \neq 0. \tag{2.7}$$

The operator $D_{\mathrm{q}}\zeta(ł)$ can be expressed as an infinite series:

$$D_{\mathrm{q}}\zeta(ł) = \sum_{n=1}^{\infty} [n]_{\mathrm{q}} c_n ł^{n-1}.$$

where $[n]_{\mathrm{q}}$ is the q-analogue and c_n is the coefficient of the series expansion.

2.8 Subclasses of Class S in q-Calculus

The subclasses of class S in q-calculus is as following:

Definition 2.8.1. The class of q-starlike function are known as S_s^*. The q-starlike function was introduced by Ismail et al [44]. Let a function ζ be analytic then belong to q-starlike function and defined as:

$$\left| \frac{ł}{\zeta(ł)} D_{\mathrm{q}}\zeta(ł)(ł) - \frac{1}{1-\mathrm{q}} \right| \leq \frac{1}{1-\mathrm{q}}, \;\; ł \in \mathbb{D}, \;\; 0 < \mathrm{q} < 1. \tag{2.8}$$

Definition 2.8.2. The class of q-convex function are denoted by C_q. Srivastava and Owa [62] defined a class of q-convex. Let a function ζ be analytic then belong to q-convex and defined as:

$$\left| \frac{ł}{\zeta(ł)} D_{\mathrm{q}}^2\zeta(ł)(ł) - \frac{1}{1-\mathrm{q}} \right| \leq \frac{1}{1-\mathrm{q}}, \;\; ł \in \mathbb{D}, \;\; 0 < \mathrm{q} < 1. \tag{2.9}$$

Definition 2.8.3. [47] q-starlike describes a function ζ with regard to symmetric points. When S_{s}^{*},q

$$\frac{2łD_{\mathrm{q}}\zeta(ł)}{\zeta(ł)-\zeta(-ł)} \prec v(ł), \;\; ł \in \mathbb{D}. \tag{2.10}$$

2.9 Henkel Determinant

Definition 2.9.1.

The determinant of the corresponding Hankel matrix is known as the Hankel determinant. For positive integers j,k. Pommerenke [63] established the Hankel determinant for the class of univalent functions. The q-Henkel determinant for $j \geq 0$ and $k \geq 1$ is as follows:

$$H_{j,k}(f) = \begin{bmatrix} \breve{a}_k & \breve{a}_{k+1} & \cdots & \breve{a}_{k+j-1} \\ \breve{a}_{k+1} & \breve{a}_{k+2} & \cdots & \breve{a}_{2k+j} \\ \vdots & \vdots & \ddots & \vdots \\ \breve{a}_{k+j_1} & \breve{a}_{k+j} & \cdots & \breve{a}_{k+2j-2} \end{bmatrix} \tag{2.11}$$

where j and k are positive integers, futher details, see [64, 63].

2.10 Initial Lemmas

The ensuing lemmas will be essential in guiding our research in the next chapters.

Lemma 2.10.1. if $p \in \mathfrak{P}$ and $h(ł) = 0$ and analytic in $\mathbb{D}$ then

$$[65] \;\; |p_n| \leq 2 \; for \;\; n \geq 1, \tag{2.12}$$

$$[65] \;\; |p_{i+j} - \varepsilon p_i p_j| \leq 2 \; for \;\; 0 \leq \varepsilon \leq 1, \tag{2.13}$$

and for complex number ε,we have

$$[66] \;\; |p_2 - \varepsilon p_1^2| \leq 2max\{1, |2\varepsilon - 1|\}. \tag{2.14}$$

Lemma 2.10.2. [67] Let $h \in \mathfrak{P}$ has power series, then

$$|\alpha_o p_1^3 - \beta_o p_1 p_2 + \gamma_o p_3| \leq 2|\alpha_o| + 2|\beta_o - 2\alpha_o| + 2|\alpha_o - \beta_o + \gamma_o|.$$

Lemma 2.10.3. [68] Let $\hat{m}_o$, $\hat{l}_o$, $\hat{n}_o$ and $\hat{a}_o$ satisfy the inequalities $0 < \hat{m}_o < 1$, $0 < \hat{r}_o < 1$,and

$$8\hat{r}_o(1-\hat{r}_o)\left[(\hat{m}_o\hat{n}_o-2\hat{l}_o)^2+(\hat{m}_o(\hat{r}_o+\hat{m}_o)-\hat{n}_o)^2\right]+\hat{m}_o(1-\hat{m}_o)(\hat{n}_o-2\hat{r}_o\hat{m}_o)^2 \leq 4\hat{m}_o^2(1-\hat{m}_o)^2\hat{r}_o(1-\hat{r}_o).$$

If $h \in \mathfrak{P}$ and has power series, then

$$|\hat{l}_o p_1^4 + \hat{r}_o p_2^2 + 2\hat{m}_o p_1 p_3 - \frac{3}{2}\hat{n}_o p_1^2 P_2 - p_4| \leq 2.$$

CHAPTER 3

ON A NEW SUBCLASSES OF STARLIKE FUNCTIONS ASSOCIATED WITH SYMMETRIC POINTS SINE FUNCTION

3.1 Introduction

The chapter's objective is to examine a number of fundamental and traditional findings that form the basis for further investigation. The review of starlike functions opens this section. The definition of these classes is based on symmetric points connected to the sine function. Futhermore, a number of important findings, including the Zalcman functional, Fekete-Szego inequality, Hankel determinants, and coefficient bounds, will be examined. The Khan *et al.* [35], introduce the class of trigonometric sine function-related Starlike functions connected to symmetric points.

Defination 3.1.1. Lets $\zeta(ł) \in A$ is in S_s^* after that

$$\frac{2ł\zeta'(ł)}{\zeta(ł)-\zeta(-ł)} \prec 1+\sin(ł),$$

for all ł$\in \mathbb{D}$.

3.2 Coefficients Estimates and Fekete-Szego Inequality

The subsequent outcomes pertaining to the specified class S_s^*

Theorem 3.2.1. If $\zeta(ł) \in S_s^*$, then

$$|\check{a}_2| \leq \frac{1}{2},\ |\check{a}_3| \leq \frac{1}{2},\ |\check{a}_4| \leq \frac{1}{4},\ |\check{a}_5| \leq \frac{3}{4}.$$

Proof. According to the definition:

$$\frac{2ł\zeta'(ł)}{\zeta(ł)-\zeta(-ł)} \prec 1+\sin(ł)$$

Since $\zeta(ł) \in S_s^*$, using the subordination techniques, then we get

$$\frac{2ł\zeta'(ł)}{\zeta(ł)-\zeta(-ł)} = 1+\sin(v(ł)). \tag{3.1}$$

Since $\zeta(ł) \in \mathscr{A}$, then $\zeta(ł)$ is in the form that as:

$$\zeta(ł) = ł + \sum_{n=2}^{\infty} \check{a}_n ł^2, \quad (ł \in \mathbb{D}). \tag{3.2}$$

So, it can be written as

$$\zeta(ł) = ł + \check{a}_2 ł^2 + \check{a}_3 ł^3 + \check{a}_4 ł^4 + \check{a}_5 ł^5 + ...,$$

it follows that

$$\zeta(-ł) = -ł + \check{a}_2 ł^2 - \check{a}_3 ł^3 + \check{a}_4 ł^4 - \check{a}_5 ł^5 + ...,$$

now,

$$\begin{aligned}\zeta(ł)-\zeta(-ł) &= (ł + \check{a}_2 ł^2 + \check{a}_3 ł^3 + \check{a}_4 ł^4 + \check{a}_5 ł^5 + ...) - (-ł + \check{a}_2 ł^2 - \check{a}_3 ł^3 + \check{a}_4 ł^4 - \check{a}_5 ł^5 + ...),\\ &= ł + \check{a}_2 ł^2 + \check{a}_3 ł^3 + \check{a}_4 ł^4 + \check{a}_5 ł^5 + ... + ł - \check{a}_2 ł^2 + \check{a}_3 ł^3 - \check{a}_4 ł^4 + \check{a}_5 ł^5 - ....,\end{aligned}$$

which implies

$$\zeta(ł)-\zeta(-ł) = 2ł + 2\check{a}_3 ł^3 + 2\check{a}_5 ł^5 + ...,$$

taking derivative of $\zeta(ł)$, then we get

$$\zeta'(ł) = 1 + 2\check{a}_2 ł + 3\check{a}_3 ł^2 + 4\check{a}_4 ł^3 + 5\check{a}_5 ł^4 + ...,$$

so, we have

$$2ł\zeta'(ł) = 2ł + 4\check{a}_2 ł^2 + 6\check{a}_3 ł^3 + 8\check{a}_4 ł^4 + ...,$$

which leads us

$$\frac{2ł\zeta'(ł)}{\zeta(ł)-\zeta(-ł)} = \frac{2ł + 4\check{a}_2 ł^2 + 6\check{a}_3 ł^3 + 8\check{a}_4 ł^4 + ...}{2ł + 2\check{a}_3 ł^3 + 2\check{a}_5 ł^5 + ...},$$

taking 2ł common on R.H.S, then we have

$$\frac{2ł\zeta'(ł)}{\zeta(ł)-\zeta(-ł)} = \frac{2ł(1+2\breve{a}_2ł+3\breve{a}_3ł^2+4\breve{a}_4ł^3+...)}{2ł(1+\breve{a}_3ł^2+\breve{a}_5ł^4+...)},$$

so, it can also be written as

$$\frac{2ł\zeta'(ł)}{\zeta(ł)-\zeta(-ł)} = (1+2\breve{a}_2ł+3\breve{a}_3ł^2+4\breve{a}_4ł^3+...)(1+\breve{a}_3ł^2+\breve{a}_5ł^4+...)^{-1}, \tag{3.3}$$

consider $(1+\breve{a}_3ł^2+\breve{a}_5ł^4+...)^{-1}$ using Binomial Theorem, then we get

$$\begin{aligned}[1+(\breve{a}_3ł^2+\breve{a}_5ł^4)]^{-1} &= 1+(-1)(\breve{a}_3ł^2+\breve{a}_5ł^4)+\frac{(-1)(-1-1)}{2!}(\breve{a}_3ł^2+\breve{a}_5ł^4)^2+...,\\ &= 1-\breve{a}_3ł^2-\breve{a}_5ł^4+\breve{a}_3{}^2ł^4+...,\end{aligned}$$

put above eqution in (3.3), then

$$\begin{aligned}\frac{2ł\zeta'(ł)}{\zeta(ł)-\zeta(-ł)} &= (1+2\breve{a}_2ł+3\breve{a}_3ł^2+4\breve{a}_4ł^3+...)(1-\breve{a}_3ł^2-\breve{a}_5ł^4-\breve{a}_3{}^2ł^4),\\ &= 1-\breve{a}_3ł^2+(\breve{a}_3{}^2-\breve{a}_5)ł^4+2\breve{a}_2ł-2\breve{a}_2\breve{a}_3ł^3+3\breve{a}_3ł^2-3\breve{a}_3{}^2ł^4+4\breve{a}_4ł^3+5\breve{a}_5ł^4+...,\end{aligned}$$

after simplification,then we will get

$$\Longrightarrow \frac{2ł'(ł)}{\zeta(ł)-\zeta(-ł)} = [1+(2\breve{a}_2)ł+(2\breve{a}_3)ł^2+(4\breve{a}_4-2\breve{a}_2\breve{a}_3)ł^3+(4\breve{a}_5-2\breve{a}_3{}^2)ł^4+......]. \tag{3.4}$$

Lets us a function

$$h(ł) = \frac{1+v(ł)}{1-v(ł)} = 1+p_1ł+p_2ł^2+....., \tag{3.5}$$

since $h(ł)\in\mathscr{P}$, then $h(ł)$ is in the form that as:

$$h(ł) = 1+\sum_{n=2}^{\infty} p_nł^n, \quad (ł\in\mathbb{D}). \tag{3.6}$$

Using the above statement then we get

$$v(ł) = \frac{h(ł)-1}{h(ł)+1}, \tag{3.7}$$

using (3.5) in (3.7) and then we have

$$v(ł) = \frac{1+p_1ł+p_2ł^2+.....-1}{1+p_1ł+p_2ł^2+.....+1},$$

so

$$v(ł) = \frac{p_1ł+p_2ł^2+p_3ł^3+p_4ł^4}{2+p_1ł+ł_2ł^2+p_3ł^3},$$

it can also be written as

$$v(\text{ł}) = (p_1\text{ł} + p_2\text{ł}^2 + p_3\text{ł}^3 + p_4\text{ł}^4)(2 + p_1\text{ł} + p_2\text{ł}^2 + p_3\text{ł}^3)^{-1},$$

after simplification we will get

$$v(\text{ł}) = \frac{p_1\text{ł}}{2} + \left(\frac{-p_1^2}{4} + \frac{p_2}{2}\right)\text{ł}^2 + \left(\frac{-p_1p_2}{2} + \frac{p_1^3}{8} + \frac{p_3}{2}\right)\text{ł}^3 + \left(\frac{-p_3p_1}{2} + \frac{3p_1^2p_2}{8} - \frac{p_1^4}{16} + \frac{p_4}{2} - \frac{p_2^2}{4}\right)\text{ł}^4 + ..., \tag{3.8}$$

as we know that

$$\sin[v(\text{ł})] = [v(\text{ł})] - \frac{[v(\text{ł})]^3}{3!} + \frac{[v(\text{ł})]^5}{5!} + ...,$$

so,

$$\sin[v(\text{ł})] = \frac{p_1\text{ł}}{2} + \left(\frac{-p_1^2}{4} + \frac{p_2}{2}\right)\text{ł}^2 + \left(\frac{-p_1p_2}{2} + \frac{5p_1^3}{8} + \frac{p_3}{2}\right)\text{ł}^3 + \left(\frac{-p_3p_1}{2} + \frac{5p_1^2p_2}{16} - \frac{p_1^4}{32} + \frac{p_4}{2} - \frac{p_2^2}{4}\right)\text{ł}^4 + ..., \tag{3.9}$$

$$\Longrightarrow 1 + \sin[v(\text{ł})] = \left[1 + \frac{p_1\text{ł}}{2} + \left(\frac{-p_1^2}{4} + \frac{p_2}{2}\right)\text{ł}^2 + \left(\frac{-p_1p_2}{2} + \frac{5p_1^3}{8} + \frac{p_3}{2}\right)\text{ł}^3 + \left(\frac{-p_3p_1}{2} + \frac{5p_1^2p_2}{16} - \frac{p_1^4}{32} + \frac{p_4}{2} - \frac{p_2^2}{4}\right)\text{ł}^4 + ...\right]. \tag{3.10}$$

Using (3.4) and (3.10) and substituting in (3.1), then we will get

$$1 + (2\check{a}_2)\text{ł} + (2\check{a}_3)\text{ł}^2 + (4\check{a}_4 - 2\check{a}_2\check{a}_3)\text{ł}^3 + (4\check{a}_5 - 2\check{a}_3{}^2)\text{ł}^4 + ... = \left[1 + \frac{p_1\text{ł}}{2} + \left(\frac{-p_1^2}{4} + \frac{p_2}{2}\right)\text{ł}^2 + \left(\frac{-p_1p_2}{2} + \frac{5p_1^3}{8} + \frac{p_3}{2}\right)\text{ł}^3 + \left(\frac{-p_3p_1}{2} + \frac{5p_1^2p_2}{16} - \frac{-p_1^4}{32} + \frac{p_4}{2} - \frac{p_2^2}{4}\right)\text{ł}^4 + ...\right]. \tag{3.11}$$

By comparing both sides power of the above equation, then we obtained

$$2\check{a}_2 = \frac{p_1}{2}, \tag{3.12}$$

$$2\check{a}_3 = \left(\frac{p_2}{2} - \frac{p_1^2}{4}\right), \tag{3.13}$$

$$4\check{a}_4 - 2\check{a}_2\check{a}_3 = \left(\frac{-p_1p_2}{2} + \frac{5p_1^3}{8} + \frac{p_3}{2}\right), \tag{3.14}$$

$$4\check{a}_5 - 2\check{a}_3{}^2 = \left(\frac{-p_3p_1}{2} + \frac{5p_1^2p_2}{16} - \frac{p_1^4}{32} + \frac{p_4}{2} - \frac{p_2^2}{4}\right). \tag{3.15}$$

Consider (3.12) and solve for the coefficient $\check{a}_2$, and then we get

$$\therefore 2\check{a}_2 = \frac{p_1}{2},$$

$$\Longrightarrow \breve{a}_2 = \frac{p_1}{4}. \tag{3.16}$$

Consider (3.13) and solve for the coefficient $\breve{a}_3$, and then we get

$$\therefore 2\breve{a}_3 = \left(\frac{p_2}{2} - \frac{p_1^2}{4}\right),$$

$$\Longrightarrow \breve{a}_3 = \frac{p_2}{4} - \frac{p_1^2}{8}. \tag{3.17}$$

Using (3.16) and (3.17), put in (3.14) and solve for the coefficient $\breve{a}_4$

$$\therefore 4\breve{a}_4 - 2\breve{a}_2\breve{a}_3 = \left(\frac{-p_1p_2}{2} + \frac{5p_1^3}{8} + \frac{p_3}{2}\right),$$

$$4\breve{a}_4 = 2\breve{a}_2\breve{a}_3 + \left(\frac{-p_1p_2}{2} + \frac{5p_1^3}{8} + \frac{p_3}{2}\right),$$

$$\breve{a}_4 = 2\left(\frac{p_1}{4}\right)\left(\frac{p_2}{4} - \frac{p_1^2}{8}\right) + \left(\frac{-p_1p_2}{2} + \frac{5p_1^3}{8} + \frac{p_3}{2}\right),$$

$$\breve{a}_4 = \frac{p_1p_2}{8} - \frac{p_1^3}{16} - \frac{-p_1p_2}{2} + \frac{5p_1^3}{8} + \frac{p_3}{2},$$

$$\Longrightarrow \breve{a}_4 = \frac{p_1^3}{96} - \frac{3p_1p_2}{32} + \frac{p_3}{8}. \tag{3.18}$$

Using (3.17), put in (3.15) and solve for the coefficient $\breve{a}_5$

$$\therefore 4\breve{a}_5 - 2\breve{a}_3{}^2 = \left(\frac{-p_3p_1}{2} + \frac{5p_1^2p_2}{16} - \frac{p_1^4}{32} + \frac{p_4}{2} - \frac{p_2^2}{4}\right),$$

$$4\breve{a}_5 = 2\breve{a}_3{}^2 + \left(\frac{-p_3p_1}{2} + \frac{5p_1^2p_2}{16} - \frac{p_1^4}{32} + \frac{p_4}{2} - \frac{p_2^2}{4}\right),$$

$$4\breve{a}_5 = 2\left(\frac{p_2}{4} - \frac{p_1^2}{8}\right)^2 + \left(\frac{-p_3p_1}{2} + \frac{5p_1^2p_2}{16} - \frac{p_1^4}{32} + \frac{p_4}{2} - \frac{p_2^2}{4}\right),$$

$$4\breve{a}_5 = \left[\frac{p_2^2}{16} + \frac{p_1^4}{64} - \frac{2p_2p_1^2}{16}\right] + \left(\frac{-p_3p_1}{2} + \frac{5p_1^2p_2}{16} - \frac{p_1^4}{32} + \frac{p_4}{2} - \frac{p_2^2}{4}\right),$$

$$4\breve{a}_5 = \frac{-p_2^2}{8} + \frac{3p_1^2p_2}{16} - \frac{p_3p_1}{2} + \frac{p_4}{8},$$

$$\Longrightarrow \breve{a}_5 = -\frac{p_2^2}{32} + \frac{3p_1^2p_2}{64} - \frac{p_3p_1}{8} + \frac{p_4}{8}. \tag{3.19}$$

Now we will find the absolute values of the coefficients.

We have (3.16).

$$\breve{a}_2 = \frac{p_1}{4},$$

taking modulus on both sides, then

$$|\breve{a}_2| = \left|\frac{p_1}{4}\right|,$$

using Lemma (2.10.1) and Equation (2.12), then we have

$$|\check{a}_2| = \left|\frac{p_1}{4}\right| \leq \frac{2}{4} \leq \frac{1}{2},$$

so,

$$|\check{a}_2| \leq \frac{1}{2}. \tag{3.20}$$

We have (3.17).

$$\check{a}_3 = \frac{p_2}{4} - \frac{p_1^2}{8},$$

taking modulus on both sides, then

$$|\check{a}_3| = \left|\frac{p_2}{4} - \frac{p_1^2}{8}\right| = \frac{1}{4}\left|p_2 - \frac{p_1^2}{2}\right|,$$

using Lemma (2.10.1) and Equation (2.14), then

$$\begin{aligned}\frac{1}{4}\left|p_2 - \frac{p_1^2}{2}\right| &\leq \frac{2}{4} m\hat{a}x\left\{1, \left|2\left(\frac{1}{2}\right) - 1\right|\right\},\\ &\leq \frac{2}{4} m\hat{a}x\{1, 0\} = \frac{1}{2},\end{aligned}$$

so,

$$|\check{a}_3| \leq \frac{1}{2}. \tag{3.21}$$

We have (3.18).

$$\check{a}_4 = \frac{p_1^3}{96} - \frac{3p_1p_2}{32} + \frac{p_3}{8},$$

taking modulus on both sides, then

$$|\check{a}_4| = \left|\frac{p_1^3}{96} - \frac{3p_1p_2}{32} + \frac{p_3}{8}\right|,$$

using triangular inequality and Lemma (2.10.2), then we have

$$\begin{aligned}\left|\frac{p_1^3}{96} - \frac{3p_1p_2}{32} + \frac{p_3}{8}\right| &\leq 2\left|\frac{1}{96}\right| + 2\left|\frac{3}{32} - \frac{2}{96}\right| + 2\left|\frac{1}{96} - \frac{3}{32} + \frac{1}{8}\right|,\\ &\leq \frac{1}{48} + \frac{7}{48} + \frac{4}{48} = \frac{1}{4},\end{aligned}$$

so,

$$|\check{a}_4| \leq \frac{1}{4}. \tag{3.22}$$

We have (3.19).

$$\check{a}_5 = -\frac{p_2^2}{32} + \frac{3p_1^2p_2}{64} - \frac{p_3p_1}{8} + \frac{p_4}{8},$$

taking modulus on both sides, then

$$|\check{a}_5| = \left| -\frac{p_2^2}{32} + \frac{3p_1^2 p_2}{64} - \frac{p_3 p_1}{8} + \frac{p_4}{8} \right|,$$

by rearranging the above equation,then we get

$$|\check{a}_5| = \left| \frac{1}{8}\left(p_4 - \frac{p_2^2}{4}\right) - \frac{p_1}{8}\left(p_3 - \frac{3p_1p_2}{8}\right) \right|,$$

using Lemma (2.10.1) and Equations (2.12) and (2.13), then we get

$$\begin{aligned}
|\check{a}_5| &\leq \frac{1}{8}|p_4 - p_1p_3| + \frac{3}{64}|p_1^2||p_2| + \frac{1}{32}|p_2^2|,\\
&\leq \frac{1}{8}(2) + \frac{3}{64}(2)^2(2) + \frac{1}{32}(2)^2,\\
&\leq \frac{1}{4} + \frac{3}{8} + \frac{1}{8} = \frac{3}{4},
\end{aligned}$$

so,

$$|\check{a}_5| \leq \frac{3}{4}. \tag{3.23}$$

Hence proved. □

Theorem 3.2.2. If $\zeta(\text{ł}) \in S_s^*$, then

$$|\check{a}_3 - \varepsilon\check{a}_2{}^2| \leq \frac{1}{2}m\hat{a}x\left\{1, \frac{|\varepsilon|}{2}\right\}.$$

Proof. Utilizing (3.16) and (3.17), then we get

$$\begin{aligned}
|\check{a}_3 - \varepsilon\check{a}_2{}^2| &= \left| \left(\frac{p_2}{4} - \frac{p_1^2}{8}\right) - \varepsilon\left(\frac{p_1}{4}\right)^2 \right|,\\
&= \left| \frac{p_2}{4} - \frac{p_1^2}{8} - \varepsilon\frac{p_1^2}{16} \right|,\\
&= \left| \frac{p_2}{4} - \left(\frac{1}{8} + \frac{\varepsilon}{16}\right)p_1^2 \right|,\\
&= \left| \frac{p_2}{4} - \left(\frac{2+\varepsilon}{16}\right)p_1^2 \right|,\\
&= \frac{1}{4}\left| p_2 - \left(\frac{2+\varepsilon}{4}\right)p_1^2 \right|,
\end{aligned}$$

using Application of Lemma (2.10.1) and Equation (2.14),then we get

$$\begin{aligned}
|\check{a}_3 - \varepsilon\check{a}_2{}^2| &\leq \frac{2}{4}m\hat{a}x\left\{1, \left|2\left(\frac{2+\varepsilon}{4}\right) - 1\right|\right\},\\
&\leq \frac{2}{4}m\hat{a}x\left\{1, \left|\frac{4}{4} + \frac{\varepsilon}{2} - 1\right|\right\},
\end{aligned}$$

so,

$$|\check{a}_3 - \varepsilon\check{a}_2{}^2| \leq \frac{1}{2}m\hat{a}x\left\{1, \frac{|\varepsilon|}{2}\right\}. \tag{3.24}$$

Hence proved. □

Corollary 3.2.2.1. If $\zeta(ł) \in S_s^*$ and $\varepsilon = 1$, then

$$|\check{a}_3 - \check{a}_2{}^2| \leq \frac{1}{2}.$$

Proof. Utilizing (3.16) and (3.17), then we get

$$\begin{aligned}|\check{a}_3 - \check{a}_2{}^2| &= \left| \left(\frac{p_2}{4} - \frac{p_1^2}{8}\right) - \left(\frac{p_1}{4}\right)^2 \right|,\\ &= \left| \frac{p_2}{4} - \frac{p_1^2}{8} - \frac{p_1^2}{16} \right|,\\ &= \left| \frac{p_2}{4} - \left(\frac{1}{8} + \frac{1}{16}\right) p_1^2 \right|,\\ &= \left| \frac{p_2}{4} - \left(\frac{2+1}{16}\right) p_1^2 \right|,\\ &= \frac{1}{4}\left| p_2 - \left(\frac{3}{4}\right) p_1^2 \right|,\end{aligned}$$

using Application of Lemma (2.10.1) and Equation (2.14),then we get

$$\begin{aligned}|\check{a}_3 - \varepsilon\check{a}_2{}^2| &\leq \frac{2}{4} m\hat{a}x \left\{ 1, \left| 2\left(\frac{3}{4}\right) - 1 \right| \right\},\\ &\leq \frac{2}{4} m\hat{a}x \left\{ 1, \frac{1}{2} \right\},\end{aligned}$$

so,

$$|\check{a}_3 - \check{a}_2{}^2| \leq \frac{1}{2}. \tag{3.25}$$

Hence proved. □

3.3 Hankel determinants

The following results are evaluted.

Theorem 3.3.1. If $\zeta(ł) \in S_s^*$, then

$$|\check{a}_2\check{a}_3 - \check{a}_4| \leq \frac{1}{4}.$$

Proof. From (3.16), (3.17) and (3.18), then we have

$$\begin{aligned}|\check{a}_2\check{a}_3 - \check{a}_4| &= \left| \left(\frac{p_1}{4}\right)\left(\frac{p_2}{4} - \frac{p_1^2}{8}\right) - \left(\frac{p_1^3}{96} - \frac{3p_1p_2}{32} + \frac{p_3}{8}\right) \right|,\\ &= \left| \frac{p_1p_2}{16} - \frac{p_1^3}{32} - \frac{p_1^3}{96} + \frac{3p_1p_2}{32} - \frac{p_3}{8} \right|,\\ &= \left| \frac{p_1^3}{24} - \frac{5p_1p_2}{32} + \frac{p_3}{8} \right|,\end{aligned}$$

implementation of triangular inequality and Lemma (2.10.2), then we get

$$\begin{aligned}\left|\frac{p_1^3}{24}-\frac{5p_1p_2}{32}+\frac{p_3}{8}\right| &\leq 2\left|\frac{1}{24}\right|+2\left|\frac{5}{32}-\frac{2}{24}\right|+2\left|\frac{1}{24}-\frac{5}{32}+\frac{1}{8}\right|,\\ &\leq 2\left|\frac{1}{24}\right|+2\left|\frac{15-8}{96}\right|+2\left|\frac{4-15+12}{96}\right|,\\ &\leq \frac{2}{24}+\frac{14}{96}+\frac{2}{96}=\frac{24}{96}=\frac{1}{4},\end{aligned}$$

so,

$$|\check{a}_2\check{a}_3-\check{a}_3{}^2|\leq\frac{1}{4}. \tag{3.26}$$

Hence proved. □

Theorem 3.3.2. If $\zeta(\text{ł})\in S_s^*$, then

$$|\check{a}_2\check{a}_4-\check{a}_3{}^2|\leq\frac{11}{16}.$$

Proof. From (3.15), (3.16) and (3.17), then we have

$$\begin{aligned}|\check{a}_2\check{a}_4-\check{a}_3{}^2| &= \left|\frac{p_1}{4}\left(\frac{p_1^3}{96}-\frac{3p_1p_2}{32}+\frac{p_3}{8}\right)-\left(\frac{p_2}{4}-\frac{p_1^2}{8}\right)^2\right|,\\ &= \left|\frac{p_1^4}{384}-\frac{3p_1^2p_2}{128}+\frac{p_1p_3}{32}-\frac{p_2^2}{16}-\frac{p_1^4}{64}+\frac{p_1^2p_2}{16}\right|,\\ &= \left|\frac{p_1p_3}{32}+\frac{5p_1^2p_2}{128}-\frac{5p_1^4}{384}-\frac{p_2^2}{16}\right|,\\ &= \left|\frac{5p_1^2}{128}\left(p_2-\frac{p_1^2}{3}\right)+\frac{1}{16}\left(p_1p_3-p_2^2\right)-\frac{p_1p_3}{32}\right|,\end{aligned}$$

using Lemma (2.10.1) and Equation (2.12),(2.13) and (2.14), then we get

$$\begin{aligned}|\check{a}_2\check{a}_4-\check{a}_3{}^2| &\leq \frac{5}{128}(2)^2(2)+\frac{1}{16}(2)^2+\frac{1}{32}(2)(2),\\ &\leq \frac{5}{16}+\frac{1}{4}+\frac{1}{8}=\frac{5+4+2}{16}=\frac{11}{16},\end{aligned}$$

so,

$$|\check{a}_2\check{a}_4-\check{a}_3{}^2|\leq\frac{11}{16}. \tag{3.27}$$

Hence proved. □

Theorem 3.3.3. If $\zeta(\text{ł})\in S_s^*$, then

$$|\mathfrak{H}_{3,1}(\zeta)|\leq\frac{25}{32}\backsimeq 0.78125.$$

Proof. Third order Hankel determinant is defined as:

$$\mathfrak{H}_{3,1}(\zeta)=\check{a}_3(\check{a}_2\check{a}_4-\check{a}_3{}^2)-\check{a}_4(\check{a}_4-\check{a}_2\check{a}_3)+\check{a}_5(\check{a}_3-\check{a}_2{}^2),$$

taking modulus on both sides, then we get

$$|\mathfrak{H}_{3,1}(\zeta)| = |\breve{a}_3||\breve{a}_2\breve{a}_4 - \breve{a}_3{}^2| - |\breve{a}_4||\breve{a}_4 - \breve{a}_2\breve{a}_3| + |\breve{a}_5||\breve{a}_3 - \breve{a}_2{}^2|,$$

by implementing results Theorem 3.2.1, Corollary 3.2.2.1, Theorem 3.3.1 and Theorem 3.3.2, then we obtained

$$\begin{aligned}|\mathfrak{H}_{3,1}(\zeta)| &\le \frac{1}{2}\left(\frac{11}{16}\right) + \frac{1}{4}\left(\frac{1}{4}\right) + \frac{3}{4}\left(\frac{1}{2}\right),\\ &\le \frac{11}{32} + \frac{1}{16} + \frac{3}{8},\\ &\le \frac{11+2+12}{32} = \frac{25}{32},\end{aligned}$$

so, we get

$$|\mathfrak{H}_{3,1}(\zeta)| \le \frac{25}{32} \backsimeq 0.78125. \tag{3.28}$$

Hence proved. □

3.4 Zalcman Functional

The result is evaluated:

Theorem 3.4.1. If $\zeta(\text{ł}) \in S_s^*$, then

$$|\breve{a}_3{}^2 - \breve{a}_5| \le \frac{1}{4}.$$

Proof. Using (3.16) and (3.18), then we get

$$\begin{aligned}|\breve{a}_3{}^2 - \breve{a}_5| &= \left|\left(\frac{p_2}{4} - \frac{p_1^2}{4}\right)^2 - \left(-\frac{p_2^2}{32} + \frac{3p_1^2p_2}{64} - \frac{p_3p_1}{8} + \frac{p_4}{8}\right)\right|,\\ &= \left|\frac{p_1^4}{8} + \frac{3p_2^2}{32} - \frac{7p_1^2p_2}{64} + \frac{p_3p_1}{8} - \frac{p_4}{8}\right|,\\ &= \frac{1}{8}\left|\frac{p_1^4}{8} + \frac{3p_2^2}{4} - \frac{7p_1^2p_2}{8} + \left(\frac{1}{2}\right)2p_3p_1 - p_4\right|,\end{aligned}$$

using Lemma (2.10.3), then we get

$$|\breve{a}_3{}^2 - \breve{a}_5| \le \frac{1}{8}(2) = \frac{1}{4},$$

so, then we have

$$|\breve{a}_3{}^2 - \breve{a}_5| \le \frac{1}{4}. \tag{3.29}$$

Hence proved. □

CHAPTER 4

ON A NEW SUBCLASSES OF STARLIKE FUNCTIONS ASSOCIATED WITH SYMMETRIC POINTS TANGENT HYPERBOLIC FUNCTION

4.1 Introduction

In this chapter, we established a new subclass of Starlike function and related to symmetric points. The hyperbolic function of trignometric tangent is intimately associated with this subclass. we explore the results of the Zalcman functional, Henkel determinants, Fekete-Szego inequality, and coefficient estimations.

Defination 4.1.1. Lets $\zeta(ł) \in A$ is in $S_s^*(1+tanh(ł))$ after that

$$\frac{2ł\zeta'(ł)}{\zeta(ł)-\zeta(-ł)} \prec 1+\tanh(ł),$$

for all ł$\in \mathbb{D}$.

4.2 Coefficients Estimates and Fekete-Szego Inequality

The subsequent outcomes pertaining to the specified class $S_s^*(1+tanh(ł))$

Theorem 4.2.1. If $\zeta(ł) \in S_s^*(1+tanh(ł))$, then

$$|\check{a}_2| \leq \frac{1}{2},\ |\check{a}_3| \leq \frac{1}{2},\ |\check{a}_4| \leq \frac{1}{4},\ |\check{a}_5| \leq \frac{3}{4}.$$

Proof. According to the definition:

$$\frac{2ł\zeta'(ł)}{\zeta(ł)-\zeta(-ł)} \prec 1+\tanh(ł)$$

Since $\zeta(ł) \in S_s^*(1+tanh(ł))$, using the subordination techniques, then we get

$$\frac{2ł\zeta'(ł)}{\zeta(ł)-\zeta(-ł)} = 1+\tanh(v(ł)). \tag{4.1}$$

Since $\zeta(ł) \in A$, then $\zeta(ł)$ is in the form that as:

$$\zeta(ł) = ł + \sum_{n=2}^{\infty} \check{a}_n ł^n, \quad (ł \in \mathbb{D}). \tag{4.2}$$

So, it can be written as

$$\zeta(ł) = ł + \check{a}_2 ł^2 + \check{a}_3 ł^3 + \check{a}_4 ł^4 + \check{a}_5 ł^5 + ...,$$

it follows that

$$\zeta(-ł) = -ł + \check{a}_2 ł^2 - \check{a}_3 ł^3 + \check{a}_4 ł^4 - \check{a}_5 ł^5 + ...,$$

now,

$$\begin{aligned}\zeta(ł)-\zeta(-ł) &= (ł + \check{a}_2 ł^2 + \check{a}_3 ł^3 + \check{a}_4 ł^4 + \check{a}_5 ł^5 + ...) - (-ł + \check{a}_2 ł^2 - \check{a}_3 ł^3 + \check{a}_4 ł^4 - \check{a}_5 ł^5 + ...),\\ &= ł + \check{a}_2 ł^2 + \check{a}_3 ł^3 + \check{a}_4 ł^4 + \check{a}_5 ł^5 + ... + ł - \check{a}_2 ł^2 + \check{a}_3 ł^3 - \check{a}_4 ł^4 + \check{a}_5 ł^5 - ....,\end{aligned}$$

which implies

$$\zeta(ł)-\zeta(-ł) = 2ł + 2\check{a}_3 ł^3 + 2\check{a}_5 ł^5 + ...,$$

taking derivative of $\zeta(ł)$, then we get

$$\zeta'(ł) = 1 + 2\check{a}_2 ł + 3\check{a}_3 ł^2 + 4\check{a}_4 ł^3 + 5\check{a}_5 ł^4 + ...,$$

so, we have

$$2ł\zeta'(ł) = 2ł + 4\check{a}_2 ł^2 + 6\check{a}_3 ł^3 + 8\check{a}_4 ł^4 + ...,$$

which leads us

$$\frac{2ł\zeta'(ł)}{\zeta(ł)-\zeta(-ł)} = \frac{2ł + 4\check{a}_2 ł^2 + 6\check{a}_3 ł^3 + 8\check{a}_4 ł^4 + ...}{2ł + 2\check{a}_3 ł^3 + 2\check{a}_5 ł^5 + ...},$$

taking 2ł common on R.H.S, then we have

$$\frac{2ł\zeta'(ł)}{\zeta(ł)-\zeta(-ł)} = \frac{2ł(1+2\breve{a}_2ł+3\breve{a}_3ł^2+4\breve{a}_4ł^3+...)}{2ł(1+\breve{a}_3ł^2+\breve{a}_5ł^4+...)},$$

so, it can also be written as

$$\frac{2ł\zeta'(ł)}{\zeta(ł)-\zeta(-ł)} = (1+2\breve{a}_2ł+3\breve{a}_3ł^2+4\breve{a}_4ł^3+...)(1+\breve{a}_3ł^2+\breve{a}_5ł^4+...)^{-1}, \tag{4.3}$$

consider $(1+\breve{a}_3ł^2+\breve{a}_5ł^4+...)^{-1}$ using Binomial Theorem, then we get

$$\begin{aligned}[1+(\breve{a}_3ł^2+\breve{a}_5ł^4)]^{-1} &= 1+(-1)(\breve{a}_3ł^2+\breve{a}_5ł^4)+\frac{(-1)(-1-1)}{2!}(\breve{a}_3ł^2+\breve{a}_5ł^4)^2+...,\\ &= 1-\breve{a}_3ł^2-\breve{a}_5ł^4+\breve{a}_3{}^2ł^4+...,\end{aligned}$$

put above eqution in (4.3), then we have

$$\begin{aligned}\frac{2ł\zeta'(ł)}{\zeta(ł)-\zeta(-ł)} &= (1+2\breve{a}_2ł+3\breve{a}_3ł^2+4\breve{a}_4ł^3+...)(1-\breve{a}_3ł^2-\breve{a}_5ł^4-\breve{a}_3{}^2ł^4),\\ &= 1-\breve{a}_3ł^2+(\breve{a}_3{}^2-\breve{a}_5)ł^4+2\breve{a}_2ł-2\breve{a}_2\breve{a}_3ł^3+3\breve{a}_3ł^2-3\breve{a}_3{}^2ł^4+4\breve{a}_4ł^3+5\breve{a}_5ł^4+...,\end{aligned}$$

after simplification,then we will get

$$\Longrightarrow \frac{2ł\zeta'(ł)}{\zeta(ł)-\zeta(-ł)} = [1+(2\breve{a}_2)ł+(2\breve{a}_3)ł^2+(4\breve{a}_4-2\breve{a}_2\breve{a}_3)ł^3+(4\breve{a}_5-2\breve{a}_3{}^2)ł^4+......]. \tag{4.4}$$

Lets us a function

$$h(ł) = \frac{1+v(ł)}{1-v(ł)} = 1+p_1ł+p_2ł^2+....., \tag{4.5}$$

since $h(ł) \in \mathfrak{P}$, then $h(ł)$ is in the form that as:

$$h(ł) = 1+\sum_{n=2}^{\infty} p_n ł^n, \quad (ł \in \mathbb{D}). \tag{4.6}$$

Using the above statement then we get

$$v(ł) = \frac{h(ł)-1}{h(ł)+1}, \tag{4.7}$$

using (4.5) in (4.7) and then we have

$$v(ł) = \frac{1+p_1ł+p_2ł^2+.....-1}{1+p_1ł+p_2ł^2+.....+1},$$

so

$$v(ł) = \frac{p_1ł+p_2ł^2+p_3ł^3+p_4ł^4}{2+p_1ł+p_2ł^2+p_3ł^3},$$

it can also be written as

$$v(\text{ł}) = (p_1\text{ł} + p_2\text{ł}^2 + p_3\text{ł}^3 + p_4\text{ł}^4)(2 + p_1\text{ł} + p_2\text{ł}^2 + p_3\text{ł}^3)^{-1},$$

after simplification we will get

$$v(\text{ł}) = \frac{p_1\text{ł}}{2} + \left(\frac{-p_1^2}{4} + \frac{p_2}{2}\right)\text{ł}^2 + \left(\frac{-p_1p_2}{2} + \frac{p_1^3}{8} + \frac{p_3}{2}\right)\text{ł}^3 + \left(\frac{-p_3p_1}{2} + \frac{3p_1^2p_2}{8} - \frac{p_1^4}{16} + \frac{p_4}{2} - \frac{p_2^2}{4}\right)\text{ł}^4 + ..., \tag{4.8}$$

as we know that

$$\tanh[v(\text{ł})] = [v(\text{ł})] - \frac{[v(\text{ł})]^3}{3} + \frac{2[v(\text{ł})]^5}{15} + ...,$$

so,

$$\tanh[v(\text{ł})] = \frac{p_1\text{ł}}{2} + \left(\frac{-p_1^2}{4} + \frac{p_2}{2}\right)\text{ł}^2 + \left(\frac{-p_1p_2}{2} + \frac{p_1^3}{12} + \frac{p_3}{2}\right)\text{ł}^3 + \left(\frac{-p_3p_1}{2} + \frac{p_1^2p_2}{4} - \frac{0p_1^4}{32} + \frac{p_4}{2} - \frac{p_2^2}{4}\right)\text{ł}^4 + ..., \tag{4.9}$$

$$\implies 1 + \tanh[v(\text{ł})] = \left[1 + \frac{p_1\text{ł}}{2} + \left(\frac{-p_1^2}{4} + \frac{p_2}{2}\right)\text{ł}^2 + \left(\frac{-p_1p_2}{2} + \frac{p_1^3}{12} + \frac{p_3}{2}\right)\text{ł}^3 + \left(\frac{-p_3p_1}{2} + \frac{5p_1^2p_2}{16} - \frac{p_1^4}{32} + \frac{p_4}{2} - \frac{p_2^2}{4}\right)\text{ł}^4 + ...\right]. \tag{4.10}$$

Using (4.4) and (4.10) and substituting in (4.1), then we will get

$$1 + (2\check{a}_2)\text{ł} + (2\check{a}_3)\text{ł}^2 + (4\check{a}_4 - 2\check{a}_2\check{a}_3)\text{ł}^3 + (4\check{a}_5 - 2\check{a}_3{}^2)\text{ł}^4 + ... = \left[1 + \frac{p_1\text{ł}}{2} + \left(\frac{-p_1^2}{4} + \frac{p_2}{2}\right)\text{ł}^2 + \left(\frac{-p_1p_2}{2} + \frac{p_1^3}{12} + \frac{p_3}{2}\right)\rho^3 + \left(\frac{-p_3p_1}{2} + \frac{5p_1^2p_2}{16} - \frac{p_1^4}{32} + \frac{p_4}{2} - \frac{p_2^2}{4}\right)\text{ł}^4 + ...\right]. \tag{4.11}$$

By comparing both sides of the above equation, we obtained

$$2\check{a}_2 = \frac{p_1}{2}, \tag{4.12}$$

$$2\check{a}_3 = \left(\frac{p_2}{2} - \frac{p_1^2}{4}\right), \tag{4.13}$$

$$4\check{a}_4 - 2\check{a}_2\check{a}_3 = \left(\frac{-p_1p_2}{2} + \frac{p_1^3}{12} + \frac{p_3}{2}\right), \tag{4.14}$$

$$4\check{a}_5 - 2\check{a}_3{}^2 = \left(\frac{-p_3p_1}{2} + \frac{p_1^2p_2}{4} - \frac{0p_1^4}{32} + \frac{p_4}{2} - \frac{p_2^2}{4}\right). \tag{4.15}$$

Consider (4.12) and solve for the coefficient $\check{a}_2$, and then we get

$$\therefore 2\check{a}_2 = \frac{p_1}{2},$$

$$\implies \check{a}_2 = \frac{p_1}{4}. \tag{4.16}$$

Consider (4.13) and solve for the coefficient $\check{a}_3$, and then we get

$$\therefore 2\check{a}_3 = \frac{p_2}{2} - \frac{p_1^2}{4},$$

$$\implies \check{a}_3 = \frac{p_2}{4} - \frac{p_1^2}{8}. \tag{4.17}$$

Using (4.16) and (4.17), put in (4.14) and solve for the coefficient a_4

$$\therefore 4\check{a}_4 - 2\check{a}_2\check{a}_3 = \left(\frac{-p_1p_2}{2} + \frac{p_1^3}{12} + \frac{p_3}{2}\right),$$

$$4\check{a}_4 = 2\check{a}_2\check{a}_3 + \left(\frac{-p_1p_2}{2} + \frac{p_1^3}{12} + \frac{p_3}{2}\right),$$

$$4\check{a}_4 = 2\left(\frac{p_1}{4}\right)\left(\frac{-p_1^2}{8} + \frac{p_2}{4}\right) - \frac{p_1p_2}{2} + \frac{p_1^3}{12} + \frac{p_3}{2},$$

$$4\check{a}_4 = \frac{p_1p_2}{8} - \frac{p_1^3}{16} - \frac{p_1p_2}{2} + \frac{p_1^3}{12} + \frac{p_3}{2},$$

$$4\check{a}_4 = \frac{-3p_1p_2}{8} + \frac{p_3}{2} + \frac{p_1^3}{48},$$

$$\implies \check{a}_4 = \frac{-3p_1p_2}{32} + \frac{p_3}{8} + \frac{p_1^3}{192} \tag{4.18}$$

Using (4.17), put in (4.15) and solve for the coefficient a_5

$$\therefore 4\check{a}_5 - 2\check{a}_3{}^2 = \left(\frac{-p_3p_1}{2} + \frac{p_1^2p_2}{4} - \frac{0p_1^4}{32} + \frac{p_4}{2} - \frac{p_2^2}{4}\right),$$

$$4\check{a}_5 = 2\check{a}_3{}^2 + \left(\frac{-p_3p_1}{2} + \frac{p_1^2p_2}{4} - \frac{0p_1^4}{32} + \frac{p_4}{2} - \frac{p_2^2}{4}\right),$$

$$4\check{a}_5 = 2\left(\frac{-p_1^2}{8} + \frac{p_2}{4}\right)^2 + \left(\frac{-p_3p_1}{2} + \frac{p_1^2p_2}{4} - \frac{0p_1^4}{32} + \frac{p_4}{2} - \frac{p_2^2}{4}\right),$$

$$4\check{a}_5 = \left[\frac{p_2^2}{8} + \frac{p_1^4}{32} - \frac{p_2p_1^2}{8}\right] - \frac{p_3p_1}{2} + \frac{p_1^2p_2}{4} - \frac{0p_1^4}{32} + \frac{p_4}{2} - \frac{p_2^2}{4},$$

$$4\check{a}_5 = \frac{-p_3p_1}{2} + \frac{p_1^2p_2}{8} + \frac{p_1^4}{32} + \frac{p_4}{2} - \frac{p_2^2}{8},$$

$$\implies \check{a}_5 = \frac{-p_3p_1}{8} + \frac{p_1^2p_2}{32} + \frac{p_1^4}{128} + \frac{p_4}{8} - \frac{p_2^2}{32}. \tag{4.19}$$

Now we will find the absolute values of the coefficients.

We have (4.16).

$$\check{a}_2 = \frac{p_1}{4},$$

taking modulus on both sides, then

$$|\check{a}_2| = \left|\frac{p_1}{4}\right|,$$

using Lemma (2.10.1) and Equation (2.12), then we get

$$|\check{a}_2| = \left|\frac{p_1}{4}\right| \leq \frac{2}{4} \leq \frac{1}{2},$$

so,

$$|\check{a}_2| \leq \frac{1}{2}. \tag{4.20}$$

We have (4.17).

$$\check{a}_3 = \frac{p_2}{4} - \frac{p_1^2}{8},$$

taking modulus on both sides, then

$$|\check{a}_3| = \left|\frac{p_2}{4} - \frac{p_1^2}{8}\right| = \frac{1}{4}\left|p_2 - \frac{p_1^2}{2}\right|,$$

using Lemma (2.10.1) and Equation (2.14), then

$$\begin{aligned}\frac{1}{4}|p_2 - \frac{p_1^2}{2}| &\leq \frac{2}{4}m\hat{a}x\{1, |2(\frac{1}{2}) - 1|\}, \\ &\leq \frac{2}{4}m\hat{a}x\{1, 0\} = \frac{1}{2},\end{aligned}$$

so,

$$|\check{a}_3| \leq \frac{1}{2}. \tag{4.21}$$

We have (4.19).

$$\check{a}_4 = \frac{p_1^3}{192} - \frac{3p_1p_2}{32} + \frac{p_3}{8},$$

taking modulus on both sides, then

$$|\check{a}_4| = \left|\frac{p_1^3}{192} - \frac{3p_1p_2}{32} + \frac{p_3}{8}\right|,$$

using triangular inequality and Lemma (2.10.2), then we get

$$\begin{aligned}\left|\frac{p_1^3}{192} - \frac{3p_1p_2}{32} + \frac{p_3}{8}\right| &\leq 2\left|\frac{1}{192}\right| + 2\left|\frac{3}{32} - \frac{2}{192}\right| + 2\left|\frac{1}{192} - \frac{3}{32} + \frac{1}{8}\right|, \\ &\leq \frac{2}{192} + \frac{2(16)}{192} + \frac{2(7)}{192} = \frac{1}{4},\end{aligned}$$

so,

$$|\check{a}_4| \leq \frac{1}{4}. \tag{4.22}$$

We have (4.19).

$$\check{a}_5 = -\frac{p_2^2}{32} + \frac{p_1^2p_2}{32} - \frac{p_3p_1}{8} + \frac{p_4}{8} + \frac{p_1^4}{128},$$

taking modulus on both sides, then

$$|\breve{a}_5| = \left| -\frac{p_2^2}{32} + \frac{p_1^2 p_2}{32} - \frac{p_3 p_1}{8} + \frac{p_4}{8} + \frac{p_1^4}{128} \right|,$$

using (2.10.1) and Equations (2.12) and (2.13), then

$$\begin{aligned} |\breve{a}_5| &\leq \frac{1}{8}|p_4 - p_1 p_3| + \frac{1}{32}|p_1^2||p_2| + \frac{1}{32}|p_2^2| + \frac{1}{128}|p_1^4|, \\ &\leq \frac{1}{8}(2) + \frac{1}{32}(2)^2(2) + \frac{1}{32}(2)^2 + \frac{1}{128}(2)^4, \\ &\leq \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{3}{4}, \end{aligned}$$

so,

$$|\breve{a}_5| \leq \frac{3}{4}. \tag{4.23}$$

Hence proved. □

Theorem 4.2.2. If $\zeta(\text{ɫ}) \in S_s^*(1 + tanh(\text{ɫ}))$, then

$$|\breve{a}_3 - \varepsilon \breve{a}_2{}^2| \leq \frac{1}{2} m\hat{a}x \left\{ 1, \frac{|\varepsilon|}{2} \right\}.$$

Proof. Utilizing (4.15) and (4.16), then we get

$$\begin{aligned} |\breve{a}_3 - \varepsilon \breve{a}_2{}^2| &= \left| \frac{p_2}{4} - \frac{p_1^2}{8} - \varepsilon \left(\frac{p_1}{4} \right)^2 \right|, \\ |\breve{a}_3 - \varepsilon \breve{a}_2{}^2| &= \left| \frac{p_2}{4} - \frac{p_1^2}{8} - \varepsilon \frac{p_1^2}{16} \right|, \\ |\breve{a}_3 - \varepsilon \breve{a}_2{}^2| &= \left| \frac{p_2}{4} - \left(\frac{1}{8} + \frac{\varepsilon}{16} \right) p_1^2 \right|, \\ &= \left| \frac{p_2}{4} - \left(\frac{2+\varepsilon}{16} \right) p_1^2 \right|, \\ &= \frac{1}{4} \left| p_2 - \left(\frac{2+\varepsilon}{4} \right) p_1^2 \right|, \end{aligned}$$

using Application of Lemma (2.10.1) and Equation (2.14),then we get

$$|\breve{a}_3 - \varepsilon \breve{a}_2{}^2| \leq \frac{2}{4} m\hat{a}x \left\{ 1, \left| 2 \left(\frac{2+\varepsilon}{4} \right) - 1 \right| \right\},$$

so,

$$|\breve{a}_3 - \varepsilon \breve{a}_2{}^2| \leq \frac{1}{2} m\hat{a}x \left\{ 1, \frac{|\varepsilon|}{2} \right\}. \tag{4.24}$$

Hence proved. □

Corollary 4.2.2.1. If $\zeta(\text{ł}) \in S_s^*(1+tanh(\text{ł}))$ and ε = 1, then

$$|\check{a}_3 - \check{a}_2{}^2| \leq \frac{1}{2}.$$

Proof. Utilizing (4.16) and (4.17), then we get

$$\begin{aligned}
|\check{a}_3 - \check{a}_2{}^2| &= \left| \left(\frac{p_2}{4} - \frac{p_1^2}{8}\right) - \left(\frac{p_1}{4}\right)^2 \right|, \\
&= \left| \frac{p_2}{4} - \frac{p_1^2}{8} - \frac{p_1^2}{16} \right|, \\
&= \left| \frac{p_2}{4} - \left(\frac{1}{8} + \frac{1}{16}\right) p_1^2 \right|, \\
&= \left| \frac{p_2}{4} - \left(\frac{2+1}{16}\right) p_1^2 \right|, \\
&= \frac{1}{4}\left| p_2 - \left(\frac{3}{4}\right) p_1^2 \right|,
\end{aligned}$$

using Application of Lemma (2.10.1) and Equation (2.14),then we get

$$\begin{aligned}
|\check{a}_3 - \varepsilon\check{a}_2{}^2| &\leq \frac{2}{4} m\hat{a}x \left\{ 1, \left| 2\left(\frac{3}{4}\right) - 1 \right| \right\}, \\
&\leq \frac{2}{4} m\hat{a}x \left\{ 1, \frac{1}{2} \right\},
\end{aligned}$$

so,

$$|\check{a}_3 - \check{a}_2{}^2| \leq \frac{1}{2}. \tag{4.25}$$

Hence proved. □

4.3 Hankel determinants

The following results are evaluted.

Theorem 4.3.1. If $\zeta(\text{ł}) \in S_s^*(1+tanh(\text{ł}))$, then

$$|\check{a}_2\check{a}_3 - \check{a}_4| \leq \frac{1}{4}.$$

Proof. From (4.16), (4.17) and (4.18), then we have

$$\begin{aligned}
|\check{a}_2\check{a}_3 - \check{a}_4| &= \left| \frac{p_1}{4}\left(\frac{p_2}{4} - \frac{p_1^2}{8}\right) - \left(\frac{p_1^3}{192} - \frac{3p_1p_2}{32} + \frac{p_3}{8}\right) \right|, \\
|\check{a}_2\check{a}_3 - \check{a}_4| &= \left| \frac{p_1p_2}{16} - \frac{p_1^3}{32} - \frac{p_1^3}{192} + \frac{3p_1p_2}{32} - \frac{p_3}{8} \right|, \\
&= \left| \frac{7p_1^3}{192} - \frac{5p_1p_2}{32} + \frac{p_3}{8} \right|,
\end{aligned}$$

implementation of triangular inequality and Lemma (2.10.2), then we get

$$\left|\frac{7p_1^3}{192}-\frac{5p_1p_2}{32}+\frac{p_3}{8}\right| \le 2\left|\frac{7}{192}\right|+2\left|\frac{5}{32}-\frac{2(7)}{192}\right|+2\left|\frac{7}{192}-\frac{5}{32}+\frac{1}{8}\right|,$$
$$\le \frac{7}{96}+\frac{16}{96}+\frac{1}{96}=\frac{24}{96}=\frac{1}{4},$$

so,

$$|\check{a}_2\check{a}_3-\check{a}_3{}^2| \le \frac{1}{4}. \tag{4.26}$$

Hence proved. □

Theorem 4.3.2. If $\zeta(\text{ł}) \in S_s^*(1+tanh(\text{ł}))$, then

$$|\check{a}_2\check{a}_4-\check{a}_3{}^2| \le \frac{11}{16}.$$

Proof. From (4.16), (4.17) and (4.18), then we have

$$|\check{a}_2\check{a}_4-\check{a}_3{}^2| = \left|\frac{p_1}{4}\left(\frac{p_1^3}{192}-\frac{3p_1p_2}{32}+\frac{p_3}{8}\right)-\left(\frac{p_2}{4}-\frac{p_1^2}{8}\right)^2\right|,$$
$$|\check{a}_2\check{a}_4-\check{a}_3{}^2| = \left|\frac{p_1^4}{768}-\frac{3p_1^2p_2}{128}+\frac{p_1p_3}{32}-\frac{p_2^2}{16}-\frac{p_1^4}{64}+\frac{p_1^2p_2}{16}\right|,$$
$$|\check{a}_2\check{a}_4-\check{a}_3{}^2| = \left|\frac{p_1p_3}{32}-\frac{p_2^2}{16}+\left(\frac{-3}{128}+\frac{1}{16}\right)p_1^2p_2+\left(\frac{1}{768}-\frac{1}{64}\right)p_1^4\right|,$$
$$= \left|\frac{p_1p_3}{32}+\frac{5p_1^2p_2}{128}-\frac{11p_1^4}{768}-\frac{p_2^2}{16}\right|,$$
$$|\check{a}_2\check{a}_4-\check{a}_3{}^2| = \left|\frac{5p_1^2}{128}\left(p_2-\frac{11p_1^2}{30}\right)+\frac{1}{32}p_1p_3-\frac{p_2^2}{16}\right|,$$

using Lemma (2.10.1) and Equations (2.12),(2.13) and (2.14), then we get

$$|\check{a}_2\check{a}_4-\check{a}_3{}^2| \le \frac{5}{128}(2)^2(2)+\frac{1}{16}(2)^2+\frac{1}{32}(2)(2),$$
$$\le \frac{5}{16}+\frac{1}{4}+\frac{1}{8}=\frac{5+4+2}{16}=\frac{11}{16},$$

so,

$$|\check{a}_2\check{a}_4-\check{a}_3{}^2| \le \frac{11}{16}. \tag{4.27}$$

Hence proved. □

Theorem 4.3.3. If $\zeta(\text{ł}) \in S_s^*(1+tanh(\text{ł}))$, then

$$|\mathfrak{H}_{3,1}(\zeta)| \le \frac{25}{32} \backsimeq 0.78125.$$

Proof. Third order Hankel determinant is defined as:

$$\mathfrak{H}_{3,1}(\zeta) = \breve{a}_3(\breve{a}_2\breve{a}_4 - \breve{a}_3{}^2) - \breve{a}_4(\breve{a}_4 - \breve{a}_2\breve{a}_3) + \breve{a}_5(\breve{a}_3 - \breve{a}_2{}^2),$$

taking modulus on both sides, then we get

$$|\mathfrak{H}_{3,1}(\zeta)| = |\breve{a}_3||(\breve{a}_2\breve{a}_4 - \breve{a}_3{}^2)| - |\breve{a}_4||(\breve{a}_4 - \breve{a}_2\breve{a}_3)| + |\breve{a}_5||(\breve{a}_3 - \breve{a}_2{}^2)|,$$

by implementing results Theorem 4.2.1, Corollary 4.2.2.1, Theorem 4.3.1 and Theorem 4.3.2, then we obtained

$$\begin{aligned} |\mathfrak{H}_{3,1}(\zeta)| &\leq \frac{1}{2}\left(\frac{11}{16}\right) + \frac{1}{4}\left(\frac{1}{4}\right) + \frac{3}{4}\left(\frac{1}{2}\right), \\ &\leq \frac{11}{32} + \frac{1}{16} + \frac{3}{8}, \\ &\leq \frac{11+2+12}{32} = \frac{25}{32}, \end{aligned}$$

so, we get

$$|\mathfrak{H}_{3,1}(\zeta)| \leq \frac{25}{32} \backsimeq 0.78125. \tag{4.28}$$

Hence proved. □

4.4 Zalcman Functional

The result is evaluated:

Theorem 4.4.1. if $\zeta(\text{ł}) \in S_s^*(1 + tanh(\text{ł}))$, then

$$|\breve{a}_3{}^2 - \breve{a}_5| \leq \frac{1}{4}.$$

Proof. Using (4.16) and (4.18), then we get

$$\begin{aligned} |\breve{a}_3{}^2 - \breve{a}_5| &= \left|\left(\frac{p_2}{4} - \frac{p_1^2}{4}\right)^2 - \left(-\frac{p_2^2}{32} + \frac{p_1^2 p_2}{32} + \frac{p_1^4}{128} - \frac{p_3 p_1}{8} + \frac{p_4}{8}\right)\right|, \\ &= \left|\left(\frac{p_2^2}{16} + \frac{p_1^4}{64} - \frac{p_2 p_1^2}{16}\right) + \frac{p_2^2}{32} - \frac{p_1^2 p_2}{32} - \frac{p_1^4}{128} + \frac{p_3 p_1}{8} - \frac{p_4}{8}\right|, \\ &= \left|\frac{p_1^4}{128} + \frac{3p_2^2}{32} - \frac{3p_1^2 p_2}{32} + \frac{p_3 p_1}{8} - \frac{p_4}{8}\right|, \\ &= \frac{1}{8}\left|\frac{p_1^4}{16} + \frac{3p_2^2}{4} - \frac{7p_1^2 p_2}{4} + \left(\frac{1}{2}\right) 2p_3 p_1 - p_4\right|, \end{aligned}$$

using Lemma (2.10.3), then we get

$$|\breve{a}_3{}^2 - \breve{a}_5| \leq \frac{1}{8}(2) = \frac{1}{4},$$

so, then we have

$$|\breve{a}_3{}^2 - \breve{a}_5| \leq \frac{1}{4}. \tag{4.29}$$

Hence proved. □

CHAPTER 5

q-EXTENSION ON A NEW SUBCLASSES OF STARLIKE FUNCTIONS ASSOCIATED WITH SYMMETRIC POINTS q-TANGENT HYPERBOLIC FUNCTION

5.1 Introduction

In this chapter, we established a q-extension on a new subclass of Starlike function and related to symmetric points. The hyperbolic function of trignometric tangent is intimately associated with this subclass. we explore the results of the Zalcman functional, Henkel determinants, Fekete-Szego inequality, and coefficient estimations.

For the following results we suppose $[n]_{\mathrm{q}}$, where $n = 1,2,3,4,....$ and q ∈ (0,1).

Defination 5.1.1. Lets $\zeta(ł) \in A$ is in $S^*_{s,\mathrm{q}}(1+\tanh(\mathrm{q}ł))$ after that

$$\frac{2łD_{\mathrm{q}}\zeta(ł)}{\zeta(ł)-\zeta(-ł)} \prec 1+\tanh(\mathrm{q}ł),$$

for all ł∈ $\mathbb{D}$.

5.2 Coefficients Estimates and Fekete-Szego Inequality

The main results for the specified class of $S^*_{s,\mathrm{q}}(1+\tanh(\mathrm{q}ł))$

Theorem 5.2.1. If $\zeta(\text{ł}) \in S^{*}_{s,\mathrm{q}}\left(1+tanh(\mathrm{q}\text{ł})\right)$, then

$$|\breve{a}_2| \leq \frac{\mathrm{q}}{1+\mathrm{q}}, \quad |\breve{a}_3| \leq \frac{\mathrm{q}^2}{\mathrm{q}(1+\mathrm{q})},$$

$$|\breve{a}_4| \leq \frac{\mathrm{q}^3}{(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}\left[\left|\frac{1}{6}-\frac{1}{4\mathrm{q}(1+\mathrm{q})}\right|+\left|\frac{1}{6}+\frac{1}{4\mathrm{q}(1+\mathrm{q})}\right|+\frac{2}{3}\right],$$

$$|\breve{a}_5| \leq \frac{\mathrm{q}^4}{\mathrm{q}(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}\left[\left|\frac{2\mathrm{q}+\mathrm{q}^2-2}{\mathrm{q}(1+\mathrm{q})}\right|+\left|\frac{\mathrm{q}+\mathrm{q}^2-1}{\mathrm{q}(1+\mathrm{q})}\right|+\left|\frac{1}{\mathrm{q}(1+\mathrm{q})}\right|+1\right].$$

Proof. According to the definition:

$$\frac{2\text{ł}D_{\mathrm{q}}\zeta(\text{ł})}{\zeta(\text{ł})-\zeta(-\text{ł})} \prec 1+\tanh(\mathrm{q}\text{ł})$$

Since $\zeta(\text{ł}) \in S^{*}_{s,\mathrm{q}}$, using the subordination techniques, then we get

$$\frac{2\text{ł}D_{\mathrm{q}}\zeta(\text{ł})}{\zeta(\text{ł})-\zeta(-\text{ł})} = 1+\tanh(v(\mathrm{q}\text{ł})). \tag{5.1}$$

Since $\zeta(\text{ł}) \in A$, then $\zeta(\text{ł})$ is in the form that as:

$$\zeta(\text{ł}) = \text{ł}+\sum_{n=2}^{\infty}\breve{a}_n\text{ł}^n, \quad (\text{ł} \in \mathbb{D}). \tag{5.2}$$

So it can also be written as:

$$\zeta(\text{ł}) = \text{ł}+\breve{a}_2\text{ł}^2+\breve{a}_3\text{ł}^3+\breve{a}_4\text{ł}^4+\breve{a}_5\text{ł}^5+....,$$

it follows that

$$\zeta(-\text{ł}) = -\text{ł}+\breve{a}_2\text{ł}^2-\breve{a}_3\text{ł}^3+\breve{a}_4\text{ł}^4-\breve{a}_5\text{ł}^5+....,$$

which implies

$$\zeta(\text{ł})-\zeta(-\text{ł}) = 2\text{ł}+2\breve{a}_3\text{ł}^3+2\breve{a}_5\text{ł}^5+....,$$

as we know that

$$D_{\mathrm{q}}\zeta(\text{ł}) = \frac{\zeta(\mathrm{q}\text{ł})-\zeta(\rho)}{\mathrm{q}\text{ł}-\text{ł}},$$

so, then we have

$$D_{\mathrm{q}}\zeta(\text{ł}) = \frac{(\mathrm{q}\text{ł}+\breve{a}_2\mathrm{q}^2\text{ł}^2+\breve{a}_3\mathrm{q}^3\text{ł}^3+\breve{a}_4\mathrm{q}^4\text{ł}^4+\breve{a}_5\mathrm{q}^5\text{ł}^5+...)-(\text{ł}+\breve{a}_2\text{ł}^2+\breve{a}_3\text{ł}^3+\breve{a}_4\text{ł}^4+\breve{a}_5\text{ł}^5+....)}{\mathrm{q}\text{ł}-\text{ł}},$$

$$= \mathrm{q}_1+2\breve{a}_2\mathrm{q}_2\text{ł}+3\breve{a}_3\mathrm{q}_3\text{ł}^2+4\breve{a}_4\mathrm{q}_4\text{ł}^3+5\breve{a}_5\mathrm{q}_5\text{ł}^4+....,$$

which leads us

$$\frac{2\text{ł}D_{\mathrm{q}}\zeta(\text{ł})}{\zeta(\text{ł})-\zeta(-\text{ł})} = \frac{2(\mathrm{q}_1\rho+\mathrm{q}_2\breve{a}_2\text{ł}^2+\mathrm{q}_3\breve{a}_3\text{ł}^3+\mathrm{q}_4\breve{a}_4\text{ł}^4+\mathrm{q}_5\breve{a}_5\text{ł}_5...)}{2\rho+2\breve{a}_3\rho^3+2\breve{a}_5\text{ł}^5+...},$$

after simplification,then we will get

$$\frac{2łD_{q}\zeta(ł)}{\zeta(ł)-\zeta(-ł)} = [q_1 + (\check{a}_2 q_2)ł + (\check{a}_3 q_3 - q_1\check{a}_3)ł^2 + (\check{a}_4 q_4 - \check{a}_2\check{a}_3 q_2)ł^3 + (q_5\check{a}_5 - q_1\check{a}_5 - \check{a}_3{}^2 q_3 + q_1\check{a}_3{}^2)ł^4 + ...]. \tag{5.3}$$

It can also be written as:

$$\frac{2\rho D_{q}\zeta(ł)}{\zeta(ł)-\zeta(-ł)} = [1 + \check{a}_2(1+q)ł + \check{a}_3 q(1+q)ł^2 + [\check{a}_4(1+q+q^2+q^3) - \check{a}_2\check{a}_3(1+q)]ł^3 + [\check{a}_5 q(1+q+q^2+q^3) - \check{a}_3{}^2 q(1+q)]ł^4 + ...]. \tag{5.4}$$

Since $h(ł) \in \mathfrak{P}$, Then $h(ł)$ is in the form that as:

$$h(ł) = 1 + \sum_{n=2}^{\infty} p_n ł^n, \quad (ł \in \mathbb{D}). \tag{5.5}$$

Similarly,

$$h(qł) = 1 + \sum_{n=2}^{\infty} p_n q_n ł^n. \tag{5.6}$$

So, lets us a function

$$h(qł) = \frac{1+v(qł)}{1-v(qł)} = 1 + p_1 qł + p_2 q^2 ł^2 + ..., \tag{5.7}$$

using the above statement then we get

$$v(qł) = \frac{h(qł)-1}{h(qł)+1}, \tag{5.8}$$

using (5.7) in (5.8) and then we have

$$\begin{aligned} v(qł) &= \frac{1 + p_1 qł + p_2 q^2 ł^2 + ... - 1}{1 + p_1 qł + p_2 q^2 ł^2 + ... + 1}, \\ &= \frac{p_1 qł + p_2 q^2 ł^2 + p_3 q^3 ł^3 ......}{2 + p_1 qł + p_2 q^2 ł^2 + ...}, \end{aligned}$$

so

$$v(qł) = (p_1 qł + p_2 q^2 ł^2 + p_3 q^3 ł^3 ......)(2 + p_1 qł + p_2 q^2 ł^2 + ...)^{-1},$$

after simplifying above equation, then we get

$$v(qł) = \frac{p_1 qł}{2} + \left(\frac{-p_1^2}{4} + \frac{p_2}{2}\right) q^2 ł^2 + \left(\frac{-p_1 p_2}{2} + \frac{p_1^3}{8} + \frac{p_3}{2}\right) q^3 ł^3 + \left(\frac{-p_3 p_1}{2} + \frac{3p_1^2 p_2}{8} - \frac{p_1^4}{16} + \frac{p_4}{2} - \frac{p_2^2}{4}\right) q^4 ł^4 + ..., \tag{5.9}$$

as we know that

$$\tanh[v(\mathrm{q}ł)] = [v(\mathrm{q}ł)] - \frac{[v(\mathrm{q}ł)]^3}{3} + \frac{2[v(\mathrm{q}ł)]^5}{15} + ...,$$

So,

$$\tanh[v(\mathrm{q}ł)] = \frac{p_1\mathrm{q}ł}{2} + \left(\frac{-p_1^2}{4} + \frac{p_2}{2}\right)\mathrm{q}^2ł^2 + \left(\frac{-p_1p_2}{2} + \frac{p_1^3}{12} + \frac{p_3}{2}\right)\mathrm{q}^3ł^3 +$$
$$\left(\frac{-p_3p_1}{2} + \frac{p_1^2p_2}{4} - \frac{0p_1^4}{32} + \frac{p_4}{2} - \frac{p_2^2}{4}\right)\mathrm{q}^4ł^4 + ..., \tag{5.10}$$

$$\Longrightarrow 1 + \tanh[v(\mathrm{q}ł)] = \left[1 + \frac{p_1\mathrm{q}ł}{2} + \left(\frac{-p_1^2}{4} + \frac{p_2}{2}\right)\mathrm{q}^2ł^2 + \left(\frac{-p_1p_2}{2} + \frac{p_1^3}{12} + \frac{p_3}{2}\right)\mathrm{q}^3ł^3 +\right.$$
$$\left.\left(\frac{-p_3p_1}{2} + \frac{5p_1^2p_2}{16} - \frac{p_1^4}{32} + \frac{p_4}{2} - \frac{p_2^2}{4}\right)\mathrm{q}^4ł^4 + ...\right]. \tag{5.11}$$

Using (5.4) and (5.11) and substituting in (5.1), then we will get

$$1 + \check{a}_2(1+\mathrm{q})ł + \check{a}_3\mathrm{q}(1+\mathrm{q})ł^2 + [\check{a}_4(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3) - \check{a}_2\check{a}_3(1+\mathrm{q})]ł^3 +$$
$$[\check{a}_5\mathrm{q}(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3) - \check{a}_3{}^2\mathrm{q}(1+\mathrm{q})]ł^4 + ... = \left[1 + \frac{p_1\mathrm{q}ł}{2} + \left(\frac{-p_1^2}{4} + \frac{p_2}{2}\right)\mathrm{q}^2ł^2 +\right.$$
$$\left.\left(\frac{-p_1p_2}{2} + \frac{p_1^3}{12} + \frac{p_3}{2}\right)\mathrm{q}^3ł^3 + \left(\frac{-p_3p_1}{2} + \frac{5p_1^2p_2}{16} - \frac{p_1^4}{32} + \frac{p_4}{2} - \frac{p_2^2}{4}\right)\mathrm{q}^4ł^4 + ...\right]. \tag{5.12}$$

By comparing both sides of the above equation, we obtained

$$\check{a}_2(1+\mathrm{q}) = \frac{p_1\mathrm{q}}{2}, \tag{5.13}$$

$$\check{a}_3\mathrm{q}(1+\mathrm{q}) = \left(\frac{p_2}{2} - \frac{p_1^2}{4}\right)\mathrm{q}^2, \tag{5.14}$$

$$\check{a}_4(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3) - a_2a_3(1+\mathrm{q}) = \left(\frac{-p_1p_2}{2} + \frac{p_1^3}{12} + \frac{p_3}{2}\right)\mathrm{q}^3, \tag{5.15}$$

$$\check{a}_5\mathrm{q}(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3) - \check{a}_3{}^2\mathrm{q}(1+\mathrm{q}) = \left(\frac{-p_3p_1}{2} + \frac{p_1^2p_2}{4} - \frac{0p_1^4}{32} + \frac{p_4}{2} - \frac{p_2^2}{4}\right)\mathrm{q}^4. \tag{5.16}$$

Consider (5.13) and solve for the coefficient $\check{a}_2$, and then we get

$$\therefore \check{a}_2(1+\mathrm{q}) = \frac{p_1\mathrm{q}}{2},$$

so,

$$\Longrightarrow \check{a}_2 = \frac{p_1\mathrm{q}}{2(1+\mathrm{q})}. \tag{5.17}$$

Consider (5.14) and solve for the coefficient $\check{a}_3$, and then we get

$$\therefore \check{a}_3\mathrm{q}(1+\mathrm{q}) = \left(\frac{p_2}{2} - \frac{p_1^2}{4}\right)\mathrm{q}^2,$$

so,

$$\Longrightarrow \check{a}_3 = \frac{q^2}{q(1+q)}\left[\frac{p_2}{2} - \frac{p_1^2}{4}\right]. \tag{5.18}$$

Consider (5.15) and solving

$$\therefore \check{a}_4(1+q+q^2+q^3) - \check{a}_2\check{a}_3(1+q) = \left(\frac{-p_1p_2}{2} + \frac{p_1^3}{12} + \frac{p_3}{2}\right)q^3,$$
$$\check{a}_4(1+q+q^2+q^3) = \check{a}_2\check{a}_3(1+q) + \left(\frac{-p_1p_2}{2} + \frac{p_1^3}{12} + \frac{p_3}{2}\right)q^3,$$

using (5.17) and (5.18), put in above statement and solve for the coefficient $\check{a}_4$

$$\begin{aligned}\check{a}_4(1+q+q^2+q^3) &= (1+q)\left[\frac{p_1 q}{2(1+q)}\right]\left[\frac{q^2}{q(1+q)}\left(\frac{-p_1^2}{4} + \frac{p_2}{2}\right)\right] + \left(\frac{-p_1p_2}{2} + \frac{p_1^3}{12} + \frac{p_3}{2}\right)q^3,\\ &= \frac{q^3}{q(1+q)}\left(\frac{-p_1^3}{8} + \frac{p_1p_2}{4}\right) + \left(\frac{-p_1p_2}{2} + \frac{p_1^3}{12} + \frac{p_3}{2}\right)q^3,\\ &= q^3\left[\left(\frac{-1}{8q(1+q)} + \frac{1}{12}\right)p_1^3 - \left(\frac{1}{2} - \frac{1}{4q(1+q)}\right)p_1p_2 + \frac{p_3}{2}\right],\end{aligned}$$

so,

$$\Longrightarrow \check{a}_4 = \frac{q^3}{1+q+q^2+q^3}\left[\left(\frac{-1}{8q(1+q)} + \frac{1}{12}\right)p_1^3 - \left(\frac{1}{2} - \frac{1}{4q(1+q)}\right)p_1p_2 + \frac{p_3}{2}\right]. \tag{5.19}$$

Consider (5.14) and solving

$$\therefore \check{a}_5 q(1+q+q^2+q^3) - \check{a}_3{}^2 q(1+q) = \left(\frac{-p_3p_1}{2} + \frac{p_1^2p_2}{4} - \frac{0p_1^4}{32} + \frac{p_4}{2} - \frac{p_2^2}{4}\right)q^4,$$
$$\check{a}_5 q(1+q+q^2+q^3) = \check{a}_3{}^2 q(1+q) + \left(\frac{-p_3p_1}{2} + \frac{p_1^2p_2}{4} - \frac{0p_1^4}{32} + \frac{p_4}{2} - \frac{p_2^2}{4}\right)q^4,$$

using (5.18), put in above statement and solve for the coefficient a_5

$$\begin{aligned}\check{a}_5 q(1+q+q^2+q^3) &= q(1+q)\left[\frac{q^2}{q(1+q)}\left(\frac{p_2}{4} - \frac{p_1^2}{8}\right)\right]^2 + \left(\frac{-p_3p_1}{2} + \frac{p_1^2p_2}{4} - \frac{0p_1^4}{32} + \frac{p_4}{2} - \frac{p_2^2}{4}\right)q^4,\\ &= q(1+q)\frac{q^4}{q^2(1+q)^2}\left(\frac{-p_1}{4} + \frac{p_2}{2}\right)^2 + \left(\frac{-p_3p_1}{2} + \frac{p_1^2p_2}{4} - \frac{0p_1^4}{32} + \frac{p_4}{2} - \frac{p_2^2}{4}\right)q^4,\\ &= q^4\left[\frac{-p_1^4}{16q(1+q)} + \frac{p_2^2}{4q(1+q)} - \frac{p_1^2p_2}{4q(1+q)} - \frac{p_3p_1}{2} + \frac{p_1^2p_2}{4}\right],\\ &= \left[\left(\frac{-1}{16q(1+q)} - 0\right)p_1^4 + \left(\frac{1}{4q(1+q)} - \frac{1}{4}\right)p_2^2 + \left(\frac{1}{4} - \frac{1}{4q(1+q)}\right)p_1^2p_2 - \right.\\ &\qquad \left.\frac{p_3p_1}{2} + \frac{p_4}{2}\right],\end{aligned}$$

so,

$$\Longrightarrow \check{a}_5 = \frac{q^4}{q(1+q+q^2+q^3)}\left[\left(\frac{1}{4} - \frac{1}{4q(1+q)}\right)p_1^2p_2 - \left(\frac{1}{4} - \frac{1}{4q(1+q)}\right)p_2^2 + \right.$$

$$\left(\frac{-1}{16\mathrm{q}(1+\mathrm{q})}\right)p_1^4-\frac{p_3p_1}{2}+\frac{p_4}{2}\Bigg]. \tag{5.20}$$

Now we will find the absolute values of the coefficients.

we have (5.17).

$$\check{a}_2=\frac{p_1\mathrm{q}}{2(1+\mathrm{q})},$$

taking modulus on both sides, then

$$|\check{a}_2|=\left|\frac{p_1\mathrm{q}}{2(1+\mathrm{q})}\right|,$$

$$|\check{a}_2|=\frac{|p_1|\mathrm{q}}{2(1+\mathrm{q})},$$

using Lemma (2.10.1) and Equation (2.12),then we get

$$|\check{a}_2|\leq\frac{|p_1|\mathrm{q}}{2(1+\mathrm{q})}=\frac{2\mathrm{q}}{2(1+\mathrm{q})}=\frac{\mathrm{q}}{1+\mathrm{q}},$$

so,

$$|\check{a}_2|\leq\frac{\mathrm{q}}{1+\mathrm{q}}. \tag{5.21}$$

We have (5.18).

$$\check{a}_3=\frac{\mathrm{q}^2}{\mathrm{q}(1+\mathrm{q})}\left[\frac{p_2}{2}-\frac{p_1^2}{4}\right],$$

taking modulus on both sides, then

$$|\check{a}_3|=\left|\frac{\mathrm{q}^2}{\mathrm{q}(1+\mathrm{q})}\left[\frac{p_2}{2}-\frac{p_1^2}{4}\right]\right|,$$

$$|\check{a}_3|=\frac{\mathrm{q}^2}{\mathrm{q}(1+\mathrm{q})}\left|\frac{p_2}{2}-\frac{p_1^2}{4}\right|,$$

$$|\check{a}_3|=\frac{\mathrm{q}^2}{2\mathrm{q}(1+\mathrm{q})}\left|p_2-\frac{p_1^2}{2}\right|,$$

using Lemma (2.10.1) and Equation (2.14), then

$$|p_2-\frac{p_1^2}{2}|\leq 2max\left\{1,\left|2\left(\frac{1}{2}\right)-1\right|\right\},$$

$$\leq 2max\{1,0\}=2,$$

so,

$$|\check{a}_3|\leq\frac{\mathrm{q}^2}{2\mathrm{q}(1+\mathrm{q})}(2),$$

then we have

$$|\check{a}_3|\leq\frac{\mathrm{q}^2}{\mathrm{q}(1+\mathrm{q})} \tag{5.22}$$

We have (5.19).

$$\check{a}_4 = \frac{q^3}{(1+q+q^2+q^3)}\left[\left(\frac{-1}{8q(1+q)}+\frac{1}{12}\right)p_1^3-\left(\frac{1}{2}-\frac{1}{4q(1+q)}\right)p_1p_2+\frac{p_3}{2}\right],$$

taking modulus on both sides, then

$$|\check{a}_4| = \left|\frac{q^3}{(1+q+q^2+q^3)}\left[\left(\frac{-1}{8q(1+q)}+\frac{1}{12}\right)p_1^3-\left(\frac{1}{2}-\frac{1}{4q(1+q)}\right)p_1p_2+\frac{p_3}{2}\right]\right|,$$

$$|\check{a}_4| = \frac{q^3}{(1+q+q^2+q^3)}\left|\left(\frac{-1}{8q(1+q)}+\frac{1}{12}\right)p_1^3-\left(\frac{1}{2}-\frac{1}{4q(1+q)}\right)p_1p_2+\frac{p_3}{2}\right|,$$

using triangular inequality and Lemma (2,10.2), then we get

$$\begin{aligned}\left|\left(\frac{-1}{8q(1+q)}+\frac{1}{12}\right)p_1^3-\left(\frac{1}{2}-\frac{1}{4q(1+q)}\right)p_1p_2+\frac{p_3}{2}\right| &\leq 2\left|\frac{1}{12}-\frac{1}{8q(1+q)}\right|+2\left|\frac{1}{2}-\frac{1}{4q(1+q)}-\right.\\ &\left.2\left(\frac{1}{12}-\frac{1}{8q(1+q)}\right)\right|+2\left|\frac{1}{12}-\frac{1}{8q(1+q)}-\frac{1}{2}+\frac{1}{4q(1+q)}+\frac{1}{2}\right|,\\ &\leq \left|\frac{1}{6}-\frac{1}{4q(1+q)}\right|+2\left|\frac{1}{2}-\frac{1}{6}\right|+\left|\frac{1}{6}+\frac{1}{4q(1+q)}\right|,\\ &\leq \left|\frac{1}{6}-\frac{1}{4q(1+q)}\right|+2\left|\frac{2}{6}\right|+\left|\frac{1}{6}+\frac{1}{4q(1+q)}\right|,\\ &\leq \left|\frac{1}{6}-\frac{1}{4q(1+q)}\right|+\left|\frac{1}{6}+\frac{1}{4q(1+q)}\right|+\frac{2}{3},\end{aligned}$$

so, then we have

$$|\check{a}_4| \leq \frac{q^3}{(1+q+q^2+q^3)}\left[\left|\frac{1}{6}-\frac{1}{4q(1+q)}\right|+\left|\frac{1}{6}+\frac{1}{4q(1+q)}\right|+\frac{2}{3}\right]. \tag{5.23}$$

We have (5.20).

$$\begin{aligned}\check{a}_5 = \frac{q^4}{q(1+q+q^2+q^3)}&\left[\left(\frac{1}{4}-\frac{1}{4q(1+q)}\right)p_1^2p_2-\left(\frac{1}{4}-\frac{1}{4q(1+q)}\right)p_2^2+\right.\\ &\left.\left(\frac{-1}{16q(1+q)}\right)p_1^4-\frac{p_3p_1}{2}+\frac{p_4}{2}\right],\end{aligned}$$

taking modulus on both sides, then

$$\begin{aligned}|\check{a}_5| = \left|\frac{q^4}{q(1+q+q^2+q^3)}\right.&\left[\left(\frac{1}{4}-\frac{1}{4q(1+q)}\right)p_1^2p_2-\left(\frac{1}{4}-\frac{1}{4q(1+q)}\right)p_2^2+\right.\\ &\left.\left.\left(\frac{-1}{16q(1+q)}\right)p_1^4-\frac{p_3p_1}{2}+\frac{p_4}{2}\right]\right|,\end{aligned}$$

$$\begin{aligned}|\check{a}_5| = \frac{q^4}{q(1+q+q^2+q^3)}&\left|\left[\left(\frac{1}{4}-\frac{1}{4q(1+q)}\right)p_1^2p_2-\left(\frac{1}{4}-\frac{1}{4q(1+q)}\right)p_2^2+\right.\right.\\ &\left.\left.\left(\frac{-1}{16q(1+q)}\right)p_1^4-\frac{p_3p_1}{2}+\frac{p_4}{2}\right]\right|,\end{aligned}$$

using Lemma (2.10.1) and Equations (2.12) and (2.13), then

$$|\breve{a}_5| \leq \frac{\text{q}^4}{\text{q}(1+\text{q}+\text{q}^2+\text{q}^3)}\left[\left|\frac{1}{4}-\frac{1}{4\text{q}(1+\text{q})}\right||p_1|^2|p_2|+\left|\frac{1}{4}-\frac{1}{4\text{q}(1+\text{q})}\right||p_2|^2+\right.$$
$$\left.\left|\frac{1}{16\text{q}(1+\text{q})}\right||p_1|^4+\frac{1}{2}|p_1p_3-p_4|\right],$$
$$|\breve{a}_5| \leq \frac{\text{q}^4}{\text{q}(1+\text{q}+\text{q}^2+\text{q}^3)}\left[\frac{1}{4}\left|1-\frac{1}{\text{q}(1+\text{q})}\right|(2)^2(2)+\frac{1}{4}\left|1-\frac{1}{\text{q}(1+\text{q})}\right|(2)^2+\right.$$
$$\left.\left|\frac{1}{16\hat{q}_o(1+\hat{q}_o)}\right|(2)^4+\frac{1}{2}(2)\right],$$

so,

$$|\breve{a}_5| \leq \frac{\text{q}^4}{\text{q}(1+\text{q}+\text{q}^2+\text{q}^3)}\left[\left|\frac{2\text{q}+2\text{q}^2-2}{\text{q}(1+\text{q})}\right|+\left|\frac{\text{q}+\text{q}^2-1}{\text{q}(1+\text{q})}\right|+\left|\frac{1}{\text{q}(1+\text{q})}\right|+1\right]. \tag{5.24}$$

The proof is finally complete. □

In this theorem, when $\text{q} \to 1^-$, we obtain the result for class S_s^* previously established by Khan *et al.* [35], as displayed in the corollary.

Corollary 5.2.1.1. If $\zeta(\text{ł}) \in S_s^*$, then

$$|\breve{a}_2| \leq \frac{1}{2},\ |\breve{a}_3| \leq \frac{1}{2},\ |\breve{a}_4| \leq \frac{1}{4},\ |\breve{a}_5| \leq \frac{3}{4}.$$

Theorem 5.2.2. If $\zeta(\text{ł}) \in S_{s,\text{q}}^*\ (1+tanh(\text{qł}))$, then

$$|\breve{a}_3-\varepsilon\breve{a}_2{}^2| \leq \frac{\text{q}^2}{\text{q}(1+\text{q})}m\hat{a}x\left\{1,\frac{|\varepsilon|\text{q}}{1+\text{q}}\right\}.$$

Proof. Utilizing (5.17) and (5.18), then we get

$$|\breve{a}_3-\varepsilon\breve{a}_2{}^2| = \left|\left(\frac{\text{q}^2}{\text{q}(1+\text{q})}\left[\frac{p_2}{2}-\frac{p_1^2}{4}\right]\right)-\varepsilon\left(\frac{p_1\text{q}}{2(1+\text{q})}\right)^2\right|,$$
$$= \left|\frac{\text{q}^2p_2}{2\text{q}(1+\text{q})}-\frac{\text{q}^2p_1^2}{4\text{q}(1+\text{q})}-\varepsilon\frac{p_1^2\text{q}^2}{4(1+\text{q})^2}\right|,$$
$$= \left|\frac{\text{q}^2p_2}{2\text{q}(1+\text{q})}-\left(\frac{1}{4\text{q}(1+\text{q})}+\frac{\varepsilon}{4(1+\text{q})^2}\right)p_1^2q^2\right],$$
$$= \left|\frac{\text{q}^2p_2}{2\text{q}(1+\text{q})}-\left(\frac{(1+\text{q})+\varepsilon\text{q}}{4\text{q}(1+\text{q})^2}\right)p_1^2\text{q}^2\right|,$$
$$= \frac{\text{q}^2}{2\text{q}(1+\text{q})}\left|p_2-\left(\frac{1+\text{q}+\varepsilon\text{q}}{2(1+\text{q})}\right)p_1^2\right|,$$

using Application of Lemma (2.10.1) and Equation (2.14), then we get

$$|\breve{a}_3-\varepsilon\breve{a}_2{}^2| \leq \frac{2\text{q}^2}{2\text{q}(1+\text{q})}m\hat{a}x\left\{1,\left|2\left(\frac{1+\text{q}+\varepsilon\text{q}}{2(1+\text{q})}\right)-1\right|\right\},$$
$$\leq \frac{\text{q}^2}{\text{q}(1+\text{q})}m\hat{a}x\left\{1,\left|\frac{1+\text{q}+\varepsilon\text{q}-(1+\text{q})}{1+\text{q}}\right|\right\},$$
$$\leq \frac{\text{q}^2}{\text{q}(1+\text{q})}m\hat{a}x\left\{1,\frac{|\varepsilon|\text{q}}{1+\text{q}}\right\},$$

so,

$$|\breve{a}_3 - \varepsilon \breve{a}_2{}^2| \leq \frac{q^2}{q(1+q)} \hat{max}\left\{1, \frac{|\varepsilon| q}{1+q}\right\}. \tag{5.25}$$

The proof is finally complete. □

In this Theorem, when $q \to 1^-$, we arrive at the well-known outcome that is stated in the following corollary.

Corollary 5.2.2.1. If $\zeta(\text{ł}) \in S_s^*$, then

$$|\breve{a}_3 - \varepsilon \breve{a}_2{}^2| \leq \frac{1}{2} \hat{max}\left\{1, \frac{|\varepsilon|}{2}\right\}.$$

Theorem 5.2.3. If $\zeta(\text{ł}) \in S_{s,q}^*\ (1 + tanh(q\text{ł}))$, then

$$|\breve{a}_3 - \breve{a}_2{}^2| \leq \frac{q^2}{q(1+q)}.$$

Proof. Utilizing (5.17) and (5.18), then we get

$$\begin{aligned}
|\breve{a}_3 - \breve{a}_2{}^2| &= \left| \left(\frac{q^2}{q(1+q)} \left[\frac{p_2}{2} - \frac{p_1^2}{4} \right] \right) - \left(\frac{p_1 q}{2(1+q)} \right)^2 \right|, \\
&= \left| \frac{q^2 p_2}{2q(1+q)} - \frac{q^2 p_1^2}{4q(1+q)} - \frac{p_1^2 q^2}{4(1+q)^2} \right|, \\
&= \left| \frac{q^2 p_2}{2q(1+q)} - \left(\frac{1}{4q(1+q)} + \frac{1}{4(1+q)^2} \right) p_1^2 q^2 \right|, \\
&= \left| \frac{q^2 p_2}{2q(1+q)} - \left(\frac{(1+q)+q}{4q(1+q)^2} \right) p_1^2 q^2 \right|, \\
&= \frac{q^2}{2q(1+q)} \left| p_2 - \left(\frac{1+2q}{2(1+q)} \right) p_1^2 \right|,
\end{aligned}$$

using Application of Lemma (2.10.1) and Equation (2.14), then we get

$$\begin{aligned}
|\breve{a}_3 - \breve{a}_2{}^2| &\leq \frac{2q^2}{2q(1+q)} \hat{max}\left\{1, \left| 2\left(\frac{1+2q}{2(1+q)} \right) - 1 \right| \right\}, \\
&\leq \frac{q^2}{q(1+q)} \hat{max}\left\{1, \left| \frac{1+2q-(1+q)}{1+q} \right| \right\}, \\
&\leq \frac{q^2}{q(1+q)} \hat{max}\left\{1, \frac{q}{1+q} \right\} = \frac{q^2}{q(1+q)}(1),
\end{aligned}$$

so,

$$|\breve{a}_3 - \breve{a}_2{}^2| \leq \frac{q^2}{q(1+q)}. \tag{5.26}$$

The proof is finally complete. □

In this Theorem, when $q \to 1^-$, we arrive at the well-known outcome that is stated in the following corollary.

Corollary 5.2.3.1. If $\zeta(\text{ł}) \in S_s^*$, then

$$|\breve{a}_3 - \breve{a}_2{}^2| \leq \frac{1}{2}.$$

5.3 Hankel determinants

The following results are evaluted.

Theorem 5.3.1. If $\zeta(\text{ł}) \in S^{*}_{s,\mathrm{q}}\ (1+tanh(\mathrm{q}\text{ł}))$, then

$$|\breve{a}_2\breve{a}_3-\breve{a}_4| \leq \mathrm{q}^3\Bigg[\Bigg|\frac{1}{4\mathrm{q}(1+\mathrm{q})^2}+\frac{1}{6(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}-\frac{1}{4\mathrm{q}(1+\mathrm{q})(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}\Bigg|+ \frac{2}{3(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}+\Bigg|\frac{-1}{4\mathrm{q}(1+\mathrm{q})^2}+\frac{1}{6(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}+ \frac{1}{4\mathrm{q}(1+\mathrm{q})(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}\Bigg|\Bigg].$$

Proof. From (5.17), (5.18) and (5.19), then we have

$$|\breve{a}_2\breve{a}_3-\breve{a}_4| = \Bigg|\left(\frac{p_1\mathrm{q}}{2(1+\mathrm{q})}\right)\left(\frac{\mathrm{q}^2}{\mathrm{q}(1+\mathrm{q})}\left[\frac{-p_1^2}{4}+\frac{p_2}{2}\right]\right)-\frac{\mathrm{q}^3}{1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3}\Bigg[\left(\frac{1}{12}- \frac{1}{8\mathrm{q}(1+\mathrm{q})}\right)p_1^3-\left(\frac{1}{2}-\frac{1}{4\mathrm{q}(1+\mathrm{q})}\right)p_1p_2+\frac{p_3}{2}\Bigg]\Bigg|,$$

$$|\breve{a}_2\breve{a}_3-\breve{a}_4| = \Bigg|\frac{p_1q^3}{2\mathrm{q}(1+\mathrm{q})}\left(\frac{-p_1^2}{4}+\frac{p_2}{2}\right)-\frac{\mathrm{q}^3p_1^3}{1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3}\left(\frac{1}{12}-\frac{1}{8\mathrm{q}(1+\mathrm{q})}\right)+ \frac{\mathrm{q}^3p_1p_2}{1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3}\left(\frac{1}{2}-\frac{1}{4\mathrm{q}(1+\mathrm{q})}\right)-\frac{\mathrm{q}^3p_3}{2(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}\Bigg|,$$

$$|\breve{a}_2\breve{a}_3-\breve{a}_4| = \mathrm{q}^3\Bigg|\frac{p_1}{2\mathrm{q}(1+\mathrm{q})}\left(\frac{-p_1^2}{4}+\frac{p_2}{2}\right)-\frac{p_1^3}{1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3}\left(\frac{1}{12}-\frac{1}{8\mathrm{q}(1+\mathrm{q})}\right)+ \frac{p_1p_2}{1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3}\left(\frac{1}{2}-\frac{1}{4\mathrm{q}(1+\mathrm{q})}\right)-\frac{p_3}{2(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}\Bigg|,$$

$$|\breve{a}_2\breve{a}_3-\breve{a}_4| = \mathrm{q}^3\Bigg|\frac{-p_1^3}{8\mathrm{q}(1+\mathrm{q})^2}+\frac{p_1p_2}{4\mathrm{q}(1+\mathrm{q})^2}-\frac{p_1^3}{1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3}\left(\frac{1}{12}-\frac{1}{8\mathrm{q}(1+\mathrm{q})}\right)+ \frac{p_1p_2}{1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3}\left(\frac{1}{2}-\frac{1}{4\mathrm{q}(1+\mathrm{q})}\right)-\frac{p_3}{2(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}\Bigg|,$$

$$|\breve{a}_2\breve{a}_3-\breve{a}_4| = \mathrm{q}^3\Bigg|-p_1^3\left(\frac{1}{8\mathrm{q}(1+\mathrm{q})^2}+\frac{1}{12(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}-\frac{1}{8\mathrm{q}(1+\mathrm{q})(1+\mathrm{q}+\mathrm{q}^2+\bar{\mathrm{q}}^3)}\right)+ p_1p_2\left(\frac{1}{4\mathrm{q}(1+\mathrm{q})^2}+\frac{1}{2(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}-\frac{1}{4\mathrm{q}(1+\mathrm{q})(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}\right)- \frac{p_3}{2(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}\Bigg|,$$

$$|\breve{a}_2a_3-\breve{a}_4| = \mathrm{q}^3\Bigg|p_1^3\left(\frac{1}{8\mathrm{q}(1+\mathrm{q})^2}+\frac{1}{12(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}-\frac{1}{8\mathrm{q}(1+\mathrm{q})(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}\right)- p_1p_2\left(\frac{1}{4\mathrm{q}(1+\mathrm{q})^2}+\frac{1}{2(1+\mathrm{q}+\mathrm{q}^2+\bar{\mathrm{q}}^3)}-\frac{1}{4\mathrm{q}(1+\mathrm{q})(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}\right)+ \frac{p_3}{2(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}\Bigg|,$$

implementation of triangular inequality and Lemma (2.10.2), then we get

$$\Bigg|p_1^3\left(\frac{1}{8\mathrm{q}(1+\mathrm{q})^2}+\frac{1}{12(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}-\frac{1}{8\mathrm{q}(1+\mathrm{q})(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}\right)-p_1p_2\left(\frac{1}{4\mathrm{q}(1+\mathrm{q})^2}+ \frac{1}{2(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}-\frac{1}{4\mathrm{q}(1+\mathrm{q})(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}\right)+\frac{p_3}{2(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}\Bigg| \leq 2\Bigg|\frac{1}{8\mathrm{q}(1+\mathrm{q})^2}+ \frac{1}{12(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}-\frac{1}{8\mathrm{q}(1+\mathrm{q})(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}\Bigg|+2\Bigg|\frac{1}{4\mathrm{q}(1+\mathrm{q})^2}+\frac{1}{2(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}- \frac{1}{4\mathrm{q}(1+\mathrm{q}(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}-2\left(\frac{1}{8\mathrm{q}(1+\mathrm{q})^2}+\frac{1}{12(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}-\frac{1}{8\mathrm{q}(1+\mathrm{q})(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}\right)\Bigg| +2\Bigg|\frac{1}{8\mathrm{q}(1+\mathrm{q})^2}+\frac{1}{12(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}-\frac{1}{8\mathrm{q}(1+\mathrm{q})(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}-\left(\frac{1}{4\mathrm{q}(1+\mathrm{q})^2}+ \frac{1}{2(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}-\frac{1}{4\mathrm{q}(1+\mathrm{q})(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}\right)+\frac{1}{2(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}\Bigg|,$$

$$\leq \Bigg|\frac{1}{4\mathrm{q}(1+\mathrm{q})^2}+\frac{1}{6(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}-\frac{1}{4\mathrm{q}(1+\mathrm{q})(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}\Bigg|+2\Bigg|\frac{1}{4\mathrm{q}(1+\mathrm{q})^2}+ \frac{1}{2(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}-\frac{1}{4\mathrm{q}(1+\mathrm{q})(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}-\frac{1}{4\mathrm{q}(1+\mathrm{q})^2}-\frac{1}{6(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}+ \frac{1}{4\mathrm{q}(1+\mathrm{q})(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}\Bigg|+\Bigg|\frac{1}{4\mathrm{q}(1+\mathrm{q})^2}+\frac{1}{6(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}-\frac{1}{4\mathrm{q}(1+\mathrm{q})(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}- \frac{1}{2\mathrm{q}(1+\mathrm{q})^2}-\frac{1}{(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}+\frac{1}{2\mathrm{q}(1+\mathrm{q}(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)})+\frac{1}{2(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}\Bigg|,$$

$$\leq \Bigg|\frac{1}{4\mathrm{q}(1+\mathrm{q})^2}+\frac{1}{6(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}-\frac{1}{4\mathrm{q}(1+\mathrm{q})(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}\Bigg|+\Bigg|\frac{1}{(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}- \frac{1}{3(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}\Bigg|+\Bigg|\frac{1}{4\mathrm{q}(1+\mathrm{q})^2}-\frac{1}{2\mathrm{q}(1+\mathrm{q})^2}+\frac{1}{6(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}- \frac{1}{4\mathrm{q}(1+\mathrm{q})(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}+\frac{1}{2\mathrm{q}(1+\mathrm{q})(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}\Bigg|,$$

$$\leq \Bigg|\frac{1}{4\mathrm{q}(1+\mathrm{q})^2}+\frac{1}{6(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}-\frac{1}{4\mathrm{q}(1+\mathrm{q})(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}\Bigg|+\frac{2}{3(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}+ \Bigg|\frac{-1}{\mathrm{q}(1+\mathrm{q})^2}+\frac{1}{6(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}+\frac{1}{4\mathrm{q}(1+\mathrm{q})^2(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}\Bigg|,$$

so, then we get

$$|\breve{a}_2\breve{a}_3-\breve{a}_4| \leq \mathrm{q}^3\Bigg[\Bigg|\frac{1}{4\mathrm{q}(1+\mathrm{q})^2}+\frac{1}{6(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}-\frac{1}{4\mathrm{q}(1+\mathrm{q})(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}\Bigg|+$$

$$\frac{2}{3(1+q+q^2+q^3)}+\left|\frac{-1}{4q(1+q)^2}+\frac{1}{6(1+q+q^2+q^3)}+\right.$$
$$\left.\frac{1}{4q(1+q)(1+q+q^2+q^3)}\right|\Bigg]. \tag{5.27}$$

This is the desired result. □

In this Theorem, when $q \to 1^-$, we arrive at the well-known outcome that is stated in the following corollary.

Corollary 5.3.1.1. If $\zeta(ł) \in S_s^*$, then

$$|\check{a}_2\check{a}_3-\check{a}_4| \leq \frac{1}{4}.$$

Theorem 5.3.2. If $\zeta(ł) \in S_{s,q}^* \ (1+tanh(qł))$, then

$$|\check{a}_2\check{a}_4-\check{a}_3{}^2| \leq \frac{q^4}{1+q}\left[\frac{1}{1+q+q^2+q^3}+\frac{1}{q^2(1+q)}+\left|\frac{(-2-3\bar{q})}{q^2(1+\bar{q})(1+q+q^2+q^3)}\right|\right].$$

Proof. From (5.17), (5.18) and (5.19), then we have

$$|\check{a}_2\check{a}_4-\check{a}_3{}^2| = \left|\left(\frac{p_1 q}{2(1+q)}\right)\left(\frac{q^3}{(1+q+q^2+q^3)}\left[\left(\frac{-1}{8q(1+q)}+\frac{1}{12}\right)p_1^3-\left(\frac{1}{2}-\right.\right.\right.\right.$$
$$\left.\left.\left.\left.\frac{1}{4q(1+q}\right)p_1p_2+\frac{p_3}{2}\right]\right)-\left(\frac{q^2}{q(1+q)}\left[\frac{-p_1^2}{4}+\frac{p_2}{2}\right]\right)^2\right|,$$

$$= \left|\frac{p_1^4 q^4}{(1+q)(1+q+q^2+q^3}\left(\frac{-1}{16q(1+q)}+\frac{1}{24}\right)-\frac{p_1^2p_2q^4}{(1+q)(1+q+q^2+q^3)}\left(\frac{1}{4}-\right.\right.$$
$$\left.\frac{1}{8q(1+q)}\right)+\frac{p_1p_3q^4}{4(1+q)(1+q+q^2+q^3)}-\left(\frac{p_1^4q^4}{16q^2(1+q)^2(1+q+q^2+q^3)}+\right.$$
$$\left.\left.\frac{q^4}{4q^2(1+q)^2}-\frac{p_1^2p_2q^4}{4q^2(1+q)^2}\right)\right|,$$

$$= \frac{q^4}{(1+q)}\left|p_1^4\left(\frac{1}{24(1+q+q^2+q^3)}-\frac{1}{16q(1+q)(1+q+q^2+q^3)}-\frac{1}{16q^2(1+q)^2}\right)-\right.$$
$$p_1^2p_2\left(\frac{1}{4(1+q+q^2+q^3)}-\frac{1}{8q(1+q)(1+q+q^2+q^3)}-\frac{1}{4q^2(1+q)}\right)+$$
$$\left.\frac{p_1p_3}{4(1+q+q^2+q^3)}-\frac{p_2^2}{4q^2(1+q)}\right|,$$

$$= \frac{q^4}{(1+q)}\left|p_1^2\left[p_2\left(\frac{1}{4(1+q+q^2+q^3)}-\frac{1}{8q(1+q)(1+q+q^2+q^3)}-\frac{1}{4q^2(1+q)}\right)-\right.\right.$$
$$\left.p_1^2\left(\frac{1}{24(1+q+q^2+q^3)}-\frac{1}{16q(1+q)(1+q+q^2+q^3)}-\frac{1}{16q^2(1+q)^2}\right)\right]+$$
$$\left.\frac{p_1p_3}{4(1+q+q^2+q^3)}-\frac{p_2^2}{4q^2(1+q)}\right|,$$

$$= \frac{q^4}{(1+q)}\left|p_1^2\left[p_2\left(\frac{-2-3q}{8q^2(1+q+q^2+q^3)}\right)-p_1^2\left(\frac{-3-6q-q^2-q^3}{48q^2(1+q+q^2+q^3)}\right)\right]+\frac{p_1p_3}{4(1+q+q^2+q^3)}-\frac{p_2^2}{4q^2(1+q)}\right|,$$

$$= \frac{q^4}{(1+q)}\left[|p_1^2|\left|p_2\left(\frac{-2-3q}{8q^2(1+q+q^2+q^3)}\right)-p_1^2\left(\frac{-3-6q-q^2-q^3}{48q^2(1+q+q^2+q^3)}\right)\right|+\left|\frac{p_1p_3}{4(1+q+q^2+q^3)}\right|+\frac{p_2^2}{4q^2(1+q)}\right|\right],$$

$$= \frac{q^4}{(1+q)}\left\{|p_1|^2\left|\left(\frac{-2-3q}{8q^2(1+q+q^2+q^3)}\right)\left[p_2-p_1^2\left(\frac{-3-6q-q^2-q^3}{-16-24q}\right)\right]\right|+\frac{|p_1||p_3|}{4(1+q+q^2+q^3)}+\frac{|p_2|^2}{4q^2(1+q)}\right|\right\},$$

$$= \frac{q^4}{(1+q)}\left[|p_1|^2\left|\frac{-2-3q}{8q^2(1+q+q^2+q^3)}\right|\left|p_2-p_1^2\left(\frac{-3-6q-q^2-q^3}{-16-24q}\right)\right|+\frac{|p_1||p_3|}{4(1+q+q^2+q^3)}+\frac{|p_2|^2}{4q^2(1+q)}\right|\right],$$

using Lemma (2.10.1) and Equations (2.12) and (2.14),then we get

$$|\check{a}_2\check{a}_4-\check{a}_3{}^2|\leq \frac{q^4}{(1+q)}\left[(2)^2\left|\frac{-2-3q}{8q^2(1+q+q^2+q^3)}\right|(2)+\frac{(2)(2)}{4(1+q+q^2+q^3)}+\frac{(2)^2}{4q^2(1+q)}\right],$$

$$|\check{a}_2\check{a}_4-\check{a}_3{}^2|\leq \frac{q^4}{(1+q)}\left[\left|\frac{-2-3q}{q^2(1+q+q^2+q^3)}\right|+\frac{1}{(1+q+q^2+q^3)}+\frac{1}{q^2(1+q)}\right],$$

so,

$$|\check{a}_2\check{a}_4-\check{a}_3{}^2|\leq \frac{q^4}{1+q}\left[\frac{1}{1+q+q^2+q^3}+\frac{1}{q^2(1+q)}+\left|\frac{(-2-3q)}{q^2(1+\bar{q})(1+q+q^2+q^3)}\right|\right]. \quad (5.28)$$

This is the desired result. □

In this Theorem, when $q\rightarrow 1^-$, we arrive at the well-known outcome that is stated in the following corollary.

Corollary 5.3.2.1. If $\zeta(\text{ł})\in S_s^*$, then

$$|\check{a}_2\check{a}_4-\check{a}_3{}^2|\leq \frac{11}{16}.$$

Theorem 5.3.3. If $\zeta(\text{ł}) \in S^*_{s,\mathrm{q}}\,(1+tanh(\mathrm{q}\text{ł}))$, then

$$|\mathfrak{H}_{3,1}(\rho)| \leq \mathrm{q}^6\Bigg[\frac{1}{\mathrm{q}(1+\mathrm{q})^2}\left(\frac{1}{1+\mathrm{q}+\mathrm{q}^2+\bar{\mathrm{q}}^3}+\frac{1}{\mathrm{q}^2(1+\mathrm{q})}+\left|\frac{(-2-3\bar{\mathrm{q}})}{\mathrm{q}^2(1+\mathrm{q})(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}\right|\right)+ \frac{1}{(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}\left(\left|\frac{1}{6}-\frac{1}{4\mathrm{q}(1+\mathrm{q})}\right|+\left|\frac{1}{6}+\frac{1}{4\mathrm{q}(1+\mathrm{q})}\right|+\frac{2}{3}\right)\left(\left|\frac{1}{4\mathrm{q}(1+\mathrm{q})^2}+ \frac{1}{6(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}-\frac{1}{4\mathrm{q}(1+\mathrm{q})(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}\right|+\frac{2}{3(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}+ \left|\frac{-1}{4\mathrm{q}(1+\mathrm{q})^2}+\frac{1}{6(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}+\frac{1}{4\mathrm{q}(1+\mathrm{q})(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}\right|\right)+ \frac{1}{\mathrm{q}^2(1+\mathrm{q})(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}\left(\left|\frac{2\mathrm{q}(1+\mathrm{q})-2}{\mathrm{q}(1+\mathrm{q})}\right|+\left|\frac{\mathrm{q}(1+\bar{\mathrm{q}})-1}{\mathrm{q}(1+\mathrm{q})}\right|+ \left|\frac{1}{\mathrm{q}(1+\mathrm{q})}\right|+1\right)\Bigg].$$

Proof. Third order Hankel determinant is defined as:

$$\mathfrak{H}_{3,1}(\zeta) = \breve{a}_3(\breve{a}_2\breve{a}_4-\breve{a}_3{}^2)-\breve{a}_4(\breve{a}_4-\breve{a}_2\breve{a}_3)+\breve{a}_5(\breve{a}_3-\breve{a}_2{}^2),$$

taking modulus on both sides, then we get

$$|\mathfrak{H}_{3,1}(\zeta)| = |\breve{a}_3||(\breve{a}_2\breve{a}_4-\breve{a}_3{}^2)|+|a_4||(a_4-a_2a_3)|+|a_5||(a_3-a_2^2)|,$$

by implementing results Theorem 5.2.1, Theorem 5.2.3, Theorem 5.3.1 and Theorem 5.3.2, then we obtained

$$|\mathfrak{H}_{3,1}(\zeta)| \leq \left(\frac{\mathrm{q}^2}{\mathrm{q}(1+\mathrm{q})}\right)\left(\frac{\mathrm{q}^4}{1+\mathrm{q}}\left[\frac{1}{1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3}+\frac{1}{\mathrm{q}^2(1+\mathrm{q})}+\left|\frac{(-2-3\mathrm{q})}{\mathrm{q}^2(1+\mathrm{q})(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}\right|\right]\right)+ \left(\frac{\mathrm{q}^3}{(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}\left[\left|\frac{1}{6}-\frac{1}{4\mathrm{q}(1+\mathrm{q})}\right|+\left|\frac{1}{6}+\frac{1}{4\mathrm{q}(1+\mathrm{q})}\right|+\frac{2}{3}\right]\right)\left(\mathrm{q}^3\left[\left|\frac{1}{4\mathrm{q}(1+\mathrm{q})^2}+ \frac{1}{6(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}-\frac{1}{4\mathrm{q}(1+\mathrm{q})(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}\right|+\frac{2}{3(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}+\left|\frac{-1}{4\mathrm{q}(1+\mathrm{q})^2}+ \frac{1}{6(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}+\frac{1}{4\mathrm{q}(1+\mathrm{q})(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}\right|\right]\right)+\left(\frac{\mathrm{q}^4}{\mathrm{q}(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)} \left[\left|\frac{2\mathrm{q}(1+\mathrm{q})-2}{\mathrm{q}(1+\mathrm{q})}\right|+\left|\frac{\mathrm{q}(1+\bar{\mathrm{q}})-1}{\mathrm{q}(1+\mathrm{q})}\right|+\left|\frac{1}{\mathrm{q}(1+\mathrm{q})}\right|+1\right]\right)\left(\frac{\mathrm{q}^2}{\mathrm{q}(1+\mathrm{q})}\right),$$

$$|\mathfrak{H}_{3,1}(\zeta)| \leq \frac{q^6}{q(1+q)^2}\left(\frac{1}{1+q+q^2+q^3}+\frac{1}{q^2(1+q)}+\left|\frac{(-2-3q)}{q^2(1+q)(1+q+q^2+q^3)}\right|\right)+$$
$$\frac{q^6}{(1+q+q^2+q^3)}\left(\left|\frac{1}{6}-\frac{1}{4q(1+q)}\right|+\left|\frac{1}{6}+\frac{1}{4q(1+q)}\right|+\frac{2}{3}\right)\left(\left|\frac{1}{4q(1+q)^2}+\right.\right.$$
$$\left.\frac{1}{6(1+q+q^2+q^3)}-\frac{1}{4q(1+q)(1+q+q^2+q^3)}\right|+\frac{2}{3(1+q+q^2+q^3)}+$$
$$\left.\left|\frac{-1}{4q(1+q)^2}+\frac{1}{6(1+q+q^2+q^3)}+\frac{1}{4q(1+q)(1+q+q^2+q^3)}\right|\right)+$$
$$\frac{q^6}{q^2(1+q)(1+q+q^2+q^3)}\left(\left|\frac{2q(1+q)-2}{q(1+q)}\right|+\left|\frac{\bar{q}(1+q)-1}{q(1+\bar{q})}\right|+\right.$$
$$\left.\left|\frac{1}{q(1+q)}\right|+1\right),$$

so,

$$|\mathfrak{H}_{3,1}(\zeta)| \leq q^6\left[\frac{1}{q(1+q)^2}\left(\frac{1}{1+q+q^2+q^3}+\frac{1}{q^2(1+q)}+\left|\frac{(-2-3q)}{q^2(1+q)(1+q+q^2+q^3)}\right|\right)+\right.$$
$$\frac{1}{(1+q+q^2+q^3)}\left(\left|\frac{1}{6}-\frac{1}{4q(1+q)}\right|+\left|\frac{1}{6}+\frac{1}{4q(1+q)}\right|+\frac{2}{3}\right)\left(\left|\frac{1}{4q(1+q)^2}+\right.\right.$$
$$\left.\frac{1}{6(1+q+q^2+q^3)}-\frac{1}{4q(1+q)(1+q+q^2+q^3)}\right|+\frac{2}{3(1+q+q^2+q^3)}+$$
$$\left.\left|\frac{-1}{4q(1+q)^2}+\frac{1}{6(1+q+q^2+q^3)}+\frac{1}{4q(1+q)(1+q+q^2+q^3)}\right|\right)+$$
$$\frac{1}{q^2(1+q)(1+q+q^2+q^3)}\left(\left|\frac{2q(1+q)-2}{q(1+q)}\right|+\left|\frac{q(1+q)-1}{q(1+q)}\right|+\right.$$
$$\left.\left.\left|\frac{1}{q(1+q)}\right|+1\right)\right]. \tag{5.29}$$

This is the desired result. □

In this Theorem, when $q \to 1^-$, we arrive at the well-known outcome that is stated in the following corollary.

Corollary 5.3.3.1. If $\zeta(ł) \in S_s^*$, then

$$|\mathfrak{H}_{3,1}(\zeta)| \leq \frac{25}{32} \backsimeq 0.78125.$$

5.4 Zalcman Functional

The result is evaluated:

Theorem 5.4.1. If $\zeta(ł) \in S^{*}_{s,q}\,(1+tanh(qł))$, then

$$|\breve{a}_3{}^2 - \breve{a}_5| \leq \frac{q^4}{\bar{q}(1+q+q^2+q^3)}.$$

Proof. Using (5.17) and (5.20), then we get

$$|\breve{a}_3{}^2 - \breve{a}_5| = \left|\left(\frac{q^2}{q(1+q)}\left[\frac{p_2}{2} - \frac{p_1^2}{4}\right]\right)^2 - \left(\frac{q^4}{q(1+q+q^2+q^3)}\left[\left(\frac{1}{4} - \frac{1}{4q(1+q)}\right)p_1^2 p_2 - \left(\frac{1}{4} - \frac{1}{4q(1+q)}\right)p_2^2 + \left(\frac{-1}{16q(1+q)}\right)p_1^4 - \frac{p_3 p_1}{2} + \frac{p_4}{2}\right]\right)\right|,$$

$$|\breve{a}_3{}^2 - \breve{a}_5| = \left|\frac{q^4}{q^2(1+q)^2}\left(\frac{p_2^2}{4} + \frac{p_1^4}{16} - \frac{p_1^2 p_2}{4}\right) - \frac{q^4}{q(1+q+q^2+q^3)}\left(\frac{1}{4} - \frac{1}{4q(1+q)}\right)p_1^2 p_2 - \frac{q^4}{q(1+q+q^2+q^3)}\left(\frac{1}{4} - \frac{1}{4q(1+q)}\right)p_2^2 - \frac{q^4 p_1^4}{16q^2(1+q)(1+q+q^2+q^3)} + \frac{q^4 p_3 p_1}{2q(1+q+q^2+q^3)} - \frac{q^4 p_4}{2q(1+q+q^2+q^3)}\right|,$$

$$|\breve{a}_3{}^2 - \breve{a}_5| = \frac{q^4}{q}\left[\left|\frac{1}{q(1+q)^2}\left(\frac{p_2^2}{4} + \frac{p_1^4}{16} - \frac{p_1^2 p_2}{4}\right) - \frac{1}{(1+q+q^2+q^3)}\left(\frac{1}{4} - \frac{1}{4q(1+q)}\right)p_1^2 p_2 - \frac{1}{q(1+q+q^2+q^3)}\left(\frac{1}{4} - \frac{1}{4q(1+q)}\right)p_2^2 - \frac{p_1^4}{16q(1+q)(1+q+q^2+q^3)} + \frac{p_3 p_1}{2(1+q+q^2+q^3)} - \frac{p_4}{2(1+q+q^2+q^3)}\right|\right],$$

$$|\breve{a}_3{}^2 - \breve{a}_5| = \frac{q^4}{q}\left[\left|\left(\frac{1}{16q(1+q)^2} - \frac{1}{16q(1+q)(1+q+q^2+q^3)}\right)p_1^4 + \left(\frac{1}{4q(1+q)^2} - \frac{1}{4q(1+q+q^2+q^3)} + \frac{1}{4q^2(1+q)(1+q+q^2+q^3)}\right)p_2^2 - \left(\frac{1}{4q(1+q)^2} + \frac{1}{4(1+q+q^2+q^3)} - \frac{1}{4q(1+q)(1+q+q^2+q^3)}\right)p_1^2 p_2 + \frac{p_3 p_1}{2(1+q+q^2+q^3)} - \frac{p_4}{2(1+q+q^2+q^3)}\right|\right],$$

$$= \frac{q^4}{q}\left[\left|\left(\frac{(1+q+q^2+q^3)-(1+q)}{16q(1+q)^2(1+q+q^2+q^3)}\right)p_1^4 + \left(\frac{q(1+q+q^2+q^3)-q(1+q)^2+(1+q)}{4q^2(1+q)^2(1+q+q^2+q^3)}\right)p_2^2 - \left(\frac{(1+q+q^2+q^3)+q(1+q)-(1+q)}{4q(1+q)^2(1+q+q^2+q^3)}\right)p_1^2 p_2 + \frac{p_3 p_1}{2(1+q+q^2+q^3)} - \frac{p_4}{2(1+q+q^2+q^3)}\right|\right],$$

$$= \frac{q^4}{2q(1+q+q^2+q^3)}\left[\left|\left(\frac{q^2+q^3}{8q(1+q)^2}\right)p_1^4 + \left(\frac{q(1+q+q^2+q^3)-q(1+q)^2+(1+q)}{2q^2(1+q)^2}\right)p_2^2 - \left(\frac{(1+q+q^2+q^3)+q(1+q)-(1+q)}{2q(1+q)^2}\right)p_1^2 p_2 + p_3 p_1 - p_4\right|\right],$$

using Lemma (2.10.3), then we get

$$|\check{a}_3{}^2 - \check{a}_5| \leq \frac{\mathrm{q}^4}{2\mathrm{q}(1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3)}(2)$$

so,

$$|\check{a}_3{}^2 - \check{a}_5| \leq \frac{\mathrm{q}^4}{\mathrm{q}(1+\bar{\mathrm{q}}+\mathrm{q}^2+\mathrm{q}^3)} \tag{5.30}$$

This is the desired result. □

In this Theorem, when $\mathrm{q} \to 1^-$, we arrive at the well-known outcome that is stated in the following corollary.

Corollary 5.4.1.1. If $\zeta(\text{ł}) \in S_s^*$, then

$$|\check{a}_3{}^2 - \check{a}_5| \leq \frac{1}{4}.$$

CHAPTER 6

CONCLUSION

The main goal of this thesis is to find bounds on the initial coefficients of analytic, univalent, normalized functions in the open unit disk.We started by reiterating key concepts and initial findings from Geometric Function Theory. These fundamental ideas serve as the basis for our groundbreaking discoveries, and we also looked at modern developments in Quantum Calculus. Apart from a detailed examination of the impact of the q-derivative operator on Geometric Function Theory. Additionally, using q-Calculus, we have created new families of analytic functions connected to symmetric points.

The research centers on a specific category of univalent functions known as starlike functions, which are related to symmetric points. Expanding on the work on the S_s^* class of starlike functions associated with symmetric locations done by Khan *et al.* [35], i had explored into expanding this class. A new class expressing starlike functions with regard to symmetric points subordinate to the hyperbolic tangent function was introduced, an extension of the original S_s^* category.

I had provided a q-extension of these function classes by constructing the $S_s^*(q)$ class, which comprises q-starlike functions subordinate to the q-hyperbolic tangent function. The q-derivative operator was employed to define these classes, and their properties were analyzed using subordination techniques.

I investigated several essential characteristics of functions within the newly defined class, including the Fekete–Szegö inequality, the Zalcman functional, and coefficient bounds. Futhermore, i had analyzed the Hankel determinants of second and third order for these functions. The results show that these new classes offer advancements above those that already exist, extending beyond previously established theorems in Geometric Function Theory. I verified the consistency of our findings with known results by examining the limit as $q \to 1^-$. I expect that this research will make a significant contribution to the field of Geometric Function Theory.

6.1 Future Work

This thesis explores a important category in univalent function theory: starlike functions related to symmetric points, specifically those subordinate to the hyperbolic tangent function. I extend these classes utilizing the idea of convexity. Additionally, we draw conclusions for the more comprehensive category of q-convex functions and examine both the geometric and analytical relationships between the functions discussed in this work and the newly introduced class.

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