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q-EXTENSION OF STARLIKE FUNCTIONS SUBORDINATED WITH COSINE HYPERBOLIC FUNCTION

By

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NATIONAL UNIVERSITY OF MODERN LANGUAGES

ISLAMABAD

August, 2024

q-EXTENSION OF STARLIKE FUNCTIONS SUBORDINATED WITH COSINE HYPERBOLIC FUNCTION

By

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Candidate of Master of Science in Mathematics at the National University of Modern Languages do hereby declare that the thesis q-EXTENSION OF STARLIKE FUNCTIONS SUBORDINATED WITH COSINE HYPERBOLIC FUNCTION submitted by me in partial fulfillment of MS degree, is my original work and has not been submitted or published earlier. I also solemnly declare that it shall not, in the future, be submitted by me for obtaining any other degree from this or any other university or institution. I also understand that if evidence of plagiarism is found in my thesis/dissertation at any stage, even after the award of a degree, the work may be canceled and the degree revoked.

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ABSTRACT

Title: $\hat{\hat{q}}$-EXTENSION OF STARLIKE FUNCTIONS SUBORDINATED WITH COSINE HYPERBOLIC FUNCTION

By merging classical mathematical principles with the innovative framework of quantum calculus, this thesis pioneer new advancements in the study of analytic functions. This thesis will advance the understanding of analytic functions by integrating classical principles with quantum calculus. It will introduce the class $S^*_{\hat{\hat{q}}cosh}$, which will extend starlike functions associated with $\hat{\hat{q}}$-cosine hyperbolic function. Through the application of the $\hat{\hat{q}}$-derivative operator and subordination techniques, the research will explore key properties such as coefficient bounds, Zalcman functional, Fekete-Szegö problem and Hankel Determinants. The results, anticipated to be validated as $\hat{\hat{q}}$ approaches to 1^-, will demonstrate significant progress beyond existing theories, enhancing both the theoretical and practical aspects of quantum calculus.

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LIST OF SYMBOLS

λ	-	Open Unit Disk
A	-	Class of Analytic Function
P	-	Class of Caratheodory Function
S^*	-	Class of Starlike Function
S^*_{cosh}	-	Class of Starlike function with respect to cosine hyperbolic function
$S^*_{\hat{\hat{q}}cosh}$	-	Class of $\hat{\hat{q}}$-Starlike function Subordinated with $\hat{\hat{q}}$-cosine hyperboic function
$\hat{\hat{q}}$	-	Quantum symbol
$D_{\hat{\hat{q}}}$	-	$\hat{\hat{q}}$-Derivative operator symbol
$\prec$	-	Subordination symbol
$\hat{\omega}(\tilde{z})$	-	Family of Schwarz functions

ACKNOWLEDGMENT

In the name of Allah, the Most Gracious, the Most Merciful, I begin this acknowledgment with the verse, "You Alone we worship; You Alone we ask for help." My journey has been a testament to Allah's boundless mercy, and I am profoundly grateful for His guidance through every step. As it is said in the Qur'an, "And He found you lost and guided [you]," I reflect on how Allah's infinite wisdom has steered me through challenges and towards enlightenment.

I extend my deepest gratitude to my family, whose unwavering support has been a constant source of strength. Their prayers and encouragement have lifted me in moments of doubt and fueled my perseverance. To my teachers, I owe a special debt of gratitude. Your wisdom, patience, and guidance have shaped my academic journey and instilled in me a love for learning that goes beyond the confines of this thesis.

In closing, I acknowledge that this achievement is not mine alone but a culmination of the collective efforts, prayers, and blessings of those around me. I humbly ask Allah to guide me on the path of knowledge and righteousness.

DEDICATION

This thesis work is dedicated to my parents, family, and the remarkable teachers who have been with me throughout my educational journey—this work is a testament to your unwavering love, support, and inspiration. Your shining examples have instilled in me the drive to work tirelessly toward my dreams.

CHAPTER 1

INTRODUCTION AND LITERATURE REVIEW

1.1 Overview

Imagine stepping into a world where math meets shapes and patterns in a way that's both captivating and practical. That's what geometric function theory is all about. It's like exploring an enchanting domain where we use complex numbers to understand how functions twist, turn, and shape the world around us. In this chapter, we will discuss the introduction along with the literature review, exploring where geometric function theory originated from and who contributed to it. I identified gaps in this literature review and began working on them. Emerging from the groundbreaking contributions of early mathematicians, namely Cauchy, Riemann and Weierstrass, it delves into the intricate mappings and transformations of shapes within the complex plane. Fundamental concepts such as holomorphicity and conformality, along with pivotal theorems like the Riemann Mapping Theorem, form the backbone of this field. Its practical applications span a wide range of disciplines, from fluid dynamics to mathematical physics.

1.2 Riemann Mapping Theorem

The foundations of geometric function theory can be originated from **Riemann Mapping Theorem** [1] in 1857. Cauchy, founding figure, embarked on his journey in function theory, making the initiation of productive career that would see him authoring over 200 papers in this domain. With remarkable precision, he clarified how to deal specific type of integrals with complex limits and introduced some key concepts such as the Cauchy Integral Theorem [2]. He investigated how to break down complex functions into simpler parts using series. Following Cauchy's lead, Riemann emerged as the second prominent figure, in function theory. Riemann built on Cauchy's ideas, especially focusing on the Cauchy-Riemann Differential Equations [3]. These equations were important for defining analytic functions, which Riemann defined as functions like $f(x+iy) = u+iv$. Through this investigation, he then formulates Riemann mapping theorem. It states that any simply connected domain in the complex plane can be conformally mapped to any other or unit disk with similar description where simply connected domain means a path-connected domain where one can continuously shrink any simple closed curve into a point while remaining in the domain. The groundwork of this theory can be followed to the 19th century when mathematicians like Augustin-Louis Cauchy and Karl Weierstrass made meaningful contributions to complex analysis. They studied properties and behavior of holomorphic functions and their mappings.

1.3 Analytic Function and Univalent Function

In 1907, **Koebe** [4] worked on univalent functions and presented a theorem known as Koebe one-quarter theorem which states that in open unit disk if function is holomorphic function $f(0) = 0$ and $f'(0) = 1$, then image of unit disk under mapping contains a disk of ¼ radius centered at $f(0)$. In the late 19th century, **Hermann Amandus Schwarz** [5] made significant contribution to the geometric theory, which involves the boundary fixed points. Schwarz lemma at the boundary is also a dynamic topic in complex analysis, different unique results have been made by him. A framework for examining how univalent functions behave near the edge of their domain is offered by Schwarz boundary fixed points and associated theorems. These findings lay the groundwork for understanding how these functions behave close to the boundary, guaranteeing both the preservation of the geometric characteristics of the functions and well-controlled mappings. Analytic function [6], an essential part of complex analysis i.e., a complex valued function which is differentiable at every point within its domain.

1.4 Biebarbach Conjecture

A German mathematician, **Ludwig Bieberbach** [7] in 1916, formulated his conjecture, this conjecture states that if $\hat{g}(\tilde{z})$ is a univalent function in the unit disk $\lambda = \{\tilde{z} : |\tilde{z} < 1\}$, Taylor series expansion is:

$$\hat{g}(\tilde{z}) = \tilde{z} + a_2\tilde{z}^2 + a_3\tilde{z}^3 + \ldots, \quad \tilde{z} \in \lambda, \tag{1.1}$$

where the coefficients and for all n ≤ 2. He proved that $|c_2| \leq 2$, it holds if and only if the function is koebe fuction. A famous math puzzle called Bieberbach conjecture has stood for sixty years. This puzzle has motivated the mathematicians to come up with new ideas, many people used different techniques to solve this conjecture. **Karl Lowener** in 1923 [8], proved that $|c_3| \leq 3$. After a long gap of 30 years, **Garabedian and Schiffer** *et al.* [9], proved fourth coefficient $|c_4| \leq 4$ in his research.

This conjecture left open for many years till 1984, proven by American mathematician **Louis de Branges** and considered as fundamental achievement theory of univalent function and complex analysis. De Branges trying to prove Bieberbach conjecture, first by the help of Lowner differential equation he proves result on bounded univalent functions which shows contracting flow on the unit disk. Then to prove his inequality, he uses this result [10].

1.5 Subclasses of Analytic and Univalent Function

In Geometric Function Theory, analytic functions plays vital role. This theory examines geometric properties and also categorized into number of classes and further broker down into subclasses. Several subclasses of analytic functions which are component of geometric function theory could be investigated and methods like subordination and -calculus are frequently employed. These subclasses, which include convex and starlike functions, are useful in mathematical physics and engineering. They are distinguished by particular geometric features.

1.6 Starlike Functions

Class of Starlike function and Convex function is defined by **Ma and Minda** [11] by using subordination,

$$S^* = \left\{ g \in A : \left(\frac{\tilde{z}g'(\tilde{z}}{g(\tilde{z})} \right) \prec \delta(\tilde{z}), \tilde{z} \in \lambda \right\}, \tag{1.2}$$

where the function delta of $\tilde{z}$ fulfill Schwarz function on unit disk.

1.7 Fekete-Szegö Inequality

Problem of Fekete-Szegö [12], relates to coefficients of a univalent analytic function and is connected to the Bieberbach conjecture. In general, Fekete-szegö inequality is expressed as $|c_3 - \alpha c_2^2|$ for some constant α, α may be real or complex. This intricate inequality is true for, $0 \leq \alpha < 1$. It is an essential result in complex analysis which provides limitations on the coefficients of specific classes. Particularly, it gives upper bound on the modulus of coefficients of a function that is normalized and analytic as well on the unit disk. Whereas Bieberbach Conjecture is involved with the coefficients of the Taylor expansion. By utilizing the Fekete-Szegö inequality, mathematicians were able to make important progress towards proving the Bieberbach Conjecture.

1.8 Hankel Determinant

The idea of Hankel Determinant originates from the study of hankel matrices, which bear the name of German mathematician Hermann Hankel (1839-1873). **Pommerenke** [13] demonstrated the concept of Hankel Determinant for particular univalent functions. In 1967, **Noon and Thomas** [14] have determined the $\hat{\hat{q}}$th -hankel determinant, the $\hat{\hat{q}}$th- Hankel determinant is defined as,

$$H_q(n) = \begin{vmatrix} c_n & c_{n+1} & c_{n+2} & \cdots & c_{n+\hat{\hat{q}}-1} \\ c_{n+1} & c_{n+2} & c_{n+3} & \cdots & c_{n+q} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{n+\hat{\hat{q}}-1} & c_{n+q} & c_{n+q+1} & \cdots & c_{n+2\hat{\hat{q}}-2} \end{vmatrix}.$$

The Hankel determinant for the starlike and convex functions was studied by **Janteng *et al.*** [15], in 2002. Hankel Determinant $H_2(2) = |c_2c_4 - c_3^2|$ for the starlike class is $|c_2c_4 - c_3^2| \leq 1$ whereas for convex class is $|c_2c_4 - c_3^2| \leq \frac{1}{8}$, obtained results were sharp. **Sokol** [16] determined third order of Hankel determinant.

$$H_3(1) = \begin{bmatrix} c_1 & c_2 & c_3 \\ c_2 & c_3 & c_4 \\ c_3 & c_4 & c_5 \end{bmatrix}.$$

His contributions to the famous classes of convex and starlike functions in the disk. The starlike functions Hankel Determinant is $|H_3(1)| \leq 16$, whereas the convex functions Hankel determinant is $|H_3(1)| \leq 15$. In 2017, **Prajapat** [17] introduced constraints for specific categories. In 2019, **Lecko *et al.*** [18] work deals with forming sharp bounds. Sharp bounds are accurate limits or inequalities that define the exterior behavior of mathematical expressions. In this case, they are investigating the most extreme values that Hankel determinant can take for this particular subclass of functions with an order ½.

1.9 $\hat{\hat{q}}-$Calculus

$\hat{\hat{q}}-$ calculus, a powerful tool for handling discrete and quantum like phenomena. It is also known as **Jackson's $\hat{\hat{q}}-$ calculus** [19] that generalizes traditional calculus and has vast applications. He was the originator in creating a systematic approach to $\hat{\hat{q}}-$integrals and $\hat{\hat{q}}-$derivatives, which are special mathematical tools. Later on, researchers explored how these $\hat{\hat{q}}$-tools are linked to quantum groups, which are a bit special mathematical family.

Euler was the first mathematician who come up with a theory in math about numbers could be break into smaller parts, and this theory is called additive analytic number theory. The theory was like the starting of another math area called $\hat{\hat{q}}-$analysis. Euler wrote a lot of math material, but it was not put all together and published until early 1800s. It was published under the name of someone named Jacobi, who was a legendary mathematician, even though Euler had always written his work in Latin. In 1829, **Jacobi** [20] came up with his own idea in math called elliptic functions, and it's kind of like $\hat{\hat{q}}-$analysis.

Another important mathematician, C.F. Gauss, who lived from 1777 to 1855, also contributed to $\hat{\hat{q}}-$calculus. He's known for inventing hypergeometric series and some related mathematical relationships in 1812. Many mathematicians worked on subclasses of Starlike function S^* and defined them in $\hat{\hat{q}}-$deformation like [21] in which $\hat{\hat{q}}-$derivative operator on convex and starlike functions are defined widely with derivation as well. Different classes of analytic function have been put forward and developed on the unit disk by the help of subordination technique. Like, **Seoudy** ***et al***. [22] used $\hat{\hat{q}}-$derivatives to found new divisions of quantum star-like operations in complex order. Estimates on coefficients for second and third coefficients of these classes have been also found in it. The $\hat{\hat{q}}-$analogue of analytic functions linked with -cosine function is defined in [23], by using subordination technique. **Chetan Swarup** [24] presented a subclass of $\hat{\hat{q}}-$ starlike functions, which are related with $\hat{\hat{q}}$- analogue of hyperbolic tangent function through subordination relation.

1.10 Starlike Functions subordinated with Cosine Hyperbolic Function

Alotaibi ***et al.*** [25], define a family of starlike functions associated to cosine hyperbolic function. He investigated different properties of these function. This research inspired us to introduce $\hat{\hat{q}}-$extension of starlike functions subordinated with cosine hyperbolic function. These functions can serve as analytical tool for solving complex problems that involve the concepts of $\hat{\hat{q}}-$calculus. It is such a comprehensive subject that it has application in different areas of applied sciences, current mathematical physics, including engineering, number theory, statistical mechanics and area of signal processing.

1.11 Preface

The goal of this study is to examine and characterize a few new analytic function subdivisions such as $\hat{\hat{q}}-$starlike functions that are associated to cosine hyperbolic function. Divided into five sections, here is brief summary of each chapters:

In **Chapter 2,** the definition of important sub-classes of univalent functions are explored and chapter concludes with foundational lemmas that will serve as the basis for future talks. It is noteworthy for not presenting new findings but providing a thorough synthesis of accepted principles.

In **Chapter 3,** we delve into the realm of starlike functions subordinated with cosine function along with an exploration of specific key findings. It's important to underscore that proper citation of reviewed literature is diligently maintained throughout this investigation.

In **Chapter 4,** $\hat{\hat{q}}-$starlike functions subordinated to cosine hyperbolic function—is the subject of this chapter. Additionally, proven findings for functions in this class are inferred in this chapter. It is shown via corollaries that the newly derived conclusions are consistent with those that other researchers have already established.

In **Chapter 5,** the coefficient bounds, Zalcman's conjecture, or the third-order Hankel determinant are discussed for our new class using lemmas and corollaries with conclusion. Also, explore the future work for new researchers.

CHAPTER 2

DEFINITIONS AND PRELIMINARY CONCEPTS

2.1 Overview

This chapter's goal is to provide basic key terminologies and classical results of geometric function theory and associated subjects. Geometric Function theory gracefully combines the intricate patterns of analytic functions and the univalent functions into a fascinating material of mathematical insight and practical applications. This chapter defines special functions, linear operator and preliminary lemmas and some recent classes of analytic functions. This study examines the captivating realm of mathematics, revealing the beauty of complex functions through the lens of geometry.

2.2 Holomporhic Function

Essential concepts in complex analysis, a field of mathematics that address complex numbers and associated functions. In this field, holomorphic functions are core concepts.

Definition 2.2.1. [26] A function is called holomorphic at some point $\check{z}_0$, if it is complex differentiable at $\check{z}_0$ and in a close region around $\check{z}_0$. In a more precise way, function exhibits holomorphic at $\check{z}_0$ if the limit persists.

$$\hat{g}'(\check{z}_0) = \lim_{h \to 0} \frac{\hat{g}(\check{z}_0 + h) - \hat{g}(\check{z}_0)}{\hat{g}(\check{z}_0)}.$$

Analytic functions are all holomorphic functions and demonstrated within some convergence boundary.

2.3 Riemann Mapping Theorem

The roots of geometric function theory originates from Riemann Mapping theorem in 1857.

Definition 2.3.1. [27] It asserts that conformal mapping is possible for any simply connected domain in the complex plane to any other or unit disk with similar description where simply connected domain means a path-connected domain where one can continuously shrink any simple closed curve into a point while remaining in the domain.

2.4 The Class A

Functions belong to class A is normalized analytic. An analytic function (additionally referred to as **Holomorphic** Function) is a function that is complex differentiable in a neighborhood of all points in its domain.

Definition 2.4.1. [28] An analytic function $f(\tilde{z})$ is one that has a derivative at every point in its domain λ. Analytic functions can be represented and expressed as an infinite sum of power,

$$g(\tilde{z}) = \tilde{z} + \sum_{m=2}^{\infty} c_m \tilde{z}^m, \tilde{z} \in \lambda, \tag{2.1}$$

where coefficients can be determined.

Definition 2.4.2. [1] Function $\tilde{z}$ is said to be Normalized Analytic Function, if it takes zero at origin $(g(0) = 0)$ and its derivative takes the value 1 at the origin $(g'(0) = 1)$.

2.5 The Class S

Functions belong to class S are analytic, normalized and univalent as well.

Definition 2.5.1. [4] Function is said to be univalent if it maps unique points in its domain to unique point in its range. These functions are also known as one -to-one or injective function. For any two distinct complex numbers $\tilde{z}_1$and $\tilde{z}_2$ in the domain,

$$g(\tilde{z}_1) \neq g(\tilde{z}_2). \tag{2.2}$$

The function in the class S is normalized by the restrictions $g(0) = 0, g'(0) = 1$.

2.6 The Class P

Functions whose real part is positive, are belong to class P.

Definition 2.6.1. [1] In Harmonic Functions, real valued functions are considered, functions whose real part is positive.

Definition 2.6.2. [1]**(Caratheodory Function)** In the framework of functions that map the unit disk to itself, the caratheodory class consists of a function that have a positive real part in the unit disk.

$$P = \{\breve{p} : \breve{p}(0) = 1, \mathbb{R}\breve{p}(\tilde{z}) > 0, \tilde{z} \in \lambda\}.$$

$$\breve{p}(\tilde{z}) = 1 + \sum_{m=1}^{\infty} c_m \breve{p}^m, \tag{2.3}$$

2.7 Sub-classes of Univalent Functions

Sub-divisions of univalent functions are essential in the analysis of Geometric Function Theory and complex analysis. these functions have number of applications in different areas of mathematics, its application include conformal mapping, potential theory and fluid dynamics. There are many sub-classes of univalent functions but our research revolves around one main class i.e. Starlike Functions. Also analyze the relationship between these classes and P class with their proven properties.

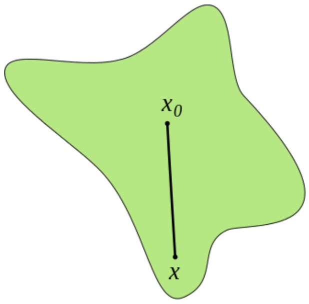

Figure 2.1: Starlike domain

2.7.1 The Class of Starlike Function

Functions defined in the unit disk in the complex plane $\lambda = \{\tilde{z} : |\tilde{z}| < 1\}$,with some definite geometric properties are known as Starlike Functions. Under the function, geometric properties are related to the shape of the image of the unit disk. Star likeness is an essential geometrical characteristic. All the points in the set are connected with a fixed point by a straight line to form a starlike domain. If all of these straight lines were fall within the domain, that particular domain becomes starlike in terms of fixed point.

Definition 2.7.1. [29] A holomorphic function defined on the unit disk **D** (where **D**= $\{\tilde{z} \in \mathbb{C} : |\tilde{z}| < 1\}$ is called starlike if it fulfills the requirements listed below:

$$S^* = \left\{ \hat{g} \in A : Re(\frac{\tilde{z}\hat{g}'(\tilde{z})}{\hat{g}(\tilde{z})}) > 0, \tilde{z} \in \mathbf{D} \right\}. \tag{2.4}$$

2.8 Subordination

In complex analysis, subordination is a strong tool for exploring the relationships between different classes of functions and insight their geometric and analytic properties. It enables the analysis of functions by associating them to simpler or better-understood functions, leading to more profound insights into their behavior and structure.

Definition 2.8.1. [30] If ϕ and ψ belong to class A, we say that, the function ϕ is considered to be subordinated to ψ ,figuratively expressed as $\phi \prec \psi$, if $\phi(\tilde{z}) = \psi(\hat{\omega}(\tilde{z})$, where $\hat{\omega}(\tilde{z})$in an open unit disk is an analytical function known as Schwarz function, fulfilling two criterion that it gives zero at origin and less than or equal to 1 at $\tilde{z}$.

2.9 Fekete-Szegö Inequality

The Fekete-szegö inequality, connected to Bieberbach conjecture and deals with coefficients of a univalent analytic function.

Definition 2.9.1. [31] The Fekete-Szegö Inequality is a finding, particularly in analysis of univalent (holomorphic and injective) functions. This inequality offers bounds for certain coefficients of these functions when they are normalized in a particular manner.

A function that is univalent in the unit disk and normalized such that in (2.1). The Fekete-Szegö Inequality includes second coefficient c_2 and higher coefficients c_3, c_4... This inequality states that for any real number α,

$$|c_3 - \alpha c_2^2| \leq 1 + |\alpha|.$$

- For $\alpha = 0$, it becomes $|c_3| \leq 1$,
- For $\alpha = 1$, it becomes $|c_3 - c_2^2| \leq 2$.

(Also known as second order Hankel determinant)

2.10 Hankel Determinant

In analytic and univalent functions, coefficient problems plays very important role. Hankel determinant is a study from linear algebra and mathematical analysis, named after the German mathematician Hermann Hankel. It has applications in different areas including moment problems, control theory .

Definition 2.10.1. [32] A square matrix with every ascending skew diagonal from left to right being constant is known as Hankel determinant. It is associated with a series of a numbers. The $\hat{q}$th-Hankel determinant is defined as,

$$H_q(n) = \begin{vmatrix} c_n & c_{n+1} & c_{n+2} & \cdots & c_{n+\hat{q}-1} \\ c_{n+1} & c_{n+2} & c_{n+3} & \cdots & c_{n+\hat{q}} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{n+\hat{q}-1} & c_{n+\hat{q}} & c_{n+\hat{q}+1} & \cdots & c_{n+2\hat{q}-2} \end{vmatrix},$$

For second order Hankel determinant put $\hat{\hat{q}} = 2$ and n=1, it becomes

$$H_2(1) = \begin{vmatrix} c_1 & c_2 \\ c_2 & c_3 \end{vmatrix} = c_1c_3 - c_2^2.$$

$$H_2(1) = c_3 - c_2^2. \tag{2.5}$$

Fekete-Szegö Inequality is mentioned as the determinant $H_2(1)$. For now,let n=2 and q=2, it becomes

$$H_2(2) = \begin{vmatrix} c_2 & c_3 \\ c_3 & c_4 \end{vmatrix} = c_2c_4 - c_3^2.$$

In 2018, **Zaprwa** [18] calculates Hankel determinant $H_2(3)$,

Now, for q=2 and n=3. it becomes

$$H_2(3) = \begin{vmatrix} c_3 & c_4 \\ c_4 & c_5 \end{vmatrix} = c_3c_5 - c_4^2.$$

Third order Hankel determined has been determined by many researchers [12, 13, 14, 15] and is given by,

$$H_3(1) = \begin{vmatrix} c_1 & c_2 & c_3 \\ c_2 & c_3 & c_4 \\ c_3 & c_4 & c_5 \end{vmatrix},$$

$$H_3(1) = c_5(c_3 - c_2^2) - c_4(c_4 - c_2c_3) + c_3(c_2c_4 - c_3^2). \tag{2.6}$$

2.11 Quantum Calculus

$\hat{\hat{q}}-$calculus [33], a field of mathematics that extends and generalizes classical calculus. In quantum calculus, functions such as the derivative and integral are revised in terms of $\hat{\hat{q}}-$analogues, arising in new properties and behaviors. These $\hat{\hat{q}}-$derivatives and $\hat{\hat{q}}-$integrals frequently diminished to the traditional calculus equivalents when $\hat{\hat{q}}-$ approaches 1.

Definition 2.11.1. [34] $\hat{\hat{q}}-$calculus or non-Newtonian calculus, is a mathematical structure which prolongs traditional calculus by presenting a parameter $\hat{\hat{q}}$. This parameter is frequently considered to be a real number close to 1.

Definition 2.11.2. [35] $\hat{\hat{q}}-$Derivative is defined in such a way that it narrows to the traditional derivative when $\hat{\hat{q}}$ approaches to 1^-. $\hat{\hat{q}}-$derivative operator is denoted as $D_{\hat{\hat{q}}}$

$$D_{\hat{\hat{q}}}g(\tilde{z}) = \begin{cases} \frac{g(\tilde{z}q)-g(\tilde{z})}{\tilde{z}q-\tilde{z}}, & \tilde{z} \neq 0 \\ g'(0), & \tilde{z} = 0. \end{cases} \tag{2.7}$$

The Maclaurin's series of q-derivative is:

$$D_{\hat{\hat{q}}}g(\tilde{z}) = \sum_{n=0}^{\infty} [n]_{\hat{\hat{q}}} a_n (\tilde{z})^{n-1}, \tag{2.8}$$

where $\hat{\hat{q}}-$Pochhammer

$$[n]_{\hat{\hat{q}}} = \begin{cases} \frac{1-\hat{\hat{q}}^n}{1-\hat{\hat{q}}}, & \hat{\hat{q}} \neq 1 \\ n, & \hat{\hat{q}} = 1. \end{cases}$$

Definition 2.11.3. [36] $\hat{\hat{q}}-$Integral series in the theory of special functions shows the inverse operation to q-derivative, which was introduced by Frank Hilton Jackson. It is defined as:

$$\int g(\tilde{z}) d_{\hat{\hat{q}}}(\tilde{z}) = (1-\hat{\hat{q}})\tilde{z} \sum_{k=0}^{\infty} \hat{\hat{q}}^k g(\hat{\hat{q}}^k \tilde{z}). \tag{2.9}$$

Definition 2.11.4. [37] $\hat{\hat{q}}-$Exponential is a $\hat{\hat{q}}$-deformation of the traditional exponential function and is used to extends exponential growth. The $\hat{\hat{q}}-$exponential function is defined as:

$$e_{\hat{\hat{q}}}(\tilde{z}) = \sum_{n=0}^{\infty} \frac{\tilde{z}^n}{[n]_{\hat{\hat{q}}}!}, \qquad 0 < \hat{\hat{q}} < 1. \tag{2.10}$$

Definition 2.11.5. [38] $\hat{\hat{q}}-$trigonometric functions are defined as

$$sin_{\hat{\hat{q}}}(\tilde{z}) = \frac{e_{\hat{\hat{q}}}^{i\tilde{z}} - e_{\hat{\hat{q}}}^{-i\tilde{z}}}{2i}. \tag{2.11}$$

$$cos_{\hat{\hat{q}}}(\tilde{z}) = \frac{e_{\hat{\hat{q}}}^{i\tilde{z}} + e_{\hat{\hat{q}}}^{-i\tilde{z}}}{2}. \tag{2.12}$$

Definition 2.11.6. [39] $\hat{\hat{q}}-$ hyperbolic trigonometric functions are defined as

$$sinh_{\hat{\hat{q}}}(\tilde{z}) = \frac{e_{\hat{\hat{q}}}^{\tilde{z}} - e_{-}\hat{\hat{q}}^{-\tilde{z}}}{2}. \tag{2.13}$$

$$cosh_{\hat{\hat{q}}}(\tilde{z}) = \frac{e_{\hat{\hat{q}}}^{\tilde{z}} + e_{-}\hat{\hat{q}}^{-\tilde{z}}}{2}. \tag{2.14}$$

2.12 Zalcman Conjecture

Zalcman presented a conjecture in 1960, for functions that are univalent whose extended form is offered by **Ma** in 1999.

Definition 2.12.1. [40] Zalcman Conjectures states that all functions belong to univalent function having form (2.1) satisfies the subsequent sharp inequality.

$$|a_n^2 - a_{2n-1}| \leq (n-1)^2, n \geq 2. \tag{2.15}$$

Definition 2.12.2. [41] Generalized Zalcman Conjecture states that the Taylor coefficient form of (2.1) satisfy the following inequality

$$|\grave{a}_{\hat{r}}\grave{a}_{\tilde{p}} - \grave{a}_{\hat{r}+\tilde{p}-1}| \leq (\hat{r}-1)(\tilde{p}-1), \forall \hat{r} \geq 2, \tilde{p} \geq 2. \tag{2.16}$$

2.13 Preliminary Lemmas

Following lemmas will be useful in producing findings in the next chapters:

Lemma 2.13.1. [42] If $\check{p}(\tilde{z}) = 1 + \sum_{n=1}^{\infty} c_n \tilde{z}^n \in \mathbf{P}$ then,

$$2c_2 = c_1^2 + \alpha(4 - c_1^2). \tag{2.17}$$

$$4c_3 = c_1^3 + 2(4-c_1^2)c_1\alpha - (4-c_1^2)c_1^2\alpha + 2(4-c_1^2)(1-|\alpha|^2)\beta, \tag{2.18}$$

for some

$$\alpha(|\alpha \leq 1), \beta(|\beta| \leq 1).$$

Lemma 2.13.2. [43] Let the function $\check{p} \in \mathbf{P}$ given by (2.1), then

$$|c_n| \leq 2, (n \in \mathbf{N}). \tag{2.19}$$

Lemma 2.13.3. [44] Let the function $\check{p} \in \mathbf{P}$ given by (2.1), then

$$|\tilde{\upsilon}c_n - c_k c_{n-k}| \leq \begin{cases} 2|2-\tilde{\upsilon}|, \tilde{\upsilon} \leq 1 \\ 2\tilde{\upsilon}, \tilde{\upsilon} \geq 1 \end{cases} \tag{2.20}$$

CHAPTER 3

CLASS OF STARLIKE FUNCTION SUBORDINATED TO COSINE HYPERBOLIC FUNCTION

3.1 Introduction

This section goal is to investigate numerous essential and established conclusions that assist as pillar for posterior study. This part initiates by examining functions that are starlike and categories established in connections with cosine hyperbolic function. Furthermore, various major conclusions will be evaluated.

Definition 3.1.1. A function $\hat{g} \in S$ is considered to be in the class of S^*_{cosh}, if it satisfies the mentioned boundaries

$$\frac{\tilde{z}\hat{g}'(\tilde{z})}{\hat{g}(\tilde{z})} \prec cosh(\tilde{z}), \tilde{z} \in \lambda. \tag{3.1}$$

That is,

$$S^*_{cosh} = \left\{ \hat{g} \in A : \frac{\tilde{z}\hat{g}'(\tilde{z})}{\hat{g}(\tilde{z})} \prec cosh(\tilde{z}) \right\}. \tag{3.2}$$

3.2 Coefficient Inequalities

The following class S^*_{cosh} is associated with following finding.

Theorem 3.2.1. If $\hat{g} \in S^*_{cosh}$ has the series form as given in (2.1). Then

$$|\grave{a}_2| \leq 0, |\grave{a}_3| \leq \frac{1}{4}, |\grave{a}_4| \leq \frac{1}{3}, |\grave{a}_5| \leq 0.2135416. \tag{3.3}$$

Proof. Let $\hat{g} \in S^*_{cosh}$ then

$$\frac{\tilde{z}\hat{g}'(\tilde{z})}{\hat{g}(\tilde{z})} = cosh(\omega(\tilde{z})), \tilde{z} \in \lambda, \tag{3.4}$$

where

$$\omega(\tilde{z}) = \frac{\breve{p}(\tilde{z}) - 1}{1 + \breve{p}(\tilde{z})}.$$

If $\breve{p}(\tilde{z})$ corresponds to (2.3), then

$$\omega(\tilde{z}) = \frac{c_1\tilde{z} + c_2\tilde{z}^2 + c_3\tilde{z}^3 + \dots}{2 + c_1\tilde{z} + c_2\tilde{z}^2 + c_3\tilde{z}^3}.$$

As we know,

$$cosh(\omega(\tilde{z})) = cosh\left(\frac{c_1\tilde{z} + c_2\tilde{z}^2 + c_3\tilde{z}^3 + \dots}{2 + c_1\tilde{z} + c_2\tilde{z}^2 + c_3\tilde{z}^3}\right). \tag{3.5}$$

So, we have,

$$cosh\omega(\tilde{z}) = 1 + \frac{c_1^2\tilde{z}^2}{8} + \frac{(-c_1^3 + 2c_1c_2)\tilde{z}^3}{8} + \frac{1}{8}\left[\frac{c_1^4}{4} + c_2^2 + 2c_1c_3 - 3c_1^2c_2 + \frac{c_1^4}{2}\right]\tilde{z}^4 + \frac{c_1^4}{384}\tilde{z}^4 + \dots,$$

Now, taking left hand side of (3.4),

$$\hat{g}(\tilde{z}) = \tilde{z} + \grave{a}_2\tilde{z}^2 + \grave{a}_3\tilde{z}^3 + \grave{a}_4\tilde{z}^4 + \dots, \tag{3.6}$$

$$\hat{g}'(\tilde{z}) = 1 + 2\grave{a}_2\tilde{z} + 3\grave{a}_3\tilde{z}^2 + 4\grave{a}_4\tilde{z}^3 + \dots, \tag{3.7}$$

which gives

$$\frac{\tilde{z}\hat{g}'(\tilde{z})}{\hat{g}(\tilde{z})} = \frac{\tilde{z}(1 + 2\grave{a}_2\tilde{z} + 3\grave{a}_3\tilde{z}^2 + 4\grave{a}_4\tilde{z}^3 + \dots)}{\tilde{z} + \grave{a}_2\tilde{z}^2 + \grave{a}_3\tilde{z}^3 + \grave{a}_4\tilde{z}^4 + \dots}.$$

It leads to

$$\frac{\tilde{z}\hat{g}'(\tilde{z})}{\hat{g}(\tilde{z})} = \frac{\tilde{z}(1 + 2\grave{a}_2\tilde{z} + 3\grave{a}_3\tilde{z}^2 + 4\grave{a}_4\tilde{z}^3 + \dots)}{\tilde{z}(1 + \grave{a}_2\tilde{z} + \grave{a}_3\tilde{z}^2 + \grave{a}_4\tilde{z}^3 + \dots}.$$

This implies that,

$$\frac{\tilde{z}g'(\tilde{z})}{\hat{g}(\tilde{z})} = (1 + 2\grave{a}_2\tilde{z} + 3\grave{a}_3\tilde{z}^2 + 4\grave{a}_4\tilde{z}^3 + \dots)(1 + (\grave{a}_2\tilde{z} + \grave{a}_3\tilde{z}^2 + \grave{a}_4\tilde{z}^3 + \dots)^{-1}.$$

After Binomial Expansion, we get

$$\frac{\tilde{z}\hat{g}'(\tilde{z})}{\hat{g}(\tilde{z})} = 1 + \grave{a}_2\tilde{z} + (2\grave{a}_3 - \grave{a}_2^2)\tilde{z}^2 + (3\grave{a}_4 - 3\grave{a}_2\grave{a}_3 + \grave{a}_2^3)\tilde{z}^3 + (4\grave{a}_5 - 2\grave{a}_3^2 - 3\grave{a}_2^4 + 4\grave{a}_2^2\grave{a}_3 - 4\grave{a}_2\grave{a}_4)\tilde{z}^4 + \dots, \tag{3.8}$$

Regarding a comparison of the coefficients of $\tilde{z}, \tilde{z}^2, \tilde{z}^3, \tilde{z}^4$, in addition with specific computation we get,

$$\grave{a}_2 = 0, \tag{3.9}$$

$$\grave{a}_3 = \frac{c_1^2}{16}, \tag{3.10}$$

$$\grave{a}_4 = \frac{c_1(2c_2 - c_1^2)}{24}, \tag{3.11}$$

$$\grave{a}_5 = \frac{5c_1^4}{192} + \frac{c_1 c_3}{16} + \frac{c_2^2}{32} - \frac{3c_1^2 c_2}{32}, \tag{3.12}$$

Using Lemma 2.16.2 to (3.15), we get

$$|\grave{a}_3| \leq \frac{1}{4}.$$

Consider,

$$|\grave{a}_4| = \left| \frac{c_1(2c_2 - c_1^2)}{24} \right|,$$

Applying Lemma 2.16.3 to (3.11) with $\upsilon = 2$, we get

$$|\grave{a}_4| \leq \frac{1}{3}.$$

Now, applying Lemma 2.16.1 in (3.12), we get

$$\grave{a}_5 = \frac{5}{192}c_1^4 + \frac{1}{64}c_1 \left(c_1^3 + 2c_1(4-c_1^2)\alpha - c_1(4-c_1^2)\alpha^2 + 2(4-c_1^2)(1-|\alpha|^2)\beta \right) + \frac{1}{128}\left(c_1^2 + \alpha(4-c_1^2)\right)^2 - \frac{3}{64}c_1^2\left(c_1^2 + \alpha(4-c_1^2)\right),$$

which results in

$$\grave{a}_5 = \frac{5}{192}c_1^4 + \frac{1}{64}\left(c_1^4 + 2(4-c_1^2)c_1^2\alpha - (4-c_1^2)c_1^2\alpha^2 + 2c_1(4-c_1^2)(1-|\alpha|^2)\beta \right) + \frac{1}{128}\left(c_1^2 + \alpha(4-c_1^2)\right)^2 - \frac{3}{64}\left(c_1^4 + c_1^2\alpha(4-c_1^2)\right).$$

This implies that

$$\grave{a}_5 = \frac{1}{384}\Big[10c_1^4 + 6\left(c_1^4 + 2(4-c_1^2)c_1^2\alpha - (4-c_1^2)c_1^2\alpha^2 + 2c_1(4-c_1^2)(1-|\alpha|^2)\beta \right) + 3\left(c_1^4 + \alpha^2(4-c_1^2)^2 + 2(4-c_1^2)c_1^2\alpha\right) - 18\left(c_1^4 + c_1^2\alpha(4-c_1^2)\right)\Big].$$

This leads us to

$$\grave{a}_5 = \frac{1}{384}\left[c_1^4 - 6(4-c_1^2)c_1^2\alpha^2 + 12(4-c_1^2)c_1(1-|\alpha|^2\beta) + 3(4-c_1^2)^2\alpha^2\right].$$

Using triangular inequality, let $c_1 = c$ and $|\alpha| = t$

$$|\grave{a}_5| \leq \frac{1}{384}\left|c^4 + 6(4-c^2)c^2t^2 + 12(4-c^2)c + 3(4-c^2)^2t^2\right|.$$

We assume that

$$\varphi(c,t) = \frac{1}{384}\left[c^4 + 6(4-c^2)c^2t^2 + 12(4-c^2)c + 3(4-c^2)^2t^2\right].$$

Upon partial differentiation, we get

$$\frac{\partial \varphi}{\partial t} = \frac{1}{384}\left[12(4-c^2)c^2t + 6(4-c^2)t\right].$$

It implies $\varphi(c,t)$ is a function that grows from [0,1]. So,

$$max(\varphi(c,t)) = \varphi(c,1) = \frac{1}{384}\left[c^4 + 6(4-c^2)c^2 + 12(4-c^2)c + 3(4-c^2)^2\right].$$

Set

$$\omega(c) = \frac{1}{384}\left[c^4 + 6(4-c^2)c^2 + 12(4-c^2)c + 3(4-c^2)^2\right].$$

Now,

$$\omega(c) = \frac{1}{384}\left[c^4 + 24c^2 - 6c^4 + 48c - 12c^3 + 48 + 3c^4 - 24c^2\right].$$

On simplifying, we get

$$\omega(c) = \frac{1}{384}[-2c^4 - 12c^3 + 48c + 48].$$

By differentiating with respect to 'c', we get

$$\omega'(c) = \frac{1}{384}[-8c^3 - 36c^2 + 48].$$

Certain calculations shows that $\omega'(c) > 0$ for $c \in [0, 1.1]$ and $q \in [0,1]$ Also, $\omega'(c) \leq [1.2, 2]$ and $q \in [0,1]$. It implies that $\omega(c)$ is increasing in [0,1] and $\omega(c)$ is decreasing in $[1.1, 2]$. This means that,

$$\omega(c) \leq \omega(1) = \frac{1}{384}\left[-2(1)^4 - 12(1)^3 + 48(1) + 48\right] \leq 0.2135416.$$

Consequently, we get

$$|\grave{a}_5| \leq 0.2135416.$$

Hence, the proof is complete.

□

3.3 Fekete-Szegö Inequality

Theorem 3.3.1. If $g \in S^*_{cosh}$ has the series form as given in (2.1). Then

$$|\grave{a}_3 - \grave{a}_2^2| \leq \frac{1}{4}. \tag{3.13}$$

Proof. From (3.9) and (3.10),

$$\grave{a}_2 = 0,$$
$$\grave{a}_3 = \frac{c_1^2}{16}.$$

Using Lemma 2,16.2, we have

$$|\grave{a}_3 - \grave{a}_2^2| = \left|\frac{2^2}{16} - 0\right|,$$
$$|\grave{a}_3 - \grave{a}_2^2| \le \frac{1}{4}.$$

Hence, proof is completed. □

Theorem 3.3.2. If $g \in S^*_{cosh}$ has the series form as given in (2.1). Then

$$|\grave{a}_4 - \grave{a}_3\grave{a}_2| \le \frac{1}{3}. \tag{3.14}$$

Proof. From (3.9) and (3.11),

$$\grave{a}_2 = 0,$$
$$\grave{a}_4 = \frac{c_1(2c_2 - c_1^2)}{24}, \tag{3.15}$$
$$|\grave{a}_4 - \grave{a}_3\grave{a}_2| = \left|\frac{c_1(2c_2 - c_1^2)}{24} - 0\right|,$$
$$= \frac{|c_1|}{24}|2c_2 - c_1^2|.$$

Applying Lemma 2.16.3 with $\upsilon = 2$, we get

$$|\grave{a}_4 - \grave{a}_3\grave{a}_2| \le \frac{1}{3}.$$

which is the needed result. □

3.4 Zalcman Functional

Theorem 3.4.1. If $g \in S^*_{cosh}$ has the series form as given in (2.1). Then

$$|\grave{a}_5 - \grave{a}_3^2| \le 0.2122. \tag{3.16}$$

Proof. From (3.9) and (3.12), we have

$$\grave{a}_3 = \frac{c_1^2}{16},$$

$$\grave{a}_5 = \frac{5c_1^4}{192} + \frac{c_1c_3}{16} + \frac{c_2^2}{32} - \frac{3c_1^2c_2}{32},$$

$$\grave{a}_5 - \grave{a}_3^2 = \frac{5c_1^4}{192} + \frac{c_1c_3}{16} + \frac{c_2^2}{32} - \frac{3c_1^2c_2}{32} - \left(\frac{c_1^2}{16}\right)^2,$$

$$\grave{a}_5 - \grave{a}_3^2 = \frac{17c_1^4}{768} + \frac{c_1c_3}{16} + \frac{c_2^2}{32} - \frac{3c_1^2c_2}{32}. \tag{3.17}$$

Applying Lemma 2.16.1 to 3.17,

$$\grave{a}_5 - \grave{a}_3^2 = \frac{17c_1^4}{768} + \frac{c_1}{16}\left[\frac{c_1^3 + 2(4-c_1^2)c_1\alpha - (4-c_1^2)c_1\alpha^2 + 2(4-c_1^2)(1-|\alpha|^2\beta)}{4}\right] + \frac{1}{32}\left[\frac{c_1^2+\alpha(4-c_1^2)}{2}\right]^2 - \frac{3c_1^2}{32}\left[\frac{c_1^2+\alpha(4-c_1^2)}{2}\right],$$

$$\grave{a}_5 - \grave{a}_3^2 = \frac{17c_1^4}{768} + \frac{1}{64}\left[c_1^4 + 2(4-c_1^2)c_1^2\alpha - (4-c_1^2)c_1^2\alpha^2 + 2(4-c_1^2)(1-|\alpha|^2\beta)c_1\right] + \frac{1}{128}\left[c_1^4 + \alpha^2(4-c_1^2)^2 + 2(4-c_1^2)c_1^2\alpha\right] - \frac{3}{64}\left[c_1^4 + \alpha(4-c_1^2)c_1^2\right],$$

$$\grave{a}_5 - \grave{a}_3^2 = \frac{1}{768}\Big[17c_1^4 + 12c_1^4 + 24(4-c_1^2)c_1^2\alpha - 12(4-c_1^2)c_1^2\alpha^2 + 24(4-c_1^2)(1-|\alpha|^2)\beta c_1 + 6[c_1^4 + (4-c_1^2)^2\alpha^2 + 2(4-c_1^2)c_1^2\alpha] - 36[c_1^4 + c_1^2\alpha(4-c_1^2)]\Big],$$

$$\grave{a}_5 - \grave{a}_3^2 = \frac{1}{768}\Big[17c_1^4 + 12c_1^4 + 24(4-c_1^2)c_1^2\alpha - 12(4-c_1^2)c_1^2\alpha^2 + 24(4-c_1^2)(1-|\alpha|^2)\beta c_1 + 6c_1^4 + 6(4-c_1^2)^2\alpha^2 + 12(4-c_1^2)c_1^2\alpha - 36c_1^4 - 36c_1^2\alpha(4-c_1^2)\Big],$$

$$\grave{a}_5 - \grave{a}_3^2 = \frac{1}{768}\Big[-c_1^4 - 12(4-c_1^2)c_1^2\alpha^2 + 24(4-c_1^2)(1-|\alpha|^2)\beta c_1 + 6(4-c_1^2)^2\alpha^2\Big].$$

Applying modulus and let $c_1 = c, |\alpha| = t$,

$$|\grave{a}_5 - \grave{a}_3^2| \le \left|\frac{1}{768}\right| \left|c^4 + 12(4-c)^2c^2t^2 + 24(4-c)^2c + 6(4-c^2)^2t^2\right|. \tag{3.18}$$

We assume that

$$X(c,t) = \frac{1}{768}\left[c^4 + 12(4-c)^2c^2t^2 + 24(4-c)^2c + 6(4-c^2)^2t^2\right]. \tag{3.19}$$

Upon partial differentiation, we get

$$\frac{\partial X}{\partial t} = \frac{1}{768}\left[24(4-c)^2c^2t + 12(4-c^2)^2t\right] > 0.$$

It indicates that X(c,t) is escalating in [0,1]. So,

$$max(X(c,t)) = X(c,1) = \frac{1}{768}\left[c^4 + 12(4-c)^2c^2 + 24(4-c)^2c + 6(4-c^2)^2\right].$$

Set

$$Z(c) = \frac{1}{768}\left[c^4 + 12(4-c)^2c^2 + 24(4-c)^2c + 6(4-c^2)^2\right].$$

$$Z(c) = \frac{1}{768}\left[c^4 + 48c^2 - 12c^4 + 96c - 24c^3 + 96 + 6c^4 - 48c^2\right].$$

$$Z(c) = \frac{1}{768}\left[-5c^4 - 24c^3 + 96c + 96\right].$$

$$Z'(c) = \frac{1}{768}\left[-20c^3 - 72c^2 + 96\right].$$

Certain calculations shows that $Z'(c) > 0$ for $c \in [0,1]$ and $Z'(c) < 0$ for $c \in [1.1,2]$. It indicates Z(c) is increasing in [0,1] and Z(c) is decreasing in [1.1,2].

$$Z(c) \le Z(1) = \frac{1}{768}\left[(-5)(1)^4 - 24(1)^3 + 96(1) + 96\right] < 0.2122.$$

Consequently, we get

$$|\grave{a}_5 - \grave{a}_3^2| \le 0.2122.$$

Hence, the proof is complete. □

3.5 Hankel Determinants

Theorem 3.5.1. If the series representation of $g \in S^*_{cosh}$ is as described in (2.1). Then

$$|H_{2,1}(g)| \le \frac{1}{4}. \tag{3.20}$$

Proof. The Hankel Determinant:

$$H_{2,1}(g) = \begin{vmatrix} \grave{a}_1 & \grave{a}_2 \\ \grave{a}_2 & \grave{a}_3 \end{vmatrix}.$$

As $\grave{a}_2 = 0$,

$$H_{2,1}(g) = \grave{a}_3.$$

Substituting the values of $\grave{a}_3$ we get

$$H_{2,1}(g) = \frac{c_1^2}{16}.$$

Applying Lemma 2.16.2, we get

$$|H_{2,1}(g)| \leq \frac{2^2}{16},$$

$$|H_{2,1}(g)| \leq \frac{1}{4}.$$

□

Theorem 3.5.2. If the series form of $g \in S^*_{cosh}$ is as stated in (2.1). Then

$$|H_{2,2}(g)| \leq \frac{1}{16}. \tag{3.21}$$

Proof. The Hankel Determinant:

$$H_{2,2}(g) = \begin{vmatrix} \grave{a}_2 & \grave{a}_3 \\ \grave{a}_3 & \grave{a}_4 \end{vmatrix}.$$

As $\grave{a}_2 = 0$, So,

$$H_{2,2}(g) = -\grave{a}_3^2.$$

On Substituting values of $\grave{a}_3$, we get

$$H_{2,2}(g) = -\left(\frac{c_1^2}{16}\right)^2,$$

$$H_{2,2}(g) = -\left(\frac{c_1^2}{16}\right)^2.$$

Applying Lemma 2.16.2, we get

$$|H_{2,2}(g)| \leq \left(\frac{2^2}{16}\right)^2,$$

$$|H_{2,2}(g)| \leq \left(\frac{4}{16}\right)^2,$$

$$|H_{2,2}(g)| \leq \frac{1}{16}.$$

□

Theorem 3.5.3. If the series representation of $g \in S^*_{cosh}$ is as described in (2.1). Then

$$|H_{3,1}(g)| \leq 0.0293. \tag{3.22}$$

Proof.

$$H_{3,1}(g) = \grave{a}_5\grave{a}_3 - \grave{a}_4^2 + \grave{a}_3^3.$$

On Substituting values of $\grave{a}_3, \grave{a}_4$, we get

$$H_{3,1}(g) = \left[\frac{5c_1^4}{192} + \frac{c_1c_3}{16} + \frac{c_2^2}{32} - \frac{3c_1^2c_2}{32}\right]\left(\frac{c_1^2}{16}\right) - \left[\frac{c_1(2c_2 - c_1^2)}{24}\right]^2 - \left[\frac{c_1^2}{16}\right]^3.$$

After simplification, we get

$$H_{3,1}(g) = -\frac{13}{36864}c_1^6 + \frac{5}{4608}c_1^4c_2 + \frac{1}{256}c_1^3c_3 - \frac{23}{4608}c_1^2c_2^2.$$

Applying Lemma 2.16.1, we get

$$\begin{aligned} H_{3,1}(g) = &-\frac{13}{36864}c_1^6 + \frac{5}{4608}c_1^4\left[\frac{c_1^2 + \alpha(4 - c_1^2}{2}\right] \\ &+ \frac{1}{256}c_1^3\left[\frac{c_1^3 + 2(4 - c_1^2)c_1\alpha - (4 - c_1^2)c_1\alpha^2 + 2(4 - c_1^2)(1 - |\alpha|^2)\beta}{4}\right] \\ &- \frac{23}{4608}c_1^2\left[\frac{c_1^2 + \alpha(4 - c_1)^2}{2}\right]^2. \end{aligned}$$

$$\begin{aligned} H_{3,1}(g) = &-\frac{13}{36864}c_1^6 + \frac{5}{4608}c_1^6 + \frac{5}{4608}(4 - c_1^2)c_1^4\alpha + \frac{1}{1024}\Big[c_1^6 + 2(4 - c_1^2)c_1^4\alpha - (4 - c_1^2)c_1^4\alpha^2 \\ &+ 2(4 - c_1^2)(1 - |\alpha|^2)\beta c_1^3\Big] - \frac{23}{18432}\Big[c_1^6 + \alpha^2(4 - c_1^2)^2c_1^2 + 2(4 - c_1^2)c_1^4\alpha\Big]. \end{aligned}$$

$$\begin{aligned} H_{3,1}(g) = \frac{1}{36864}\Big[&-13c_1^6 + 20c_1^6 + 20(4 - c_1^2)c_1^4\alpha + 36\big[c_1^6 + 2(4 - c_1^2)c_1^4\alpha - (4 - c_1^2)c_1^4\alpha^2 \\ &+ 2(4 - c_1^2)(1 - |\alpha|^2)\beta c_1^3\big] - 46\big[c_1^6 + (4 - c_1^2)^2c_1^2\alpha^2 + 2(4 - c_1^2)c_1^4\alpha\big]\Big]. \end{aligned}$$

$$\begin{aligned} H_{3,1}(g) = \frac{1}{36864}\Big[&-13c_1^6 + 20c_1^6 + 20(4 - c_1^2)c_1^4\alpha + 36c_1^6 + 72(4 - c_1^2)c_1^4\alpha - 36(4 - c_1^2)c_1^4\alpha^2 \\ &+ 72(4 - c_1^2)(1 - |\alpha|^2)\beta c_1^3 - 46c_1^6 - 46(4 - c_1^2)^2c_1^2\alpha^2 - 92(4 - c_1^2)c_1^4\alpha\Big]. \end{aligned}$$

$$H_{3,1}(g) = \frac{1}{36864}\Big[-3c_1^6 - 36(4 - c_1^2)c_1^4\alpha^2 + 72(4 - c_1^2)(1 - |\alpha|^2)\beta c_1^3 - 46(4 - c_1^2)^2c_1^2\alpha^2\Big].$$

Applying modulus on both sides, let $c_1 = c$ and $|\alpha| = t$

$$|H_{3,1}(g)| \leq \left|\frac{1}{36864}\right|\left|3c^6 + 36(4 - c^2)c^4t^2 + 72(4 - c^2)c^3 + 46(4 - c^2)^2c^2t^2\right|.$$

Set

$$\phi(t, c) = \frac{1}{36864}\left[3c^6 + 36(4 - c^2)c^4t^2 + 72(4 - c^2)c^3 + 46(4 - c^2)^2c^2t^2\right].$$

By differentiating w.r.t. 't', we get

$$\frac{\partial \phi}{\partial t} = \frac{1}{36864}\left[72(4-c^2)c^4 t + 92(4-c^2)^2 c^2 t\right] > 0.$$

Clearly, $\frac{\partial \phi}{\partial t} > 0$, for $c \in [0,2]$. Hence, ϕ is increasing in [0,1].

$$max(\phi(t,c)) = \phi(1,c) = \frac{1}{36864}\left[3c^6 + 36(4-c^2)c^4 + 72(4-c^2)c^3 + 46(4-c^2)^2 c^2\right].$$

Say,

$$\Omega(c) = \frac{1}{36864}\left[3c^6 + 144c^4 - 36c^6 + 288c^3 - 72c^5 + 46(16 + c^4 - 8c^2)^2 c^2\right].$$

$$\Omega(c) = \frac{1}{36864}\left[13c^6 - 72c^5 - 224c^4 + 288c^3 + 736c^2\right].$$

Upon differentiating w.r.t. 'c', we get

$$\Omega'(c) = \frac{1}{36864}\left[78c^5 - 360c^4 - 896c^3 + 864c^2 + 1472c\right].$$

Some calculations shows that $\Omega'(c) > 0$ for $c \in [0,1.4]$ and $\Omega'(c) < 0$ for $c \in [1.5,2]$, which implies that $\Omega(c)$ is an increasing function in $c \in [0,1.4]$ and decreasing in $c \in [1.5,2]$. So,

$$\Omega(c) \leq \Omega(1.4) = \frac{1}{36864}\left[78(1.4)^5 - 360(1.4)^4 - 896(1.4)^3 + 864(1.4)^2 + 1472(1.4)\right] < 0.0293.$$

Accordingly, we get

$$|H_{3,1}(g)| \leq 0.0293.$$

Hence, the proof is complete. □

CHAPTER 4

$\hat{\hat{q}}-$STARLIKE FUNCTIONS SUBORDINATED TO $\hat{\hat{q}}-$COSINE HYPERBOLIC FUNCTION

4.1 Introduction

This section goal is to examine various essential and traditional results that assist as pillars for future study. Specifically, it delves into fundamental theorems and findings that have historically underpinned advancements in this field. The section begins with an investigation of $\hat{\hat{q}}-$Starlike Functions, exploring their properties and the classification systems that have been established in subordinating with the $\hat{\hat{q}}-$cosine hyperbolic function . These functions are critical in understanding the broader implications of $\hat{\hat{q}}-$calculus in geometric function theory. Moreover, this section will discuss multiple major conclusions derived from these investigations.

Definition 4.1.1. A function $g \in S$ is considered part of $S^*_{\hat{\hat{q}}cosh}$ class, if it meets the following criterion:

$$\frac{\tilde{z}D_{\hat{\hat{q}}}g(\tilde{z})}{g(\tilde{z})} \prec cosh_{\hat{\hat{q}}}(\tilde{z}), \tilde{z} \in \lambda. \tag{4.1}$$

Which is,

$$S^*_{\hat{\hat{q}}cosh} = \left\{ g \in A : \frac{\tilde{z}D_{\hat{\hat{q}}}g(\tilde{z})}{g(\tilde{z})} \prec cosh_{\hat{\hat{q}}}(\tilde{z}) \right\}. \tag{4.2}$$

4.2 Coefficient Inequalities

The subsequent results are associated to class of $S^*_{\hat{\hat{q}}cosh}$.

Theorem 4.2.1. If the series representation of $g \in S^*_{\hat{\hat{q}}cosh}$ is as described in (2.1). Then

$$|\grave{a}_2| \leq 0, |\grave{a}_3| \leq \frac{1}{\hat{\hat{q}}(1+\hat{\hat{q}})^2}, |\grave{a}_4| \leq \frac{2}{\hat{\hat{q}}(1+\hat{\hat{q}}+\hat{\hat{q}}_2)(1+\hat{\hat{q}})}, \tag{4.3}$$

$$|\grave{a}_5| \leq \frac{27\hat{\hat{q}}^7+81\hat{\hat{q}}^6+135\hat{\hat{q}}^5+163\hat{\hat{q}}^4+136\hat{\hat{q}}^3+84\hat{\hat{q}}^2+29\hat{\hat{q}}+1}{16\hat{\hat{q}}^2(1+\hat{\hat{q}}^2)^2(1+\hat{\hat{q}}+\hat{\hat{q}}^2)(1+\hat{\hat{q}})^4}. \tag{4.4}$$

Proof. By definition,

$$S^*_{\hat{\hat{q}}cosh} = \left\{ g \in A : \frac{\tilde{z}D_{\hat{\hat{q}}}g(\tilde{z})}{g(\tilde{z})} \prec cosh_{\hat{\hat{q}}}(\tilde{z}) \right\}. \tag{4.5}$$

Using subordination principle, we have

$$\frac{\tilde{z}D_{\hat{\hat{q}}}g(\tilde{z})}{g(\tilde{z})} = cosh_{\hat{\hat{q}}}(\hat{\omega}(\tilde{z})), \quad \tilde{z} \in \lambda. \tag{4.6}$$

where

$$\hat{\omega}(\tilde{z}) = \frac{\breve{p}(\tilde{z})-1}{1+\breve{p}(\tilde{z})}.$$

If$\breve{p}(\tilde{z})$ follows the form of (2.3), then

$$\hat{\omega}(\tilde{z}) = \frac{c_1\tilde{z}+c_2\tilde{z}^2+c_3\tilde{z}^3+...}{2+c_1\tilde{z}+c_2\tilde{z}^2+c_3\tilde{z}^3+...},$$

$$cosh_{\hat{\hat{q}}}(\hat{\omega}(\tilde{z})) = cosh_{\hat{\hat{q}}}\left(\frac{c_1\tilde{z}+c_2\tilde{z}^2+c_3\tilde{z}^3+...}{2+c_1\tilde{z}+c_2\tilde{z}^2+c_3\tilde{z}^3+...}\right).$$

Let

$$M = \frac{c_1\tilde{z}+c_2\tilde{z}^2+c_3\tilde{z}^3+...}{2+c_1\tilde{z}+c_2\tilde{z}^2+c_3\tilde{z}^3+...}.$$

Then,

$$cosh_{\hat{\hat{q}}}(\hat{\omega}(\tilde{z})) = 1+\frac{M^2}{[2]_{\hat{\hat{q}}}!}+\frac{M^4}{[4]_{\hat{\hat{q}}}!}+...,$$

$$= 1+\frac{1}{[2]_{\hat{\hat{q}}}!}\left[\frac{c_1\tilde{z}+c_2\tilde{z}^2+c_3\tilde{z}^3+...}{2+c_1\tilde{z}+c_2\tilde{z}^2+c_3\tilde{z}^3+...}\right]^2+\frac{1}{[4]_{\hat{\hat{q}}}!}\left[\frac{c_1\tilde{z}+c_2\tilde{z}^2+c_3\tilde{z}^3+...}{2+c_1\tilde{z}+c_2\tilde{z}^2+c_3\tilde{z}^3+...}\right]^4+...,$$

$$= 1+\frac{1}{[2]_{\hat{\hat{q}}}!}\left[\frac{c_1\tilde{z}+c_2\tilde{z}^2+c_3\tilde{z}^3+...}{2(1+\frac{c_1\tilde{z}}{2}+\frac{c_2\tilde{z}^2}{2}+...)}\right]^2+\frac{1}{[4]_{\hat{\hat{q}}}!}\left[\frac{c_1\tilde{z}+c_2\tilde{z}^2+c_3\tilde{z}^3+...}{2(1+\frac{c_1\tilde{z}}{2}+\frac{c_2\tilde{z}^2}{2}+...)}\right]^4+...,$$

$$= 1+\frac{1}{4[2]_{\hat{\hat{q}}}!}\left[(c_1\tilde{z}+c_2\tilde{z}^2+c_3\tilde{z}^3+...)\left(1+\frac{c_1\tilde{z}}{2}+\frac{c_2\tilde{z}^2}{2}+...\right)^{-1}\right]^2+$$

$$\frac{1}{16[4]_{\hat{\hat{q}}}!}\left[(c_1\tilde{z}+c_2\tilde{z}^2+c_3\tilde{z}^3+...)\left(1+\frac{c_1\tilde{z}}{2}+\frac{c_2\tilde{z}^2}{2}+...\right)^{-1}\right]^4+...,$$

$$= 1+\frac{1}{4[2]_{\hat{q}}!}\left[(c_1\tilde{z}+c_2\tilde{z}^2+c_3\tilde{z}^3+\ldots)\left(1-\frac{c_1\tilde{z}}{2}-\frac{c_2\tilde{z}^2}{2}-\frac{c_3\tilde{z}^3}{2}+\frac{c_1\tilde{z}^2}{4}+\frac{c_2^2\tilde{z}^4}{4}+\frac{c_3^2\tilde{z}^6}{4}+\right.\right.$$

$$\left.\left.2\left(\frac{c_1\tilde{z}}{2}\right)\left(\frac{(c_2\tilde{z}^2}{2}\right)+2\left(\frac{c_1\tilde{z}}{2}\right)\left(\frac{c_3\tilde{z}^3}{2}\right)+2\left(\frac{c_2\tilde{z}^2}{2}\right)\left(\frac{c_3\tilde{z}^3}{2}\right)+\ldots\right)\right]^2+\ldots,$$

$$= 1+\frac{1}{4[2]_{\hat{q}}!}\left[c_1\tilde{z}-\frac{c_1^2\tilde{z}^2}{2}-\frac{c_1c_2\tilde{z}^3}{2}+\frac{c_1^3\tilde{z}^3}{4}+c_2\tilde{z}^2-\frac{c_1c_2\tilde{z}^3}{2}+c_3\tilde{z}^3+\ldots\right]^2+\frac{c_1^4\tilde{z}^4}{16[4]_{\hat{q}}!}+\ldots,$$

$$= 1+\frac{1}{4[2]_{\hat{q}}!}\left[c_1^2\tilde{z}^2+2c_1c_2\tilde{z}^3-c_1^3\tilde{z}^3+\frac{c_1^4\tilde{z}^4}{4}+\frac{c_1^4\tilde{z}^4}{2}-3c_1^2c_2\tilde{z}^4+c_2^2\tilde{z}^4+2c_1c_3\tilde{z}^4+\ldots\right]+\frac{c_1^4\tilde{z}^4}{16[4]_{\hat{q}}!}+\ldots,$$

Now, taking left hand side of (4.5), gives

$$\frac{\tilde{z}D_{\hat{q}}g(\tilde{z})}{g(\tilde{z})}=\frac{\tilde{z}(\tilde{z}+\grave{a}_2\tilde{z}^2+\grave{a}_3\tilde{z}^3+\ldots)-(\hat{q}\tilde{z}+\hat{q}\grave{a}_2\tilde{z}^2+\hat{q}\grave{a}_3\tilde{z}^3+\ldots)}{\tilde{z}(1-\hat{q})(\tilde{z}+\grave{a}_2\tilde{z}^2+\grave{a}_3\tilde{z}^3+\ldots)},$$

$$\frac{\tilde{z}D_{\hat{q}}g(\tilde{z})}{g(\tilde{z})}=\grave{a}_2\hat{q}\tilde{z}+1+\left[-\hat{q}\grave{a}_2^2+\hat{q}(1+\hat{q})\grave{a}_3\right]\tilde{z}^2+\left[-(2+\hat{q})\hat{q}\grave{a}_2\grave{a}_3+\hat{q}\grave{a}_2^3+(1+\hat{q}+\hat{q}^2)\hat{q}\grave{a}_4\right]\tilde{z}^3$$

$$+\hat{q}\left[\grave{a}_5\hat{q}^3+(-\grave{a}_2\grave{a}_4+\grave{a}_5)\hat{q}^2+(-\grave{a}_3^2+\grave{a}_3\grave{a}_2^2-\grave{a}_2\grave{a}_4+\grave{a}_5)\hat{q}+3\grave{a}_3\grave{a}_2^2-2\grave{a}_2\grave{a}_4-\grave{a}_2^4-\grave{a}_3^2+\grave{a}_5\right]\tilde{z}^4+\ldots,$$

By matching coefficients of $\tilde{z},\tilde{z}^2,\tilde{z}^3 and \tilde{z}^4$, , we get

$$\grave{a}_2=0, \tag{4.7}$$

$$\grave{a}_3=\frac{c_1^2}{4\hat{q}(1+\hat{q})[2]_{\hat{q}}!}, \tag{4.8}$$

$$\grave{a}_4=\frac{c_1(2c_2-c_1^2)}{4\hat{q}(1+\hat{q}+\hat{q}^2)[2]_{\hat{q}}!}, \tag{4.9}$$

$$\grave{a}_5=\frac{1}{16\hat{q}^2(\hat{q}^4+2\hat{q}^3+2\hat{q}^2+2\hat{q}+1)[2]_{\hat{q}}!^2[4]_{\hat{q}}!}\Big(\big([2]_{\hat{q}}!\hat{q}^2([2]_{\hat{q}}!+3[4]_{\hat{q}}!)+[2]_{\hat{q}}!\hat{q}([2]_{\hat{q}}!$$

$$+3[4]_{\hat{q}}!)+[4]_{\hat{q}}!\big)c_1^4-12[2]_{\hat{q}}![4]_{\hat{q}}!\hat{q}(1+\hat{q})c_1^2c_2+8[2]_{\hat{q}}![4]_{\hat{q}}!\hat{q}(1+\hat{q})c_1c_3+4[2]_{\hat{q}}![4]_{\hat{q}}!\hat{q}(1+\hat{q})c_2^2\Big). \tag{4.10}$$

Using Lemma 2.16.2 to (4.8), we get

$$|\grave{a}_3|\leq\frac{1}{\hat{q}(1+\hat{q})^2}. \tag{4.11}$$

Using Lemma 2.16.3 to (4.9) with $\upsilon=2$, which gives

$$|\grave{a}_4|\leq\frac{2}{\hat{q}(1+\hat{q}+\hat{q}^2)(1+\hat{q})}. \tag{4.12}$$

Now, applying Lemma 2.16.1 to in (4.10) and let $c_1 = c, |\alpha| = t$, gives us

$$|\grave{a}_5| = \left|\frac{1}{16\hat{\hat{q}}^2(\hat{\hat{q}}^4+2\hat{\hat{q}}^3+2\hat{\hat{q}}^2+2\hat{\hat{q}}+1)[2]_{\hat{\hat{q}}}!^2[4]_{\hat{\hat{q}}}!}\right| \Big| \big[([2]^2_{\hat{\hat{q}}}!\hat{\hat{q}}([2]_{\hat{\hat{q}}}!+3[4]_{\hat{\hat{q}}}!)c_1^4+2c_1^2t^2(4-c_1^2)$$

$$[2]_{\hat{\hat{q}}}![4]_{\hat{\hat{q}}}!\hat{\hat{q}}(1+\hat{\hat{q}})+4c_1(4-c_1^2)[2]_{\hat{\hat{q}}}![4]_{\hat{\hat{q}}}!\hat{\hat{q}}(1+\hat{\hat{q}})+[2]_{\hat{\hat{q}}}![4]_{\hat{\hat{q}}}!\hat{\hat{q}}(1+\hat{\hat{q}})(4-c_1^2)^2t^2\big]\Big| := \chi_q(c,t)$$

Upon differentiating, we get

$$\frac{\partial\chi_q}{\partial t}(c,t) = \frac{1}{16\hat{\hat{q}}^2[2]_{\hat{\hat{q}}}!^2(\hat{\hat{q}}^4+2\hat{\hat{q}}^3+2\hat{\hat{q}}^2+2\hat{\hat{q}}+1)[4]_{\hat{\hat{q}}}!}\big[4[4]_{\hat{\hat{q}}}!\hat{\hat{q}}(1+\hat{\hat{q}})[2]_{\hat{\hat{q}}}!(4-c^2)c^2t$$
$$+2[2]_{\hat{\hat{q}}}![4]_{\hat{\hat{q}}}!\hat{\hat{q}}(1+\hat{\hat{q}})(4-c^2)^2t\big] > 0.$$

It implies that $\frac{\partial\chi_q}{\partial t}(c,t)$ is an increasing [0,1].So,

$$max(\chi_q(c,t)) = \chi_q(c,1) = \frac{1}{16\hat{\hat{q}}^2[2]_{\hat{\hat{q}}}!^2(\hat{\hat{q}}^4+2\hat{\hat{q}}^3+2\hat{\hat{q}}^2+2\hat{\hat{q}}+1)[4]_{\hat{\hat{q}}}!}\big[([2]^2_{\hat{\hat{q}}}!\hat{\hat{q}}([2]_{\hat{\hat{q}}}!+3[4]_{\hat{\hat{q}}}!)c^4$$

$$+2c^2(4-c^2)[2]_{\hat{\hat{q}}}![4]_{\hat{\hat{q}}}!\hat{\hat{q}}(1+\hat{\hat{q}})+4c(4-c^2)[2]_{\hat{\hat{q}}}![4]_{\hat{\hat{q}}}!\hat{\hat{q}}(1+\hat{\hat{q}})+[2]_{\hat{\hat{q}}}![4]_{\hat{\hat{q}}}!\hat{\hat{q}}(1+\hat{\hat{q}})(4-c^2)^2\big];=\xi(c)$$

. By differentiating w.r.t 'c', we get

$$\xi'(c) = \frac{1}{16\hat{\hat{q}}^2(\hat{\hat{q}}^4+2\hat{\hat{q}}^3+2\hat{\hat{q}}^2+2\hat{\hat{q}}+1)[2]_{\hat{\hat{q}}}!^2[4]_{\hat{\hat{q}}}!}\big[(4[2]^2_{\hat{\hat{q}}}!\hat{\hat{q}}(1+\hat{\hat{q}})+4[4]_{\hat{\hat{q}}}!-4[2]_{\hat{\hat{q}}}![4]_{\hat{\hat{q}}}!\hat{\hat{q}}(1+\hat{\hat{q}}))$$
$$c^3-12[2]_{\hat{\hat{q}}}![4]_{\hat{\hat{q}}}!\hat{\hat{q}}(1+\hat{\hat{q}})^2+16[2]_{\hat{\hat{q}}}![4]_{\hat{\hat{q}}}!\hat{\hat{q}}(1+\hat{\hat{q}})\big].$$

Calculations shows that $\xi'(c) > 0$ for $c \in [0,1]$ and $\xi'(c) < 0$ for $c \in [1.1,2]$. This indicates that $\xi(c)$ is increasing in $c \in [0,1]$ and decreasing in $c \in [1.1,2]$ for $q \in [0,1]$. So,

$$|\grave{a}_5| \le \xi(1) = \frac{27\hat{\hat{q}}^7+81\hat{\hat{q}}^6+135\hat{\hat{q}}^5+163\hat{\hat{q}}^4+136\hat{\hat{q}}^3+84\hat{\hat{q}}^2+29\hat{\hat{q}}+1}{16\hat{\hat{q}}^2(1+\hat{\hat{q}}^2)^2(1+\hat{\hat{q}}+\hat{\hat{q}}^2)(1+\hat{\hat{q}})^4}.$$

we get the needed result. □

When $\hat{\hat{q}}$ approaches 1^-, the result above simplifies to the following:

Corollary 4.2.1.1. If $g \in S^*_{cosh}$ has the series form as given in (2.1). Then,

$$|\grave{a}_3| \le \frac{1}{4}, |\grave{a}_4| \le \frac{1}{3}, |\grave{a}_5| \le \frac{41}{192}. \tag{4.13}$$

4.3 Fekete-Szegö Inequality

This inequality is investigated for the class $S^*_{\hat{\hat{q}}cosh}$.

Theorem 4.3.1. If the series representation of $g \in S^*_{\hat{\hat{q}}cosh}$ is as described in (2.1). Then

$$|\grave{a}_3 - \grave{a}_2^2| \leq \frac{1}{\hat{\hat{q}}(1+\hat{\hat{q}})^2}. \tag{4.14}$$

Proof. From (4.16) and (4.17),

$$|\grave{a}_3 - \grave{a}_2^2| = \left| \frac{c_1^2}{4[2]_{\hat{\hat{q}}}!\hat{\hat{q}}(1+\hat{\hat{q}})} \right|.$$

Using Lemma 2.16.2, we have

$$|\grave{a}_3 - \grave{a}_2^2| \leq \frac{1}{\hat{\hat{q}}(1+\hat{\hat{q}})^2}.$$

□

When $\hat{\hat{q}}$ approaches 1^-, the result above simplifies to the following:

Corollary 4.3.1.1. If the series representation of $g \in S^*_{cosh}$ is as described in (2.1). Then

$$|\grave{a}_3 - \grave{a}_2^2| \leq \frac{1}{4}. \tag{4.15}$$

Theorem 4.3.2. If the series representation of $g \in S^*_{\hat{\hat{q}}cosh}$ is as described in (2.1). Then

$$|\grave{a}_4 - \grave{a}_3\grave{a}_2| \leq \frac{2}{\hat{\hat{q}}(1+\hat{\hat{q}})(1+\hat{\hat{q}}+\hat{\hat{q}}^2)}. \tag{4.16}$$

Proof. From (4.16), (4.17), (4.18),

$$|\grave{a}_4 - \grave{a}_3\grave{a}_2| = \frac{|c_1||(2c_2 - c_1^2)|}{4[2]_{\hat{\hat{q}}}!\hat{\hat{q}}(1+\hat{\hat{q}}+\hat{\hat{q}}^2)}.$$

Using Lemma 2.16.2 and 2.16.3 with $\upsilon = 2$,

$$|\grave{a}_4 - \grave{a}_3\grave{a}_2| \leq \frac{2}{\hat{\hat{q}}(1+\hat{\hat{q}}+\hat{\hat{q}}^2)(1+\hat{\hat{q}})}.$$

□

When $\hat{\hat{q}}$ approaches 1^-, the result above simplifies to the following:

Corollary 4.3.2.1. If $g \in S^*_{cosh}$ has the series form as given in (2.1). Then,

$$|\grave{a}_4 - \grave{a}_3\grave{a}_2| \leq \frac{1}{3}. \tag{4.17}$$

4.4 Zalcman Functional

Theorem 4.4.1. If $g \in S^{*}_{\hat{\hat{q}}cosh}$ is expressed in the series form as shown in (2.1), then,

$$|\grave{a}_5 - \grave{a}_3^2| \leq 0.2130. \tag{4.18}$$

Proof. From (4.17) and (4.20),

$$\grave{a}_5 - \grave{a}_3^2 = \frac{1}{16\hat{q}[2]_{\hat{q}}!^2(\hat{q}^4+2\hat{q}^3+2\hat{q}^2+2\hat{q}+1)[4]_{\hat{q}}!}\Bigg((3\hat{q}^9+14\hat{q}^8+33\hat{q}^7+51\hat{q}^6+57\hat{q}^5 + 48\hat{q}^4+30\hat{q}^3+12\hat{q}^2+\hat{q}-1)c_1^4 - 12[2]_{\hat{q}}![4]_{\hat{q}}!\hat{q}(1+\hat{q})c_1^2c_2 + 8[2]_{\hat{q}}![4]_{\hat{q}}!\hat{q}(1+\hat{q})c_1c_3 + 4[2]_{\hat{q}}![4]_{\hat{q}}!\hat{q}(1+\hat{q})c_2^2\Bigg).$$

Applying Lemma 2.16.1, letting $c_1 = c$ and $|\alpha| = t$, we have

$$|\grave{a}_5 - \grave{a}_3^2| \leq \left|\frac{1}{16\hat{q}[2]_{\hat{q}}!^2(\hat{q}^4+2\hat{q}^3+2\hat{q}^2+2\hat{q}+1)[4]_{\hat{q}}!}\right|\Big|(q^8+3\hat{q}^7+6\hat{q}^6+9\hat{q}^5+9\hat{q}^4+3\hat{q}^3+3\hat{q}^2 + \hat{q}+1)c^4 + 2[2]_{\hat{q}}![4]_{\hat{q}}!\hat{q}(1+\hat{q})(4-c^2)c^2t^2 + 4[2]_{\hat{q}}![4]_{\hat{q}}!\hat{q}(1+\hat{q})(4-c^2)c + [2]_{\hat{q}}![4]_{\hat{q}}!\hat{q}(1+\hat{q})(4-c^2)^2t^2)\Big| := \psi_{\hat{q}}(c,t).$$

Differentiating with respect to 't', it is clearly seen that $\frac{\partial \psi_{\hat{q}}}{\partial t}>0$, which indicates $\psi_{\hat{q}(c,t)}$ is increasing in $[0,1]$.

$$max(\psi_{\hat{q}}(c,t)) = (\psi_{\hat{q}}(c,1)) = \frac{1}{16\hat{q}[2]_{\hat{q}}!^2(\hat{q}^4+2\hat{q}^3+2\hat{q}^2+2\hat{q}+1)[4]_{\hat{q}}!}\Bigg((\hat{q}^8+3\hat{q}^7+6\hat{q}^6 + 9\hat{q}^5+9\hat{q}^4+3\hat{q}^3+3\hat{q}^2+\hat{q}+1)c^4 + 2[2]_{\hat{q}}![4]_{\hat{q}}!\hat{q}(1+\hat{q})(4-c^2)c^2 + 4[2]_{\hat{q}}![4]_{\hat{q}}!\hat{q}(1+\hat{q}) (4-c^2)c + [2]_{\hat{q}}![4]_{\hat{q}}!\hat{q}(1+\hat{q})(4-c^2)^2)\Bigg) := \Upsilon_{\hat{q}}(c).$$

By Differentiating w.r.t 'c',

$$\Upsilon'_{\hat{q}}(c) = \frac{1}{16\hat{q}[2]_{\hat{q}}!^2(\hat{q}^4+2\hat{q}^3+2\hat{q}^2+2\hat{q}+1)[4]_{\hat{q}}!}(Ac^3+Bc^2+C),$$

where

$A = -4\hat{q}^9 - 19\hat{q}^8 - 45\hat{q}^7 - 70\hat{q}^6 - 79\hat{q}^5 - 67\hat{q}^4 - 29\hat{q}^3 - 17\hat{q}^2 - 19.$

$B = -12[2]_{\hat{q}}![4]_{\hat{q}}!\hat{q}(1+\hat{q}).$

$C = 16[2]_{\hat{q}}![4]_{\hat{q}}!\hat{q}(1+\hat{q}).$

Certain calculations show that $\Upsilon'_{\hat{\hat{q}}}(c) \geq 0$ for $c \in [0,1]$ and $\Upsilon'_{\hat{\hat{q}}}(c) \leq 0$ for $c \in (1,2]$. This implies that $\Upsilon_{\hat{\hat{q}}}(c)$ is increasing in [0,1] and $\Upsilon_{\hat{\hat{q}}}(c)$ is decreasing in (1,2).

$$\Upsilon_{\hat{\hat{q}}}(c) \leq \Upsilon_{\hat{\hat{q}}}(1) = 0.2130.$$

Consequently, we get

$$|\grave{a}_5 - \grave{a}_3^2| \leq 0.2130$$

which is the needed result. □

4.5 Hankel Determinants

Theorem 4.5.1. If $g \in S^*_{\hat{\hat{q}}cosh}$ has the series form as given in (2.1). Then,

$$|H_{2,1}(g)| \leq \frac{1}{\hat{\hat{q}}(1+\hat{\hat{q}})^2}. \tag{4.19}$$

Proof. As

$$H_{2,1}(g) = \grave{a}_3 - \grave{a}_2^2.$$

Using (4.7) and (4.8), we get

$$\grave{a}_3 - \grave{a}_2^2 = \frac{c_1^2}{4[2]_{\hat{\hat{q}}}!\hat{\hat{q}}(1+\hat{\hat{q}})}. \tag{4.20}$$

Applying Lemma 2.16.2 to (4.20),

$$|H_{2,1}(g)| \leq \frac{1}{\hat{\hat{q}}(1+\hat{\hat{q}})^2}.$$

which is the needed result. □

When $\hat{\hat{q}}$ approaches 1^-, the result above simplifies to the following:

Corollary 4.5.1.1. If $g \in S^*_{cosh}$ has the series form as given in (2.1). Then,

$$|H_{2,1}(g)| \leq \frac{1}{4}. \tag{4.21}$$

Theorem 4.5.2. If $g \in S^*_{cosh}$ is expressed in the series form provided in (2.1), then,

$$|H_{2,2}(g)| \leq \frac{1}{\hat{\hat{q}}^2(1+\hat{\hat{q}})^4}. \tag{4.22}$$

Proof. As

$$H_{2,2}(g) = \grave{a}_2\grave{a}_4 - \grave{a}_3^2. \tag{4.23}$$

Using (4.7),(4.8) and (4.9), we get

$$\grave{a}_2\grave{a}_4 - \grave{a}_3^2 = 0 - \left[\frac{c_1^2}{4[2]_{\hat{\hat{q}}}!\hat{\hat{q}}(1+\hat{\hat{q}})}\right]^2,$$

,

$$\grave{a}_2\grave{a}_4 - \grave{a}_3^2 = -\frac{c_1^4}{16[2]_{\hat{\hat{q}}}!^2\hat{\hat{q}}^2(1+\hat{\hat{q}})^2}. \tag{4.24}$$

. Applying Lemma 2.16.2 to (4.24) and by taking modulus, we get

$$|H_{2,2}(g)| \leq \frac{1}{\hat{\hat{q}}^2(1+\hat{\hat{q}})^4}.$$

which is the needed result. □

When $\hat{\hat{q}}$ approaches 1^-, the result above simplifies to the following:

Corollary 4.5.2.1. If $g \in S^*_{cosh}$ has the series form as given in (2.1). Then,

$$|H_{2,2}(g)| \leq \frac{1}{16}. \tag{4.25}$$

Theorem 4.5.3. If $g \in S^*_{\hat{\hat{q}}cosh}$ has the series form as given in (2.1). Then,

$$\begin{aligned} |H_{3,1}| \leq{} & \frac{\hat{\hat{q}}^7 + 5\hat{\hat{q}}^6 + 7\hat{\hat{q}}^5 + 10\hat{\hat{q}}^4 + 12\hat{\hat{q}}^3 + 9\hat{\hat{q}}^2 + 5\hat{\hat{q}} + 3}{64\hat{\hat{q}}^2\hat{\hat{q}}^4 + 2\hat{\hat{q}}^3 + 2\hat{\hat{q}}^2 + 2\hat{\hat{q}} + 1)[2]_{\hat{\hat{q}}}!^2[4]_{\hat{\hat{q}}}!(1+\hat{\hat{q}}+\hat{\hat{q}}^2)}(1.4)^6 - \\ & \frac{1}{16\hat{\hat{q}}^2[2]_{\hat{\hat{q}}}!^2(\hat{\hat{q}}^4 + 2\hat{\hat{q}}^3 + 2\hat{\hat{q}}^2 + 2\hat{\hat{q}} + 1)}(1.4)^5 - \\ & \frac{\hat{\hat{q}}^4 + 2\hat{\hat{q}}^3 + \hat{\hat{q}}^2 + 2\hat{\hat{q}} + 1}{4\hat{\hat{q}}^2[2]_{\hat{\hat{q}}}!^2(\hat{\hat{q}}^4 + 2\hat{\hat{q}}^3 + 2\hat{\hat{q}}^2 + 2\hat{\hat{q}} + 1)(1+\hat{\hat{q}}+\hat{\hat{q}}^2)^2}(1.4)^4 + \\ & \frac{1}{4\hat{\hat{q}}^2[2]_{\hat{\hat{q}}}!^2(\hat{\hat{q}}^4 + 2\hat{\hat{q}}^3 + 2\hat{\hat{q}}^2 + 2\hat{\hat{q}} + 1)}(1.4)^3 + \\ & \frac{3\hat{\hat{q}}^4 + 6\hat{\hat{q}}^3 + 5\hat{\hat{q}}^2 + 6\hat{\hat{q}} + 3}{4\hat{\hat{q}}^2[2]_{\hat{\hat{q}}}!^2(\hat{\hat{q}}^4 + 2\hat{\hat{q}}^3 + 2\hat{\hat{q}}^2 + 2\hat{\hat{q}} + 1)(1+\hat{\hat{q}}+\hat{\hat{q}}^2)^2}(1.4)^2. \end{aligned} \tag{4.26}$$

Proof. By (2.20), we get

$$H_{3,1}(g) = \grave{a}_5\grave{a}_3 - \grave{a}_4^2 - \grave{a}_3^3. \tag{4.27}$$

$$H_{3,1}(g) = -\frac{\hat{q}^7+5\hat{q}^6+7\hat{q}^5+10\hat{q}^4+12\hat{q}^3+9\hat{q}^2+5\hat{q}+3}{64\hat{q}(\hat{q}^4+2\hat{q}^3+2\hat{q}^2+2\hat{q}+1)[2]_{\hat{q}}!^2[4]_{\hat{q}}!(1+\hat{q}+\hat{q}^2)}c_1^6$$
$$+\frac{\hat{q}^4+2\hat{q}^3-\hat{q}^2+2\hat{q}+1}{16\hat{q}^2(\hat{q}^4+2\hat{q}^3+2\hat{q}^2+2\hat{q}+1)[2]_{\hat{q}}!^2(1+\hat{q}+\hat{q}^2)^2}c_1^4c_2$$
$$+\frac{1}{8\hat{q}^2[2]_{\hat{q}}!^2(\hat{q}^4+2\hat{q}^3+2\hat{q}^2+2\hat{q}+1)}c_1^3c_3$$
$$-\frac{3\hat{q}^4+6\hat{q}^3+5\hat{q}^2+6\hat{q}+3}{16\hat{q}^2[2]_{\hat{q}}!^2(\hat{q}^4+2\hat{q}^3+2\hat{q}^2+2\hat{q}+1)(1+\hat{q}+\hat{q}^2)^2}c_1^2c_2^2.$$

Using Lemma 2.16.1, let $c_1 = c$ and $|\alpha| = t$, gives us

$$|H_{3,1}(g)| = \Bigg|\frac{(\hat{q}^7+2\hat{q}^6+4\hat{q}^5+4\hat{q}^4+3\hat{q}^3-\hat{q}-1)}{64\hat{q}^2(\hat{q}^4+2\hat{q}^3+2\hat{q}^2+2\hat{q}+1)[2]_{\hat{q}}!^2[4]_{\hat{q}}!(1+\hat{q}+\hat{q}^2)}c^6 -$$
$$\frac{(4-c^2)c^4t^2}{32[2]_{\hat{q}}!^2\hat{q}^2(\hat{q}^4+2\hat{q}^3+2\hat{q}^2+2\hat{q}+1)} -$$
$$\frac{(3\hat{q}^4+6\hat{q}^3+5\hat{q}^2+6\hat{q}+3)(4-c^2)^2c^2t^2}{64[2]_{\hat{q}}!^2\hat{q}^2(\hat{q}^4+2\hat{q}^3+2\hat{q}^2+2\hat{q}+1)(1+\hat{q}+\hat{q}^2)^2} +$$
$$\frac{(4-c^2)c^3}{16[2]_{\hat{q}}!^2\hat{q}^2(\hat{q}^4+2\hat{q}^3+2\hat{q}^2+2\hat{q}+1)}\Bigg| := \Delta(c,t).$$

By differentiating w.r.t 't', it is clearly seen that $\frac{\partial\hat{\Delta}}{\partial t} > 0$. Hence, $\hat{\Delta}(c,t)$ is increasing in $[0,1]$.

$$max(\hat{\Delta}(c,t)) = \hat{\Delta}(c,1) = \frac{(\hat{q}^7+2\hat{q}^6+4\hat{q}^5+4\hat{q}^4+3\hat{q}^3-\hat{q}-1)}{64\hat{q}^2(\hat{q}^4+2\hat{q}^3+2\hat{q}^2+2\hat{q}+1)[2]_{\hat{q}}!^2[4]_{\hat{q}}!(1+\hat{q}+\hat{q}^2)}c^6 -$$
$$\frac{(4-c^2)c^4}{32\hat{q}^2(\hat{q}^4+2\hat{q}^3+2\hat{q}^2+2\hat{q}+1)[2]_{\hat{q}}!^2} - \frac{(3\hat{q}^4+6\hat{q}^3+5\hat{q}^2+6\hat{q}+3)(4-c^2)^2c^2}{64\hat{q}^2(\hat{q}^4+2\hat{q}^3+2\hat{q}^2+2\hat{q}+1)[2]_{\hat{q}}!^2(1+\hat{q}+\hat{q}^2)^2} +$$
$$\frac{(4-c^2)c^3}{16\hat{q}^2(\hat{q}^4+2\hat{q}^3+2\hat{q}^2+2\hat{q}+1)[2]_{\hat{q}}!^2} := \rho(c).$$

By differentiating w.r.t 'c', certain calculations show that $\rho'_{\hat{q}}(c) \geq 0$ for $c \in [0,1.4]$ and $\rho'_{\hat{q}}(c) \leq 0$ for $c \in [1.5,2]$. This implies that $\rho_{\hat{q}}(c)$ is increasing in [0,1.4] and $\rho_{\hat{q}}(c)$ is decreasing in (1.5,2).

$$
\begin{aligned}
|H_{3,1}| \leq \rho(1.4) = &\frac{\hat{\hat{q}}^7+5\hat{\hat{q}}^6+7\hat{\hat{q}}^5+10\hat{\hat{q}}^4+12\hat{\hat{q}}^3+9\hat{\hat{q}}^2+5\hat{\hat{q}}+3}{64\hat{\hat{q}}^2[2]_{\hat{\hat{q}}}!^2[4]_{\hat{\hat{q}}}!(\hat{\hat{q}}^4+2\hat{\hat{q}}^3+2\hat{\hat{q}}^2+2\hat{\hat{q}}+1)(1+\hat{\hat{q}}+\hat{\hat{q}}^2)}(1.4)^6- \\
&\frac{1}{16\hat{\hat{q}}^2[2]_{\hat{\hat{q}}}!^2(\hat{\hat{q}}^4+2\hat{\hat{q}}^3+2\hat{\hat{q}}^2+2\hat{\hat{q}}+1)}(1.4)^5- \\
&\frac{\hat{\hat{q}}^4+2\hat{\hat{q}}^3+\hat{\hat{q}}^2+2\hat{\hat{q}}+1}{4[2]_{\hat{\hat{q}}}!^2\hat{\hat{q}}^2(\hat{\hat{q}}^4+2\hat{\hat{q}}^3+2\hat{\hat{q}}^2+2\hat{\hat{q}}+1)(1+\hat{\hat{q}}+\hat{\hat{q}}^2)^2}(1.4)^4+ \\
&\frac{1}{4[2]_{\hat{\hat{q}}}!^2\hat{\hat{q}}^2(\hat{\hat{q}}^4+2\hat{\hat{q}}^3+2\hat{\hat{q}}^2+2\hat{\hat{q}}+1)}(1.4)^3+ \\
&\frac{3\hat{\hat{q}}^4+6\hat{\hat{q}}^3+5\hat{\hat{q}}^2+6\hat{\hat{q}}+3}{4[2]_{\hat{\hat{q}}}!^2\hat{\hat{q}}^2(\hat{\hat{q}}^4+2\hat{\hat{q}}^3+2\hat{\hat{q}}^2+2\hat{\hat{q}}+1)(1+\hat{\hat{q}}+\hat{\hat{q}}^2)^2}(1.4)^2.
\end{aligned}
$$

which is the needed result. □

When $\hat{\hat{q}}$ approaches 1^-, the result above simplifies to the following:

Corollary 4.5.3.1. If $g \in S^*_{cosh}$ has the series form as given in (2.1). Then,

$$|H_{3,1}| \leq 0.0293. \tag{4.28}$$

CHAPTER 5

CONCLUSION AND FUTURE WORK

5.1 Conclusion

By smoothly integrating classical principles with cutting-edge advancement in Quantum Calculus, this thesis not only advances our understanding of analytic functions but also sets the stage for future investigation and application in mathematical analysis. A key part of this research is to examine the classes of analytic functions by utilizing the tools and techniques of Quantum calculus.The study begins by introducing essential definitions, preliminary results and unfolds coefficient bounds of functions as well.

The focus of this research is to first examine and then extend the class of starlike function associated with cosine hyperbolic function.Introducing the class $S^*_{\hat{\hat{q}}cosh}$, which include starlike functions subordinated to $\hat{\hat{q}}-$cosine hyperbolic functions. This extended class was established using $\hat{\hat{q}}-$ derivative operator also executed subordination technique to briefly examine their properties.We have identified several key characteristics within the newly defined class, coefficient limits, Zalcman functional and renowned Fekete-Szegö problem. Additionally, this research expanded to examine Hankel Determinants for this innovative class.

To ensure the accuracy of our results, we approach limit as $\hat{\hat{q}} \to 1^-$, which confirmed the alignment of our results with known outcomes.Our analysis demonstrates that the new class offer a refined perspective compared to previously developed ones. The results we have acquired show notable advancements beyond earlier theorems presented by researchers. Our research combines traditional principles with modern developments in Quantum Calculus to push the boundaries of current understanding in this field. The findings not only elevate the theoretical structure but also addition to the practical applications of these mathematical concepts.

5.2 Future Work

Building upon the findings revealed in this thesis, various exciting path for future research emerge. One promising direction is the exploration of additional function classes that develop the concepts of starlike function further into the realm of $\hat{\hat{q}}-$calculus. This could involve developing new sub-classes and examining their unique properties, especially in relation to other trigonometric functions. Results for the refined classes could be explored.

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