On a New Class of q̃-Starlike Function Associated with Sine Inverse Hyperbolic Function

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Candidate of Master of Science in Mathematics at the National University of Modern Languages do hereby declare that the thesis On a new class of q̃-Starlike Function Associated with Sine Inverse Hyperbolic Function submitted by me in partial fulfillment of MSMA degree, is my original work and has not been submitted or published earlier. I also solemnly declare that it shall not, in the future, be submitted by me for obtaining any other degree from this or any other university or institution. I also understand that if evidence of plagiarism is found in my thesis/dissertation at any stage, even after the award of a degree, the work may be cancelled and the degree revoked.

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ABSTRACT

Title: On a new class of q̃-Starlike Function Associated with Sine Inverse Hyperbolic Function

The aim of this thesis is to present and describe new subclass of univalent functions within the open unit disk. The q-extension of the starlike functions subordinated to inverse sine hyperbolic function will be determined by applying q-calculus. Furthermore, we will look into noteworthy properties, including coefficient bounds, the Fekete-Szeg*ö* inequality, and the Zalcman functional. The upper bounds on Hankel Determinants for functions belong to this newly defined class will also be explored. It will be demonstrated that recently acquired results are more advanced than those previously obtained by a large number of researchers in the field of Geometric Function Theory. The special cases of newly derived results will be presented in the form of corollaries.

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Ψ	-	Open unit disk
$\mathbb{A}$	-	Class of Analytic functions
$\mathbb{S}$	-	Class of Univalent functions
$\mathbb{C}$	-	Class of Convex functions
$\mathbb{K}$	-	Class of Close-to-Convex functions
$\mathbb{P}$	-	Class of Carathéodory functions
$\mathbb{S}^*$	-	Class of Starlike functions
$\prec$	-	Subordination symbol
$\mathbb{S}^*_{\rho\tilde{q}_e}$	-	Class of $\tilde{q}_e$-Starlike functions Subordinated with $\tilde{q}_e$-sine inverse hyperbolic function
$\mathbb{D}_{\tilde{q}_e}$	-	q-Derivative operator symbol
$\mathbb{H}$	-	Hankel Determinant symbol

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In the name of Allah, the Most Gracious and Most Merciful, I begin this acknowledgment with the verse, " You Alone we Worship; You Alone we ask for Help." I am deeply grateful to Allah, the Most Wise, for His countless blessings that have guided me through this academic journey. I am reminded of the Hadith of the Prophet Muhammad (peace be upon him) who said, "Seek knowledge from the cradle to the grave."

I would like to express my deep appreciation to my family for their unwavering support, to my teachers for their invaluable guidance, for it is through your collective encouragement and inspiration that I stand here today, humbly acknowledging the blessings and opportunities Allah has bestowed upon me.

May Allah accept our efforts and guide us on the path of righteousness and wisdom.

DEDICATION

This thesis work is dedicated to my parents, family, and my teachers throughout my education career who have not only loved me unconditionally but whose good examples have taught me to work hard for the things that I aspire to achieve.

CHAPTER 1

INTRODUCTION AND LITERATURE REVIEW

1.1 Overview

This chapter offers a thorough overview of the introduction and extensive review of literature of Geometric Function Theory. It includes primary concepts and ideas within classes and their subclasses of this theory. Since the class which contains functions which are analytic, univalent work as infrastructure in Geometric Function Theory, so this chapter explores the foundation of these classes and investigation of related subclasses. Hankel determinant, Fekete-Szeg*ö* inequality and coefficient bounds for the classes are also briefly introduced. This chapter also turns over the crucial ideas of quantum calculus.

1.2 Pioneers of Function Theory

Geometric Function Theory mainly concerned with analytic functions, is an intriguing area of mathematics developed around 20th century. It is concerned with the geometric assets of analytic functions. The functions whose ranges define star, close-to-star, convex, close-to-convex etc geometries are called geometric functions and the study of geometric functions is called Geometric Function Theory. The prominent contributors known for their endeavors to develop the framework of function theory are Cauchy, Riemann and Weierstrass. Their ideas and methods are unique in a way that they explicate analytic functions by offering details with some distinct perspectives to give an advanced pattern to complex analysis.

1.3 Riemann Mapping Theorem

A theorem that Bernard Riemann came up with in 1851, known as Riemann Mapping Theorem [1], is the setting stone of this theory simply defined as any arbitrary domain (of analytic functions) can be replaced or substituted with the open unit disk $\Psi = \{ \ \hat{r} : |\hat{r}| < 1 \ \}$. This theorem serves a primary role in the base of Geometric Function Theory, see [2]. C. F. Gauss (1777-1855) [3], who had already established the basic ideas associated with function theory which includes Cauchy Integral Theory and complex integration [4], was the mathematician most significantly inspired Riemann in the development of function theory.

1.4 Analytic and Univalent Functions

In Geometric Function Theory, functions are categorized into various classes and sub-classes based upon their properties. All functions which are analytic and normalized belong to class $\mathbb{A}$, the most famous class in this theory. Analytic functions are those functions which are differentiable at a point as well as in the neighbourhood of that point and if their derivative exists at each point in a region then they will be called analytic in that region. Base of modern function theory is laid on analytic functions defined in a chosen domain. By normalization, it means that the function maps origin to origin means $\phi(0)$=0 and its derivative at origin is 1 that is $\phi'(0)$=1. Whenever two functions (let us say $\hat{g}$ and $\hat{k}$) are in class $\mathbb{A}$, we say that $\hat{g}$ is subordinated to $\hat{k}$, mathematically symbolized as $\hat{g} \prec \hat{k}$ if $\hat{g}(\hat{r}) = \hat{k}(\omega(\hat{r}))$ such that $\omega(\hat{r})$ is an analytic function, with the conditions that it maps origin to origin i.e ($\omega(0)$=0) and $|\omega(\hat{r})| \leq 1$, see [5]. The most significant class in this theory is $\mathbb{S}$, having normalized univalent functions. In 1907, Koebe studied about univalent functions in unit disk Ψ [6]. The analytic functions which assume no more than one value in open unit disk (i.e. there exists a one-one mapping from any arbitrary domain to open unit disk) are referred as univalent functions. Various nice geometries have been possessed by ranges of these functions and hence geometry of image plays a vital role in their detailed analysis, see [2].

1.5 Subclasses of Analytic and Univalent Functions

A well-known subject in complex analysis, the idea of univalent functions was established by Koebe [7] in 1907. According to what he proposed, functions in class $\mathbb{S}$ are analytic, univalent in Ψ and also fulfill requirements of normalization. Class $\mathbb{S}$ has been categorized into the class containing convex functions denoted by $\mathbb{C}$, $\mathbb{S}^*$ (this class deals with the functions having star geometry i.e. starlike functions), the class having functions possessing close-to-convex geometry is represented by $\mathbb{K}$ and $\mathbb{C}^*$ (quasi-convex functions' class). There exists a well-known Alexander relation [8] among $\mathbb{S}^*$ (class containing starlike functions) and $\mathbb{C}$ (the class which has convex functions) given in 1915. Later Libra defined an operator and these two classes were shown to be closed under this operator.

1.6 Class of Starlike Functions

Starlikeness is a fundamental geometrical characteristics. A part of the complex plane or the domain is said to possess star geometry with respect to the reference (fixed) point, say $\hat{r}_o$, if each point in it is associated with $\hat{r}_o$, by a straight line and all such line segments entirely fall within the domain [6, 7]. Geometrically, the domain will be star-shaped or starlike when comparing to that fixed point if all points of the domain can be seen from that reference point. The star-like univalent functions' class is comprised of all those functions belong to the class $\mathbb{S}$, satisfying the requirement $\hat{r}$ real of $\frac{\hat{r}\phi'(\hat{r})}{\phi(\hat{r})}$ is positive where $\hat{r}$ belongs to open unit disk Ψ, see [9]. The univalent function ϕ satisfying Re $\phi(\Psi) > 0$ was proposed by two researchers Ma and Minda [10] where $\phi(\Psi)$ possesses star geometry satisfying $\phi(0)$-1=0 and ϕ' at zero is positive. Additionally $\phi(\Psi)$ is symmetric about the real axis. They defined the general sub-class of starlike function as:

$$\mathbb{S}^*(\phi) = \left\{ h \in \mathbb{S} : \left(\frac{\hat{r}h'(\hat{r})}{h(\hat{r})} \right) \prec \phi(\hat{r}) \right\}$$

Starlike classes for different functions $\phi(\hat{r})$ were took into consideration by a number of researchers and their study investigated the geometrical traits, radius findings and coefficient estimations for the functions of those classes. $\mathbb{S}^*_L$, the class introduced and investigated by Sokol and Stankiewicz [11], consists of normalized analytic functions (let say b) in Ψ that satisfy the following criteria:

$$\left|\left(\left[\frac{\hat{r}b'(\hat{r})}{b(\hat{r})}\right]^2 - 1\right)\right| < 1, \qquad \hat{r} \in \Psi \tag{1.1}$$

A function in the class $\mathbb{S}^*_L$ is called Sokol-Stankiewicz starlike function. For the function $\phi(\hat{r})$ if we consider $1+sin(\hat{r})$, we get a new class $\mathbb{S}^*(1+sin(\hat{r}))$ which was examined by Cho *et al* [12]. The function class given by $\mathbb{S}^*_e$ was considered by Mendiratta *et al* and this class was generalized by Srivastava *et al*, see [13]. Inspired by the classes defined above, K . Arora and S .S. Kumar [14] considered the function $1+sinh^{-1}(\hat{r})$ and introduced a new class of starlike function.

1.7 Coefficient Bounds

The problem of discovering coefficient bounds for the analysis of geometry of complex-valued functions contributes a fundamental role in this theory. Throughout the long history a celebrated result is the *Bieberbach* conjecture [2] posed in 1916 and taken up by many researchers from 1916-1985. It states that for each function $\phi \in \mathbb{S}$, $|a_n| \leq n$ for n = 2,3,... Bieberbach, himself proved that $|a_2| \leq 2$ whereas Loewner [15] settled third in 1923. In 1955, Garabedian and Schiffer [16] solved the fourth coefficient bound whereas an easy illustration of the same conclusion was presented by Charzynski and Schiffer [2] in 1960. The proofs for the 5th and 6th were appeared over a few years later. Then in 1985 de-branges defined the known coefficient assumption for the function, for detail see [17]. To put it concisely, numerous attempts of researchers have solved this conjecture for a few values of n and all values of n for certain sub-classes of univalent functions. Bieberbach postulated in 1916 that the univalent function's nth coefficient is smaller than or equals to that of the famous Koebe function. In 1932, Littlewood and Parley proved that for odd univalent function, for each n, the modulus of $\tilde{c}$ (being coefficients of analytic function) is less than an absolute constant B, whose value by their method came to be less than 14 [2]. But this conjecture was not supposed to be hold true for certain subclasses of $\mathbb{S}$ and become a source of problem called as Fekete-Szeg$\ddot{o}$ problem.

1.8 Fekete-Szeg$\ddot{o}$ Problem

An inequality found by Fekete and Szeg$\ddot{o}$ in 1933 for the coefficients of functions which are univalent analytic has connection to the conjecture proposed by Bieberbach is called Fekete-Szeg$\ddot{o}$ inequality [18] and this problem is referred to as calculating similar estimates for different various classes of functions. The inequality becomes an equality for some functions making it sharp and hence termed as extremal functions. Extremal functions serve an important part in the analyzation of functions which are analytic in open unit disk.

1.9 Hankel Determinant

Since coefficient problems are essential for the study of analytic and univalent functions, Hankel determinant is one of that coefficient problem. Herman Hankel presented a Hankel matrix, a square matrix with the property that coefficients are arranged in ascending order from left to right as well as top to bottom. Hankel determinant assists a researcher in identifying the properties of the function. The determinant of the Hankel matrix is called Hankel determinant problem. The Hankel matrix is given by as follow:

$$\begin{bmatrix} \check{p}_q & \check{p}_{q+1} & \cdots & \check{p}_{q+y-1} \\ \check{p}_{q+1} & \check{p}_{q+2} & \cdots & \check{p}_{q+y} \\ \vdots & \vdots & \ddots & \vdots \\ \check{p}_{q+y-1} & \check{p}_{q+y} & \cdots & \check{p}_{q+2y-2} \end{bmatrix} \qquad (q, y \in \mathbb{N}) \tag{1.2}$$

Alongwith Fekete-Szeg$\ddot{o}$ problem, many researchers investigated and calculated the upper bounds of the determinant of Hankel matrix for different subclasses of functions which are analytic and univalent.

In 1967, Pommerenk formally presented Hankel determinant of the univalent functions [19]. The Hankel determinant of order third and its upper bound for the subclass of class $\mathbb{S}$ was initially investigated by Babalola [2]. Noor also discussed this problem for other intriguing classes containing analytic functions [20]. The Hankel determinant for exponential polynomials was studied by Ehrenborg [21] in 2000 and with the help of his work Hankel transform was thorough presented by Layman [22]. Research of determinant of Hankel matrix for functions which are starlike and convex was performed by Janteng *et al*, see [23].

The third order Hankel determinant for starlike functions of order $\frac{1}{2}$ and also its sharp bound was presented by Lecko *et al* in 2009 [24]. Joshi *et al* [25] in 2022 discovered the Hankel determinant of order third of starlike functions' class subordinated with exponential functions.

1.10 Quantum Calculus

Quantum theory is a vital tool for handling intricate and challenging information. Quantum calculus, called as q-calculus, is a traditional infinitesimal calculus but it revolves around the derivation of q-analogous results without the notion of limits [26]. q-calculus appeared as an influential topic of research due to demand of mathematics that models quantum computing. It emphasizes on generalization of integration and differentiation procedures. The core ingredient of quantum calculus is nothing but Bernoulli and Euler function. There is a significant interest in implementing q-calculus as it has vast applications in several areas like quantum mechanics, analytic number theory, multiple hypergeometric functions. Because of its numerous uses, it has caused utmost importance to research in q-calculus.

Here derivatives are considered as differences while antiderivatives are referred as sums. The systematic introduction of q-calculus, q-derivatives and q-integral concepts, all credited to Jackson [27, 28]. In 1740s the concept of partitions, ways of expressing a number as sums of non-negative integer, was developed by Euler. Then in 1748, he introduced an operator which lead to the q-difference operator. This is the root of q-calculus but till 1800s his work was not gathered and published as the work of Euler was written in Latin, see [29]. The results of Euler was then generalized by Gauβ in many areas of mathematics. Saeed *et al* [30] in 2020 presented the q-analog of the functions that are starlike associated to the famous trigonometric *sine* function.

With the help of q-difference operator, a number of subclasses of $\mathbb{A}$ consisting of analytic functions were studied and examined with elaboration. The usage of q-difference operator in this theory was initiated by a researcher named Ismail *et al* who defined and created the starlike functions' class [31]. Through the use of difference operator with an appropriate alteration in regard with domain of the function, he produced a newly made class called as the q-starlike functions' class. Characteristics of functions belonging to q-starlike and q-convex classes were examined and explored. Apart from exploring the famous inequality named as Fekete-Szeg*ö*, he additionally develop the set of q-starlike functions subordinated with some trigonometric functions. Different authors investigated some properties, such as the renowed problem of Fekete-Szeg*ö*, the conditions which are both necessary and sufficient, the growth and distortion bound, radii of starlikeness including the problem with partial sum for this class.

1.11 Preface

To specify and look into some subclasses of analytic functions with the help of subordination technique is the core aim of this thesis. It is carved up into five chapters. Below is the concise prolution of every chapter.

In **Chapter 2**, the primary concern is on fundamental ideas in Geometric Function Theory, which builds an adequate framework for the subsequent chapters. It begins by looking at the concepts of functions which are analytic as well as univalent functions following conditions of normaliztaion within open unit disk and then followed by providing an overview of their numerous subclasses. Preliminary lemmas that will be used in later chapters are introduced at the end of this chapter. Interestingly, this chapter also presents a thorough analysis of theories and concepts related to this field.

Chapter 3 includes the examination of the class of starlike function associated with inverse sine hyperbolic function. It involves investigation of certain main results related to coefficient bounds, Fekete-Szeg*ö* and Hankel determinant problem.

Chapter 4 focuses on new class of q-starlike functions associated with inverse sine hyperbolic function. The chapter will incorporate conclusions from research for functions in this class. It will be demonstrated via corollaries that the recently derived results are consistent with those that have already been established by other scholars.

In **Chapter 5**, conclusions of the aforementioned research will be presented.

CHAPTER 2

PRELIMINARY CONCEPTS

2.1 Overview

This chapter's goal is to go over some important terms and traditional findings in an effort to lay the groundwork for future research. Normalized analytical univalent functions and the Caratheodory functions will be examined through a thorough analysis in this study. We'll take into account a few special functions, linear operators, some initial lemmas alongwith a quick review of q-calculus. As analysis describes functions and their properties and geometry studies about spatial structures so the relationship between geometry and analysis is probably the exciting part of Geometric Function Theory.

2.2 Analytic Functions

The main goal of this theory, as its name suggests, is the geometric attributes of analytic functions. It usually looks at how complex function behaviour distorts shapes and how they translate one geometric form to another. Moreover, Geometric Function Theory always pointed towards a certain domain, known as open unit disk. An open unit disk is referred as a disk centered at the origin, having radius 1 without including its boundary points.

Definition 2.2.1. [32] A function which is complex-valued $\phi(\hat{r})$ is called to be an analytic function at $\hat{r}_o$ in a domain if it is defined and has a derivative at and in neighbourhood of $\hat{r}_o$. A function $\phi(\hat{r})$ whose derivative exists at all points of domain is considered to be analytic in that domain.

Because of analyticity they can be expressed in Taylor series. Their series representation form centred at $\hat{r}_o$ is as follow:

$$\phi(\hat{r}) = \sum_{k=0}^{\infty} \frac{\phi^k(\hat{r}_0)}{k!}(\hat{r}-\hat{r}_0)^k$$

Definition 2.2.2. [33] A function which is analytic and satisfy the conditions of normalization which are $\phi(0) = 0$ and $\phi'(0) - 1 = 0$ in a domain which is an open unit disk Ψ is said to be in class $\mathbb{A}$. They have their series form as:

$$\phi(\hat{r}) = \hat{r} + \sum_{n=2}^{\infty} \check{a}_n \hat{r}^n$$

where $\hat{r} \in \Psi$ and $\check{a}_1 = 1$.

2.3 Univalent Functions

In 1907, Koebe studied about univalent functions in unit disk Ψ.

Definition 2.3.1. [34] A function which is analytic and an injective function is known as univalent function or univalent mapping. Preisely, a function(let say ϕ) defined in a domain is said to be univalent if it maps distinct element of its domain to distinct elements of its codomain.

In mathematical form, for every two distinguish complex numbers $\hat{r}_1$ and $\hat{r}_2$ in a domain, we have $\phi(\hat{r}_1) \neq \phi(\hat{r}_2)$. Alternatively, we can also say that no different images of domain can ever be mapped to the same output under ϕ, being univalent mapping. The large family of univalent functions is leaded by its member, the Koebe function.

2.4 Normalized Univalent Functions' class, $\mathbb{S}$

The functions that are analytic as well as univalent in a domain which is open unit disk Ψ belong to the most significant class, denoted by $\mathbb{S}$. Also, the functions satisfy $\phi(0) = 0$ and

$\phi'(0) = 1$ which make them normalized univalent functions, see [34]. Koebe function is this class's extremal function.

$$k(\hat{r}) = \frac{\hat{r}}{(1-\hat{r})^2}$$

It maps $|\hat{r}| < 1$ to the entire complex plane without $(-\infty, \frac{-1}{4}]$.

2.5 Caratheodory Functions

The Caratheodory functions' class contains all such functions having real part positive. Many subclasses of univalent functions are likely to be related with this class.

Definition 2.5.1. [33] Consider $p(\hat{r}) \in \mathbb{P}$ to be an analytic function with positive real part i.e. Real $(p(\hat{r})) > 0$ and $(\hat{r} \in \Psi)$ and represented in the form

$$p(\hat{r}) = 1 + \sum_{j=1}^{\infty} p_j \hat{r}^j$$

This class was initially created by Caratheodory known as Caratheodory class or class of functions having real part positive.

The function $p_o(\hat{r}) = \frac{1+\hat{r}}{1-\hat{r}}$ known as Mobius function plays a crucial role compare favourably to the Koebe function for the class $\mathbb{S}$. Many several kinds of functions having geometric qualities like starlikeness, convexity, close-to-convexity are intimately related to this class and they are of vital importance in the study of univalent functions.

2.6 Certain subclasses of univalent functions

Since Geometric Function Theory is related to geometry of functions, so univalent functions are divided into different subclasses based upon their geometries. For the sake of refinement, smoothness and improvement of many known results-especially to facilitate the development of a new subclass-the numerous research topics are being reinvestigated for various classes of functions. The main subclass, Starlike functions' class, has been defined here.

Definition 2.6.1. [33, 32] A function $\phi(\hat{r})$ in $\mathbb{S}$ is called to be starlike(with reference to origin) if $tw \in \Psi$ whenever $t \in [0,1]$ and $w \in \phi(\Psi)$ where Ψ represents open unit disc. Analytically, we say that $\phi \in \mathbb{S}^*$ if and only if real part of quantity $\frac{\hat{r}\phi'(\hat{r})}{\phi(\hat{r})}$ is positive i.e.

$$Real\left[\frac{\hat{r}\phi(\hat{r})}{\phi(\hat{r})}\right] > 0$$

The starlike functions' class of order α represented by $\mathbb{S}^*(\alpha)$ where $0 \leq \alpha \leq 1$ is simply a generalization of $\mathbb{S}^*$ [35]. Indeed, a function $\phi(\hat{r}) \in \mathbb{S}^*(\alpha)$ iff

$$Real\left(\frac{\hat{r}\phi(\hat{r})}{\phi(\hat{r})}\right) > \alpha$$

in Ψ. We set $\mathbb{S}^*(0) = \mathbb{S}^*$ which is recognized as starlike functions's class with respect to the origin.

2.7 Subordination

The notion of subordination was adopted for defining many of classes of functions examined in Geometric Function Theory. The subordination relation helps us in a way that one can start from a given function (let say $\hat{k}$) and examine the behaviour of all functions subordinated to $\hat{k}$. It is defined with the help of Schwarz function.

Definition 2.7.1. [36] Whenever two functions (let us say $\hat{g}$ and $\hat{k}$) are in class $\mathbb{A}$, we say that $\hat{g}$ is subordinated to $\hat{k}$, mathematically symbolized as $\hat{g} \prec \hat{k}$ if $\hat{g}(\hat{r}) = \hat{k}(\omega(\hat{r}))$ such that $\omega(\hat{r})$ is an analytic function in unit disk, following the criteria that ω (0)= 0 and $|\ \omega(\hat{r})\ | \leq 1$.

2.8 Subclass of Starlike Function Subordinated to Inverse Sine Hyperbolic Function

In 2022, Kush Arora and S. Sivaprasad Kumar introduced a new class $\mathbb{S}^*_\rho$ of starlike functions associated with $1 + sinh^{-1}(\hat{r})$ defined as under:

$$\mathbb{S}^*_\rho = \left\{\frac{\hat{r}\phi'(\hat{r})}{\phi(\hat{r})} \prec 1 + sinh^{-1}(\hat{r}), \quad \forall \hat{r} \in \Psi\right\} \tag{2.1}$$

2.9 Quantum Calculus

The modern name for the study of calculus without bounds is quantum calculus. In the early 20th century, Frank Hilton Jackson introduced the quantum calculus, commonly referred as the q-calculus, however Jacobi and Euler had already solved this sort of calculus.

This calculus without limits introduced q-version of series, or power series containing q-version of the common operations of calculus. Such series have applications in theory of numbers and combinatories (q-binomial theorem) [37].

2.10 $\tilde{q}_e$-derivative

The $\tilde{q}_e$-derivative operator or Jackson derivative, symbolized by $\mathbb{D}_{\tilde{q}_e}$, is a basic idea in $\tilde{q}_e$-calculus, an area of mathematics which defines a parameter $\tilde{q}_e$ in order to generalize ordinary calculus. $\mathbb{D}_{\tilde{q}_e}$ is a $\tilde{q}_e$-analog of the ordinary derivative. From now onwards, $\tilde{q}_e \in (0,1)$ unless otherwise stated.

Definition 2.10.1. [37] Consider $\phi(\hat{r})$ to be a complex-valued function. Then the $\tilde{q}_e$-derivative of $\phi(\hat{r})$ is defined as follow:

$$\mathbb{D}_{\tilde{q}_e}(\phi(\hat{r})) = \frac{\phi(\tilde{q}_e\hat{r}) - \phi(\hat{r})}{(\tilde{q}_e - 1)\hat{r}}, \quad \tilde{q}_e \in (0,1)$$

and

$$\mathbb{D}_{\tilde{q}_e}(\phi(\hat{r})) = \frac{d\phi(\hat{r})}{d\hat{r}}, \quad \tilde{q}_e = 1$$

It can be seen clearly that if $\tilde{q}_e$ approaches to 1^-, we will have

$$\lim_{\tilde{q}_e \to 1^-} \mathbb{D}_{\tilde{q}_e}(\phi(\hat{r})) = \lim_{\tilde{q}_e \to 1^-} \frac{\phi(\tilde{q}_e\hat{r}) - \phi(\hat{r})}{(\tilde{q}_e - 1)\hat{r}} = \phi'(\hat{r})$$

The function $\mathbb{D}_{\tilde{q}_e}(\phi(\hat{r}))$ has its Maclaurin series represented as:

$$\mathbb{D}_{\tilde{q}_e}(\phi(\hat{r})) = \sum_{h=1}^{\infty} [h]_{\tilde{q}_e} \check{a}_h \hat{r}^{h-1}$$

where

$$[h]_{\tilde{q}_e} = \begin{cases} \frac{\tilde{q}_e{}^h - 1}{\tilde{q}_e - 1} & , h \in \mathbb{C} \\ \sum_{h=0}^{h-1} \tilde{q}_e{}^h & , h \in \mathbb{N}. \end{cases}$$

2.11 $\tilde{q}_e$-Starlike Function

Definition 2.11.1. A complex-valued function $\phi \in \mathbb{S}$ is called to be in $\tilde{q}_e$-starlike functions's class represented by $\mathbb{S}^*_{\tilde{q}_e}$, in Ψ if it satisfies

$$\phi(0) = 0$$

and

$$\phi'(0) - 1 = 0$$

Also

$$\left| \frac{\hat{r}\mathscr{D}_{\tilde{q}_e}(\phi(\hat{r}))}{\phi(\hat{r})} - \frac{1}{1-\tilde{q}_e} \right| \leq \frac{1}{1-\tilde{q}_e} \quad (\hat{r} \in \Psi)$$

2.12 Fekete-Szegö Inequality

In complex analysis, there are various significant ramifications and uses for the Fekete-Szegö inequality. For particular subclasses, like the classes which contain starlike or convex functions, their coefficient bounds can be derived using it. There are certain functions for which this inequality turns out to be an equality implies that the Fekete-Szegö inequality is sharp. These functions, known as extremal functions, greatly aided in analyzing the behaviour of functions that are analytic in a domain which is open unit disk.

Definition 2.12.1. [38] A famous result $|\check{a}_3 - \eta \check{a}_2^2| \leq 1 + 2e^{\frac{-2\eta}{1-\eta}}, 0 \leq \eta \leq 1$ in complex analysis is called as Fekete-Szegö inequality. For a specific group of analytic functions, it gives an upper constraint on the determinant's absolute value, and pertains specifically to functions that are specified in $\Psi = \{|\hat{r}| \leq 1, \hat{r} \in \mathbb{C}\}$ and meet the criteria $\phi(0) = 0$ and $\phi'(0) = 1$.

2.13 Hankel Determinant

The Hankel matrix's determinant is known as the Hankel determinant. While studying power series with integer coefficient and in the concept of singularities, Hankel determinants can be beneficial. In 1966, Pommerenke [39] conducted an extensive study on the determinants of Hankel matrices of starlike functions. The study of the Hankel determinant of exponential functions was carried by Ehrenborg [21]. The Hankel determinant of 2nd order for univalent functions was investigated by Hayman [40] in 1968. Noonan and Thomas [41] defined $\tilde{q}_e$-th version of Hankel determinant as:

$$|\mathbb{H}_{\tilde{q}_e,s}(\phi)| = \begin{vmatrix} \check{y}_s & \check{y}_{s+1} & \check{y}_{s+2} & \cdots & \check{y}_{s+\tilde{q}_e-1} \\ \check{y}_{s+1} & \check{y}_{s+2} & \check{y}_{s+3} & \cdots & \check{y}_{s+\tilde{q}} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \check{y}_{s+\tilde{q}_e-1} & \check{y}_{s+\tilde{q}_e} & \check{y}_{s+\tilde{q}_e+1} & \cdots & \check{y}_{s+2\tilde{q}_e-2} \end{vmatrix} \quad (\check{y}_1 = 1, \tilde{q}_e, s \in \mathbb{N})$$

In particular,

$$\mathbb{H}_{2,1}(\phi) = \begin{vmatrix} \check{y}_1 & \check{y}_2 \\ \check{y}_2 & \check{y}_3 \end{vmatrix} = \check{y}_1\check{y}_3 - \check{y}_2^2 = \check{y}_3 - \check{y}_2^2$$

Finding $\mathbb{H}_{2,1}$ for numerous subclasses of univalent and multiunivalent functions have been the focus of investigation by various authors. A problem of determining a sharp upper bound of $|\check{p}_3 - \eta \check{p}_2^2|$ is occasionally referred to as the Fekete-Szeg*ö* problem, where η is real or complex. Some familiar classes of univalent functions, notably starlike and convex functions, have had the problem of finding functional's sharp bound $|\check{p}_3 - \eta \check{p}_2^2|$ entirely resolved, see [42, 43, 44]. Janteng *et al* [23] have determined the sharp upper bound to $|\mathbb{H}_{2,1}|$ for the family of functions $\{\phi \in \mathbb{A} : Re(\phi'(\hat{r})) > 0, \hat{r} \in \Psi\}$. Also,

$$\mathbb{H}_{2,2}(\phi) = \begin{vmatrix} \check{y}_2 & \check{y}_3 \\ \check{y}_3 & \check{y}_4 \end{vmatrix} = \check{y}_2\check{y}_4 - \check{y}_3^2$$

Similarly, by using Hankel matrix the third order Hankel determinant [45] is calculated as follow:

$$\mathbb{H}_{3,1}(\phi) = \begin{vmatrix} \check{y}_1 & \check{y}_2 & \check{y}_3 \\ \check{y}_2 & \check{y}_3 & \check{y}_4 \\ \check{y}_3 & \check{y}_4 & \check{y}_5 \end{vmatrix} = \check{y}_5(\check{y}_3\check{y}_1 - \check{y}_2.\check{y}_2) - \check{y}_4(\check{y}_1\check{y}_4 - \check{y}_2\check{y}_3) + \check{y}_3(\check{y}_2\check{y}_4 - \check{y}_3.\check{y}_3)$$

Hence

$$\mathbb{H}_{3,1}(\phi) = \check{y}_5(\check{y}_3 - \check{y}_2^2) - \check{y}_4(\check{y}_4 - \check{y}_2\check{y}_3) + \check{y}_3(\check{y}_2\check{y}_4 - \check{y}_3^2) \ \ , \check{y}_1 = 1$$

This implies that

$$|\mathbb{H}_{3,1}(\phi)| = |\check{y}_5||(\check{y}_3 - \check{y}_2^2)| + |\check{y}_4||(\check{y}_4 - \check{y}_2\check{y}_3)| + |\check{y}_3||(\check{y}_2\check{y}_4 - \check{y}_3^2)|$$

For more details, see [58,59].

2.14 Zalcman Functional

In 1960, Zalcman [34] presented a surprising conjecture for univalent functions, the modified version of which was proved by Ma [46] in 1999. Zalcman conjecture states that for any $t \geq 2$, coefficients of univalent functions on the domain which is open unit disk fulfill the following sharp inequality.

$$|\breve{a}_t^2 - \breve{a}_{2t-1}| \leq (t-1)^2$$

where $\breve{a}_t$ and $\breve{a}_{2t-1}$ represent the coefficients of function and this equality holds exclusively for the well-known famous function of class $\mathbb{S}$, Koebe function.

2.15 Preliminary Lemmas

These are the lemmas that will be necessary in order to obtain results in the chapters that follow.

Lemma 2.15.1. [47, 6] If $p(\hat{r}) = 1 + \sum_{h=1}^{\infty} \breve{d}_h \hat{r}^h \in \mathbb{P}$, then

$$2\breve{d}_2 = \breve{d}_1^2 + \psi(4 - \breve{d}_1^2),$$

$$4\breve{d}_3 = \breve{d}_1^3 + 2\breve{d}_1(4 - \breve{d}_1^2)\psi - \breve{d}_1(4 - \breve{d}_1^2)\psi^2 + 2(4 - \breve{d}_1^2)(1 - |\psi|^2)\Upsilon,$$

for some $\psi(|\psi| \leq 1), \Upsilon(|\Upsilon| \leq 1)$.

Lemma 2.15.2. [6] Consider the function $p(\hat{r}) \in \mathbb{P}$ whose series form is $p(\hat{r}) = 1 + \sum_{h=1}^{\infty} \breve{d}_h \hat{r}^h$. Then,

$$|\breve{d}_h| \leq 2 \quad (h \in \mathbb{N})$$

Lemma 2.15.3. [6] Suppose the function $p(\hat{r}) \in \mathbb{P}$ defined as $p(\hat{r}) = 1 + \sum_{h=1}^{\infty} \breve{d}_h \hat{r}^h$. Then,

$$|\lambda \breve{d}_h - \breve{d}_k \breve{d}_{h-k}| \leq \begin{cases} 2|2-\lambda| & ,\lambda \leq 1 \\ 2\lambda & \lambda \geq 1. \end{cases}$$

CHAPTER 3

CLASS OF STARLIKE FUNCTION SUBORDINATED WITH SINE INVERSE HYPERBOLIC FUNCTION

3.1 Overview

The objective of this chapter is to examine a number of fundamental and classical findings that act as essential components for further investigation. The first part of the chapter goes over starlike function associated with the inverse sine hyperbolic function. Added to that, a number of substantial findings will be investigated such as the bounds of the coefficients, the widely acknowledged inequality proposed by Fekete and Szeg*ö*, the Zalcmann functional, and the famous Hankel determinants.

The class containing starlike functions subordinated with inverse sine hyperbolic function was defined by Kush Arora and S. Sivaprasad Kumar [14].

Definition 3.1.1. A function ϕ belongs to $\mathbb{A}$ is in $\mathbb{S}^*_\rho$ then

$$\left(\frac{\hat{r}\phi'(\hat{r})}{\phi(\hat{r})}\right) \prec 1+sinh^{-1}(\hat{r}), \qquad \forall\ \hat{r}\in\Psi \tag{3.1}$$

3.2 Coefficient Inequalities

For the class $\mathbb{S}^*_\rho$ the related results are as follow.

Theorem 3.2.1. If $\phi(\hat{r}) \in \mathbb{S}^*_\rho$ then

$$|\breve{a}_2| \leq 1,\ \ |\breve{a}_3| \leq \frac{1}{2} = 0.5,\ \ |\breve{a}_4| \leq 0.5767,\ \ |\breve{a}_5| \leq 0.6070.$$

Proof. By definition

$$\left(\frac{\hat{r}\phi'(\hat{r})}{\phi(\hat{r})}\right) \prec 1 + sinh^{-1}(\hat{r})$$

Since $\phi \in \mathbb{S}^*_\rho$, by using the technique of subordination, we can write

$$\left(\frac{\hat{r}\phi'(\hat{r})}{\phi(\hat{r})}\right) = 1 + sinh^{-1}(\sigma(\hat{r})). \tag{3.2}$$

Consider the function defined as

$$p(\hat{r}) = \frac{1+\sigma(\hat{r})}{1-\sigma(\hat{r})} = 1 + \breve{d}_1\hat{r} + \breve{d}_r\hat{r}^2 + \breve{d}_3\hat{r}^3 + \breve{d}_4\hat{r}^4 + ..., \tag{3.3}$$

where $p(\hat{r})$ is an analytic function in Ψ following the condition that $p(0) = 1$. The above implies as

$$\sigma(\hat{r}) = (p(\hat{r}) - 1)(p(\hat{r}) + 1)^{-1}, \tag{3.4}$$

Simplification will give us the following as:

$$\sigma(\hat{r}) = \left(\frac{\breve{d}_1}{2}\right)\hat{r} + \left(\frac{\breve{d}_2}{2} - \frac{\breve{d}_1^2}{4}\right)\hat{r}^2 + \left(\frac{\breve{d}_3}{2} + \frac{\breve{d}_1^3}{8} - \frac{\breve{d}_1\breve{d}_2}{2}\right)\hat{r}^3 + \left(\frac{3\breve{d}_1^2 d_2}{8} - \frac{\breve{d}_1\breve{d}_3}{2} - \frac{\breve{d}_1^4}{16} - \frac{\breve{d}_2^2}{4} + \frac{\breve{d}_4}{2}\right)\hat{r}^4 + ... \tag{3.5}$$

As we have,

$$1 + sinh^{-1}(\sigma(\hat{r})) = 1 + \sigma(\hat{r}) - \frac{(\sigma(\hat{r}))^3}{6} + \frac{3(\sigma(\hat{r}))^5}{40} - ...$$

So, we will get

$$1 + sinh^{-1}(\sigma(\hat{r})) = 1 + \left(\frac{\breve{d}_1}{2}\right)\hat{r} + \left(\frac{\breve{d}_2}{2} - \frac{\breve{d}_1^2}{4}\right)\hat{r}^2 + \left(\frac{\breve{d}_3}{2} - \frac{\breve{d}_1\breve{d}_2}{2} + \frac{5\breve{d}_1^3}{48}\right)\hat{r}^3 + \left(\frac{\breve{d}_4}{2} - \frac{\breve{d}_1\breve{d}_3}{2} + \frac{5\breve{d}_1^2\breve{d}_2}{16} - \frac{\breve{d}_1^4}{32} - \frac{\breve{d}_2^2}{4}\right)\hat{r}^4 + ... \tag{3.6}$$

Also,we have

$$\frac{\hat{r}\phi'(\hat{r})}{\phi(\hat{r})} = 1 + (\breve{a}_2)\hat{r} + (2\breve{a}_3 - \breve{a}_2^2)\hat{r}^2 + (3\breve{a}_4 - 3\breve{a}_2\breve{a}_3 + \breve{a}_2^3)\hat{r}^3 + (4\breve{a}_5 - 4\breve{a}_2\breve{a}_4 - 2\breve{a}_3^2 - \breve{a}_2^4 + 4\breve{a}_3\breve{a}_2^2)\hat{r}^4 + ...$$

Now substitute values in (4.2) we will have,

$$1+(\breve{a}_2)\hat{r}+(2\breve{a}_3-\breve{a}_2^2)\hat{r}^2+(3\breve{a}_4-3\breve{a}_2\breve{a}_3+\breve{a}_2^3)\hat{r}^3+(4\breve{a}_5-4\breve{a}_2\breve{a}_4-2\breve{a}_3^2-\breve{a}_2^4+4\breve{a}_3\breve{a}_2^2)\hat{r}^4+\ldots$$
$$=1+\left(\frac{\breve{d}_1}{2}\right)\hat{r}+\left(\frac{\breve{d}_2}{2}-\frac{\breve{d}_1^2}{4}\right)\hat{r}^2+\left(\frac{\breve{d}_3}{2}-\frac{\breve{d}_1\breve{d}_2}{2}+\frac{5\breve{d}_1^3}{48}\right)\hat{r}^3$$
$$+\left(\frac{\breve{d}_4}{2}-\frac{\breve{d}_1\breve{d}_3}{2}+\frac{5\breve{d}_1^2\breve{d}_2}{16}-\frac{\breve{d}_1^4}{32}-\frac{\breve{d}_2^2}{4}\right)\hat{r}^4+\ldots$$

Comparison of both sides of above equation will result in,

$$\breve{a}_2=\frac{\breve{d}_1}{2}, \tag{3.7}$$

$$2\breve{a}_3-\breve{a}_2^2=\frac{\breve{d}_2}{2}-\frac{\breve{d}_1^2}{4}, \tag{3.8}$$

$$3\breve{a}_4-3\breve{a}_2\breve{a}_3+\breve{a}_2^3=\frac{\breve{d}_3}{2}-\frac{\breve{d}_1\breve{d}_2}{2}+\frac{5\breve{d}_1^3}{48}, \tag{3.9}$$

$$4\breve{a}_5-4\breve{a}_2\breve{a}_4-2\breve{a}_3^2-\breve{a}_2^4+4\breve{a}_3\breve{a}_2^2=\frac{\breve{d}_4}{2}-\frac{\breve{d}_1\breve{d}_3}{2}+\frac{5\breve{d}_1^2\breve{d}_2}{16}-\frac{\breve{d}_1^4}{32}-\frac{\breve{d}_2^2}{4}, \tag{3.10}$$

From (3.7) we get coefficient $\breve{a}_2$ as

$$\breve{a}_2=\frac{\breve{d}_1}{2}. \tag{3.11}$$

On solving (3.8) we get $\breve{a}_3$,

$$\breve{a}_3=\frac{\breve{d}_2}{4}. \tag{3.12}$$

By solving (3.9) we determine $\breve{a}_4$

$$3\breve{a}_4=3\breve{a}_2\breve{a}_3-\breve{a}_2^3+\frac{\breve{d}_3}{2}-\frac{\breve{d}_1\breve{d}_2}{2}+\frac{5\breve{d}_1^3}{48},$$

Now substitute the values of $\breve{a}_2$ and $\breve{a}_3$ in $\breve{a}_4$, we get

$$\breve{a}_4=\frac{\breve{d}_3}{6}-\frac{\breve{d}_1\breve{d}_2}{24}-\frac{\breve{d}_1^3}{144}. \tag{3.13}$$

Now for $\breve{a}_5$, we will use equation (3.10),

$$4\breve{a}_5=4\breve{a}_2\breve{a}_4+2\breve{a}_3^2+\breve{a}_2^4-4\breve{a}_3\breve{a}_2^2+\frac{\breve{d}_4}{2}-\frac{\breve{d}_1\breve{d}_3}{2}+\frac{5\breve{d}_1^2\breve{d}_2}{16}-\frac{\breve{d}_1^4}{32}-\frac{\breve{d}_2^2}{4}$$

Now we will substitute values of $\breve{a}_2$, $\breve{a}_3$ and $\breve{a}_4$ in $\breve{a}_5$, we will get

$$\breve{a}_5=\frac{\breve{d}_4}{8}-\frac{\breve{d}_1\breve{d}_3}{24}-\frac{\breve{d}_1^2\breve{d}_2}{192}+\frac{5\breve{d}_1^4}{1152}-\frac{\breve{d}_2^2}{32}. \tag{3.14}$$

Now from (3.7) we have

$$\breve{a}_2=\frac{\breve{d}_1}{2}$$

Applying Lemma 2.15.2, we will get

$$|\breve{a}_2| = \left|\frac{\breve{d}_1}{2}\right| \leq \frac{|\breve{d}_1|}{2} \leq \frac{2}{2} = 1$$

$$|\breve{a}_2| \leq 1. \tag{3.15}$$

From (3.8), we have:

$$\breve{a}_3 = \frac{\breve{d}_2}{4}$$

Using Lemma 2.15.2 leads us to

$$|\breve{a}_3| = \left|\frac{\breve{d}_2}{4}\right| = \frac{|\breve{d}_2|}{4} \leq \frac{2}{4} = \frac{1}{2}$$

$$|\breve{a}_3| \leq \frac{1}{2}. \tag{3.16}$$

From (3.13) we have

$$\breve{a}_4 = \frac{\breve{d}_3}{6} - \frac{\breve{d}_1\breve{d}_2}{24} - \frac{\breve{d}_1^3}{144}$$

Taking modulus on both sides, we get

$$|\breve{a}_4| = \left|\frac{\breve{d}_3}{6} - \frac{\breve{d}_1\breve{d}_2}{24} - \frac{\breve{d}_1^3}{144}\right|$$

Now applying Lemma 2.15.1, we have

$$|\breve{a}_4| = \left|\frac{\breve{d}_1^3 + 2\breve{d}_1(4-\breve{d}_1^2)\psi - (4-\breve{d}_1^2)\breve{d}_1\psi^2 + 2(4-\breve{d}_1^2)(1-|\psi|^2)\Upsilon}{24} \right.$$
$$\left. - \frac{\breve{d}_1}{24}\left(\frac{\breve{d}_1^2 + \psi(4-\breve{d}_1^2}{2}\right) - \frac{\breve{d}_1^3}{144}\right|$$

After doing simplification and combining like terms,we will get

$$|\breve{a}_4| = \left|\frac{\breve{d}_1^3}{72} + \frac{(4-\breve{d}_1^2)\breve{d}_1\psi}{16} - \frac{(4-\breve{d}_1^2)\breve{d}_1\psi^2}{24} + \frac{2(4-\breve{d}_1^2)(1-|\psi|^2)\Upsilon}{24}\right|$$

Using triangular inequality and let $\breve{d}_1 = \breve{d} \in [0,2]$, $|\psi| = t$, we have

$$|\breve{a}_4| \leq \frac{\breve{d}^3}{72} + (4-\breve{d}^2)\left(\frac{\breve{d}t}{16} + \frac{\breve{d}t^2}{24} + \frac{1}{12}\right)$$

We assume that

$$G(\breve{d},t) = \frac{\breve{d}^3}{72} + \frac{(4-\breve{d}^2)\breve{d}t}{16} + \frac{(4-\breve{d}^2)\breve{d}t^2}{24} + \frac{(4-\breve{d}^2)}{12}$$

Upon partial differentiation we get

$$\frac{\partial G}{\partial t} = \frac{2(4-\breve{d}^2)\breve{d}}{24} + \frac{2t\breve{d}(4-\breve{d}^2)}{24} + \frac{\breve{d}(4-\breve{d}^2)}{48} > 0$$

It implies that $G(\breve{d},t)$ is an increasing function in [0,1]. So,

$$max(G(\breve{d},t)) = (G(\breve{d},1)) = \frac{\breve{d}^3}{72} + \frac{(4-\breve{d}^2)\breve{d}}{16} + \frac{(4-\breve{d}^2)\breve{d}}{24} + \frac{(4-\breve{d}^2)}{12}$$

Set

$$K(\breve{d}) = \frac{\breve{d}^3}{72} + \frac{5\breve{d}(4-\breve{d}^2)}{48} + \frac{(4-\breve{d}^2)}{12}$$

Now differentiating the above equation,we get

$$K'(\breve{d}) = \frac{-13\breve{d}^2}{48} - \frac{\breve{d}}{6} + \frac{20}{48}$$

If we set $K'(\breve{d}) = 0$, we get two roots that is $\breve{d}_1 = \frac{-4+2\sqrt{69}}{13}$ and $\breve{d}_2 = \frac{-4-2\sqrt{69}}{13}$. Calculations show that $K(\breve{d})$ has its maximum value at a root $\breve{d}_1 = \frac{-4+2\sqrt{69}}{13}$. So on substitution of $\breve{d}_1$ in the function $K(\breve{d})$, we get

$$|\breve{a}_4| \leq K(\breve{d}_1) = 0.5767$$

Hence we have:

$$|\breve{a}_4| \leq 0.5767$$

Now from (3.14) we have

$$\breve{a}_5 = \frac{\breve{d}_4}{8} - \frac{\breve{d}_1\breve{d}_3}{24} - \frac{\breve{d}_1^2\breve{d}_2}{192} + \frac{5\breve{d}_1^4}{1152} - \frac{\breve{d}_2^2}{32}.$$

By using Lemma 2.15.1 and substituting values of $\breve{d}_2$ and $\breve{d}_3$, we will get

$$\breve{a}_5 = \frac{\breve{d}_4}{8} - \frac{\breve{d}_1}{24}\left(\frac{\breve{d}_1^3 + 2(4-\breve{d}_1^2)\breve{d}_1\psi - (4-\breve{d}_1^2)\breve{d}_1\psi^2 + 2(4-\breve{d}_1^2)(1-|\psi|^2)\Upsilon}{4}\right) - \frac{\breve{d}_1^2}{192}\left(\frac{\breve{d}_1^2 + \psi(4-\breve{d}_1^2)}{2}\right) + \frac{5\breve{d}_1^4}{1152} - \frac{1}{32}\left(\frac{\breve{d}_1^2 + \psi(4-\breve{d}_1^2)}{2}\right)^2$$

Now after doing simplification and combining like terms, $\breve{a}_5$ will become

$$\breve{a}_5 = \frac{\breve{d}_4}{8} - \frac{19\breve{d}_1^4}{1152} - \frac{5\breve{d}_1^2\psi(4-\breve{d}_1^2)}{128} + \frac{\psi^2\breve{d}_1^2(4-\breve{d}_1^2)}{96} - \frac{\psi^2(4-\breve{d}_1^2)^2}{128} - \frac{2\breve{d}_1(4-\breve{d}_1^2)(1-|\psi|^2)\Upsilon}{96}$$

Taking modulus on both sides and let $\breve{d}_1 = \breve{d}$,we have

$$|\breve{a}_5| = \left|\frac{\breve{d}_4}{8} - \frac{19\breve{d}^4}{1152} - \frac{5\breve{d}^2\psi(4-\breve{d}^2)}{128} + \frac{\psi^2\breve{d}^2(4-\breve{d}^2)}{96} - \frac{\psi^2(4-\breve{d}^2)^2}{128} - \frac{2\breve{d}(4-\breve{d}^2)(1-|\psi|^2)\Upsilon}{96}\right|$$

By using Lemma 2.15.2 on above and let $|\psi| = t$

$$|\breve{a}_5| \leq \frac{1}{4} + \frac{19\breve{d}^4}{1152} + \frac{5\breve{d}^2 t(4-\breve{d}^2)}{128} + \frac{t^2\breve{d}^2(4-\breve{d}^2)}{96} + \frac{t^2(4-\breve{d}^2)^2}{128} + \frac{2\breve{d}(4-\breve{d}^2)}{96}$$

Suppose that

$$H(\breve{d},t) = \frac{1}{4} + \frac{19\breve{d}^4}{1152} + \frac{5\breve{d}^2 t(4-\breve{d}^2)}{128} + \frac{t^2\breve{d}^2(4-\breve{d}^2)}{96} + \frac{t^2(4-\breve{d}^2)^2}{128} + \frac{\breve{d}(4-\breve{d}^2)}{48}$$

On partial differentiation of $H(\breve{d},t)$, we have

$$\frac{\partial H}{\partial t} = \frac{5\breve{d}^2(4-\breve{d}^2)}{128} + \frac{2t\breve{d}^2(4-\breve{d}^2)}{96} + \frac{2t(4-\breve{d}^2)^2}{128}$$

As $\frac{\partial H}{\partial t} > 0$, clearly implies that in [0,1] the function $H(\breve{d},t)$ will increase. So,

$$max(H(\breve{d},t)) = H(\breve{d},1) = \frac{1}{4} + \frac{19\breve{d}^4}{1152} + \frac{5\breve{d}^2(4-\breve{d}^2)}{128} + \frac{\breve{d}^2(4-\breve{d}^2)}{96} + \frac{(4-\breve{d}^2)^2}{128} + \frac{\breve{d}(4-\breve{d}^2)}{48}$$

Set

$$I(\breve{d}) = \frac{1}{4} + \frac{19\breve{d}^4}{1152} + \frac{19\breve{d}^2(4-\breve{d}^2)}{384} + \frac{(4-\breve{d}^2)^2}{128} + \frac{\breve{d}(4-\breve{d}^2)}{48}$$

Differentiating the above equation, we get

$$I'(\breve{d}) = \frac{-29\breve{d}^3}{288} - \frac{3\breve{d}^2}{48} + \frac{13\breve{d}}{48} + \frac{4}{48}$$

Put $I'(\breve{d}) = 0$, we get three roots as $\breve{d}_1 = -1.8387$, $\breve{d}_2 = 1.5151$ and $\breve{d}_3 = -0.2971$. As $\breve{d}$ belongs to the interval [0,2] so the values $\breve{d}_1 = -1.8387$ and $\breve{d}_3 = -0.2971$ will not be considered as critical points. Calculations show that $I(\breve{d})$ has its maximum value at its root $\breve{d}_2 = 1.5151$.

$$|\breve{a}_5| \leq I(\breve{d}_2) = 0.6070$$

Hence we have

$$|\breve{a}_5| \leq 0.6070$$

□

3.3 Fekete-Szegö Inequality

The investigation of inequality given by Fekete and Szegö for the class $\mathbb{S}^*_\rho$ is as follow:

Theorem 3.3.1. If $\phi(\hat{r}) \in \mathbb{S}^*_\rho$, then we have $|\breve{a}_3 - \breve{a}_2^2| \leq \frac{1}{2}$

Proof. Using (3.7) and (3.8), we will get

$$|\breve{a}_3 - \breve{a}_2^2| = \left|\frac{\breve{d}_2}{4} - \frac{\breve{d}_1^2}{4}\right| = \frac{1}{4}|\breve{d}_2 - \breve{d}_1^2|$$

Using Lemma 2.15.3 with $\lambda = 1$, h=2 and k=1, we have

$$|\breve{a}_3 - \breve{a}_2^2| \leq \frac{2}{4} = \frac{1}{2}$$

So

$$|\breve{a}_3 - \breve{a}_2^2| \leq \frac{1}{2} \tag{3.17}$$

Hence proof is completed. □

3.4 Hankel Determinants

For the class S^*_ρ, the following are the related results:

Theorem 3.4.1. If $\phi(\hat{r}) \in \mathbb{S}^*_\rho$, then $|\breve{a}_2\breve{a}_3 - \breve{a}_4| \leq 0.4254$

Proof. From (3.7), (3.8) and (3.14),we obtain:

$$|\breve{a}_2\breve{a}_3 - \breve{a}_4| = \left|\frac{\breve{d}_1}{2}(\frac{\breve{d}_2}{4}) - \left(\frac{\breve{d}_3}{6} - \frac{\breve{d}_1\breve{d}_2}{24} - \frac{\breve{d}_1^3}{144}\right)\right|$$

$$|\breve{a}_2\breve{a}_3 - \breve{a}_4| = \left|\frac{\breve{d}_1^3}{144} + \frac{\breve{d}_1\breve{d}_2}{6} - \frac{\breve{d}_3}{6}\right|$$

Substitution of $\breve{d}_2$ and $\breve{d}_3$ by considering Lemma 2.15.1, we have

$$|\breve{a}_2\breve{a}_3 - \breve{a}_4| = \left|\frac{\breve{d}_1^3}{144} + \frac{\breve{d}_1}{6}\left(\frac{\breve{d}_1^2 + \psi(4-\breve{d}_1^2)}{2}\right)\right.$$
$$\left. - \frac{1}{6}\left(\frac{\breve{d}_1^3 + 2(4-\breve{d}_1^2)\breve{d}_1\psi - (4-\breve{d}_1^2)\breve{d}_1\psi^2 + 2(4-\breve{d}_1^2)(1-|\psi|^2)\Upsilon}{4}\right)\right|$$

On solving and combining like terms, we get

$$|\breve{a}_2\breve{a}_3 - \breve{a}_4| = \left|\frac{7\breve{d}_1^3}{144} + \frac{(4-\breve{d}_1^2)\breve{d}_1\psi^2}{24} - \frac{(4-\breve{d}_1^2)(1-|\psi|^2)\Upsilon}{12}\right|$$

Now by applying triangular inequality and let $\breve{d}_1 = \breve{d}$, and $|\psi| = t$

$$|\breve{a}_2\breve{a}_3 - \breve{a}_4| \leq \frac{7\breve{d}^3}{144} + (4-\breve{d}^2)\left(\frac{\breve{d}t^2}{24} + \frac{1}{12}\right)$$

We assume that

$$J(\breve{d},t) = \frac{1}{144}\left(7\breve{d}^3 + 6(4-\breve{d}^2)\breve{d}t^2 + 12(4-\breve{d}^2)\right)$$

Upon partial differentiation, we get

$$\frac{\partial J}{\partial t} = \frac{\breve{d}t(4-\breve{d}^2)}{12} > 0$$

It clearly shows about increasing function of $J(\breve{d},t)$ in [0,1]. So,

$$max(J(\breve{d},t)) = J(\breve{d},1) = \frac{1}{144}\left(7\breve{d}^3 + 6(4-\breve{d}^2)\breve{d} + 12(4-\breve{d}^2)\right)$$

Set

$$L(\breve{d}) = \frac{1}{144}\left(7\breve{d}^3 + 6(4-\breve{d}^2)\breve{d} + 12(4-\breve{d}^2)\right)$$

Differentiating the above, we have

$$L'(\breve{d}) = \frac{1}{144}\left(3\breve{d}^2 - 24\breve{d} + 24\right)$$

Set $L'(\breve{d}) = 0$,we get two roots as $\breve{d}_1 = 4+2\sqrt{2}$ and $\breve{d}_2 = 4-2\sqrt{2}$ On checking we come to know that $L(\breve{d})$ has maximum value at $\breve{d}_2$. So,

$$|\breve{a}_2\breve{a}_3 - \breve{a}_4| \leq L(\breve{d}_2) = 0.4254$$

Hence, we have

$$|\breve{a}_2\breve{a}_3 - \breve{a}_4| \leq 0.4254$$

□

Theorem 3.4.2. If $\phi(\hat{r}) \in \mathbb{S}^*_\rho$, then $|\breve{a}_2\breve{a}_4 - \breve{a}_3^2| \leq 0.3382$

Proof. From (3.7), (3.8) and (3.14),we obtain:

$$|\breve{a}_2\breve{a}_4 - \breve{a}_3^2| = \left|\frac{\breve{d}_1}{2}\left(\frac{\breve{d}_3}{6} - \frac{\breve{d}_1\breve{d}_2}{24} - \frac{\breve{d}_1^3}{144}\right) - \left(\frac{\breve{d}_2}{4}\right)^2\right|$$

$$|\breve{a}_2\breve{a}_4 - \breve{a}_3^2| = \left|\frac{\breve{d}_1\breve{d}_3}{12} - \frac{\breve{d}_1^2\breve{d}_2}{48} - \frac{\breve{d}_1^4}{288} - \frac{\breve{d}_2^2}{16}\right|$$

By using Lemma 2.15.1 substitute values of $\breve{d}_2$ and $\breve{d}_3$ we have

$$|\breve{a}_2\breve{a}_4-\breve{a}_3^2| = \left|\frac{\breve{d}_1}{12}\left(\frac{\breve{d}_1^3+2(4-\breve{d}_1^2)\breve{d}_1\psi-(4-\breve{d}_1^2)\breve{d}_1\psi^2+2(4-\breve{d}_1^2)(1-|\psi|^2)\Upsilon}{4}\right)-\right.$$
$$\left.\frac{\breve{d}_1^2}{48}\left(\frac{\breve{d}_1^2+\psi(4-\breve{d}_1^2)}{2}\right)-\frac{\breve{d}_1^4}{288}-\frac{1}{16}\left(\frac{\breve{d}_1^2+\psi(4-\breve{d}_1^2)}{2}\right)^2\right|$$

After simplification,we get

$$|\breve{a}_2\breve{a}_4-\breve{a}_3^2| = \left|\frac{-5\breve{d}_1^4}{576}-\frac{(4-\breve{d}_1^2)\breve{d}_1^2\psi^2}{48}+\frac{\breve{d}_1(4-\breve{d}_1^2)(1-|\psi|^2)\Upsilon}{24}-\frac{\psi^2(4-\breve{d}_1^2)^2}{64}\right|$$

$$|\breve{a}_2\breve{a}_4-\breve{a}_3^2| = \frac{1}{576}\left|-5\breve{d}_1^4-12\breve{d}_1^2\psi^2(4-\breve{d}_1^2)+24\breve{d}_1(4-\breve{d}_1^2)(1-|\psi|^2)\Upsilon-9\psi^2(4-\breve{d}_1^2)^2\right|$$

Applying Lemma 2.15.2 and letting $\breve{d}_1=\breve{d}$ and $|\psi|=t$, we get

$$|\breve{a}_2\breve{a}_4-\breve{a}_3^2| \le \frac{1}{576}\left(5\breve{d}^4+12\breve{d}^2t^2(4-\breve{d}^2)+24\breve{d}(4-\breve{d}^2)+9t^2(4-\breve{d}^2)^2\right)$$

Suppose that

$$H(\breve{d},t) = \frac{1}{576}\left(5\breve{d}^4+12\breve{d}^2t^2(4-\breve{d}^2)+24\breve{d}(4-\breve{d}^2)+9t^2(4-\breve{d}^2)^2\right)$$

Upon partial differentiation, we get

$$\frac{\partial H}{\partial t} = \frac{(4-\breve{d}_1^2)}{576}\left(12\breve{d}^2(2t)+9(2t)(4-\breve{d}^2)\right)$$

$$\frac{\partial H}{\partial t} = \frac{(4-\breve{d}_1^2)}{576}\left(24\breve{d}^2t+18t(4-\breve{d}^2)\right) > 0$$

So $H(\breve{d},t)$ is an increasing function in [0,1]. Hence,

$$max(H(\breve{d},t)) = H(\breve{d},1) = \frac{1}{576}\left(5\breve{d}^4+12\breve{d}^2(4-\breve{d}^2)+24\breve{d}(4-\breve{d}^2)+9(4-\breve{d}^2)^2\right)$$

Set

$$M(\breve{d}) = \frac{1}{576}\left(5\breve{d}^4+12\breve{d}^2(4-\breve{d}^2)+24\breve{d}(4-\breve{d}^2)+9(4-\breve{d}^2)^2\right)$$

Now after differentiating and solving,we get $M'(\breve{d})$ as follow:

$$M'(\breve{d}) = \frac{1}{576}\left(8\breve{d}^3-72\breve{d}^2-48\breve{d}+96\right)$$

Put $M'(\breve{d})=0$, we have

$$8\breve{d}^3-72\breve{d}^2-48\breve{d}+96=0$$

On solving we will get three roots as $\breve{d}_1 = -1.4006$, $\breve{d}_2 = 9.4987$ and $\breve{d}_3 = 0.9020$. Since $\breve{d}$ belongs to interval [0,2] so $\breve{d}_1 = -1.4006$, $\breve{d}_2 = 9.4987$ will not be considered as critical values and calculations show that the maximum value of the function exists at $\breve{d}_3 = 0.9020$

$$|\breve{a}_2\breve{a}_4 - \breve{a}_3^2| \leq M(\breve{d}_3) = 0.3382$$

Hence

$$|\breve{a}_2\breve{a}_4 - \breve{a}_3^2| \leq 0.3382$$

□

Theorem 3.4.3. If $\phi(\hat{r}) \in \mathbb{S}^*_{\rho}$, then $|\mathbb{H}_{3,1}(\phi)| \leq 0.6271$

Proof. The upper bound of third order Hankel determinant is given by:

$$|\mathbb{H}_{3,1}(\phi)| \leq ||(\breve{a}_3 - \breve{a}_2^2)||\breve{a}_5| + |(\breve{a}_4 - \breve{a}_2\breve{a}_3)||\breve{a}_4| + |(\breve{a}_2\breve{a}_4 - \breve{a}_3^2)||\breve{a}_3|$$

Substitute the value of $|\breve{a}_2|, |\breve{a}_3|, |\breve{a}_4| and |\breve{a}_5|$ from Theorem 3.2.1, use $|(\breve{a}_3 - \breve{a}_2^2)|$ from Theorem 3.3.1, put $|(\breve{a}_4 - \breve{a}_2\breve{a}_3)|$ from Theorem 3.4.1 and $(\breve{a}_2\breve{a}_4 - \breve{a}_3^2)|$ by using Theorem 3.4.2, we have

$$|\mathbb{H}_{3,1}(\phi)| \leq 0.4257(\frac{1}{2}) + 0.5767(0.4257) + \frac{1}{2}(0.3382)$$

Calculations gives value of $\mathbb{H}_{3,1}(\phi)$ as

$$|\mathbb{H}_{3,1}(\phi)| \leq 0.6271$$

This completes the proof. □

3.5 Zalcman Functional

Theorem 3.5.1. If $\phi(\hat{r}) \in \mathbb{S}^*_{\rho}$, then $|\breve{a}_5 - \breve{a}_3^2| \leq 0.8570$.

Proof. From (3.8) and (3.14), we have the following:

$$\breve{a}_5 - \breve{a}_3^2 = \frac{\breve{d}_4}{8} - \frac{\breve{d}_1\breve{d}_3}{24} - \frac{\breve{d}_1^2\breve{d}_2}{192} + \frac{5\breve{d}_1^4}{1152} - \frac{\breve{d}_2^2}{32} - \frac{\breve{d}_2^2}{16}$$

$$\breve{a}_5 - \breve{a}_3^2 = \frac{\breve{d}_4}{8} - \frac{\breve{d}_1\breve{d}_3}{24} - \frac{\breve{d}_1^2\breve{d}_2}{192} + \frac{5\breve{d}_1^4}{1152} - \frac{3\breve{d}_2^2}{32}$$

Substitution of $\breve{d}_2$ and $\breve{d}_3$ by using Lemma 2.15.1 give us:

$$\breve{a}_5-\breve{a}_3^2=\left[\frac{\breve{d}_4}{8}-\frac{\breve{d}_1}{24}\left(\frac{\breve{d}_1^3+2(4-\breve{d}_1^2)\breve{d}_1\psi-(4-\breve{d}_1^2)\breve{d}_1\psi^2+2(4-\breve{d}_1^2)(1-|\psi|^2)\Upsilon}{4}\right)-\right.$$
$$\left.\frac{\breve{d}_1^2}{192}\left(\frac{\breve{d}_1^2+\psi(4-\breve{d}_1^2)}{2}\right)+\frac{5\breve{d}_1^4}{1152}-\frac{3}{32}\left(\frac{\breve{d}_1^2+\psi(4-\breve{d}_1^2)}{2}\right)^2\right]$$

After simplification, we will get

$$\breve{a}_5-\breve{a}_3^2=\left[\frac{\breve{d}_4}{8}-\frac{37\breve{d}_1^4}{1152}-\frac{9(4-\breve{d}_1^2)\breve{d}_1^2\psi}{128}+\frac{\breve{d}_1^2\psi^2(4-\breve{d}_1^2)}{96}+\frac{\breve{d}_1(4-\breve{d}_1^2)(1-|\psi|^2)\Upsilon}{48}\right.$$
$$\left.-\frac{3\psi^2(4-\breve{d}_1^2)^2}{128}\right]$$

Now taking modulus on both sides and let $\breve{d}_1=d$,we have

$$|\breve{a}_5-\breve{a}_3^2|=\left|\frac{\breve{d}_4}{8}-\frac{37\breve{d}^4}{1152}-\frac{9(4-\breve{d}^2)\breve{d}^2\psi}{128}+\frac{\breve{d}^2\psi^2(4-\breve{d}^2)}{96}+\frac{\breve{d}(4-\breve{d}^2)(1-|\psi|^2)\Upsilon}{48}\right.$$
$$\left.-\frac{3\psi^2(4-\breve{d}^2)^2}{128}\right|$$

An application of Lemma 2.15.2 on above and take $|\psi|=t$ we will have the following:

$$|\breve{a}_5-\breve{a}_3^2|\leq\frac{1}{4}+\frac{37\breve{d}^4}{1152}+\frac{9(4-\breve{d}^2)\breve{d}^2t}{128}+\frac{\breve{d}^2t^2(4-\breve{d}^2)}{96}+\frac{\breve{d}(4-\breve{d}^2)}{48}+\frac{3t^2(4-\breve{d}^2)^2}{128}$$

We assume that

$$L(\breve{d},t)=\frac{1}{4}+\frac{37\breve{d}^4}{1152}+\frac{9(4-\breve{d}^2)\breve{d}^2t}{128}+\frac{\breve{d}^2t^2(4-\breve{d}^2)}{96}+\frac{\breve{d}(4-\breve{d}^2)}{48}+\frac{3t^2(4-\breve{d}^2)^2}{128}$$

Upon partial differentiation, we get

$$\frac{\partial L}{\partial t}=\frac{9\breve{d}^2(4-\breve{d}^2)}{128}+\frac{2t(4-\breve{d}^2)\breve{d}^2}{96}+\frac{6t(4-\breve{d}^2)^2}{128}$$

As $\frac{\partial L}{\partial t}>0$, so it implies of $L(\breve{d},t)$ being an increasing function in [0,1]. Hence

$$max(L(\breve{d},t))=L(\breve{d},1)$$

$$L(\breve{d},1)=\frac{1}{4}+\frac{37\breve{d}^4}{1152}+(4-\breve{d}^2)\breve{d}^2\left(\frac{9}{128}+\frac{1}{96}\right)+\frac{\breve{d}(4-\breve{d}^2)}{48}+\frac{3(4-\breve{d}^2)^2}{128}$$

Set

$$B(\breve{d})=\frac{1}{4}+\frac{37\breve{d}^4}{1152}+(4-\breve{d}^2)\breve{d}^2\left(\frac{31}{384}\right)+\frac{\breve{d}(4-\breve{d}^2)}{48}+\frac{3(4-\breve{d}^2)^2}{128}$$

Simplification of $B(\breve{d})$ yields

$$B(\breve{d})=\frac{1}{1152}\left(-29\breve{d}^4-24\breve{d}^3+156\breve{d}^2+96\breve{d}+720\right)$$

Differentiate $B(\breve{d})$ and we will get

$$B'(\breve{d}) = \frac{1}{1152}\left(-116\breve{d}^3 - 72\breve{d}^2 + 312\breve{d} + 96\right)$$

Set $B'(\breve{d}) = 0$

$$-116\breve{d}^3 - 72\breve{d}^2 + 312\breve{d} + 96 = 0$$

From the above equation, we will get three roots as $\breve{d}_1 = -1.8387$, $\breve{d}_2 = 1.5151$ and $\breve{d}_3 = -0.2971$. $\breve{d}_1$ and $\breve{d}_3$ will not be considered as critical points as $\breve{d} \in [0,2]$ and hence calculations show that the function $B(\breve{d})$ has maximum value at its critical point $\breve{d}_2$.

$$|\breve{a}_5 - \breve{a}_3^2| \leq B(\breve{d}_2) = 0.8570$$

Hence

$$|\breve{a}_5 - \breve{a}_3^2| \leq 0.8570$$

This completes the proof. □

CHAPTER 4

$\tilde{q}_e$-STARLIKE FUNCTION ASSOCIATED WITH $\tilde{q}_e$-SINE INVERSE HYPERBOLIC FUNCTION

4.1 Overview

The aim of this chapter is to introduce a new class of univalent functions which is $\tilde{q}_e$-starlike function correspond to $\tilde{q}_e$-series of inverse sine hyperbolic function. This chapter provides a number of significant findings such as the bounds on the coefficients, the widely acknowledged inequality proposed by Fekete and Szeg$\ddot{o}$, the Zalcmann functional, and the famous Hankel determinants.

The novel q̃-starlike functions' class subordinated with q̃-inverse sine hyperbolic function is given as follow.

Definition 4.1.1. A function ϕ belongs to $\mathbb{A}$ is in $\mathbb{S}^*_{\rho\tilde{q}_e}$ then

$$\left(\frac{\hat{r}\mathbb{D}_{\tilde{q}_e}\phi(\hat{r})}{\phi(\hat{r})}\right) \prec 1+sinh^{-1}(\tilde{q}_e\hat{r}) \qquad ,\tilde{q}_e \in (0,1) \tag{4.1}$$

for all $\hat{r} \in \Psi$.

4.2 Coefficient Inequalities

For the class $\mathbb{S}^*_{\rho\tilde{q}_e}$, the obtained results of coefficient bounds for function are as follow.

Theorem 4.2.1. If $\phi(\hat{r}) \in \mathbb{S}^*_{\rho\tilde{q}_e}$ then $\tilde{q}_e$

$$|\breve{a}_2| \leq 1, \ \ |\breve{a}_3| \leq \frac{1}{(1+\tilde{q}_e)}, \ \ |\breve{a}_4| \leq \frac{(94332+72732\tilde{q}_e - 729\tilde{q}_e{}^2 - 729\tilde{q}_e{}^3)}{48000(1+\tilde{q}_e)(1+\tilde{q}_e+\tilde{q}_e{}^2)}$$

$$|\breve{a}_5| \leq \frac{(1536+6732\tilde{q}_e+8343\tilde{q}_e{}^2+4953\tilde{q}_e{}^3+540\tilde{q}_e{}^4+270\tilde{q}_e{}^5)}{1536(\tilde{q}_e{}^3+\tilde{q}_e{}^2+\tilde{q}_e+1)(1+\tilde{q}_e+\tilde{q}_e{}^2)(1+\tilde{q}_e)}$$

Proof. By definition

$$\left(\frac{\hat{r}\mathbb{D}_{\tilde{q}_e}\phi(\hat{r})}{\phi(\hat{r})}\right) \prec 1+sinh^{-1}(\tilde{q}_e\hat{r})$$

Since $\phi \in \mathbb{S}^*_{\rho\tilde{q}_e}$, so by using the principle of subordination, we have

$$\left(\frac{\hat{r}\mathbb{D}_{\tilde{q}_e}\phi(\hat{r})}{\phi(\hat{r})}\right) = 1+sinh^{-1}(\tilde{q}_e(\sigma(\hat{r}))). \tag{4.2}$$

Consider the function

$$p(\hat{r}) = \frac{1+\sigma(\hat{r})}{1-\sigma(\hat{r})} = 1+\breve{d}_1\hat{r}+\breve{d}_2\hat{r}^2+\breve{d}_3\hat{r}^3+\breve{d}_4\hat{r}^4+..., \tag{4.3}$$

where $p(\hat{r})$ is an analytic function in Ψ following the condition $p(0) = 1$.We can have

$$\sigma(\hat{r}) = \frac{(p(\hat{r})-1)}{(p(\hat{r})+1)} \tag{4.4}$$

Simplification of above yields,

$$\sigma(\hat{r}) = \left(\frac{\breve{d}_1}{2}\right)\hat{r}+\left(\frac{\breve{d}_2}{2}-\frac{\breve{d}_1^2}{4}\right)\hat{r}^2+\left(\frac{\breve{d}_3}{2}+\frac{\breve{d}_1^3}{8}-\frac{\breve{d}_1\breve{d}_2}{2}\right)\hat{r}^3+ \left(\frac{3\breve{d}_1^2 d_2}{8}-\frac{\breve{d}_1\breve{d}_3}{2}-\frac{\breve{d}_1^4}{16}-\frac{\breve{d}_2^2}{4}+\frac{\breve{d}_4}{2}\right)\hat{r}^4+... \tag{4.5}$$

As we know,

$$1+sinh^{-1}(\tilde{q}_e(\sigma(\hat{r}))) = 1+\tilde{q}_e(\sigma(\hat{r}))-\frac{(\tilde{q}_e(\sigma(\hat{r})))^3}{6}+\frac{3(\tilde{q}_e(\sigma(\hat{r})))^5}{40}-...$$

So,we will get

$$1+sinh^{-1}(\tilde{q}_e(\sigma(\hat{r}))) = 1+\left(\frac{\breve{d}_1}{2}\right)\tilde{q}_e\hat{r}+\left(\frac{\breve{d}_2}{2}-\frac{\breve{d}_1^2}{4}\right)\tilde{q}_e\hat{r}^2+\left(\frac{\tilde{q}_e\breve{d}_3}{2}-\frac{\tilde{q}_e\breve{d}_1\breve{d}_2}{2}+\frac{\tilde{q}_e\breve{d}_1^3}{8} -\frac{\tilde{q}_e{}^3\breve{d}_1^3}{48}\right)\hat{r}^3+\left(\frac{\tilde{q}_e\breve{d}_4}{2}-\frac{\tilde{q}_e\breve{d}_2^2}{4}-\frac{\tilde{q}_e\breve{d}_1\breve{d}_3}{2}+\frac{3\tilde{q}_e\breve{d}_1^2\breve{d}_2}{8}-\frac{\tilde{q}_e\breve{d}_1^4}{16}-\frac{3\tilde{q}_e{}^3\breve{d}_1^2\breve{d}_2}{48}+\frac{3\tilde{q}_e{}^3\breve{d}_1^4}{96}\right)\hat{r}^4+...$$

As

$$\mathbb{D}_{\tilde{q}_e}(\phi(\hat{r})) = \frac{\phi(\hat{r})-\phi(\tilde{q}_e\hat{r})}{(1-\tilde{q}_e)\hat{r}},$$

So we get

$$\frac{\hat{r}\mathbb{D}_{\tilde{q}_e}\phi(\hat{r})}{\phi(\hat{r})} = 1+(\breve{a}_2)\tilde{q}_e\hat{r}+(\tilde{q}_e(1+\tilde{q}_e)\breve{a}_3-\tilde{q}_e\breve{a}_2^2)\hat{r}^2+\Bigg(\tilde{q}_e(1+\tilde{q}_e+\tilde{q}_e{}^2)\breve{a}_4-\tilde{q}_e(2+\tilde{q}_e)\breve{a}_2\breve{a}_3 +\tilde{q}_e\breve{a}_2^3\Bigg)\hat{r}^3+\tilde{q}_e\Bigg(\tilde{q}_e{}^3\breve{a}_5+(-\breve{a}_2\breve{a}_4+\breve{a}_5)\tilde{q}_e{}^2+(-\breve{a}_3^2+\breve{a}_3\breve{a}_2^2-\breve{a}_2\breve{a}_4+\breve{a}_5)\tilde{q}_e+3\breve{a}_3\breve{a}_2^2-2\breve{a}_2\breve{a}_4 -\breve{a}_2^4-\breve{a}_3^2+\breve{a}_5\Bigg)\hat{r}^4+\ldots$$

Substitution of values in (4.2) will give us

$$1+(\breve{a}_2)\tilde{q}_e\hat{r}+(\tilde{q}_e(1+\tilde{q}_e)\breve{a}_3-\tilde{q}_e\breve{a}_2^2)\hat{r}^2+(\tilde{q}_e(1+\tilde{q}_e+\tilde{q}_e{}^2)\breve{a}_4-\tilde{q}_e(2+\tilde{q}_e)\breve{a}_2\breve{a}_3+\tilde{q}_e\breve{a}_2^3)\hat{r}^3+ \tilde{q}_e\Bigg(\tilde{q}_e{}^3\breve{a}_5+(-\breve{a}_2\breve{a}_4+\breve{a}_5)\tilde{q}_e{}^2+(-\breve{a}_3^2+\breve{a}_3\breve{a}_2^2-\breve{a}_2\breve{a}_4+\breve{a}_5)\tilde{q}_e+3\breve{a}_3\breve{a}_2^2-2\breve{a}_2\breve{a}_4-\breve{a}_2^4-\breve{a}_3^2 +\breve{a}_5\Bigg)\hat{r}^4+\ldots = 1+\left(\frac{\breve{d}_1}{2}\right)\tilde{q}_e\hat{r}+\left(\frac{\breve{d}_2}{2}-\frac{\breve{d}_1^2}{4}\right)\tilde{q}_e\hat{r}^2+\left(\frac{\tilde{q}_e\breve{d}_3}{2}-\frac{\tilde{q}_e\breve{d}_1\breve{d}_2}{2}+\frac{\tilde{q}_e\breve{d}_1^3}{8}-\frac{\tilde{q}_e{}^3\breve{d}_1^3}{48}\right) \hat{r}^3+\left(\frac{\tilde{q}_e\breve{d}_4}{2}-\frac{\tilde{q}_e\breve{d}_2^2}{4}-\frac{\tilde{q}_e\breve{d}_1\breve{d}_3}{2}+\frac{3\tilde{q}_e\breve{d}_1^2\breve{d}_2}{8}-\frac{\tilde{q}_e\breve{d}_1^4}{16}-\frac{3\tilde{q}_e{}^3\breve{d}_1^2\breve{d}_2}{48}+\frac{3\tilde{q}_e{}^3\breve{d}_1^4}{96}\right)\hat{r}^4+\ldots \quad (4.6)$$

Comparison of both sides of above equation will result in :

$$\breve{a}_2\tilde{q}_e = \frac{\tilde{q}_e\breve{d}_1}{2}, \quad (4.7)$$

$$\tilde{q}_e(1+\tilde{q}_e)\breve{a}_3-\tilde{q}_e\breve{a}_2^2 = \tilde{q}_e\left(\frac{\breve{d}_2}{2}-\frac{\breve{d}_1^2}{4}\right), \quad (4.8)$$

$$\tilde{q}_e\Bigg((1+\tilde{q}_e+\tilde{q}_e{}^2)\breve{a}_4-(2+\tilde{q}_e)\breve{a}_2\breve{a}_3+\breve{a}_2^3\Bigg) = \tilde{q}_e\left(\frac{\breve{d}_3}{2}-\frac{\breve{d}_1\breve{d}_2}{2}+\frac{\breve{d}_1^3}{8}-\frac{\tilde{q}_e{}^2\breve{d}_1^3}{48}\right), \quad (4.9)$$

$$\tilde{q}_e\Bigg(\breve{a}_5\tilde{q}_e{}^3-\breve{a}_2\breve{a}_4\tilde{q}_e{}^2+\breve{a}_5\tilde{q}_e{}^2-\breve{a}_3^2\tilde{q}_e+\breve{a}_3\breve{a}_2^2\tilde{q}_e-\breve{a}_2\breve{a}_4\tilde{q}_e+\breve{a}_5\tilde{q}_e+3\breve{a}_3\breve{a}_2^2-2\breve{a}_2\breve{a}_4-\breve{a}_2^4 -\breve{a}_3^2+\breve{a}_5\Bigg) = \tilde{q}_e\left(\frac{\breve{d}_4}{2}-\frac{\breve{d}_2^2}{4}-\frac{\breve{d}_1\breve{d}_3}{2}+\frac{3\breve{d}_1^2\breve{d}_2}{8}-\frac{\breve{d}_1^4}{16}-\frac{3\tilde{q}_e{}^2\breve{d}_1^2\breve{d}_2}{48}+\frac{3\breve{d}_1^4\tilde{q}_e{}^2}{96}\right) \quad (4.10)$$

From (4.7) we get coefficient $\breve{a}_2$ as

$$\breve{a}_2 = \frac{\breve{d}_1}{2} \quad (4.11)$$

On solving (4.8) we get $\breve{a}_3$,

$$\breve{a}_3 = \frac{\breve{d}_2}{2(1+\tilde{q}_e)} \quad (4.12)$$

By solving (4.10) we determine $\breve{a}_4$

$$\tilde{q}_e(1+\tilde{q}_e+\tilde{q}_e{}^2)\breve{a}_4 = \tilde{q}_e\left((2+\tilde{q}_e)\breve{a}_2\breve{a}_3-\breve{a}_2^3+\frac{\breve{d}_3}{2}-\frac{\breve{d}_1\breve{d}_2}{2}+\frac{\breve{d}_1^3}{8}-\frac{\tilde{q}_e{}^2\breve{d}_1^3}{48}\right), \quad (4.13)$$

Now substitute the values of $\breve{a}_2$ and $\breve{a}_3$ in $\breve{a}_4$, we get

$$\breve{a}_4 = \frac{1}{((1+\tilde{q}_e+\tilde{q}_e{}^2)}\left(\frac{\breve{d}_3}{2} - \frac{\tilde{q}_e\breve{d}_1\breve{d}_2}{4(1+\tilde{q}_e)} - \frac{\tilde{q}_e{}^2\breve{d}_1^3}{48}\right) \tag{4.14}$$

Now for $\breve{a}_5$,we will use equation (4.10),

$$\breve{a}_5(\tilde{q}_e{}^4+\tilde{q}_e{}^3+\tilde{q}_e{}^2+\tilde{q}_e) = \Big(\breve{a}_2\breve{a}_4\tilde{q}_e{}^3 + \breve{a}_3^2\tilde{q}_e{}^2 - \breve{a}_3\breve{a}_2^2\tilde{q}_e{}^2 + \breve{a}_2\breve{a}_4\tilde{q}_e{}^2 - 3\breve{a}_3\breve{a}_2^2\tilde{q}_e + 2\breve{a}_2\breve{a}_4\tilde{q}_e +$$
$$\breve{a}_2^4\tilde{q}_e + \breve{a}_3^2\tilde{q}_e + \frac{\tilde{q}_e\breve{d}_4}{2} - \frac{\tilde{q}_e\breve{d}_2^2}{4} - \frac{\tilde{q}_e\breve{d}_1\breve{d}_3}{2} + \frac{3\tilde{q}_e\breve{d}_1^2\breve{d}_2}{8} - \frac{\tilde{q}_e\breve{d}_1^4}{16} - \frac{3\tilde{q}_e{}^3\breve{d}_1^2\breve{d}_2}{48} + \frac{3\breve{d}_1^4\tilde{q}_e{}^3}{96}\Big)$$

Now we will substitute values of $\breve{a}_2$, $\breve{a}_3$ and $\breve{a}_4$ in $\breve{a}_5$, and after calculations we will get

$$\breve{a}_5 = \frac{1}{(\tilde{q}_e{}^3+\tilde{q}_e{}^2+\tilde{q}_e+1)}\Big(\frac{\breve{d}_4}{2} - \frac{\tilde{q}_e\breve{d}_2^2}{4(1+\tilde{q}_e)} - \frac{\tilde{q}_e(1+\tilde{q}_e)\breve{c}_1\breve{d}_3}{4(1+\tilde{q}_e+\tilde{q}_e{}^2)} + \frac{\breve{d}_1^2\breve{d}_2\tilde{q}_e{}^2(3-6\tilde{q}_e{}^2-3\tilde{q}_e{}^3)}{48(1+\tilde{q}_e)(1+\tilde{q}_e+\tilde{q}_e{}^2)} +$$
$$\frac{\tilde{q}_e{}^2(2\tilde{q}_e{}^2+2\tilde{q}_e+1)\breve{d}_1^4}{96(1+\tilde{q}_e+\tilde{q}_e{}^2)}\Big)$$

Now from (4.12) we have

$$\breve{a}_2 = \frac{\breve{d}_1}{2}$$

Applying Lemma 2.15.2 ,we will get

$$|\breve{a}_2| = \left|\frac{\breve{d}_1}{2}\right| \le \frac{|\breve{d}_1|}{2} \le \frac{2}{2} = 1$$
$$|\breve{a}_2| \le 1. \tag{4.15}$$

From (4.13), we have:

$$\breve{a}_3 = \frac{\breve{d}_2}{2(1+\tilde{q}_e)}$$

Using Lemma 2.15.2 leads us to

$$|\breve{a}_3| = \left|\frac{\breve{d}_2}{2(1+\tilde{q}_e)}\right| = \frac{|\breve{d}_2|}{2(1+\tilde{q}_e)} \le \frac{2}{2(1+\tilde{q}_e)} = \frac{1}{1+\tilde{q}_e}$$
$$|\breve{a}_3| \le \frac{1}{1+\tilde{q}_e} \tag{4.16}$$

From (4.15) we have

$$\breve{a}_4 = \frac{1}{((1+\tilde{q}_e+\tilde{q}_e{}^2)(1+\tilde{q}_e)}\left(\frac{\breve{d}_3(1+\tilde{q}_e)}{2} - \frac{\tilde{q}_e\breve{d}_1\breve{d}_2}{4} - \frac{\tilde{q}_e{}^2(1+\tilde{q}_e)\breve{d}_1^3}{48}\right)$$

Applying Lemma 2.15.1 on above equation, we get

$$\breve{a}_4 = \frac{1}{(1+\tilde{q}_e+\tilde{q}_e{}^2)}\Bigg[\left(\frac{\breve{d}_1^3 + 2(4-\breve{d}_1^2)\breve{d}_1\psi - (4-\breve{d}_1^2)\breve{d}_1\psi^2 + 2(4-\breve{d}_1^2)(1-|\psi|^2)\Upsilon}{8}\right) -$$
$$\frac{\tilde{q}_e\breve{d}_1}{4(1+\tilde{q}_e)}\left(\frac{\breve{d}_1^2+\psi(4-\breve{d}_1^2}{2}\right) - \frac{\tilde{q}_e{}^2\breve{d}_1^3}{48}\Bigg]$$

After doing simplification and combining like terms,we will get

$$\breve{a}_4 = \frac{1}{48(1+\tilde{q}_e)(1+\tilde{q}_e+\tilde{q}_e{}^2)}\Big((6-\tilde{q}_e{}^2-\tilde{q}_e{}^3)\breve{d}_1^3+6(2+\tilde{q}_e)(4-\breve{d}_1^2)\breve{d}_1\psi-6(1+\tilde{q}_e)$$
$$(4-\breve{d}_1^2)\breve{d}_1\psi^2+12(1+\tilde{q}_e)((4-\breve{d}_1^2)(1-|\psi|^2)\Upsilon\Big)$$

Taking modulus on both sides and let $\breve{d}_1 = \breve{d}$, we have

$$|\breve{a}_4| \leq \frac{1}{48(1+\tilde{q}_e)(1+\tilde{q}_e+\tilde{q}_e{}^2)}\Big|(6-\tilde{q}_e{}^2-\tilde{q}_e{}^3)\breve{d}^3+6(2+\tilde{q}_e)(4-\breve{d}^2)\breve{d}\psi-6(1+\tilde{q}_e)$$
$$(4-\breve{d}^2)\breve{d}\psi^2+12(1+\tilde{q}_e)((4-\breve{d}^2)(1-|\psi|^2)\Upsilon\Big|$$

Now $|\psi| = t$, we have

$$|\breve{a}_4| \leq \frac{1}{48(1+\tilde{q}_e)(1+\tilde{q}_e+\tilde{q}_e{}^2)}\Big((6-\tilde{q}_e{}^2-\tilde{q}_e{}^3)\breve{d}^3+6(2+\tilde{q}_e)(4-\breve{d}^2)\breve{d}t+6(1+\tilde{q}_e)$$
$$(4-\breve{d}^2)\breve{d}t^2+12(1+\tilde{q}_e)((4-\breve{d}^2)\Big)$$

We assume that

$$G_{\tilde{q}_e}(\breve{d},t) = \frac{1}{48(1+\tilde{q}_e)(1+\tilde{q}_e+\tilde{q}_e{}^2)}\Big((6-\tilde{q}_e{}^2-\tilde{q}_e{}^3)\breve{d}^3+6(2+\tilde{q}_e)(4-\breve{d}^2)\breve{d}t+6(1+\tilde{q}_e)$$
$$(4-\breve{d}^2)\breve{d}t^2+12(1+\tilde{q}_e)((4-\breve{d}^2)\Big)$$

Upon partial differentiation, we get

$$\frac{\partial G_{\tilde{q}_e}}{\partial t} = \frac{1}{48(1+\tilde{q}_e)(1+\tilde{q}_e+\tilde{q}_e{}^2)}\Big(6(2+\tilde{q}_e)(4-\breve{d}^2)\breve{d}+12(1+\tilde{q}_e)(4-\breve{d}^2)\breve{d}t\Big) > 0$$

This implies that $G_{\tilde{q}_e}(\breve{d},t)$ is an increasing function in [0,1]. So,

$$max(G_{\tilde{q}_e}(\breve{d},t)) = G_{\tilde{q}_e}(\breve{d},1)$$

$$G_{\tilde{q}_e}(\breve{d},1) = \frac{1}{48(1+\tilde{q}_e)(1+\tilde{q}_e+\tilde{q}_e{}^2)}\Big((6-\tilde{q}_e{}^2-\tilde{q}_e{}^3)\breve{d}^3+6(2+\tilde{q}_e)(4-\breve{d}^2)\breve{d}+6(1+\tilde{q}_e)$$
$$(4-\breve{d}^2)\breve{d}+12(1+\tilde{q}_e)((4-\breve{d}^2)\Big)$$

Set

$$K_{\tilde{q}_e}(\breve{d}) = \frac{1}{48(1+\tilde{q}_e)(1+\tilde{q}_e+\tilde{q}_e{}^2)}\Big((6-\tilde{q}_e{}^2-\tilde{q}_e{}^3)\breve{d}^3+6(2+\tilde{q}_e)(4-\breve{d}^2)\breve{d}+6(1+\tilde{q}_e)$$
$$(4-\breve{d}^2)\breve{d}+12(1+\tilde{q}_e)((4-\breve{d}^2)\Big)$$

Upon Differentiation and simplification of above we will have

$$K'_{\tilde{q}_e}(\breve{d}) = \frac{1}{48(1+\tilde{q}_e)(1+\tilde{q}_e+\tilde{q}_e{}^2)}\left(-(36+3\tilde{q}_e{}^2+3\tilde{q}_e{}^3+36\tilde{q}_e)\breve{d}^2-24(1+\tilde{q}_e)\breve{d}+72+48\tilde{q}_e\right)$$

We can write above as,

$$K'_{\tilde{q}_e}(\breve{d}) = \frac{1}{48(1+\tilde{q}_e)(1+\tilde{q}_e+\tilde{q}_e{}^2)}\left(A\breve{d}^2+B\breve{d}+C\right)$$

where

$$A = -(36+3\tilde{q}_e{}^2+3\tilde{q}_e{}^3+36\tilde{q}_e)$$

$$B = -24(1+\tilde{q}_e)$$

and

$$C = 72+48\tilde{q}_e$$

After calculations, it can be observed that $\left(A\breve{d}^2+B\breve{d}+C\right) > 0$ for $\breve{d} \in (0,0.9]$ and $\tilde{q}_e \in (0,1)$. Also, $\left(A\breve{d}^2+B\breve{d}+C\right) < 0$ for $\breve{d} \in (0.9,2)$ and $\tilde{q}_e \in (0,1)$. This implies that $K'_{\tilde{q}_e}(\breve{d}) > 0$ for $\breve{d} \in (0,0.9]$ and $\tilde{q}_e \in (0,1)$ and $K'_{\tilde{q}_e}(\breve{d}) < 0$ for $\breve{d} \in (0.9,2)$ and $\tilde{q}_e \in (0,1)$. It implies $K_{\tilde{q}_e}(\breve{d})$is increasing function in (0,0.9] and decreasing in (0.9,2). Hence,

$$|\breve{a}_4| \leq K_{\tilde{q}_e}(0.9) = \frac{1}{48(1+\tilde{q}_e)(1+\tilde{q}_e+\tilde{q}_e{}^2)}\Big((6-\tilde{q}_e{}^2-\tilde{q}_e{}^3)(0.9)^3+6(2+\tilde{q}_e)(4-(0.9)^2)) \\ (0.9)+6(1+\tilde{q}_e)(4-(0.9)^2)(0.9)+12(1+\tilde{q}_e)((4-(0.9)^2)\Big)$$

After solving all terms inside, we will get $|\breve{a}_4|$ as,

$$|\breve{a}_4| \leq \frac{1}{48000(1+\tilde{q}_e)(1+\tilde{q}_e+\tilde{q}_e{}^2)}\left(94332+72732\tilde{q}_e-729\tilde{q}_e{}^2-729\tilde{q}_e{}^3\right) \tag{4.17}$$

By solving (4.11) we will get $\breve{a}_5$ as:

$$\breve{a}_5 = \frac{1}{96(\tilde{q}_e{}^3+\tilde{q}_e{}^2+\tilde{q}_e+1)(1+\tilde{q}_e+\tilde{q}_e{}^2)(1+\tilde{q}_e)}\Big[48\breve{d}_4(1+\tilde{q}_e)(1+\tilde{q}_e+\tilde{q}_e{}^2)-24\tilde{q}_e\breve{d}_2^2 \\ (1+\tilde{q}_e+\tilde{q}_e{}^2)-24\tilde{q}_e(1+\tilde{q}_e)^2\breve{d}_1\breve{d}_3+2\tilde{q}_e{}^2\breve{d}_1^2\breve{d}_2(3-6\tilde{q}_e{}^2-3\tilde{q}_e{}^3)+\tilde{q}_e{}^2(1+\tilde{q}_e) \\ (2\tilde{q}_e{}^2+2\tilde{q}_e+1)\breve{d}_1^4\Big]$$

Now by using lemma 2.15.1, substitute value of $\breve{d}_2$ and $\breve{d}_3$ and we will get the following:

$$\breve{a}_5 = \frac{1}{96(\tilde{q}_e{}^3+\tilde{q}_e{}^2+\tilde{q}_e+1)(1+\tilde{q}_e+\tilde{q}_e{}^2)(1+\tilde{q}_e)}\Big[48(1+2\tilde{q}_e+2\tilde{q}_e{}^2+\tilde{q}_e{}^3)\breve{d}_4-24\tilde{q}_e\Big(1+\tilde{q}_e \\ +\tilde{q}_e{}^2\Big)\left(\frac{\breve{d}_1^2+\psi(4-\breve{d}_1^2)}{2}\right)^2-6\tilde{q}_e(1+\tilde{q}_e)^2\breve{d}_1\Big(\breve{d}_1^3+2(4-\breve{d}_1^2)\breve{d}_1\psi-(4-\breve{d}_1^2)\breve{d}_1\psi^2+2(4-\breve{d}_1^2) \\ (1-|\psi|^2)\Upsilon\Big)+2\tilde{q}_e{}^2(3-6\tilde{q}_e{}^2-3\tilde{q}_e{}^3)\breve{d}_1^2\left(\frac{\breve{d}_1^2+\psi(4-\breve{d}_1^2)}{2}\right)+\tilde{q}_e{}^2(1+\tilde{q}_e)(2\tilde{q}_e{}^2+2\tilde{q}_e+1)\breve{d}_1^4\Big]$$

Solving all terms inside brackets we will get

$$\check{a}_5 = \frac{1}{96(\tilde{q}_e^3+\tilde{q}_e^2+\tilde{q}_e+1)(1+\tilde{q}_e+\tilde{q}_e^2)(1+\tilde{q}_e)}\Big(48(1+2\tilde{q}_e+2\tilde{q}_e^2+\tilde{q}_e^3)\check{d}_4-\check{d}_1^4(12\tilde{q}_e+$$
$$14\tilde{q}_e^2+9\tilde{q}_e^3+2\tilde{q}_e^4+\tilde{q}_e^5)-6\tilde{q}_e(1+\tilde{q}_e+\tilde{q}_e^2)\psi^2(4-\check{d}_1^2)^2-\psi\check{d}_1^2(4-\check{d}_1^2)(24\tilde{q}_e+33\tilde{q}_e^2+24\tilde{q}_e^3$$
$$+6\tilde{q}_e^4+3\tilde{q}_e^5)+6\tilde{q}_e(1+\tilde{q}_e^2+2\tilde{q}_e)\check{d}_1^2\psi^2(4-\check{d}_1^2)-12\tilde{q}_e(1+\tilde{q}_e)^2\check{d}_1(4-\check{d}_1^2)(1-|\psi|^2)\Upsilon\Big)$$

Now by applying triangular inequality with lemma 2.15.2 and let $\check{d}_1=\check{d}$ and $|\psi|=t$

$$|\check{a}_5| \leq \frac{1}{96(\tilde{q}_e^3+\tilde{q}_e^2+\tilde{q}_e+1)(1+\tilde{q}_e+\tilde{q}_e^2)(1+\tilde{q}_e)}\Big(48(1+2\tilde{q}_e+2\tilde{q}_e^2+\tilde{q}_e^3).2+\check{d}^4(12\tilde{q}_e+$$
$$14\tilde{q}_e^2+9\tilde{q}_e^3+2\tilde{q}_e^4+\tilde{q}_e^5)+6\tilde{q}_e(1+\tilde{q}_e+\tilde{q}_e^2)t^2(4-\check{d}^2)^2+t\check{d}^2(4-\check{d}^2)(24\tilde{q}_e+33\tilde{q}_e^2+24\tilde{q}_e^3$$
$$+6\tilde{q}_e^4+3\tilde{q}_e^5)+6\tilde{q}_e(1+\tilde{q}_e^2+2\tilde{q}_e)\check{d}^2t^2(4-\check{d}^2)+12\tilde{q}_e(1+\tilde{q}_e)^2\check{d}(4-\check{d}^2)\Big)$$

We assume that

$$H_{\tilde{q}_e}(\check{d},t) = \frac{1}{96(\tilde{q}_e^3+\tilde{q}_e^2+\tilde{q}_e+1)(1+\tilde{q}_e+\tilde{q}_e^2)(1+\tilde{q}_e)}\Big(48(1+2\tilde{q}_e+2\tilde{q}_e^2+\tilde{q}_e^3).2+\check{d}^4(12$$
$$\tilde{q}_e+14\tilde{q}_e^2+9\tilde{q}_e^3+2\tilde{q}_e^4+\tilde{q}_e^5)+6\tilde{q}_e(1+\tilde{q}_e+\tilde{q}_e^2)t^2(4-\check{d}^2)^2+t\check{d}^2(4-\check{d}^2)(24\tilde{q}_e+33\tilde{q}_e^2$$
$$+24\tilde{q}_e^3+6\tilde{q}_e^4+3\tilde{q}_e^5)+6\tilde{q}_e(1+\tilde{q}_e^2+2\tilde{q}_e)\check{d}^2t^2(4-\check{d}^2)+12\tilde{q}_e(1+\tilde{q}_e)^2\check{d}(4-\check{d}^2)\Big)$$

Upon partial differentiation we have

$$\frac{\partial H_{\tilde{q}_e}}{\partial t} = \frac{1}{96(\tilde{q}_e^3+\tilde{q}_e^2+\tilde{q}_e+1)(1+\tilde{q}_e+\tilde{q}_e^2)(1+\tilde{q}_e)}\Big(0+0+(2t)6\tilde{q}_e(1+\tilde{q}_e+\tilde{q}_e^2)(4-\check{d}^2)^2$$
$$+\check{d}^2(4-\check{d}^2)(24\tilde{q}_e+33\tilde{q}_e^2+24\tilde{q}_e^3+6\tilde{q}_e^4+3\tilde{q}_e^5)+6\tilde{q}_e(1+\tilde{q}_e^2+2\tilde{q}_e)\check{d}^2(2t)(4-\check{d}^2)\Big)>0$$

So $H_{\tilde{q}_e}(\check{d},t)$ is clearly an increasing function in [0,1]. Hence,

$$maxH_{\tilde{q}_e}(\check{d},t) = H_{\tilde{q}_e}(\check{d},1)$$

$$H_{\tilde{q}_e}(\check{d},1) = \frac{1}{96(\tilde{q}_e^3+\tilde{q}_e^2+\tilde{q}_e+1)(1+\tilde{q}_e+\tilde{q}_e^2)(1+\tilde{q}_e)}\Big(96(1+2\tilde{q}_e+2\tilde{q}_e^2+\tilde{q}_e^3)+\check{d}^4(12\tilde{q}_e$$
$$+14\tilde{q}_e^2+9\tilde{q}_e^3+2\tilde{q}_e^4+\tilde{q}_e^5)+6\tilde{q}_e(1+\tilde{q}_e+\tilde{q}_e^2)(4-\check{d}^2)^2+\check{d}^2(4-\check{d}^2)(24\tilde{q}_e+33\tilde{q}_e^2$$
$$+24\tilde{q}_e^3+6\tilde{q}_e^4+3\tilde{q}_e^5)+6\tilde{q}_e(1+\tilde{q}_e^2+2\tilde{q}_e)\check{d}^2(4-\check{d}^2)+12\tilde{q}_e(1+\tilde{q}_e)^2\check{d}(4-\check{d}^2)\Big)$$

Let us assume

$$I_{\tilde{q}_e}(\check{d}) = \frac{1}{96(\tilde{q}_e^3+\tilde{q}_e^2+\tilde{q}_e+1)(1+\tilde{q}_e+\tilde{q}_e^2)(1+\tilde{q}_e)}\Big(96(1+2\tilde{q}_e+2\tilde{q}_e^2+\tilde{q}_e^3)+\check{d}^4(12\tilde{q}_e$$
$$+14\tilde{q}_e^2+9\tilde{q}_e^3+2\tilde{q}_e^4+\tilde{q}_e^5)+6\tilde{q}_e(1+\tilde{q}_e+\tilde{q}_e^2)(4-\check{d}^2)^2+\check{d}^2(4-\check{d}^2)(24\tilde{q}_e+33\tilde{q}_e^2$$
$$+24\tilde{q}_e^3+6\tilde{q}_e^4+3\tilde{q}_e^5)+6\tilde{q}_e(1+\tilde{q}_e^2+2\tilde{q}_e)\check{d}^2(4-\check{d}^2)+12\tilde{q}_e(1+\tilde{q}_e)^2\check{d}(4-\check{d}^2)\Big)$$

Differentiating above, we have $I'_{\tilde{q}_e}(\check{d})$ as following:

$$I'_{\tilde{q}_e}(\check{c}) = \frac{1}{96(\tilde{q}_e{}^3+\tilde{q}_e{}^2+\tilde{q}_e+1)(1+\tilde{q}_e+\tilde{q}_e{}^2)(1+\tilde{q}_e)}\Big(4\check{d}^3(12\tilde{q}_e+14\tilde{q}_e{}^2+9\tilde{q}_e{}^3+2\tilde{q}_e{}^4+\tilde{q}_e{}^5)$$
$$+6\tilde{q}_e(1+\tilde{q}_e+\tilde{q}_e{}^2)2(4-\check{d}^2)(-2\check{d})+(24\tilde{q}_e+33\tilde{q}_e{}^2+24\tilde{q}_e{}^3+6\tilde{q}_e{}^4+3\tilde{q}_e{}^5+6\tilde{q}_e+6\tilde{q}_e{}^3$$
$$+12\tilde{q}_e{}^2)(2\check{d}(4-\check{d}^2)+\check{d}^2(-2\check{d}))+12\tilde{q}_e(1+\tilde{q}_e)^2((4-\check{d}^2)+\check{d}(-2\check{d})\Big)$$

Simplification will give us

$$I'_{\tilde{q}_e}(\check{d}) = \frac{1}{96(\tilde{q}_e{}^3+\tilde{q}_e{}^2+\tilde{q}_e+1)(1+\tilde{q}_e+\tilde{q}_e{}^2)(1+\tilde{q}_e)}\Big(\check{d}^3(-48\tilde{q}_e-100\tilde{q}_e{}^2-60\tilde{q}_e{}^3-16\tilde{q}_e{}^2-$$
$$8\tilde{q}_e{}^5)-36\check{d}^2\tilde{q}_e(1+\tilde{q}_e)^2+\check{d}(144\tilde{q}_e+264\tilde{q}_e{}^2+144\tilde{q}_e{}^3+48\tilde{q}_e{}^4+24\tilde{q}_e{}^5)+48\tilde{q}_e(1+\tilde{q}_e)^2\Big)$$

We will have

$$I'_{\tilde{q}_e}(\check{d}) = \frac{1}{96(\tilde{q}_e{}^3+\tilde{q}_e{}^2+\tilde{q}_e+1)(1+\tilde{q}_e+\tilde{q}_e{}^2)(1+\tilde{q}_e)}\left(A\check{d}^3+B\check{d}^2+C\check{d}+D\right)$$

where

$$A = (-48\tilde{q}_e-100\tilde{q}_e{}^2-60\tilde{q}_e{}^3-16\tilde{q}_e{}^2-8\tilde{q}_e{}^5)$$

$$B = -36\tilde{q}_e(1+\tilde{q}_e)^2$$

$$C = (144\tilde{q}_e+264\tilde{q}_e{}^2+144\tilde{q}_e{}^3+48\tilde{q}_e{}^4+24\tilde{q}_e{}^5)$$

and

$$D = 48\tilde{q}_e(1+\tilde{q}_e)^2$$

Certain calculations show that $A\check{d}^3+B\check{d}^2+C\check{d}+D$ is greater than zero for $\check{d}\in(0,1.5]$ and $\tilde{q}_e$ belongs to (0,1). Also $A\check{d}^3+B\check{d}^2+C\check{d}+D$ is less than zero for $\check{d}\in(1.6,2)$. This means $I'_{\tilde{q}_e}(\check{d})>0$ for $\check{d}\in(0,1.5]$ and $I'_{\tilde{q}_e}(\check{d})<0$ for $\check{d}\in(1.6,2)$. This means that

$$|\check{a}_5| \le I_{\tilde{q}_e}(1.5)$$

From $I_{\tilde{q}_e}(\check{d})$, we will get the following form after simplication of terms inside paranthesis:

$$I_{\tilde{q}_e}(\check{d}) = \frac{1}{96(\tilde{q}_e{}^3+\tilde{q}_e{}^2+\tilde{q}_e+1)(1+\tilde{q}_e+\tilde{q}_e{}^2)(1+\tilde{q}_e)}\Big((96+288\tilde{q}_e+288\tilde{q}_e{}^2+192\tilde{q}_e{}^3)$$
$$+\check{d}^4(-12\tilde{q}_e-25\tilde{q}_e{}^2-15\tilde{q}_e{}^3-4\tilde{q}_e{}^4-2\tilde{q}_e{}^5)-\check{d}^3(12\tilde{q}_e(1+\tilde{q}_e)^2)$$
$$+\check{d}^2(72\tilde{q}_e+132\tilde{q}_e{}^2+72\tilde{q}_e{}^3+24\tilde{q}_e{}^4+12\tilde{q}_e{}^5)+\check{d}(48\tilde{q}_e(1+\tilde{q}_e)^2)\Big)$$

Hence

$$I_{\tilde{q}_e}(1.5) = \frac{1}{96(\tilde{q}_e{}^3+\tilde{q}_e{}^2+\tilde{q}_e+1)(1+\tilde{q}_e+\tilde{q}_e{}^2)(1+\tilde{q}_e)}\left(96+\frac{1683\tilde{q}}{4}+\frac{8343\tilde{q}_e{}^2}{16}+\frac{4953\tilde{q}_e{}^3}{16}+\frac{135\tilde{q}_e{}^4}{4}+\frac{135\tilde{q}_e{}^5}{8}\right)$$

Further simplification gives us:

$$I_{\tilde{q}_e}(1.5) = \frac{(1536+6732\tilde{q}_e+8343\tilde{q}_e{}^2+4953\tilde{q}_e{}^3+540\tilde{q}_e{}^4+270\tilde{q}_e{}^5)}{1536(\tilde{q}_e{}^3+\tilde{q}_e{}^2+\tilde{q}_e+1)(1+\tilde{q}_e+\tilde{q}_e{}^2)(1+\tilde{q}_e)}$$

So

$$|\breve{a}_5| \leq \frac{(1536+6732\tilde{q}_e+8343\tilde{q}_e{}^2+4953\tilde{q}_e{}^3+540\tilde{q}_e{}^4+270\tilde{q}_e{}^5)}{1536(\tilde{q}_e{}^3+\tilde{q}_e{}^2+\tilde{q}_e+1)(1+\tilde{q}_e+\tilde{q}_e{}^2)(1+\tilde{q}_e)}$$

□

Hence, the proof is completed.

If we take $\tilde{q}_e \to 1^-$ in the results that we have proved above, it will lead us to the results that are already proved for the class $\mathbb{S}^*_\rho$ which can be seen in the following corollary.

Corollary 4.2.1.1. If $\phi(\hat{r}) \in \mathbb{S}^*_\rho$ then

$$|\breve{a}_2| \leq 1, \ |\breve{a}_3| \leq \frac{1}{2}, \ |\breve{a}_4| \leq 0.5767, \ |\breve{a}_5| \leq 0.6070.$$

4.3 Fekete-Szeg$\ddot{o}$ Inequality

For the class $\mathbb{S}^*_{\rho\tilde{q}_e}$, this inequality will be investigated as follow:

Theorem 4.3.1. If $\phi(\hat{r}) \in \mathbb{S}^*_{\rho\tilde{q}_e}$, then $|\breve{a}_3 - \breve{a}_2^2| \leq \frac{1}{\tilde{q}_e+1}$

Proof. By using (4.12) and (4.13), we have

$$|\breve{a}_3 - \breve{a}_2^2| = \left|\frac{\breve{d}_2}{2(\tilde{q}_e+1)} - \frac{\breve{d}_1^2}{4}\right| = \frac{1}{4(\tilde{q}_e+1)}\left|2\breve{d}_2 - \breve{d}_1^2(\tilde{q}_e+1)\right|$$

Using Lemma 2.15.3 with $\lambda = 2$, h=2 and k=1 and for $\tilde{q}_e \in (0,1)$, we have

$$|\breve{a}_3 - \breve{a}_2^2| \leq \frac{2(2)}{4(\tilde{q}_e+1)} = \frac{4}{4(\tilde{q}_e+1)} = \frac{1}{\tilde{q}_e+1}$$

So

$$|\breve{a}_3 - \breve{a}_2^2| \leq \frac{1}{\tilde{q}_e+1} \tag{4.18}$$

Hence proof is completed. □

Take $\tilde{q}_e \to 1^-$ in the above results and it will give us the already proved result as can be seen in the following corollary.

Corollary 4.3.1.1. If $\phi(\hat{r}) \in \mathbb{S}^*_\rho$, then $|\breve{a}_3 - \breve{a}_2^2| \leq \frac{1}{2}$.

4.4 Hankel Determinants

For the class $\mathbb{S}^*_{\rho\tilde{q}_e}$, we have the following results:

Theorem 4.4.1. If $\phi(\hat{r}) \in \mathbb{S}^*_{\rho\tilde{q}_e}$, then

$$|\breve{a}_2\breve{a}_3 - \breve{a}_4| \leq \frac{\left(\tilde{q}_e{}^3 + 25\tilde{q}_e{}^2 + 60\tilde{q}_e + 36\right)}{48(1+\tilde{q}_e)(1+\tilde{q}_e+\tilde{q}_e{}^2)}$$

Proof. From (4.12), (4.13) and (4.15), we obtain:

$$|\breve{a}_2\breve{a}_3 - \breve{a}_4| = \left|\frac{\breve{d}_1}{2}\left(\frac{\breve{d}_2}{2(1+\tilde{q}_e)}\right) - \frac{1}{(1+\tilde{q}_e+\tilde{q}_e{}^2)}\left(\frac{\breve{d}_3}{2} - \frac{\tilde{q}_e\breve{d}_1\breve{d}_2}{4(1+\tilde{q}_e)} - \frac{\tilde{q}_e{}^2\breve{d}_1^3}{48}\right)\right|$$

$$|\breve{a}_2\breve{a}_3 - \breve{a}_4| = \left|\frac{\breve{d}_1\breve{d}_2}{4(1+\tilde{q}_e)} - \frac{1}{(1+\tilde{q}_e+\tilde{q}_e{}^2)}\left(\frac{24(1+\tilde{q}_e)\breve{d}_3 - 12\tilde{q}_e\breve{d}_1\breve{d}_2 - \tilde{q}_e{}^2(1+\tilde{q}_e)\breve{d}_1^3}{48(1+\tilde{q}_e)}\right)\right|$$

Simplification will give us:

$$|\breve{a}_2\breve{a}_3 - \breve{a}_4| = \frac{1}{48(1+\tilde{q}_e)(1+\tilde{q}_e+\tilde{q}_e{}^2)}\Big|12\breve{d}_1\breve{d}_2(1+2\tilde{q}_e) + 12\tilde{q}_e{}^2\breve{d}_1\breve{d}_2 - 24(1+\tilde{q}_e)\breve{d}_3 + \tilde{q}_e{}^2(1+\tilde{q}_e)\breve{d}_1^3\Big|$$

Now substitute values of $\breve{d}_2$ and $\breve{d}_3$ using lemma 2.15.1, we will get:

$$|\breve{a}_2\breve{a}_3 - \breve{a}_4| = \frac{1}{48(1+\tilde{q}_e)(1+\tilde{q}_e+\tilde{q}_e{}^2)}\Bigg|12\breve{d}_1(1+2\tilde{q}_e)\left(\frac{\breve{d}_1^2 + \psi(4-\breve{d}_1^2)}{2}\right) + 6\tilde{q}_e{}^2\breve{d}_1\left(\breve{d}_1^2 + \psi(4-\breve{d}_1^2)\right) - 24(1+\tilde{q}_e)\left(\frac{\breve{d}_1^3 + 2(4-\breve{d}_1^2)\breve{d}_1\psi - (4-\breve{d}_1^2)\breve{d}_1\psi^2 + 2(4-\breve{d}_1^2)(1-|\psi|^2)\Upsilon}{4}\right) + \tilde{q}_e{}^2(1+\tilde{q}_e)\breve{d}_1^3\Bigg|$$

Simplifying terms inside we will have:

$$|\breve{a}_2\breve{a}_3 - \breve{a}_4| = \frac{1}{48(1+\tilde{q}_e)(1+\tilde{q}_e+\tilde{q}_e{}^2)}\Big|\breve{d}_1^3(\tilde{q}_e{}^3 + 7\tilde{q}_e{}^2 + 6\tilde{q}_e) + \breve{d}_1\psi(4-\breve{d}_1^2)(6\tilde{q}_e{}^2 - 6) + 6(1+\tilde{q}_e)\breve{d}_1\psi^2(4-\breve{d}_1^2) - 12(1+\tilde{q}_e)(4-\breve{d}_1^2)(1-|\psi|^2)\Upsilon\Big|$$

Using triangluar inequality and let $\breve{d}_1 = \breve{d}$, $|\psi| = t$

$$|\breve{a}_2\breve{a}_3 - \breve{a}_4| \leq \frac{1}{48(1+\tilde{q}_e)(1+\tilde{q}_e+\tilde{q}_e{}^2)}\Big(\breve{d}^3(\tilde{q}_e{}^3+7\tilde{q}_e{}^2+6\tilde{q}_e)+\breve{d}t(4-\breve{d}^2)(6\tilde{q}_e{}^2-6)$$
$$+6(1+\tilde{q}_e)\breve{d}t^2(4-\breve{d}^2)+12(1+\tilde{q}_e)(4-\breve{d}^2)\Big)$$

We assume that

$$J_{\tilde{q}_e}(\breve{d},t) = \frac{1}{48(1+\tilde{q}_e)(1+\tilde{q}_e+\tilde{q}_e{}^2)}\Big(\breve{c}^3(\tilde{q}_e{}^3+7\tilde{q}_e{}^2+6\tilde{q}_e)+\breve{d}t(4-\breve{d}^2)(6\tilde{q}_e{}^2-6)$$
$$+6(1+\tilde{q}_e)\breve{d}t^2(4-\breve{d}^2)+12(1+\tilde{q}_e)(4-\breve{d}^2)\Big)$$

Using partial differentiation of above, we have

$$\frac{\partial J_{\tilde{q}_e}}{\partial t} = \frac{1}{48(1+\tilde{q}_e)(1+\tilde{q}_e+\tilde{q}_e{}^2)}\Big(\breve{d}(4-\breve{d}^2)(6\tilde{q}_e{}^2-6)+12\breve{d}t(1+\tilde{q}_e)(4-\breve{d}^2)\Big) > 0$$

It implies $J_{\tilde{q}_e}(\breve{d},t)$ is an increasing function in [0,1]. So,

$$max(J_{\tilde{q}_e}(\breve{d},t)) = J_{\tilde{q}_e}(\breve{d},1)$$

Substituting 1 in place of t in $J_{\tilde{q}_e}(\breve{d},t)$, we will get the following:

$$J_{\tilde{q}_e}(\breve{d},1) = \frac{1}{48(1+\tilde{q}_e)(1+\tilde{q}_e+\tilde{q}_e{}^2)}\Big(\breve{d}^3(\tilde{q}_e{}^3+7\tilde{q}_e{}^2+6\tilde{q}_e)+\breve{d}(4-\breve{d}^2)(6\tilde{q}_e{}^2-6)$$
$$+6(1+\tilde{q}_e)\breve{d}(4-\breve{d}^2)+12(1+\tilde{q}_e)(4-\breve{d}^2)\Big)$$

Combining like terms inside brackets, we have the following form:

$$J_{\tilde{q}_e}(\breve{d},1) = \frac{1}{48(1+\tilde{q}_e)(1+\tilde{q}_e+\tilde{q}_e{}^2)}\Big(\breve{d}^3(\tilde{q}_e{}^3+7\tilde{q}_e{}^2+6\tilde{q}_e)+\breve{d}(4-\breve{d}^2)(6\tilde{q}_e{}^2+6\tilde{q}_e)$$
$$+12(1+\tilde{q}_e)(4-\breve{d}^2)\Big)$$

Since it is only a function of $\breve{d}$, so we consider

$$L_{\tilde{q}_e}(\breve{d}) = \frac{1}{48(1+\tilde{q}_e)(1+\tilde{q}_e+\tilde{q}_e{}^2)}\Big(\breve{d}^3(\tilde{q}_e{}^3+7\tilde{q}_e{}^2+6\tilde{q}_e)+\breve{d}(4-\breve{d}^2)(6\tilde{q}_e{}^2+6\tilde{q}_e)$$
$$+12(1+\tilde{q}_e)(4-\breve{d}^2)\Big)$$

Differentiation of $L_{\tilde{q}_e}(\breve{d})$ gives us

$$L'_{\tilde{q}_e}(\breve{d}) = \frac{3\breve{c}^2(\tilde{q}_e{}^3+7\tilde{q}_e{}^2+6\tilde{q}_e)+(4-3\breve{d}^2)(6\tilde{q}_e{}^2+6\tilde{q}_e)-24\breve{d}(1+\tilde{q}_e)}{48(1+\tilde{q}_e)(1+\tilde{q}_e+\tilde{q}_e{}^2)}$$

Again simplifying terms inside brackets, we get the following $L'_{\tilde{q}_e}(\breve{d})$

$$L'_{\tilde{q}_e}(\breve{d}) = \frac{\breve{d}^2(3\tilde{q}_e^{\,3}+3\tilde{q}_e^{\,2}) - 24\breve{d}(1+\tilde{q}_e) + 24\tilde{q}_e(1+\tilde{q}_e)}{48(1+\tilde{q}_e)(1+\tilde{q}_e+\tilde{q}_e^{\,2})}$$

$$L'_{\tilde{q}_e}(\breve{d}) = \frac{1}{48(1+\tilde{q}_e)(1+\tilde{q}_e+\tilde{q}_e^{\,2})}(A\breve{d}^2+B\breve{d}+C)$$

where

$$A = (3\tilde{q}_e^{\,3}+3\tilde{q}_e^{\,2})$$

$$B = -24(1+\tilde{q}_e)$$

and

$$C = 24\tilde{q}_e(1+\tilde{q}_e)$$

Certain calculations show that $A\breve{d}^2+B\breve{d}+C>0$ for $\breve{d}\in(0,1]$ and $\tilde{q}_e\in(0,1)$. Also $A\breve{d}^2+B\breve{d}+C<0$ for $\breve{d}\in(1,2)$. This means $L'_{\tilde{q}_e}(\breve{d})>0$ for $\breve{d}\in(0,1]$ and $L'_{\tilde{q}_e}(\breve{d})\le 0$ for $\breve{d}\in(1,2)$. This implies $L_{\tilde{q}_e}(\breve{d})$ is increasing in (0,1] and decreasing in (1,2). Hence,

$$|\breve{a}_2\breve{a}_3-\breve{a}_4| \le L_{\tilde{q}_e}(1) = \frac{7\tilde{q}_e^{\,2}+\tilde{q}_e^{\,3}+6\tilde{q}_e+18\tilde{q}_e^{\,2}+18\tilde{q}_e+36+36\tilde{q}_e}{48(1+\tilde{q}_e)(1+\tilde{q}_e+\tilde{q}_e^{\,2})} \quad ,\tilde{q}_e\in(0.9,1)$$

Hence

$$|\breve{a}_2\breve{a}_3-\breve{a}_4| \le \frac{\tilde{q}_e^{\,3}+25\tilde{q}_e^{\,2}+60\tilde{q}_e+36}{48(1+\tilde{q}_e)(1+\tilde{q}_e+\tilde{q}_e^{\,2})} \tag{4.19}$$

□

This completes the proof.

As we take $\tilde{q}_e$ approaches to 1^- in the above proof, we will get the result which is already proved for the class $\mathbb{S}^*_\rho$ as shown in the preceeding corollary.

Corollary 4.4.1.1. If $\phi(\hat{r})\in\mathbb{S}^*_\rho$, then $|\breve{a}_2\breve{a}_3-\breve{a}_4|\le 0.4254$.

Theorem 4.4.2. If $\phi(\hat{r})\in\mathbb{S}^*_{\rho\tilde{q}_e}$, then

$$|\breve{a}_2\breve{a}_4-\breve{a}_3^2| \le \frac{(0.6561\tilde{q}_e^{\,4}+1.3122\tilde{q}_e^{\,3}+100.1013\tilde{q}_e^{\,2}+176.4708\tilde{q}_e+111.012)}{96(1+\tilde{q}_e)^2(1+\tilde{q}_e+\tilde{q}_e^{\,2})}$$

Proof. From (4.12), (4.13) and (4.15), we get the following:

$$|\breve{a}_2\breve{a}_4-\breve{a}_3^2| = \left|\frac{\breve{d}_1}{2(1+\tilde{q}_e+\tilde{q}_e^{\,2})}\left(\frac{\breve{d}_3}{2}-\frac{\tilde{q}_e\breve{d}_1\breve{d}_2}{4(1+\tilde{q}_e)}-\frac{\tilde{q}_e^{\,2}\breve{d}_1^3}{48}\right)-\frac{\breve{d}_2^2}{4(1+\tilde{q}_e)^2}\right|$$

$$|\breve{a}_2\breve{a}_4-\breve{a}_3^2| = \left|\frac{1}{2(1+\tilde{q}_e+\tilde{q}_e^{\,2})}\left(\frac{\breve{d}_1\breve{d}_3}{2}-\frac{\tilde{q}_e\breve{d}_1^2\breve{d}_2}{4(1+\tilde{q}_e)}-\frac{\tilde{q}_e^{\,2}\breve{d}_1^4}{48}\right)-\frac{\breve{d}_2^2}{4(1+\tilde{q}_e)^2}\right|$$

$$|\breve{a}_2\breve{a}_4-\breve{a}_3^2| = \left|\frac{1}{2(1+\tilde{q}_e+\tilde{q}_e{}^2)}\left(\frac{24(1+\tilde{q}_e)\breve{d}_1\breve{d}_3-12\tilde{q}_e\breve{d}_1^2\breve{d}_2-\tilde{q}_e{}^2(1+\tilde{q}_e)\breve{d}_1^4}{48(1+\tilde{q}_e)}\right)-\frac{\breve{d}_2^2}{4(1+\tilde{q}_e)^2}\right|$$

$$|\breve{a}_2\breve{a}_4-\breve{a}_3^2| = \frac{1}{48(1+\tilde{q}_e)^2 2(1+\tilde{q}_e+\tilde{q}_e{}^2)}\Big|24(1+\tilde{q}_e)^2\breve{d}_1\breve{d}_3-12\tilde{q}_e(1+\tilde{q}_e)\breve{d}_1^2\breve{d}_2-\tilde{q}_e{}^2$$
$$(1+\tilde{q}_e)^2\breve{d}_1^4-24\breve{d}_2^2(1+\tilde{q}_e+\tilde{q}_e{}^2)\Big|$$

Now put value of $\breve{d}_2$ and $\breve{d}_3$, we get:

$$|\breve{a}_2\breve{a}_4-\breve{a}_3^2| = \frac{1}{96(1+\tilde{q}_e)^2(1+\tilde{q}_e+\tilde{q}_e{}^2)}\Big|6(1+\tilde{q}_e)^2\breve{d}_1\Big(\breve{d}_1^3+2(4-\breve{d}_1^2)\breve{d}_1\psi-(4-\breve{d}_1^2)\breve{d}_1\psi^2$$
$$+2(4-\breve{d}_1^2)(1-|\psi|^2)\Upsilon\Big)-12\tilde{q}_e(1+\tilde{q}_e)\breve{d}_1^2\left(\frac{\breve{d}_1^2+\psi(4-\breve{d}_1^2)}{2}\right)-\tilde{q}_e{}^2(1+\tilde{q}_e)^2\breve{d}_1^4$$
$$-24(1+\tilde{q}_e+\tilde{q}_e{}^2)\left(\frac{\breve{d}_1^2+\psi(4-\breve{d}_1^2)}{2}\right)\Big|$$

$$|\breve{a}_2\breve{a}_4-\breve{a}_3^2| = \frac{1}{96(1+\tilde{q}_e)^2(1+\tilde{q}_e+\tilde{q}_e{}^2)}\Big|6(1+\tilde{q}_e)^2\breve{d}_1^4+12(1+\tilde{q}_e)^2(4-\breve{d}_1^2)\breve{d}_1^2\psi-6(1+\tilde{q}_e)^2$$
$$(4-\breve{d}_1^2)\breve{d}_1^2\psi^2+12(1+\tilde{q}_e)^2\breve{d}_1(4-\breve{d}_1^2)(1-|\psi|)^2\Upsilon-6\tilde{q}_e(1+\tilde{q}_e)\breve{d}_1^4-6\tilde{q}_e(1+\tilde{q}_e)\breve{d}_1^2\psi(4-\breve{d}_1^2)$$
$$-\tilde{q}_e{}^2(1+\tilde{q}_e)^2\breve{d}_1^4-6(1+\tilde{q}_e+\tilde{q}_e{}^2)\breve{d}_1^4-6(1+\tilde{q}_e+\tilde{q}_e{}^2)\psi^2(4-\breve{d}_1^2)^2-12(1+\tilde{q}_e+\tilde{q}_e{}^2)\breve{d}_1^2\psi(4-\breve{d}_1^2)\Big|$$

$$|\breve{a}_2\breve{a}_4-\breve{a}_3^2| = \frac{1}{96(1+\tilde{q}_e)^2(1+\tilde{q}_e+\tilde{q}_e{}^2)}\Big|(6(1+\tilde{q}_e)^2-6\tilde{q}_e(1+\tilde{q}_e)-\tilde{q}_e{}^2(1+\tilde{q}_e)^2-6(1+\tilde{q}_e+$$
$$\tilde{q}_e{}^2))\breve{d}_1^4+(12(1+\tilde{q}_e)^2-6\tilde{q}_e(1+\tilde{q}_e)-12(1+\tilde{q}_e+\tilde{q}_e{}^2))\breve{d}_1^2\psi(4-\breve{d}_1^2)-6(1+\tilde{q}_e)^2(4-\breve{d}_1^2)\breve{d}_1^2$$
$$\psi^2-6(1+\tilde{q}_e+\tilde{q}_e{}^2)\psi^2(4-\breve{d}_1^2)^2+12(1+\tilde{q}_e)^2\breve{d}_1(4-\breve{d}_1^2)(1-|\psi|^2)\Upsilon\Big|$$

After simplification of terms inside brackets, we will get

$$|\breve{a}_2\breve{a}_4-\breve{a}_3^2| = \frac{1}{96(1+\tilde{q}_e)^2(1+\tilde{q}_e+\tilde{q}_e{}^2)}\Big|-(7\tilde{q}_e{}^2+2\tilde{q}_e{}^3+\tilde{q}_e{}^4)\breve{d}_1^4+(6\tilde{q}_e-6\tilde{q}_e{}^2)\breve{d}_1^2(4-\breve{d}_1^2)$$
$$-6(1+\tilde{q}_e{}^2+2\tilde{q}_e)(4-\breve{d}_1^2)\breve{d}_1^2\psi^2-6(1+\tilde{q}_e+\tilde{q}_e{}^2)\psi^2(4-\breve{d}_1^2)^2$$
$$+12(1+\tilde{q}_e{}^+2\tilde{q}_e)\breve{d}_1(4-\breve{d}_1^2)(1-|\psi|^2)\Upsilon\Big|$$

Now by applying triangular inequality and let $\breve{d}_1=\breve{d}$ and $|\psi|=t$, we have:

$$|\breve{a}_2\breve{a}_4-\breve{a}_3^2| \leq \frac{1}{96(1+\tilde{q}_e)^2(1+\tilde{q}_e+\tilde{q}_e{}^2)}\Big((7\tilde{q}_e{}^2+2\tilde{q}_e{}^3+\tilde{q}_e{}^4)\breve{d}^4+(6\tilde{q}_e-6\tilde{q}_e{}^2)\breve{d}^2t(4-\breve{d}^2)$$
$$+6(1+\tilde{q}_e{}^2+2\tilde{q}_e)(4-\breve{d}^2)\breve{d}^2t^2+6(1+\tilde{q}_e+\tilde{q}_e{}^2)t^2(4-\breve{d}^2)^2$$
$$+12(1+\tilde{q}_e{}^2+2\tilde{q}_e)\breve{d}(4-\breve{d}^2)\Big)$$

We assume that

$$L_{\tilde{q}}(\check{d},t) = \frac{1}{96(1+\tilde{q}_e)^2(1+\tilde{q}_e+\tilde{q}_e^2)}\Big((7\tilde{q}_e{}^2+2\tilde{q}_e{}^3+\tilde{q}_e{}^4)\check{d}^4+(6\tilde{q}_e-6\tilde{q}_e{}^2)\check{d}^2t(4-\check{d}^2)$$
$$+6(1+\tilde{q}_e{}^2+2\tilde{q}_e)(4-\check{d}^2)\check{d}^2t^2+6(1+\tilde{q}_e+\tilde{q}_e{}^2)t^2(4-\check{d}^2)^2$$
$$+12(1+\tilde{q}_e{}^2+2\tilde{q}_e)\check{d}(4-\check{d}^2)\Big)$$

Upon partial differentiation, we get:

$$\frac{\partial L_{\tilde{q}_e}}{\partial t} = \frac{1}{96(1+\tilde{q}_e)^2(1+\tilde{q}_e+\tilde{q}_e{}^2)}\Big((6\tilde{q}_e-6\tilde{q}_e{}^2)\check{d}^2(4-\check{d}^2)+12(1+\tilde{q}_e{}^2+2\tilde{q}_e)(4-\check{d}^2)\check{d}^2t$$
$$+12(1+\tilde{q}_e+\tilde{q}_e{}^2)t(4-\check{d}^2)^2\Big)>0$$

So $L_{\tilde{q}_e}(\check{d},t)$ is an increasing function in [0,1]. So,

$$max(L_{\tilde{q}_e}(\check{d},t)) = L_{\tilde{q}}(\check{d},1)$$

Simplifying all terms, we have:

$$L_{\tilde{q}_e}(\check{d},1) = \frac{1}{96(1+\tilde{q}_e)^2(1+\tilde{q}_e+\tilde{q}_e{}^2)}\Big((\tilde{q}_e{}^4+2\tilde{q}_e{}^3+13\tilde{q}_e{}^2-12\tilde{q}_e)\check{d}^4-((12\tilde{q}_e{}^2+24\tilde{q}_e+12)\check{d}^3$$
$$+(-48\tilde{q}_e{}^2+24\tilde{q}_e-24)\check{d}^2+(48\tilde{q}_e{}^2+96\tilde{q}_e+48)\check{d}+(96\tilde{q}_e{}^2+96\tilde{q}_e+96)\Big)$$

Let us consider

$$M_{\tilde{q}_e}(\check{d})) = \frac{1}{96(1+\tilde{q}_e)^2(1+\tilde{q}_e+\tilde{q}_e{}^2)}\Big((\tilde{q}_e{}^4+2\tilde{q}_e{}^3+13\tilde{q}_e{}^2-12\tilde{q}_e)\check{d}^4-((12\tilde{q}_e{}^2+24\tilde{q}_e+12)\check{d}^3$$
$$+(-48\tilde{q}_e{}^2+24\tilde{q}_e-24)\check{d}^2+(48\tilde{q}_e{}^2+96\tilde{q}_e+48)\check{d}+(96\tilde{q}_e{}^2+96\tilde{q}_e+96)\Big)$$

Now,

$$M'_{\tilde{q}_e}(\check{d}) = \frac{1}{96(1+\tilde{q}_e)^2(1+\tilde{q}_e+\tilde{q}_e{}^2)}\Big((4\tilde{q}_e{}^4+8\tilde{q}_e{}^3+52\tilde{q}_e{}^2-48\tilde{q}_e)\check{d}^3-(36\tilde{q}_e{}^2+72\tilde{q}_e+36)\check{d}^2$$
$$+(-96\tilde{q}_e{}^2+48\tilde{q}_e-48)\check{d}+(48\tilde{q}_e{}^2+96\tilde{q}_e+48)\Big)$$

$$M'_{\tilde{q}_e}(\check{d}) = \frac{1}{96(1+\tilde{q}_e)^2(1+\tilde{q}_e+\tilde{q}_e{}^2)}(A\check{d}^3+B\check{d}^2+C\check{d}+D)$$

where

$$A = (4\tilde{q}_e{}^4+8\tilde{q}_e{}^3+52\tilde{q}_e{}^2-48\tilde{q}_e)$$

$$B = -(36\tilde{q}_e{}^2+72\tilde{q}_e+36)$$

$$C = (-96\tilde{q}_e{}^2 + 48\tilde{q}_e - 48)$$

$$D = (48\tilde{q}_e{}^2 + 96\tilde{q}_e + 48)$$

Certain calculations show that $A\check{d}^3 + B\check{d}^2 + C\check{d} + D$ is greater than zero for $\check{d} \in (0, 0.9]$ and $\tilde{q}_e$ belongs to $(0.95, 1)$. Also, $A\check{d}^3 + B\check{d}^2 + C\check{d} + D$ is less than zero for $\check{d} \in (0.9, 2)$. This means $M'_{\tilde{q}_e}(\check{c}) > 0$ for $\check{d} \in (0, 0.9]$ and $M'_{\tilde{q}_e}(\check{d}) \leq 0$ for $\check{d} \in (0.9, 2)$. So, this implies that $M_{\tilde{q}_e}(\check{d})$ is increasing in (0,0.9] and decreasing in (0.9,2). This means that

$$|\check{a}_2\check{a}_4 - \check{a}_3^2| \leq M_{\tilde{q}_e}(0.9) = \frac{1}{96(1+\tilde{q}_e)^2(1+\tilde{q}_e+\tilde{q}_e{}^2)}\Big((\tilde{q}_e{}^4 + 2\tilde{q}_e{}^3 + 13\tilde{q}_e{}^2 - 12\tilde{q}_e)(0.9)^4 - (12\tilde{q}_e{}^2 + 24\tilde{q}_e + 12)(0.9)^3 + (-48\tilde{q}_e{}^2 + 24\tilde{q}_e - 24)(0.9)^2 + (48\tilde{q}_e{}^2 + 96\tilde{q}_e + 48)(0.9) (96\tilde{q}_e{}^2 + 96\tilde{q}_e + 96)\Big)$$

Hence

$$|\check{a}_2\check{a}_4 - \check{a}_3^2| \leq \frac{(0.6561\tilde{q}_e{}^4 + 1.3122\tilde{q}_e{}^3 + 100.1013\tilde{q}_e{}^2 + 176.4708\tilde{q}_e + 111.012)}{96(1+\tilde{q}_e)^2(1+\tilde{q}_e+\tilde{q}_e{}^2)} \tag{4.20}$$

□

So, the proof is completed.

If we take $\tilde{q}_e$ approaches to 1^- in the above proof, we will get the result which has already been proved for $\mathbb{S}^*_\rho$ as illustrated in the following corollary.

Corollary 4.4.2.1. If $\phi(\hat{r}) \in \mathbb{S}^*_\rho$, then $|\check{a}_2\check{a}_4 - \check{a}_3^2| \leq 0.3382$.

Theorem 4.4.3. If $\phi(\hat{r}) \in \mathbb{S}^*_{\rho\tilde{q}}$ then

$$|\mathbb{H}_{3,1}(\phi)| \leq \frac{1536 + 6732\tilde{q}_e + 8343\tilde{q}_e{}^2 + 4953\tilde{q}_e{}^3 + 540\tilde{q}_e{}^4 + 270\tilde{q}_e{}^5}{1536(\tilde{q}_e{}^3 + \tilde{q}_e{}^2 + \tilde{q}_e + 1)(1+\tilde{q}_e+\tilde{q}_e{}^2)(1+\tilde{q}_e)^2} + \frac{\left(94332 + 72732\tilde{q}_e - 729\tilde{q}_e{}^2 - 729\tilde{q}_e{}^3\right)\left(\tilde{q}_e{}^3 + 25\tilde{q}_e{}^2 + 60\tilde{q}_e + 36\right)}{2304000(1+\tilde{q}_e)^2(1+\tilde{q}_e+\tilde{q}_e{}^2)^2} + \frac{(0.6561\tilde{q}_e{}^4 + 1.3122\tilde{q}_e{}^3 + 100.1013\tilde{q}_e{}^2 + 176.4708\tilde{q}_e + 111.012)}{96(1+\tilde{q}_e)^3(1+\tilde{q}_e+\tilde{q}_e{}^2)}.$$

Proof. The upper bound of third order Hankel Determinant is given as follow:

$$|\mathbb{H}_{3,1}(\phi)| \leq |\check{a}_5||(\check{a}_3 - \check{a}_2^2)| + |\check{a}_4||(\check{a}_4 - \check{a}_2\check{a}_3)| + |\check{a}_3||(\check{a}_2\check{a}_4 - \check{a}_3^2)|$$

Now substitute values in the above inequality from Theorem 4.2.1, Theorem 4.4.1 and Theorem 4.4.2, we will get the following:

$$|\mathbb{H}_{3,1}(\phi)| \leq \frac{1536+6732\tilde{q}_e+8343\tilde{q}_e{}^2+4953\tilde{q}_e{}^3+540\tilde{q}_e{}^4+270\tilde{q}_e{}^5}{1536(\tilde{q}_e{}^3+\tilde{q}_e{}^2+\tilde{q}_e+1)(1+\tilde{q}_e+\tilde{q}_e{}^2)(1+\tilde{q}_e)^2}+ \frac{\left(94332+72732\tilde{q}_e-729\tilde{q}_e{}^2-729\tilde{q}_e{}^3\right)\left(\tilde{q}_e{}^3+25\tilde{q}_e{}^2+60\tilde{q}_e+36\right)}{2304000(1+\tilde{q}_e)^2(1+\tilde{q}_e+\tilde{q}_e{}^2)^2}+ \frac{(0.6561\tilde{q}_e{}^4+1.3122\tilde{q}_e{}^3+100.1013\tilde{q}_e{}^2+176.4708\tilde{q}_e+111.012)}{96(1+\tilde{q}_e)^3(1+\tilde{q}_e+\tilde{q}_e{}^2)}.$$

which is our required proof. □

If we take $\tilde{q}_e \to 1^-$ in the proof above, it yields us to the already proved results, as shown in the corollary.

Corollary 4.4.3.1. If $\phi(\hat{r}) \in \mathbb{S}^*_\rho$, then $|\mathbb{H}_{3,1}(\phi)| \leq 0.6271$.

4.5 Zalcman Functional

Theorem 4.5.1. If $\phi(\hat{r}) \in \mathbb{S}^*_{\rho\tilde{q}_e}$, then

$$|\check{a}_5 - \check{a}_3^2| \leq \frac{(16.875\tilde{q}_e{}^5+129.75\tilde{q}_e{}^4+405.5625\tilde{q}_e{}^3+713.4375\tilde{q}_e{}^2+516.75\tilde{q}_e+192)}{96((1+\tilde{q}_e)^2(1+\tilde{q}_e{}^2)(1+\tilde{q}_e+\tilde{q}_e{}^2)}$$

Proof. Put values from (4.13) and (4.17) and do simplification, we will get the following:

$$\check{a}_5-\check{a}_3^2 = \frac{1}{96(1+\tilde{q}_e)^2(1+\tilde{q}_e+\tilde{q}_e{}^2)(1+\tilde{q}_e{}^2)}\Big[48\check{d}_4(1+\tilde{q}_e)(1+\tilde{q}_e+\tilde{q}_e{}^2)-24\tilde{q}_e(1+\tilde{q}_e)^2\check{d}_1\check{d}_3+ 2\tilde{q}_e{}^2(3-6\tilde{q}^2-3\tilde{q}_e{}^3)\check{d}_1^2\check{d}_2+\tilde{q}_e{}^2(1+\tilde{q}_e)(2\tilde{q}_e{}^2+2\tilde{q}_e+1)\check{d}_1^4-(1+\tilde{q}_e+\tilde{q}_e{}^2)(24\tilde{q}_e{}^2+24\tilde{q}_e+24)\check{d}_2^2\Big]$$

Using Lemma 2.15.1 in the above equation and it will become as follows:

$$\check{a}_5-\check{a}_3^2 = \frac{1}{96(1+\tilde{q}_e)^2(1+\tilde{q}_e+\tilde{q}_e{}^2)(1+\tilde{q}_e{}^2)}\Bigg[48\check{d}_4(1+\tilde{q}_e)(1+\tilde{q}_e+\tilde{q}_e{}^2)-24\tilde{q}_e(1+\tilde{q}_e)^2\check{d}_1 \left(\frac{\check{d}_1^3+2(4-\check{d}_1^2)\check{d}_1\psi-(4-\check{d}_1^2)\check{d}_1\psi^2+2(4-\check{d}_1^2)(1-|\psi|^2)\Upsilon}{4}\right)+2\tilde{q}_e{}^2(3-6\tilde{q}_e{}^2-3\tilde{q}_e{}^3)\check{d}_1^2 \left(\frac{\check{d}_1^2+\psi(4-\check{d}_1^2)}{2}\right)+\tilde{q}_e{}^2(1+\tilde{q}_e)(2\tilde{q}_e{}^2+2\tilde{q}_e+1)\check{d}_1^4-(1+\tilde{q}_e+\tilde{q}_e{}^2)(24\tilde{q}_e{}^2+24\tilde{q}_e+24) \left(\frac{\check{d}_1^2+\psi(4-\check{d}_1^2)}{2}\right)^2\Bigg]$$

After calculations of like terms, apply triangular inequality on both sides and use Lemma 2.15.2 and let $\check{d}_1 = \check{d}, |\psi| = t$

$$|\check{a}_5 - \check{a}_3^2| \leq \frac{1}{96(1+\tilde{q}_e)^2(1+\tilde{q}_e+\tilde{q}_e{}^2)(1+\tilde{q}_e{}^2)}\Big[96(1+\tilde{q}_e)(1+\tilde{q}_e+\tilde{q}_e{}^2)+\check{c}^4(\tilde{q}_e{}^5+8\tilde{q}_e{}^4+15\tilde{q}_e{}^3$$
$$+26\tilde{q}_e{}^2+18\tilde{q}_e+6)+(4-\check{d}^2)\check{d}^2t(3\tilde{q}_e{}^5+18\tilde{q}_e{}^4+36\tilde{q}_e{}^3+57\tilde{q}_e{}^2+36\tilde{q}_e+12)$$
$$+6\tilde{q}_e(1+\tilde{q}_e)^2(4-\check{d}^2)\check{d}^2t^2+12\tilde{q}_e(1+\tilde{q}_e)^2\check{d}(4-\check{d}^2)+6(1+\tilde{q}_e+\tilde{q}_e{}^2)^2t^2(4-\check{d}^2)^2\Big]$$

Consider

$$P_{\tilde{q}}(\check{d},t) = \frac{1}{96(1+\tilde{q}_e)^2(1+\tilde{q}_e+\tilde{q}_e{}^2)(1+\tilde{q}_e{}^2)}\Big[96(1+\tilde{q}_e)(1+\tilde{q}_e+\tilde{q}_e{}^2)+\check{d}^4(\tilde{q}_e{}^5+8\tilde{q}_e{}^4+15\tilde{q}_e{}^3$$
$$+26\tilde{q}_e{}^2+18\tilde{q}_e+6)+(4-\check{d}^2)\check{d}^2t(3\tilde{q}_e{}^5+18\tilde{q}_e{}^4+36\tilde{q}_e{}^3+57\tilde{q}_e{}^2+36\tilde{q}_e+12)$$
$$+6\tilde{q}_e(1+\tilde{q}_e)^2(4-\check{d}^2)\check{d}^2t^2+12\tilde{q}_e(1+\tilde{q}_e)^2\check{d}(4-\check{d}^2)+6(1+\tilde{q}_e+\tilde{q}_e{}^2)^2t^2(4-\check{d}^2)^2\Big]$$

Partially differentiate the above with respect to t and we will have:

$$\frac{\partial P_{\tilde{q}_e}(\check{d},t)}{\partial t} = \frac{1}{96(1+\tilde{q}_e)^2(1+\tilde{q}_e+\tilde{q}_e{}^2)(1+\tilde{q}_e{}^2)}\Big[(4-\check{c}^2)\check{d}^2(3\tilde{q}_e{}^5+18\tilde{q}_e{}^4+36\tilde{q}_e{}^3+57\tilde{q}_e{}^2+$$
$$36\tilde{q}_e+12)+12\tilde{q}_e(1+\tilde{q}_e)^2(4-\check{d}^2)\check{d}^2t+12(1+\tilde{q}_e+\tilde{q}_e{}^2)^2t(4-\check{d}^2)^2\Big]$$

Clearly $P_{\tilde{q}_e}(\check{d},t)$ turns out to be an increasing function in the interval [0,1]. So,

$$max(P_{\tilde{q}_e}(\check{d},t)) = P_{\tilde{q}_e}(\check{d},1) = P_{\tilde{q}_e}(\check{d})$$

Hence combining like terms inside and simplied them, we will get

$$P_{\tilde{q}_e}(\check{d}) = \frac{1}{96(1+\tilde{q}_e)^2(1+\tilde{q}_e+\tilde{q}_e{}^2)(1+\tilde{q}_e{}^2)}\Big[96(1+\tilde{q}_e)(1+\tilde{q}_e+\tilde{q}_e{}^2)+\check{d}^4(-2\tilde{q}_e{}^5-4\tilde{q}_e{}^4-$$
$$15\tilde{q}_e{}^3-25\tilde{q}_e{}^2-12\tilde{q}_e)-\check{d}^3(12\tilde{q}_e(1+\tilde{q}_e)^2)+\check{d}^2(12\tilde{q}_e{}^5+24\tilde{q}_e{}^4+72\tilde{q}_e{}^3+132\tilde{q}_e{}^2+72\tilde{q}_e)$$
$$+48\tilde{q}_e(1+\tilde{q}_e)^2\check{d}+96(1+\tilde{q}_e+\tilde{q}_e{}^2)^2\Big]$$

Differentiating above will lead us to the following:

$$P'_{\tilde{q}_e}(\check{d}) = \frac{1}{96(1+\tilde{q}_e)^2(1+\tilde{q}_e+\tilde{q}_e{}^2)(1+\tilde{q}_e{}^2)}\Big[\check{d}^3(A)+\check{d}^2(B)+\check{d}(C)+D\Big]$$

where

$$A = -8\tilde{q}_e{}^5-16\tilde{q}_e{}^4-60\tilde{q}_e{}^3-100\tilde{q}_e{}^2-48\tilde{q}_e$$

,

$$B = -36\tilde{q}_e(1+\tilde{q}_e)^2)$$

,

$$C = (24\tilde{q}_e^{\,5} + 48\tilde{q}_e^{\,4} + 144\tilde{q}_e^{\,3} + 264\tilde{q}_e^{\,2} + 144\tilde{q}_e)$$

and

$$D = 48\tilde{q}_e(1+\tilde{q}_e)^2.$$

After caclculations, it is observed that $\breve{d}^3(A) + \breve{d}^2(B) + \breve{d}(C) + D$ is greater than zero for $\breve{d} \in (0, 1.5]$ and $\tilde{q}_e \in (0,1)$. Also, $\breve{d}^3(A) + \breve{d}^2(B) + \breve{d}(C) + D < 0$ for $\breve{d} \in (1.6, 2)$. This means that $P_{\tilde{q}_e}(\breve{d})$ is increasing in (0,1.5] and decreasing in (1.6,2). So,

$$|\breve{a}_5 - \breve{a}_3^2| \leq P_{\tilde{q}_e}(1.5) = \frac{1}{96(1+\tilde{q}_e)^2(1+\tilde{q}_e+\tilde{q}_e^{\,2})(1+\tilde{q}_e^{\,2})}\Big[96(1+\tilde{q}_e)(1+\tilde{q}_e+\tilde{q}_e^{\,2}) + (1.5)^4$$
$$(-2\tilde{q}_e^{\,5} - 4\tilde{q}_e^{\,4} - 15\tilde{q}_e^{\,3} - 25\tilde{q}_e^{\,2} - 12\tilde{q}_e) - (1.5)^3(12\tilde{q}_e(1+\tilde{q}_e)^2) + (1.5)^2(12\tilde{q}_e^{\,5} + 24\tilde{q}_e^{\,4} + 72\tilde{q}_e^{\,3}$$
$$+ 132\tilde{q}_e^{\,2} + 72\tilde{q}_e) + 48\tilde{q}_e(1+\tilde{q}_e)^2(1.5) + 96(1+\tilde{q}_e+\tilde{q}_e^{\,2})^2\Big]$$

$$|\breve{a}_5 - \breve{a}_3^2| \leq \frac{(16.875\tilde{q}_e^{\,5} + 129.75\tilde{q}_e^{\,4} + 405.5625\tilde{q}_e^{\,3} + 713.4375\tilde{q}_e^{\,2} + 516.75\tilde{q}_e + 192)}{96(1+\tilde{q}_e)^2(1+\tilde{q}_e^{\,2})(1+\tilde{q}_e+\tilde{q}_e^{\,2})}$$

Hence, the proof is completed. □

Take $\tilde{q}_e \to 1^-$ in the above result and it gives us the already proved result for the class $\mathbb{S}_\rho^*$ which can be seen in the corollary below.

Corollary 4.5.1.1. If $\phi(\hat{r}) \in \mathbb{S}_\rho^*$, then $|\breve{a}_5 - \breve{a}_3^2| \leq 0.8570$.

CHAPTER 5

CONCLUSION

This thesis principally investigates the coefficient bounds of analytic, univalent and normalized functions in the open unit disk. Some initail findings from the study of Geometric Function Theory and primary concepts of q-calculus are examined, with a thorough analysis of the use of q-derivative operator to analytic functions. By utilizing q-difference operator, a new q-starlike functions's class is presented.

This research explored the q-starlike class $\mathbb{S}^*_{\rho}$ associated with inverse hyperbolic sine function and the q-extension of this class. This class was introduced by Kush Arora and S. Sivaprasad Kumar and in this research the work is further extended using concepts of q-calculus. The class $\mathbb{S}^*_{\rho\tilde{q}_e}$ is presented, which formally signifies q-starlike functions suborniate to the q-series of inverse sine hyperbolic function. This class is introduced using q-difference operator and the geometric techniques are used to investigate the properties of this class.

The notable characteristics of the functions of our recently introduced class are explored, encompassing coefficient bounds, the famous Fekete-Szeg$\ddot{o}$ problem, Hankel determinanats of second and third order, and the Zalcman functional. Observations have clearly demonstrated that our newly defined class is more advanced and an extensive class providing a refinement when contrasted to the existing one. The results presented advancements over theorems which have already proved by various researchers in the area of Geometric Function Theory. In order to validate the obtained results or findings limit $\tilde{q}_e \to 1^-$ has been taken which yields the already known results. It is anticipated that this study will significantly contribute in the field of Geometric Function Theory and provides a pathway for subsequent advances and breakthroughs in this field.

5.1 Future Work

The main focus of this study is the exploration of class of q-starlike functions associated to q-version of inverse hyperbolic sine function. This class can be explored further by presenting results for class of convex functions, quasi and close-to-convex functions by using subordination technique. Further, utilizing the concepts of quantum calculus results that are obtained in this study can be found for the classes containing q-convex functions, or the classes having q-close-to-convex functions and functions belong to q-quasi class.

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Theorem[section] Corollary[theorem] [theorem]Lemma [english]babel amsthm Definition[section] Remark Definition[section]