On a New Class of q-Starlike Functions with respect to Boundary Point

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NATIONAL UNIVERSITY OF MODERN LANGUAGES

ISLAMABAD

May, 2024

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MS-Math, National University of Modern Languages, Islamabad, 2024

A THESIS SUBMITTED IN PARTIAL FULFILMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

In **Mathematics**

To

FACULTY OF ENGINEERING & COMPUTING

NATIONAL UNIVERSITY OF MODERN LANGUAGES ISLAMABAD

THESIS AND DEFENSE APPROVAL FORM

The undersigned certify that they have read the following thesis, examined the defense, are satisfied with overall exam performance, and recommend the thesis to the Faculty of Engineering and Computer Sciences for acceptance.

Thesis Title: On a New Class of q-Starlike Functions with respect to Boundary Point

Submitted By: Tayyab Munir **Registration #:** 37 MS/MATH/F21

Master of Science in Mathematics

Title of the Degree

Mathematics

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Candidate of Master of Science in Mathematics at the National University of Modern Languages do hereby declare that the thesis On a New Class of q-Starlike Functions with respect to Boundary Point submitted by me in partial fulfillment of MSMA degree, is my original work and has not been submitted or published earlier. I also solemnly declare that it shall not, in the future, be submitted by me for obtaining any other degree from this or any other university or institution. I also understand that if evidence of plagiarism is found in my thesis/dissertation at any stage, even after the award of a degree, the work may be canceled and the degree revoked.

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ABSTRACT

Title: On a New Class of q-Starlike Functions with respect to Boundary Point

The aim of this research is to introduce and discuss properties of new subclasses of analytic functions in the open unit disc. The concepts of q-calculus will be used to define the q-extensions of already existing results for starlike functions. We will investigate the results thoroughly which are previously find by the researcher such as integral representation theorem, Fekete-Szegö Inequality, coefficient bounds and differential subordination results related to the class of starlike functon with respect to a boundary point. The new class of q-starlike functions with respect to a boundary point subordinated with exponential function will be introduced. Coefficient estimates, integral representation theorem, Fekete-Szegö inequality, covering and differential subordination results will be examined for our new class. The relevant connections of our new classes and results to known ones are also pointed out.

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LIST OF SYMBOLS

$\mathbb{C}$	Set of complex numbers
M	Open unit disc
A	The class of normalized analytic functions
S	Class of univalent functions
H	Class of holomorphic functions
S^*	Class of starlike functions
$\prec$	Subordination symbol
P	Class of Caratheodory functions
D_q	q-derivative
$w(\eta)$	Family of Schwarz functions
S_q^*	Class of q-starlike functions
C_q	Class of q-convex functions
S_b^*	Class of starlike functions with respect to a boundary point
G_e	Class of starlike functions with respect to a boundary point associated with exponential function
$S_{b,q}^*$	Class of q-starlike functions with respect to a boundary point

ACKNOWLEDGMENT

First of all, I wish to express my gratitude and deep appreciation to Almighty Allah, who made this study possible and successful. This study would not be accomplished unless the honest espousal was extended from several sources for which I would like to express my sincere thankfulness and gratitude. Yet, there were significant contributors to my attained success and I cannot forget their input, especially my research supervisor, Dr. Sadia Riaz, who did not leave any stone unturned to guide me during my research journey. I really appreciate Dr. Sadia Riaz, who is also our HOD, for providing us with a research environment and kind support. I really want to say thanks to our respected teachers, Dr. Muhammad Rizwan, Dr. Anum Naseem, Dr. Hadia Tariq, Dr. Asia Anjum, and other teachers, for their guidance and support.

I shall also acknowledge the extended assistance from the administration of the Department of Mathematics, who supported me all through my research experience and simplified the challenges I faced. For all whom I did not mention but shall not neglect their significant contribution, thanks for everything.

DEDICATED

To my parents, whose endless support kept me going. To my professors, whose guidance shaped this work, and to everyone who motivates me to do my best.

CHAPTER 1

INTRODUCTION AND LITERATURE REVIEW

1.1 Overview

This chapter provides a complete introduction and literature review, with an emphasis on key concepts in Geometric Function Theory. It includes the classes of analytic and univalent functions, and it also covers investigation of relevant subclasses. A brief discussion is also given to the Hankel determinant and coefficient bounds for these classes. This chapter also reviews basic concepts of quantum calculus.

1.2 Riemann Mapping Theorem

In 1851, Bernard Riemann's discover a result known as the Riemann Mapping Theorem, see [1]. As a result of this discovery, we can use an open unit disc $M = \{\eta \in \mathbb{C} : |\eta| < 1\}$ as a domain rather than a complex arbitrary domain. This theorem is essential as the foundation of Geometric Function Theory. The foundation of univalent function theory was developed in the $19th$ century by the important contributions of Cauchy, Riemann, and Weierstrass.

1.3 Analytic Function and Univalent Function

In 1907, Koebe [2, 3] proposed analytical and univalent functions in M. Within an open unit disc M, he discovered functions that are analytic, univalent and normalized. Researchers have studied these functions comprehensively, see [1, 2, 3]. In 1983, Duren [1] defined analytic functions for the first time. He proposed the class A, comprising functions that are analytic and normalized such that $g(0) = 0$ and $g'(0) = 1$. A function in class A that are analytic can be represented in series form as, $g(\eta) = \eta + \sum_{k=2}^{\infty} c_k \eta^k, \eta \in M$, where $M = \{\eta \in \mathbb{C} : |\eta| < 1\}$. In Geometric Function Theory, analytic functions are essential because they provide a comprehensive framework for studying how functions behave in the complex plane. If a function is differentiable at every point in a region, that region is said to be analytic for that function, see [1]. Analytic functions are classified into numerous classes, which are then subclassified depending on the structure of their image domains and other geometry. The geometric shape of the image domain is extremely important in the comprehensive analysis of analytic functions. Therefore, it has been a topic of discussion among scholars to introduce and investigate new geometrical structures related to analytic functions. In 1936, Robertson [4] advances the ideas behind the theory of univalent functions. A function is considered univalent in a region, if it is injective that is it takes distinct values for different inputs within that region. In 1964, MacGregor [5] presented a brand-new category of univalent functions.

1.4 Univalent and Analytic Function's Subclasses

Koebe [6] discovered the univalent function theory, a classic Analysis that is complex topic, in 1907. He proposed that the functions that are analytic, univalent in M, and satisfy normalization conditions are in class S of functions. The main focus of this current analysis will be on the subclasses of class S functions. In Geometric Function Theory, the class S of standardized univalent processes plays a significant role. Class K comprises close-to-convex functions, class C comprises convex functions, class C* comprises quasi-convex functions, and class S* comprises starlike functions. These are the four main subclasses of class S. This classification began during efforts to provide evidence for the Bieberbach conjecture [7]. Alexander [4] established a relationship known as the Alexander relation in 1915 to connect two groups of convex and starlike functions. In 1921, Nevanlinna [8] developed starlike functions in M. Univalent functions of order α that are convex of order α and starlike with negative coefficients are determined by Silverman [9] in 1975.

1.5 Coefficient Bounds

The problem of determining coefficient bounds is important part in Geometric Function Theory, and functions are further classified into several subfamilies of class A. The Bierberbach theorem, first published in 1916 by German mathematician Ludwig Bierberbach, is an important component of class S. He computed the second coefficient c_2 for univalent functions in class S. This theorem set the groundwork for Bieberbach's conjecture, which resulted in significant advances in the industry, see [7].

The widely recognized coefficient conjecture regarding the function g within the class S states that, if $g \in S$, then k^{th} coefficient of g, c_k holds the inequality $|c_k| \leq k$ for $k = \{2, 3, ...\}$, where Koebe function and one of its rotation gives sharp results. In 1916, Bieberbach [10, 11] proved the second coefficient c_2 of g, which holds the inequality $|c_2| \leq 2$ and Koebe is the extremal function for this coefficient inequality. While mathematicians have made multiple attempts to demonstrate this idea, it has proven to be a challenging task. In 1923, Karl Loewner [12] demonstrated the third coefficient, c_3 of univalent function g as $|c_3| \leq 3$. The fourth coefficient problem stayed unsolved till 1955, when Garabedian and Schiffer [13] showed that the fourth coefficient c_4 of g holds the inequality $|c_4| \leq 4$. Later on, mathematician Louis de Branges [14] successfully proved the general form of the Bieberbach conjecture in 1985, which states that $|a_k| \leq k$, where $k = \{2, 3, ...\}$, when g is a rotation of a Koebe function then the inequality is strict for every k, see [15]. In 2016, Darus [16] collected the class of second and third coefficient estimations of q-starlike and q-convex functions. Seoudy et al. [17] obtained the second and third coefficient valuations for the category of q-starlike and q-convex functions of intricate arrangement in 2016.

1.6 Hankel Determinant

The Hankel determinant refers to the determinant of the Hankel matrix. Pommerenke [18] introduced the concept of the Hankel determinant for specific univalent functions, in 1967. The Hankel determinant is important for investigating singularities and the analyzing power series with integral coefficients. A well-known Fekete-Szegö inequality for univalent function g is $|c_3 - c_2^2| = H_2(1)$. In general, it is written as $|c_3 - \lambda c_2^2|$ for a specified λ, which might be real or complex. Fekete-Szegö developed a challenging inequality for $0 \leq \lambda < 1$. The Fekete-Szegö problem involves determining the optimal constant λ and ensuring that the inequality remains less than or equal to λ for all analytic functions. In 1976, Hankel determinant of certain analytic functions were studied by Noonan and Thomas [19]. In 1967, the Hankel determinant for the coefficients of analytic function, $g(\eta) = \eta + \sum_{k=2}^{\infty} c_k \eta^k, \eta \in M$, where $M = \{\eta \in \mathbb{C} : |\eta| < 1\}$ is defined by Pommerenke [18]. The qth Hankel determinant for $q \geq 1$ and $k \geq 0$ is defined as,

$$H_q(k) = \begin{vmatrix} c_k & c_{k+1} & \cdot & \cdot & \cdot & c_{k+q-1} \\ c_{k+1} & c_{k+2} & \cdot & \cdot & \cdot & c_{k+q} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{k+q-1} & c_{k+q} & \cdot & \cdot & \cdot & c_{k+2(q-1)} \end{vmatrix}, \tag{1.1}$$

where q and k are positive integers, for more details, see [18, 20]. Hayman [21] investigated the second Hankel determinant for univalent functions in 1968. Growth rate of $H_q(k)$ for a univalent analytical function with defined bounds was determined in 1983 by Noor [22] as $k \to \infty$..

In 2000, Ehrenborg [23] conducted research on the Hankel determinant of exponential polynomials. Following that, in 2007, Janteng et al. [24] established a sharp upper limit for the functional $|c_2 c_4 - c_3^2|$ concerning starlike and convex functions. Then, in 2009, Babalola [25] introduced the Hankel determinant of order 3 for the renowned classes of starlike and convex functions within the disc M. The Hankel determinant of a convex function is $H_3(1) \leq \frac{15}{24}$, while that of a starlike function is $H_3(1) \leq 16$.

In 2019, Lecko et al. [26] discovered the acute turn for the Three-order Hankel determinant for the starlike function of hierarchy $\frac{1}{2}$ and obtained a sharp inequality. In 2022, Shi et al. [27] discovered the bounds of Hankel's determinant of order three in an open unit disc for particular subfamilies of convex and star-shaped operations connected with exponential functions. In 2022, Joshi et al. [28] found the third Hankel determinant in the open unit disc M for the class of starlike functions associated with exponential functions.

1.7 q-Calculus

Quantum Calculus can be described as a variant of ordinary calculus that operates without the concept of limits. The q-derivative and q-integral are the major techniques that Jackson [29, 30] introduced for the first time in 1909 and 1910 respectively. The theory of partitions was developed by Euler in 1740 which was the origin of q-analysis. C. F. Gauß (1777–1855) made major contributions to the development of calculus of quantum. The history of quantum computation can be linked to the contributions of Euler and Bernoulli. Researchers have expressed a strong interest in quantum calculus because of its multiple uses in Mathematics, mechanics, and physics.

Quantum Calculus is a significant subject of research in classical mathematical analysis. It aims to provide a theoretical overview of the integration and differentiation procedures. A broad field of mathematical study with ancient roots is quantum calculus. It is more difficult than other math disciplines due to the complex calculations and computations. This research aims to evaluate the geometric properties of analytic functions by employing the q-derivative. Ismail et al. [31] defined q-starlike functions using the q-derivative in 1990. This is the first time q-calculus is introduced in Geometric Function Theory. He achieved this through the use of the difference operator. When he first presented his class, he called it a "class of q-starlike functions". In 1999, Dziok and Srivastava [32] used the function of generalized hypergeometric to present a class of analytic functions with negative coefficients.

In 2011, Srivastava [33] investigated generalizations and Extensions of q in the Bernoulli [34], Genocchi [35] and Euler [36] Polynomials, laying the structure for applying Geometric Function Theory's q-calculus. Purohit [37] introduced a new class of multivalently analytic functions in the open unit disc. He also researched coefficient inequalities and several distortion theorems related to this function class. He was also the pioneer in utilizing a specific operator of the q-derivative in his research. His contribution to the field of analytic function theory was significant as he provided q-extensions for many results. In 2013, Aldweby and Darus [38] use a generalized operator involving a basic hypergeometric function to analyze a class of complex-valued harmonic univalent functions. They also provide the required coefficient conditions for functions in their class. Darus [16] introduces the class involving q-starlike and q-convex function in 2016 including Operator derivative of q. In 2016, Seoudy et al. [17] introduced novel classes of q-starlike and q-convex functions of complex order that include the q-derivative operator. Recently, q-calculus has been used by several scholars to make significant contributions in the field of Geometric Functions Theory, for more details, see [39, 40, 41, 42].

1.8 Starlike Function with respect to a Boundary Point

In 1981, Robertson [43] advances the description of the class of starlike functions and proposed a new class of starlike functions with regard to a boundary point. It is the geometric description of a function $g \in A$ such that $g(M)$ is starlike in relation to the boundary point and lies in the half plane. Silverman and Silvia [44] defined the class of univalent functions on M whose image is star-shaped relative to the boundary point in 1990. Starlike function with regard to a boundary point that are defined through subordination are studied by Mohd and Darus [45] in 2012 and results related to these functions are also proved by them. Lyzzaik [46] was the first to validate the characterisation of this class in 1984. In 1986, Todorov [47] associated this category with the functional $g(\eta)/(1-\eta)$ and derived a systematic formula along with coefficient estimates for the above mentioned class. Then, Silverman and Silvia [44] provided a comprehensive definition of the class of univalent functions on M, the image of which is star-shaped with regard to the boundary point. Subsequently, this class of starlike functions has attracted the attention of geometric function theorists and other scholars. Recently, scholars such as Noor et al. [48, 49], Bulut [50] and Ramachandran et al. [51] have employed $q-$calculus inside the domain of Geometric Function Theory to create new results.

The aforementioned scholar's research inspired us to introduce a new q-starlike class $S^*_{b,q}$ functions relative to a boundary point subordinate to the exponential function. Fekete-Szegö inequality, coefficient inequalities, covering results and differential subordination results will be find for our newly defined class. We will show that our newly defined class will be advancement of the class G_e of starlike function with regards to boundary point subordinated to exponential function. It will also be demonstrated that our new findings are an improvement of previously obtained results in the article [52]. Analytic methodologies, To achieve the desired outcomes,

q-calculus concepts and subordination techniques will be used.

1.9 Preface

This research focuses on the class of functions that are starlike in relation to a boundary point connected with exponential functions. The brief chapter-wise summary of the current thesis is as below:

In **chapter 2**, several key ideas of Geometric Function Theory are covered. The categories of univalent and analytical functions that are standardized in the disc M are explained. The subcategories of functions with one value and the group P of caratheodory functions are studied. The subordination technique is used to derive results that is pivotal in the study of analytic functions. The class of starlike functions in relation to a boundary point involving exponential functions is also given. Fekete-Szego inequality and Hankel determinant have been discussed in this chapter. A brief introduction to q-calculus and the q-function classes is presented. The concepts and outcomes discussed in this chapter are widely recognized and appropriately cited, with no advance results.

In **chapter 3**, the class of starlike functions with respect to a boundary point subordinated to exponential functions G_e will be presented. The results in accordance to this class like integral representation theorem, Fekete-Szegö problem, covering and differential subordination results will be investigated.

In **chapter 4**, a new class will be introduced by using the ideas of q-calculus, that is the class $S^*_{b,q}$ of q-starlike functions with regard to a boundary point. The properties that are related to our newly defined class will be discussed. The results like integral representation theorem, coefficient inequalities, Fekete-Szegö problem, covering and differential subordination results corresponding to the class $S^*_{b,q}$ will be derived.

In **chapter 5**, our research is concluded.

CHAPTER 2

PRELIMINARY CONCEPTS

2.1 Overview

This chapter aims at providing a foundation for further research by covering key terminologies and classical results. The research looks into a detailed discussion of normalized analytic univalent functions and caratheodory functions. The study also covers special functions, linear operators, and preliminary lemmas. Additionally, the preliminaries of q-calculus are briefly discussed, followed by an analysis of some recent classes of analytic functions.

2.2 Analytic Functions

In 1907, analytic and univalent functions are discovered by Koebe [2, 3] iFor the first time, M appears on the open unit disc. Analytical operations are essential because they provide a comprehensive framework for studying how functions behave in the complex plane.

Definition 2.2.1. [1] A function with complex values $g(\eta)$ is regarded as analytical at one point η_0 if its derivative is present elsewhere other than at that particular location η_0 but also in the neighbourhood of η_0. If a function $g(\eta)$ is analytic at every point within its domain, it is termed as an analytic function in that domain.

Definition 2.2.2. [1] An analytic function $g(\eta)$ that is normalized by the condition, $g(0) = 0$

and $g'(0) = 1$ is said to be in class A of functions and are represented as :

$$g(\eta) = \eta + \sum_{k=2}^{\infty} c_k \eta^k, \eta \in M. \tag{2.1}$$

In this thesis, $g \in A$ unless and until mentioned separately.

Definition 2.2.3. (**Riemann Mapping Theorem**) [3] Suppose η_0 be a point in a simply connected domain D and is a proper subset of a complex plane. Next, there's a unique function g that conformally represents the domain D into the accessible unit disk M with certain properties as, $g(0) = 0$ and $g'(\eta) > 1$. Instead of using an arbitrary domain D, we use an open unit disc M in Geometric Functions Theory. The Riemann Mapping Theorem offered a strong framework for investigating Geometric Function Theory.

2.3 Univalent Functions

In Geometric Function Theory, univalent functions have a special role. If a function is injective that is, it takes different values for different inputs inside the region, it is called univalent in that region. These functions are of particular interest due to their conformal mapping.

Definition 2.3.1. [3, 6] Assume that g be a complex-valued analytic function that is defined on the complex plane's domain D. Then, for any two unique points η_1 and η_2 in D, g is said to be univalent in D if $g(\eta_1)$ and $g(\eta_2)$ are also distinct. Also, if $g(\eta_1) = g(\eta_2)$, then $\eta_1 = \eta_2$. If a function $g(\eta)$ is univalent throughout the complex plane, it is referred to as univalent function in the entire plane.

2.4 The Univalent Function's Class S

The functions which are analytical and univalent within the visible unit disk M and normalized are in group S of operations. The S class in which normalized univalent operations play crucial role in Geometric Function Theory.

Definition 2.4.1. Suppose g be a function that belong to the class A of functions and is univalent in the open unit disc M, then $g \in S$. Koebe function is the famous example of class S of functions and is defined by,

$$K(\eta) = \frac{1}{4}\left(\frac{1+\eta}{1-\eta}\right)^2 - \frac{1}{4} = \frac{\eta}{(1-\eta)^2} = \sum_{k=1}^{k=\infty} k\eta^k, \eta \in M. \tag{2.2}$$

2.5 Catheodory Function's Class P

Functions with a positive real portion, called caratheodory functions, are included in the class P. Numerous subclasses of univalent functions are developed based on this class. We look on fundamental ideas that are relevant to class P and are necessary for our work.

Definition 2.5.1. Suppose that p be an analytic function in the open unit disk M, with $p(0) = 1$, $Re(p(\eta)) > 0$ and have the Taylor series expansion as:

$$p(\eta) = 1 + \sum_{k=1}^{\infty} p_k \eta^k, \eta \in M. \tag{2.3}$$

The Mobius function is a common example of a function in class P,

$$M_0(\eta) = \frac{1+\eta}{1-\eta} = 1 + 2\sum_{k=1}^{\infty} \eta^k, \eta \in M. \tag{2.4}$$

2.6 Subordination

The theory of subordination was initially presented by Lindelof [53] in 1909. Later, Littlewood [54] and Rogosinski [55, 56] made additional advancements. In Geometric Function Theory, subordination is a useful method for connecting two functions that are defined on distinct domains.

Definition 2.6.1. [57] Suppose f_1 and f_2 are two functions in the class A. Then f_1 is subordinate to f_2, written as $f_1 \prec f_2$ if there exist a Schwarz function $w(\eta)$ that is analytic and univalent in M with $w(0) = 1$ and $|w(\eta)| < 1$ such that $f_1(\eta) = f_2(w(\eta))$, $\eta \in M$.

2.7 Certain Subclasses of The Class S

The S class contains several subcategories, including the quasi-convex, convex, close-to-convex, and starlike functions classes. This section delves into two fundamental subclasses, starlike and convex functions. The discussion also explores the connection between these classes and Caratheodory functions, along with their established properties.

2.7.1 Convex and Starlike Functions

S^* represents the category of star-shaped functions, which has the following definition:

Definition 2.7.1. [3, 1] A starlike function maps an open unit disc, denoted by M, into a complex domain D, which has a geometric property known as starlike domain relative to the origin. A complex domain D in the complex plane is classified as starlike when there exists a complex number η_0 (typically symbolizing the origin, $\eta_0 = 0$), such that the line segment linking η_0 to any other point η lies wholly inside D. In simpler words, D is considered starlike if, for every point η within D, the line segment extending from the origin (or some designated central point) to η is completely contained within D. That is, $\forall \eta \in D, \lambda\eta \in D$, where $0 \leq \lambda \leq 1$, if $\eta_0 \in D$. The analytic description of starlike functions is as follows [8]:

$$S^* = \left\{ g \in A : Re\frac{\eta g'(\eta)}{g(\eta)} > 0, \eta \in M \right\}. \tag{2.5}$$

The C of convex functions' class is defined as stated below:

Definition 2.7.2. [3, 1] A convex function transfer an open unit disc M onto a complex domain D, which possesses a specific geometric property known as convexity with respect to the origin. In simpler terms, a complex domain D is considered convex when the entire line segment connecting any two points η_1 and η_2 within D is entirely contained within D. That is, $[\lambda\eta_1 + (1 - \lambda\eta_2)] \in D, \forall \eta \in D$, where η_1 and η_2 are both in D with $0 \leq \lambda \leq 1$.

The anlytical description of convex function's class is as follows:

$$C = \left\{ g \in A : Re\frac{(\eta g'(\eta))'}{g'(\eta)} > 0, \eta \in M \right\}. \tag{2.6}$$

In 1915, Alexander [58] proposed a relation between these two classes and is given by,

$$g \in C \Leftrightarrow \eta g'(\eta) \in S^*. \tag{2.7}$$

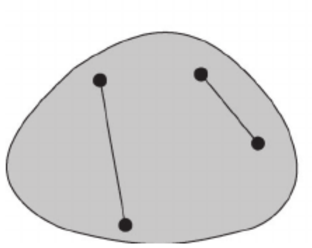

Figure 2.1: Convex Domain

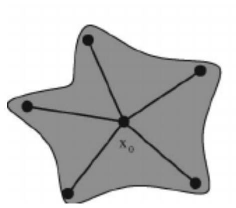

Figure 2.2: Starlike Domain

Figure 2.1 shows convex domain while figure 2.2 shows that domain is starlike.

2.7.2 Subclass of Convex Function

The convex function's class that are subordinated to the exponential function and satisfy the Schwarz function is defined by Mendiratta et al. [59], denoted as,

$$C_e = \left\{ g \in A : 1 + \frac{\eta g''(\eta)}{g'(\eta)} \prec e^{\eta} \right\}. \tag{2.8}$$

2.7.3 Class of Starlike Functions

Mendiratta and associates. [59] specified a subclass of a starlike functions connected with exponential function and is denoted by S_e^*. Its mathematical representation is as under:

$$S_e^* = \left\{ g \in A : \frac{\eta g'(\eta)}{g(\eta)} \prec e^{\eta} \right\}, \eta \in M. \tag{2.9}$$

2.7.4 Starlike Function with respect to a boundary point

In 1981, Robertson [43] proposed a new class S_b^* of starlike function with regards to a boundary point. Mathematically, it is given by:

$$S_b^* := \left\{ g \in A : Re\left\{ \frac{2\eta g'(\eta)}{g(\eta)} + \frac{1+\eta}{1-\eta} \right\} > 0 \right\}. \tag{2.10}$$

2.7.5 Starlike Function with respect to a boundary point associated with exponential function

In 2022, Lecko et al. [52], for $g \in A$, defined the class G_e, which is as under

$$G_e = \left\{ g \in A : \frac{2\eta g'(\eta)}{g(\eta)} + \frac{1+\eta}{1-\eta} \prec e^{\eta}, \eta \in M \right\}. \tag{2.11}$$

2.8 Hankel Determinant

The Hankel determinant refers to the determinant of a Hankel matrix. Pommerenke [18] proposed the Hankel determinant for certain univalent functions. The Hankel determinant is useful for studying singularities and analyzing power series with integrals coefficients. Noonan and Thomas [19] determined the Hankel determinant of certain analytic functions. In 1968, Hayman [21] studied the hankel determinant of second order for univalent functions. Bounds regarding the third Hankel determinant for particular kinds of analytical functions were determined by Prajapat and associates. [60, 61]. The qth Hankel determinant for $q \geq 1$ and $k \geq 0$ is defined as,

$$H_q(k) = \begin{vmatrix} c_k & c_{k+1} & \cdot & \cdot & \cdot & c_{k+q-1} \\ c_{k+1} & c_{k+2} & \cdot & \cdot & \cdot & c_{k+q} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{k+q-1} & c_{k+q} & \cdot & \cdot & \cdot & c_{k+2(q-1)} \end{vmatrix}, \tag{2.12}$$

where q and k are positive integers, for more details, see [18, 20].

The Hankel determinant of different order was discovered for various values of q and k. Given $q = 2$ and $k = 1$, the Hankel determinant is given by

$$H_2(1) = \begin{vmatrix} c_1 & c_2 \\ c_2 & c_3 \end{vmatrix}, = (c_1c_3 - c_2^2), (c_1 = 1).@ \tag{2.13}$$

This determinant is a a special method for determining a function's maximum value $|c_3 - \mu c_2^2|$ where μ is either complex or real in S. We call this the Fekete-Szego issue. Researchers have looked into Fekete-Szego inequalities for various classes of univalent analytic functions, see [62, 63, 64, 65, 66, 67]. Now for $q = 2$ and $k = 2$, the Hankel determinant is as under

$$H_2(2) = \begin{vmatrix} c_2 & c_3 \\ c_3 & c_4 \end{vmatrix}, = (c_2c_4 - c_3^2). \tag{2.14}$$

In 2018, Zaprawa [68] finds the Hankel determinant $H_2(3)$ for the coefficients of a function g which relates to the class S.

$$H_2(3) = \begin{vmatrix} c_3 & c_4 \\ c_4 & c_5 \end{vmatrix}, = (c_3c_5 - c_4^2). \tag{2.15}$$

Researchers have determined the formula for the Hankel determinant of third order in their respective studies, see [69, 70, 71] and is given by

$$H_3(1) = \begin{vmatrix} c_1 & c_2 & c_3 \\ c_2 & c_3 & c_4 \\ c_3 & c_4 & c_5 \end{vmatrix}, \tag{2.16}$$

it follows that

$$H_3(1) = c_1(c_3c_5 - c_4^2) - c_2(c_2c_5 - c_3c_4) + c_3(c_2c_4 - c_3^2). \tag{2.17}$$

2.9 Quantum Calculus or q-Calculus

Quantum calculus involves the q-analogs of mathematical concepts that able to retrieved as the limit $q \to 1^-$. The Euler made notable contributions to the advancement of q-calculus in the 18th century, and it was further advanced by Jackson [29, 30] before the 20th century. Researchers are becoming interested in this field because of its potential uses in Mathematics and Physics. Jackson [29, 30] systematically developed the q-derivative and q-integral.

In Geometric Function Theory, researchers examined subclasses of univalent functions utilizing q-calculus's preliminary steps. Ismail [31] developed and analyzed generalized starlike functions using q-calculus. The q-close-to-convex functions were recently defined, and various interesting findings were achieved, see [72]. Raghavendar and Swaminathan [73] examined some of the fundamental characteristics of these functions. The q-analogues of integral transforms have also been introduced. The genesis of fractional q-difference calculus are explained by Agarwal [74] and Al-Salam [75]. Later, Selvakumaran et al. [76], Ramachandranet. [77], Purohit and Raina [72] employed the fractional q-operator to describe various classes of analytic functions and their convexity properties. Subsequently, the fractional q-operator is utilized to characterize different classes of analytic functions and investigate their convexity properties, for details, see [76, 77, 72].

2.10 q-Derivative

Jackson [29] introduced the q-derivative in 1909.

Definition 2.10.1. Suppose that $g(\eta) \in A$, then q-derivative of g is defined as

$$D_q g(\eta) = \left\{ \frac{g(\eta) - g(q\eta)}{(1-q)\eta}, \eta \neq 0 \right\}, and D_q g(0) = g'(0), 0 < q < 1. \tag{2.18}$$

It is observed that if q approches to 1^-, $[n]_q \to n$ and $D_q g(\eta) \to g'(\eta)$, where $g'(\eta)$ is ordinary derivative.

$$\lim_{q \to 1^-} D_q g(\eta) = \lim_{q \to 1^-} \frac{g(\eta) - g(q\eta)}{(1-q)\eta} = g'(\eta), \tag{2.19}$$

(2.18) and (2.1) gives

$$D_q g(\eta) = \sum_{k=1}^{\infty} [k]_q c_k \eta^{k-1}, \tag{2.20}$$

where

$$[k]_q = \frac{1-q^k}{1-q}, k \in \mathbb{N}. \tag{2.21}$$

Definition 2.10.2 (**q-exponential function**). A q-analog of the $q-$exponential function is the ordinary exponential function. It is defined in terms of the q-exponential series and is often denoted as $exp_q(\eta)$. The q-exponential function is defined as :

$$exp_q(\eta) = \sum_{k=0}^{\infty} \frac{\eta^k}{[k]_q!}. \tag{2.22}$$

2.11 $q-$Convex Function

In 1989, Srivastava and Owa [78] discovered the class of $q-$convex function.

Definition 2.11.1. Suppose that $g \in A$. Then $g \in C_q$ if

$$\left| \frac{\eta D_q^2 g(\eta)}{D_q g(\eta)} - \frac{1}{1-q} \right| \leq \frac{1}{1-q}, \eta \in M, 0 < q < 1. \tag{2.23}$$

2.12 q-Starlike Function

The q-starlike function's class was proposed by Ismail et al. [31] in 1990.

Definition 2.12.1. Let $g \in A$, then $g \in S_q^*$ if

$$\left| \frac{\eta D_q g(\eta)}{g(\eta)} - \frac{1}{1-q} \right| \leq \frac{1}{1-q}, \eta \in M, 0 < q < 1. \tag{2.24}$$

The class $S_q^*(\alpha)$ of q-starlike functions of order α were discovered in 2016, by Wongsaijai and Sukantamala [79], where $0 \leq \alpha < 1$.

$$S_q^*(\alpha) = \left\{ g \in A : Re\left\{ \frac{\eta D_q g(\eta)}{g(\eta)} \right\} > \alpha, \eta \in M \right\}. \tag{2.25}$$

2.13 Preliminary Lemmas

Lemma 2.13.1. [80] Let $p \in P$ as in (2.3). Then $|p_k| \leq 2, \forall\, k \geq 1$.

For this result, Mobius function given in (2.4) is the extremal function.

Lemma 2.13.2. [81] Suppose that $g \in S^*(\psi)$ and $|\eta_0| = r < 1$. Then

$$-f_\psi(-r) \leq |g(\eta_0)| \leq f_\psi(r). \tag{2.26}$$

Equality holds for some $\eta_0 = 0$ iff g is a rotation of f_ψ.

Lemma 2.13.3. [81] Let $g \in S^*(\psi)$ and $|\eta_0| = r < 1$. Then

$$\left| arg\left\{ \frac{g(\eta_0)}{\eta_0} \right\} \right| \leq \max_{|\eta|=r} \left\{ arg\left\{ \frac{f_\psi(\eta)}{\eta} \right\} \right\}. \tag{2.27}$$

Equality holds for some $\eta_0 = 0$ iff g is a rotation of f_ψ.

Lemma 2.13.4. [81] If $p \in P$, then for $\gamma \in \mathbb{C}$,

$$|p_2 - \gamma p_1^2| \leq 2max\{1, |2\gamma - 1|\}, \tag{2.28}$$

if $\gamma \in \mathbb{R}$, then

$$|p_2 - \gamma p_1^2| \leq \left\{ \begin{array}{c} \text{-}4\gamma + 2, \gamma \leq 0 \\ 2, 0 \leq \gamma \leq 1 \\ 4\gamma - 2, \gamma \geq 0 \end{array} \right\}. \tag{2.29}$$

Lemma 2.13.5. [82] Let $p(\eta) \in P$ as given in (2.3). If $0 \leq \beta \leq 1$ and $\beta(2\beta - 1) \leq \delta \leq \beta$, then

$$|p_3 - 2\beta p_1 p_2 + \delta p_1^3| \leq 2. \tag{2.30}$$

Lemma 2.13.6. [83] Suppose g be a function with $g(0) = 1$. Then $g \in P$, if and only if

$$g(\eta) \prec \frac{1 + A\eta}{1 + B\eta}, \tag{2.31}$$

where $-1 \leq B < A \leq 1$.

Lemma 2.13.7. [84] Let $w(\eta)$ be analytic in M with $w(0) = 0$. Then if $|w(\eta)|$ achieves its maximum value on the circle $|\eta| = r < 1$ at a point η_0, we can write

$$\eta_0 w'(\eta_0) = kw(\eta_0), \tag{2.32}$$

where $k \in \mathbb{R}$ and $k \geq 1$.

Lemma 2.13.8. [85] Consider an analytic function $\phi(\eta)$ defined in M with $\phi(0) = 0$. Assume that $|\phi(\eta)|$ reaches its maximum value on the circle $|\eta| = r$ at a point $\eta_1 \in M$, then it follows

$$\eta_1 D_q \phi(\eta_1) = k\phi(\eta_1), \tag{2.33}$$

where $k \in \mathbb{R}$ and $k \geq 1$.

CHAPTER 3

STARLIKE FUNCTION WITH RESPECT TO A BOUNDARY POINT

In this chapter, we will analyse the class of starlike functions related to boundary points that are subordinated to exponential function and is given by:

$$G_e = \left\{ g \in A : \frac{2\eta g'(\eta)}{g(\eta)} + \frac{1+\eta}{1-\eta} \prec e^{\eta}, \eta \in M \right\}. \tag{3.1}$$

Results related to this class will be discussed more briefly which are recently found by Lecko et al. [52].

3.1 Representation Result

In this section, representation result related to class G_e will be evaluated.

Theorem 3.1.1. A function $g \in G_e$ if and only if there exists $p \in A$ such that $p \prec e^{\eta}$ and

$$g(\eta) = (1-\eta) exp\left(\frac{1}{2}\int_0^{\eta} \frac{P(t)-1}{t} dt\right). \tag{3.2}$$

Proof. Suppose that $g \in G_e$, then p is defined by

$$P(\eta) = 2\eta \frac{g'(\eta)}{g(\eta)} + \frac{1+\eta}{1-\eta}, \tag{3.3}$$

is holomorphic and $p \prec e^{\eta}$. Also, (3.3) gives us

$$P(\eta) - 1 = 2\eta \frac{g'(\eta)}{g(\eta)} + \frac{1+\eta}{1-\eta} - 1, \tag{3.4}$$

that is

$$p(\eta)-1=2\eta\frac{g'(\eta)}{g(\eta)}+\frac{2\eta}{1-\eta}.$$

This implies that

$$\frac{P(\eta)-1}{\eta}=2\frac{g'(\eta)}{g(\eta)}+\frac{2}{1-\eta}, \tag{3.5}$$

which results in

$$\int_0^\eta \frac{P(t)-1}{t}dt=\int_0^\eta 2\frac{g'(t)}{g(t)}dt+\int_0^\eta \frac{2}{1-t}dt. \tag{3.6}$$

This indicates that

$$\int_0^\eta \frac{P(t)-1}{t}dt=2log((g(\eta))-2log(1-\eta). \tag{3.7}$$

This precedes to

$$\int_0^\eta \frac{P(t)-1}{t}dt=log\left(\frac{g(\eta)}{1-\eta}\right)^2. \tag{3.8}$$

This leads us to

$$exp\left(\int_0^\eta \frac{P(t)-1}{t}dt\right)=\left(\frac{g(\eta)}{1-\eta}\right)^2, \tag{3.9}$$

which implies that

$$[g(\eta)]^2=[(1-\eta)]^2exp\left(\int_0^\eta \frac{P(t)-1}{t}dt\right). \tag{3.10}$$

This implements that

$$g(\eta)=(1-\eta)exp\left(\frac{1}{2}\int_0^\eta \frac{P(t)-1}{t}dt\right). \tag{3.11}$$

which is our required result.

Now conversely, let (3.2) holds that is

$$g(\eta)=(1-\eta)exp\left(\frac{1}{2}\int_0^\eta \frac{P(t)-1}{t}dt\right), \tag{3.12}$$

which results in

$$\frac{g(\eta)}{1-\eta}=exp\left(\frac{1}{2}\int_0^\eta \frac{P(t)-1}{t}dt\right). \tag{3.13}$$

It leads to

$$\ln\left(\frac{g(\eta)}{1-\eta}\right)=ln\left(exp\left(\frac{1}{2}\int_0^\eta \frac{P(t)-1}{t}dt\right)\right), \tag{3.14}$$

which implies

$$ln(g(\eta))-ln(1-\eta)=\frac{1}{2}\int_0^\eta \frac{P(t)-1}{t}dt. \tag{3.15}$$

Taking derivative on both sides of (3.15) results in

$$\frac{g'(\eta)}{g(\eta)}+\frac{1}{1-\eta}=\frac{1}{2}\left(\frac{P(\eta)-1}{\eta}\right).$$

This gives us

$$\frac{2\eta g'(\eta)}{g(\eta)} + \frac{2\eta}{1-\eta} = P(\eta) - 1. \tag{3.16}$$

By simplifying (3.16), we get

$$p(\eta) = \frac{2\eta g'(\eta)}{g(\eta)} + \frac{1+\eta}{1-\eta}, \tag{3.17}$$

which completes the theorem. □

3.2 Covering Results

In this section, covering results related to class G_e will be investigated.

Theorem 3.2.1. Let $0 < r < 1$. If $g \in G_e$, then

I. $\sqrt{\frac{-\psi_e(-r)}{r}}(1-r) \leq |g(\eta)| \leq \sqrt{\frac{\psi_e(r)}{r}}(1+r), |\eta| = r.$

II. $|arg\frac{g(\eta_0)}{(1-\eta_0)}| \leq \frac{1}{2}\max_{|\eta|=r}\left\{arg\frac{\psi_e(\eta)}{\eta}\right\}, |\eta_0| = r.$

Proof. As $g \in G_e$,

I. Suppose a function

$$h(\eta) = \frac{\eta(g(\eta))^2}{(1-\eta)^2}, \eta \in M. \tag{3.18}$$

Clearly, h is a holomorphic function in M. Taking the derivative of (3.18), we will get

$$\begin{aligned} h'(\eta) &= \frac{(1-\eta)^2\frac{d}{d\eta}[\eta(g(\eta))^2] - \eta(g(\eta))^2 2(1-\eta)(-1)}{(1-\eta)^4} \\ &= \frac{(1-\eta)(g(\eta))^2 + 2\eta(1-\eta)(g(\eta))(g'(\eta)) + 2\eta(g(\eta))^2}{(1-\eta)^3} \\ &= \frac{(g(\eta))^2(1+\eta) + 2\eta(1-\eta)(g(\eta))(g'(\eta))}{(1-\eta)^3}. \end{aligned} \tag{3.19}$$

By using (3.18) and (3.19), one can easily get,

$$\begin{aligned} \frac{\eta h'(\eta)}{h(\eta)} &= \frac{\eta(1+\eta)(g(\eta))^2 + 2\eta^2(1-\eta)g(\eta)g'(\eta)}{\eta(g(\eta))^2(1-\eta)} \\ &= \frac{1+\eta}{1-\eta} + \frac{2\eta g'(\eta)}{g(\eta)}. \end{aligned} \tag{3.20}$$

It is straight forward from (3.20) that $g \in G_e$ iff

$$\frac{\eta h(\eta)}{h(\eta)} \prec e^{\eta}. \tag{3.21}$$

By using Lemma 2.13.2, we obtain

$$-\psi_e(-r) \le |h(\eta)| \le \psi_e(r), |\eta| = r. \tag{3.22}$$

By using (3.18), we have

$$-\psi_e(-r) \le \left|\frac{\eta(g(\eta))^2}{(1-\eta)^2}\right| \le \psi_e(r), |\eta| = r. \tag{3.23}$$

This implies that

$$\frac{-\psi_e(-r)}{r} \le \left|\frac{(g(\eta)}{(1-\eta)}\right|^2 \le \frac{\psi_e(r)}{r}, |\eta| = r.$$

It follows

$$|1-\eta|\sqrt{\frac{-\psi_e(-r)}{r}} \le |g(\eta)| \le |1-\eta|\sqrt{\frac{\psi_e(r)}{r}}, |\eta| = r.$$

Now applying the properties of 'mod', we will get

$$\sqrt{\frac{-\psi_e(-r)}{r}}(1-r) \le |g(\eta)| \le \sqrt{\frac{\psi_e(r)}{r}}(1+r), |\eta| = r. \tag{3.24}$$

which is our required result.

II. By using (3.21), a function h that is denoted by (3.18) belongs to the S^* class. By using Lemma 2.13.3, we can write

$$\left|arg \frac{\eta_0(g(\eta_0))^2}{\eta_0(1-\eta_0)^2}\right| \le \max_{|\eta|=r}\left\{arg\left(\frac{\psi_e(\eta)}{\eta}\right)\right\}, \tag{3.25}$$

that is

$$\left|arg \frac{g(\eta_0)}{(1-\eta_0)}\right|^2 \le \max_{|\eta|=r}\left\{arg\left(\frac{\psi_e(\eta)}{\eta}\right)\right\}. \tag{3.26}$$

It follows that

$$\begin{aligned}\left|arg \frac{g(\eta_0)}{(1-\eta_0)}\right| &\le \max_{|\eta|=r}\left\{arg\left(\frac{\psi_e(\eta)}{\eta}\right)^{\frac{1}{2}}\right\} \\ &\le \frac{1}{2}\max_{|\eta|=r}\left\{arg\left(\frac{\psi_e(\eta)}{\eta}\right)\right\},\end{aligned} \tag{3.27}$$

which is the required result.

□

3.3 Coefficient Bounds and Fekete-Szegö Inequalities

In this section, coefficient bounds and Fekete-Szegö inequalities related to class G_e will be derived.

Theorem 3.3.1. If $g \in G_e$ is of the form

$$g(\eta) = 1 + \sum_{k=1}^{\infty} \theta_k \eta^k, \eta \in M. \tag{3.28}$$

Then

I. $|\theta_1 + 1| \leq \frac{1}{2}$,

II. $|\theta_1| \leq \frac{3}{2}$,

III. $|2\theta_2 - \theta_1^2 + 1| \leq \frac{1}{2}$,

IV. $|\theta_2| \leq \frac{3}{4}$,

V. $|3\theta_3 - 3\theta_1\theta_2 + \theta_1^3 + 1| \leq \frac{1}{2}$.

Proof. By using (2.11), we have

$$G_e = \left\{ g \in A : \frac{2\eta g'(\eta)}{g(\eta)} + \frac{1+\eta}{1-\eta} \right\} \prec e^{\eta}, \tag{3.29}$$

then there exist $w(\eta)$ in A such that

$$\frac{2\eta g'(\eta)}{g(\eta)} + \frac{1+\eta}{1-\eta} = e^{w(\eta)}. \tag{3.30}$$

By using (3.28) to the left side of (3.30), we will get

$$\frac{2\eta g'(\eta)}{g(\eta)} + \frac{1+\eta}{1-\eta} = 1 + 2(\theta_1 + 1)\eta + 2[2\theta_2 - \theta_1^2 + 1]\eta^2 + 2[3\theta_3 - 3\theta_1\theta_2 + \theta_1^3 + 1]\eta^3 + \ldots\ldots \tag{3.31}$$

Define a new function $p(\eta)$ by

$$p(\eta) = \frac{1+w(\eta)}{1-w(\eta)} = 1 + p_1\eta + p_2\eta^2 + p_3\eta^3 + \ldots., \eta \in D. \tag{3.32}$$

It is obvious that $p(\eta) \in P$. Moreover, (3.32) follows

$$(1 - w(\eta))p(\eta) = 1 + w(\eta).$$

It follows

$$p(\eta)-1=w(\eta)(1+p(\eta)).$$

This implies

$$w(\eta)=\frac{p(\eta)-1}{p(\eta)+1}. \tag{3.33}$$

By using the value of $p(\eta)$ from (3.32) in (3.33), we will get

$$w(\eta)=\frac{p_1}{2}\eta+\left(\frac{p_2}{2}-\frac{p_1^2}{4}\right)\eta^2+\left(\frac{p_3}{2}-\frac{p_1p_2}{2}+\frac{p_1^3}{8}\right)\eta^3+.....,\eta\in M. \tag{3.34}$$

As, we know that

$$e^{w(\eta)}=1+w(\eta)+\frac{(w(\eta))^2}{2}+\frac{(w(\eta))^3}{6}+.....,\eta\in M. \tag{3.35}$$

By substituting (3.34) into (3.35), we will get

$$e^{w(\eta)} = 1 + \frac{p_1}{2}\eta + \left(\frac{p_2}{2}-\frac{p_1^2}{8}\right)\eta^2 + \left(\frac{p_3}{2}-\frac{p_1p_2}{4}+\frac{p_1^3}{48}\right)\eta^3 + ... \tag{3.36}$$

Substituting (3.31) and (3.36) in (3.30), we will get

$$1+2(\theta_1+1)\eta+2[2\theta_2-\theta_1^2+1]\eta^2+2[3\theta_3-3\theta_1\theta_2+\theta_1^3+1]\eta^3+.....=1+\frac{p_1}{2}\eta$$
$$+\left(\frac{p_2}{2}-\frac{p_1^2}{8}\right)\eta^2+\left(\frac{p_3}{2}-\frac{p_1p_2}{4}+\frac{p_1^3}{48}\right)\eta^3+..... \tag{3.37}$$

Through a comparison of the respective coefficients in equation (3.37), we will get

$$2(\theta_1+1)=\frac{p_1}{2}, \tag{3.38}$$

$$2(2\theta_2-\theta_1^2+1)=\frac{p_2}{2}-\frac{p_1^2}{8}, \tag{3.39}$$

$$2(3\theta_3-3\theta_1\theta_2+\theta_1^3+1)=\frac{p_3}{2}-\frac{p_1p_2}{4}+\frac{p_1^3}{48}. \tag{3.40}$$

I. From (3.38), we have

$$\theta_1+1=\frac{p_1}{4}. \tag{3.41}$$

Taking 'mod' on both sides of (3.41), we will get

$$|\theta_1+1|=\frac{|p_1|}{4}. \tag{3.42}$$

By using Lemma 2.13.1, (3.42) becomes

$$|\theta_1+1|\leq\frac{1}{2}, \tag{3.43}$$

which is the required inequality.

II. Taking 'mod' on both sides of (3.41), and using the properties of 'mod', we will obtain

$$|\theta_1| \leq \frac{|p_1|}{4} + 1. \tag{3.44}$$

By using Lemma 2.13.1, (3.44) will become

$$|\theta_1| \leq \frac{2}{4} + 1, \tag{3.45}$$

it follows

$$|\theta_1| \leq \frac{3}{2}, \tag{3.46}$$

which is the required result.

III. Using (3.39), we have

$$2(2\theta_2 - \theta_1^2 + 1) = \frac{p_2}{2} - \frac{p_1^2}{8}. \tag{3.47}$$

Taking 'mod' on both sides of (3.47) gives us

$$2(|2\theta_2 - \theta_1^2 + 1|) = \left|\frac{p_2}{2} - \frac{p_1^2}{8}\right|. \tag{3.48}$$

It follows

$$|2\theta_2 - \theta_1^2 + 1| = \frac{1}{4}\left|p_2 - \frac{p_1^2}{4}\right|. \tag{3.49}$$

By using Lemma 2.13.4, we will get

$$|2\theta_2 - \theta_1^2 + 1| = \frac{1}{4}(2), \tag{3.50}$$

which will give

$$|2\theta_2 - \theta_1^2 + 1| = \frac{1}{2}, \tag{3.51}$$

which is the required result.

IV. By using (3.38), we can write

$$\theta_1 = \frac{p_1}{4} - 1. \tag{3.52}$$

Using (3.52) in (3.39), we have

$$4\theta_2 = \frac{p_2}{2} - p_1, \tag{3.53}$$

it follows

$$\begin{aligned} 4|\theta_2| &= \left|\frac{p_2}{2} - p_1\right| \\ &\leq \left|\frac{p_2}{2}\right| + |p_1|. \end{aligned} \tag{3.54}$$

By using Lemma 2.13.1, we will get

$$\begin{aligned} 4|\theta_2| &\leq \frac{2}{2}+2 \\ &\leq 3. \end{aligned} \tag{3.55}$$

This implies

$$|\theta_2| \leq \frac{3}{4}, \tag{3.56}$$

which is the required inequality.

V. By using (3.40), we have

$$2(3\theta_3 - 3\theta_1\theta_2 + \theta_1^3 + 1) = \frac{p_3}{2} - \frac{p_1 p_2}{4} + \frac{p_1^3}{48}. \tag{3.57}$$

It follows

$$\left|3\theta_3 - 3\theta_1\theta_2 + \theta_1^3 + 1\right| = \frac{1}{4}\left|p_3 - \frac{2p_1 p_2}{4} + \frac{p_1^3}{24}\right|. \tag{3.58}$$

By applying Lemma 2.13.5, we will get

$$\begin{aligned} \left|3\theta_3 - 3\theta_1\theta_2 + \theta_1^3 + 1\right| &\leq \frac{1}{4}(2) \\ &\leq \frac{1}{2}, \end{aligned} \tag{3.59}$$

which completes the proof.

□

3.4 Differential Subordination Results

In this section, differential subordination results related to class G_e will be find.

Theorem 3.4.1. Assume that $g(0) = 1$. If g satisfy the subordination condition,

$$\frac{2\eta g'(\eta)}{g(\eta)} + \frac{1+\eta}{1-\eta} \prec \frac{1+\eta}{1-\eta}, \eta \in M. \tag{3.60}$$

Then,

$$p(\eta) = \frac{\eta\,(g(\eta))^2}{(1-\eta)^2} \prec e^{\eta}, \eta \in M. \tag{3.61}$$

Proof. Suppose,

$$p(\eta) = \frac{\eta\,(g(\eta))^2}{(1-\eta)^2} = e^{w(\eta)}, \eta \in M. \tag{3.62}$$

This gives us,

$$\frac{p'(\eta)}{p(\eta)} = \frac{1+\eta}{\eta(1-\eta)} + \frac{2g'(\eta)}{g(\eta)} = w'(\eta), \tag{3.63}$$

$$\frac{\eta p'(\eta)}{p(\eta)} = \frac{1+\eta}{(1-\eta)} + \frac{2\eta g'(\eta)}{g(\eta)} = \eta w'(\eta). \tag{3.64}$$

It can be written as,

$$\frac{1+\eta}{(1-\eta)} + \frac{2\eta g'(\eta)}{g(\eta)} = \eta w'(\eta). \tag{3.65}$$

On contrary, we assume that $|w(\eta_0)| = 1$ such that $\eta_0 w'(\eta_0) = kw(\eta_0)$, where $k \geq 1$. If $w(\eta_0) = e^{\imath\theta}$, then by applying Lemma 2.13.7, we have

$$\begin{aligned} Re\left(\frac{2\eta g'(\eta)}{g(\eta)} + \frac{1+\eta}{1-\eta}\right) &= kw(\eta_0) \\ &= ke^{\imath\theta} \\ &= k(cos\theta + \imath sin\theta). \end{aligned} \tag{3.66}$$

For $\theta = \pi$

$$Re\left(\frac{2\eta g'(\eta)}{g(\eta)} + \frac{1+\eta}{1-\eta}\right) = -k < 0, \tag{3.67}$$

which is contradiction to our hypothesis because $k \geq 1$.

By employing Lemma 2.13.6, we have

$$Re\left(\frac{2\eta g'(\eta)}{g(\eta)} + \frac{1+\eta}{1-\eta}\right) > 0, \tag{3.68}$$

which equivalently gives (3.60). □

Theorem 3.4.2. Let $g(0) = 1$. If g fulfills the requirement of subordination,

$$\frac{2\eta g'(\eta)}{g(\eta)} + \frac{1+\eta}{1-\eta} \prec e^{\eta} + \eta, \eta \in M. \tag{3.69}$$

Then,

$$p(\eta) = \eta\left(\frac{g(\eta)}{1-\eta}\right)^2 \left(\int_0^{\eta} \left(\frac{g(t)}{1-t}\right)^2 dt\right)^{-1} \prec e^{\eta}, \eta \in M. \tag{3.70}$$

Proof. Suppose

$$p(\eta) = \eta\left(\frac{g(\eta)}{1-\eta}\right)^2 \left(\int_0^{z} \left(\frac{g(t)}{1-t}\right)^2 dt\right)^{-1} \prec e^{\eta}, \eta \in M, \tag{3.71}$$

$$p(\eta)=\eta\left(\frac{g(\eta)}{1-\eta}\right)^{2}\left(\int_{0}^{\eta}\left(\frac{g(t)}{1-t}\right)^{2}dt\right)^{-1}=e^{w(\eta)},\eta\in M, \tag{3.72}$$

where $w(\eta)$ is analytic in M, (3.72) gives us

$$\begin{aligned}\frac{p'(\eta)}{p(\eta)}&=\frac{1}{\eta}+\frac{2}{1-\eta}+\frac{2g'(\eta)}{g(\eta)}-\left(\frac{g(\eta)}{1-\eta}\right)^{2}\left(\int_{0}^{\eta}\left(\frac{g(t)}{1-t}\right)^{2}dt\right)^{-1}=w'(\eta)\\&=\frac{1+\eta}{\eta(1-\eta)}+\frac{2g'(\eta)}{g(\eta)}-\left(\frac{g(\eta)}{1-\eta}\right)^{2}\left(\int_{0}^{\eta}\left(\frac{g(t)}{1-t}\right)^{2}dt\right)^{-1}=w'(\eta).\end{aligned} \tag{3.73}$$

Using (3.70) in (3.73), we have

$$\frac{p'(\eta)}{p(\eta)}=\frac{1+\eta}{\eta(1-\eta)}+\frac{2g'(\eta)}{g(\eta)}-\frac{p(\eta)}{\eta}=w'(\eta). \tag{3.74}$$

It follows that

$$\frac{\eta p'(\eta)}{p(\eta)}+p(\eta)=\frac{1+\eta}{(1-\eta)}+\frac{2\eta g'(\eta)}{g(\eta)}=\eta w'(\eta). \tag{3.75}$$

We can also write (3.75) as

$$\frac{1+\eta}{(1-\eta)}+\frac{2\eta g'(\eta)}{g(\eta)}=\eta w'(\eta). \tag{3.76}$$

On contrary, we assume that $\eta_0\in M$, such that $|w(\eta_0)|=1$, $\eta_0 w'(\eta_0)=kw(\eta_0)$, where $k\geq 1$. By using Lemma 2.13.7, if $w(\eta_0)=e^{\iota\theta}$, then

$$\begin{aligned}Re\left(\frac{2\eta g'(\eta)}{g(\eta)}+\frac{1+\eta}{1-\eta}\right)&=kw(\eta_0)\\&=ke^{\iota\theta}\\&=k(cos\theta+\iota sin\theta).\end{aligned} \tag{3.77}$$

For $\theta=\pi$, we have

$$\begin{aligned}Re\left(\frac{2\eta g'(\eta)}{g(\eta)}+\frac{1+\eta}{1-\eta}\right)&=k(-1+0),\\&=-k<0,\end{aligned} \tag{3.78}$$

which is contradiction to our given hypothesis, since $k\geq 1$. By using Lemma 2.13.6, we will get

$$Re\left(\frac{2\eta g'(\eta)}{g(\eta)}+\frac{1+\eta}{1-\eta}\right)>0, \tag{3.79}$$

which implies (3.69), which is the required result.

□

CHAPTER 4

q-STARLIKE FUNCTIONS WITH RESPECT TO A BOUNDARY POINT

Using q-calculus principles, a new class of q-starlike functions $S^*_{b,q}$ in relation to a boundary point will be introduced. The properties of our newly defined class will be examined. We will investigate the results for the class $S^*_{b,q}$ like coefficient estimates, Fekete-Szegö problem, differential subordination results and many others.

4.1 q-Starlike Functions with respect to a Boundary Point

The noval class of q-starlike functions with respect to a boundary point is given by :

Definition 4.1.1. One describes a function g as being in class $S^*_{b,q}$ such that

$$2\eta\frac{D_q[g(\eta)]}{g(\eta)}+\frac{1+\eta}{1-\eta}\prec e^{q\eta},\eta\in M. \tag{4.1}$$

4.2 Representation Theorem

For class $S^*_{b,q}$, the representation result will be derived in this section.

Theorem 4.2.1. A function $g \in S^*_{b,q}$ if and only if there exists $p \in A$ such that $p \prec e^{q\eta}$ and

$$g(\eta) = (1-\eta)\left(exp\int_0^{\eta} \frac{P_q(t)-1}{t} d_qt\right)^{\frac{lnq}{2(q-1)}}. \tag{4.2}$$

Proof. Suppose $g \in S^*_{b,q}$, then a function p is given by,

$$P_q(\eta) = 2\eta\frac{D_q[g(\eta)]}{g(\eta)} + \frac{1+\eta}{1-\eta}, \tag{4.3}$$

is holomorphic and $p \prec e^{q\eta}$. Also, (4.3) leads us to

$$\begin{aligned} P_q(\eta)-1 &= 2\eta\frac{D_qg(\eta)}{g(\eta)} + \frac{1+\eta}{1-\eta} - 1 \\ &= 2\eta\frac{D_qg(\eta)}{g(\eta)} + \frac{2\eta}{1-\eta}. \end{aligned} \tag{4.4}$$

The equation (4.4) can be written as

$$\frac{P_q(\eta)-1}{\eta} = 2\frac{D_qg(\eta)}{g(\eta)} + \frac{2}{1-\eta}. \tag{4.5}$$

Taking q-integral on both sides of (4.5), we will get

$$\begin{aligned} \int_0^{\eta} \frac{P_q(t)-1}{t} d_qt &= \int_0^{\eta} 2\frac{D_qg(t)}{g(t)} d_qt + \int_0^{\eta} \frac{2}{1-t} d_qt \\ &= 2\frac{(q-1)}{lnq} log\left(g(\eta)\right) - 2\frac{(q-1)}{lnq} log(1-\eta) \\ &= log\left(g(\eta)\right)^{2\frac{(q-1)}{lnq}} - log(1-\eta)^{2\frac{(q-1)}{lnq}} \\ &= log\left(\frac{g(\eta)}{1-\eta}\right)^{\frac{2(q-1)}{lnq}}. \end{aligned} \tag{4.6}$$

Taking exponential on both sides of (4.6) leads us to

$$\left(\frac{g(\eta)}{1-\eta}\right)^{\frac{2(q-1)}{lnq}} = exp\int_0^{\eta} \frac{P_q(t)-1}{t} d_qt. \tag{4.7}$$

It gives us

$$\frac{g(\eta)}{1-\eta} = \left(exp\int_0^{\eta} \frac{P_q(t)-1}{t} d_qt\right)^{\frac{lnq}{2(q-1)}}. \tag{4.8}$$

It follows that,

$$g(\eta) = (1-\eta)\left(exp\int_0^{\eta} \frac{P_q(t)-1}{t} d_qt\right)^{\frac{lnq}{2(q-1)}}, \tag{4.9}$$

which is our required equation.

Now conversely, let (4.2) holds that is

$$g(\eta) = (1-\eta)\left(exp\int_0^{\eta} \frac{P_q(t)-1}{t} d_qt\right)^{\frac{lnq}{2(q-1)}}. \tag{4.10}$$

The equation (4.2) can also be written as

$$\frac{g(\eta)}{1-\eta} = \left(exp\int_0^\eta \frac{P_q(t)-1}{t}d_qt\right)^{\frac{lnq}{2(q-1)}}. \tag{4.11}$$

Taking 'ln' on both sides of (4.11), we will get

$$\ln\left(\frac{g(\eta)}{1-\eta}\right) = ln\left(exp\int_0^z \frac{P_q(t)-1}{t}d_qt\right)^{\frac{lnq}{2(q-1)}}, \tag{4.12}$$

this results in

$$ln(g(\eta)) - ln(1-\eta) = \frac{lnq}{2(q-1)}\int_0^\eta \frac{P_q(t)-1}{t}d_qt, \tag{4.13}$$

(4.13) becomes

$$D_q\left(ln(g(\eta))\right) - D_q\left(ln(1-\eta)\right) = \frac{lnq}{2(q-1)}D_q\left(\int_0^z \frac{P_q(t)-1}{t}d_qt\right),$$

we get

$$\frac{lnq}{q-1}\left(\frac{D_qg(\eta)}{g(\eta)}\right) - \frac{lnq}{q-1}\left(\frac{(-1)}{1-\eta}\right) = \frac{lnq}{2(q-1)}\left(\frac{P_q(\eta)-1}{\eta}\right). \tag{4.14}$$

Simplifying (4.14), we will get

$$\frac{lnq}{q-1}\left(\frac{D_qg(\eta)}{g(\eta)} - \frac{(-1)}{1-\eta}\right) = \frac{lnq}{2(q-1)}\left(\frac{P_q(\eta)-1}{\eta}\right),$$

implies that

$$\frac{D_qg(\eta)}{g(\eta)} + \frac{1}{1-\eta} = \frac{P_q(\eta)-1}{2\eta}.$$

This gives us

$$2\eta\frac{D_qg(\eta)}{g(\eta)} + \frac{2\eta}{1-\eta} = P_q(\eta)-1. \tag{4.15}$$

It follows that,

$$P_q(\eta) = 2\eta\frac{D_q[g(\eta)]}{g(\eta)} + \frac{1+\eta}{1-\eta}, \tag{4.16}$$

which imply (4.3).

□

Taking limit q tends to 1^- in the above result give the result proven in [52], as shown below:

Corollary 4.2.1.1. [52] If $g \in G_e$. Then, $g(\eta) = (1-\eta)exp\left(\frac{1}{2}\int_0^\eta \frac{P(t)-1}{t}dt\right)$.

4.3 Covering Results

This section will discover the covering results for class $S^*_{b,q}$.

Theorem 4.3.1. Let $0 < r < 1$. If $g \in S^*_{b,q}$, then

I. $\sqrt{\frac{-\psi_e(-r)}{r}}(1-r) \leq |g(\eta)| \leq \sqrt{\frac{\psi_e(r)}{r}}(1+r), |\eta| = r.$

II. $|arg \frac{g(\eta_0)}{(1-\eta_0)}| \leq \frac{1}{2} \max_{|\eta|=r} \left\{ arg \frac{\psi_e(\eta)}{\eta} \right\}, |\eta_0| = r.$

Proof. Let $g \in S^*_{b,q}$,

I. Suppose a function

$$h(\eta) = \frac{\eta(g(\eta))^2}{(1-\eta)^2}, \eta \in M. \tag{4.17}$$

Clearly $h \in H$, taking the q-derivative of (4.17), we will get

$$D_q h(\eta) = \frac{q(1-\eta)(g(\eta))^2 + q(1+q)\eta(1-\eta)g(\eta)D_q g(\eta) + (1+q)\eta(g(\eta))^2}{(1-\eta)(1-q\eta)^2}. \tag{4.18}$$

By using (4.17) and (4.18), we will easily get

$$\frac{\eta D_q h(\eta)}{h(\eta)} = \frac{q(1+q)\eta(1-\eta)^2 D_q g(\eta)}{g(\eta)(1-q\eta)^2} + \frac{(1-\eta)(q+\eta)}{(1-q\eta)^2}. \tag{4.19}$$

It is straight forward from above equation that $g \in S^*_{b,q}$ iff

$$\frac{\eta D_q h(\eta)}{h(\eta)} \prec e^{q\eta}. \tag{4.20}$$

By using the result of Lemma 2.13.2, we obtain

$$-\psi_e(-r) \leq |h(\eta)| \leq \psi_e(r), |\eta| = r. \tag{4.21}$$

By using (4.17) in (4.21), we have

$$-\psi_e(-r) \leq \left| \frac{\eta(g(\eta))^2}{(1-\eta)^2} \right| \leq \psi_e(r), |\eta| = r. \tag{4.22}$$

This implies that

$$\frac{-\psi_e(-r)}{r} \leq \left| \frac{(g(\eta)}{(1-\eta)} \right|^2 \leq \frac{\psi_e(r)}{r}, |\eta| = r.$$

It follows

$$|1-\eta| \sqrt{\frac{-\psi_e(-r)}{r}} \leq |g(\eta)| \leq |1-\eta| \sqrt{\frac{\psi_e(r)}{r}}, |\eta| = r.$$

Now applying the properties of mod, we will get

$$\sqrt{\frac{-\psi_e(-r)}{r}}(1-r) \leq |g(\eta)| \leq \sqrt{\frac{\psi_e(r)}{r}}(1+r), |\eta| = r, \tag{4.23}$$

which is our required result.

II. By using (4.20), a function h defined by (4.17) belongs to the class S. By using Lemma 2.13.3, we will get

$$\left| arg \frac{\eta_0(g(\eta_0))^2}{\eta_0(1-\eta_0)^2} \right| \leq \max_{|\eta|=r} \left\{ arg \left(\frac{\psi_e(\eta)}{\eta} \right) \right\}, \tag{4.24}$$

which gives us

$$\left| arg \frac{g(\eta_0)}{(1-\eta_0)} \right|^2 \leq \max_{|\eta|=r} \left\{ arg \left(\frac{\psi_e(\eta)}{\eta} \right) \right\}. \tag{4.25}$$

This implies

$$\begin{aligned} \left| arg \frac{g(\eta_0)}{(1-\eta_0)} \right| &\leq \max_{|\eta|=r} \left\{ arg \left(\frac{\psi_e(\eta)}{\eta} \right)^{\frac{1}{2}} \right\} \\ &\leq \frac{1}{2} \max_{|\eta|=r} \left\{ arg \left(\frac{\psi_e(\eta)}{\eta} \right) \right\}, \end{aligned} \tag{4.26}$$

which proofs the theorem.

□

4.4 Coefficient Bounds and Fekete-Szegö Inequalities

This section will describe the coefficient bounds and Fekete-Szegö inequalities for class $S^*_{b,q}$.

Theorem 4.4.1. If $g \in S^*_{b,q}$ is of the form

$$g(\eta) = 1 + \sum_{k=1}^{\infty} \theta_k \eta^k, \eta \in M. \tag{4.27}$$

Then

I. $|\theta_1 + 1| \leq \frac{q}{2}$,

II. $|\theta_1| \leq \frac{q}{2} + 1$,

III. $|(1+q)\theta_2 - \theta_1^2 + 1| \leq \frac{q}{2}$,

IV. $|\theta_2| \leq \frac{3q}{2(1+q)} + \frac{q(q-1)}{2(1+q)}$,

V. $|(1+q+q^2)\theta_3 - (2+q)\theta_1\theta_2 + \theta_1^3 + 1| \leq \frac{q}{2}$.

Proof. In view of

$$S^*_{b,q} = \left\{ g \in H : \frac{2\eta D_q g(\eta)}{g(\eta)} + \frac{1+\eta}{1-\eta} \right\} \prec exp(q\eta), \tag{4.28}$$

there exists $w \in H$ such that

$$\frac{2\eta D_q g(\eta)}{g(\eta)} + \frac{1+\eta}{1-\eta} = exp(qw(\eta)). \tag{4.29}$$

As from (4.27), we have

$$g(\eta) = 1 + \sum_{k=1}^{\infty} \theta_k \eta^k. \tag{4.30}$$

By using (4.30) in the left side of (4.29), by simple computation we can easily obtain

$$\frac{2\eta D_q g(\eta)}{g(\eta)} + \frac{1+\eta}{1-\eta} = 1 + 2(\theta_1+1)\eta + 2[(1+q)\theta_2 - \theta_1^2 + 1]\eta^2 + 2[(1+q+q^2)\theta_3 - (2+q)\theta_1\theta_2 + \theta_1^3 + 1]\eta^3 + \dots.., \eta \in M. \tag{4.31}$$

Define a new Function p by

$$p(\eta) = \frac{1+w(\eta)}{1-w(\eta)} = 1 + p_1\eta + p_2\eta^2 + p_3\eta^3 + \dots., \eta \in M. \tag{4.32}$$

It is clear that $p \in P$. Moreover, by rearranging (4.32), we get

$$w(\eta) = \frac{p(\eta)-1}{p(\eta)+1}. \tag{4.33}$$

By using the value of $p(\eta)$ from (4.32) in (4.33), we will get

$$w(\eta) = \frac{p_1}{2}\eta + \left(\frac{p_2}{2} - \frac{p_1^2}{4}\right)\eta^2 + \left(\frac{p_3}{2} - \frac{p_1p_2}{2} + \frac{p_1^3}{8}\right)\eta^3 + \dots.., \eta \in M. \tag{4.34}$$

As

$$e^{qw(\eta)} = 1 + qw(\eta) + \frac{q^2(w(\eta))^2}{2} + \frac{q^3(w(\eta))^3}{6} + \dots.., \eta \in M. \tag{4.35}$$

By substituting (4.34) into (4.35), it gives

$$e^{qw(\eta)} = 1 + \frac{qp_1}{2}\eta + \left[q\left(\frac{p_2}{2} - \frac{p_1^2}{4}\right) + \frac{q^2p_1^2}{8}\right]\eta^2 + \left[q\left(\frac{p_3}{2} - \frac{p_1p_2}{2} + \frac{p_1^3}{8}\right) + \frac{q^2}{2}\left(\frac{p_1p_2}{2} - \frac{p_1^3}{4}\right) + \frac{q^3p_1^3}{48}\eta^3 + \dots.., \eta \in M. \tag{4.36}$$

Using (4.31) and (4.36) in (4.29), we will get

$$1+2(\theta_1+1)\eta+2[(1+q)\theta_2-\theta_1^2+1]\eta^2+2[(1+q+q^2)\theta_3-(2+q)\theta_1\theta_2+\theta_1^3+1]\eta^3+\ldots..=1+\frac{qp_1}{2}\eta$$
$$+\left[q\left(\frac{p_2}{2}-\frac{p_1^2}{4}\right)+\frac{q^2p_1^2}{8}\right]\eta^2+\left[q\left(\frac{p_3}{2}-\frac{p_1p_2}{2}+\frac{p_1^3}{8}\right)+\frac{q^2}{2}\left(\frac{p_1p_2}{2}-\frac{p_1^3}{4}\right)+\frac{q^3p_1^3}{48}\right]\eta^3+\ldots..,\eta\in M. \tag{4.37}$$

Upon comparing the coefficients of (4.37), we will get

$$2(\theta_1+1)=\frac{qp_1}{2}, \tag{4.38}$$

$$2[(1+q)\theta_2-\theta_1^2+1]=q\left(\frac{p_2}{2}-\frac{p_1^2}{4}\right)+\frac{q^2p_1^2}{8}, \tag{4.39}$$

$$2[(1+q+q^2)\theta_3-(2+q)\theta_1\theta_2+\theta_1^3+1]=q\left(\frac{p_3}{2}-\frac{p_1p_2}{2}+\frac{p_1^3}{8}\right)+\frac{q^2}{2}\left(\frac{p_1p_2}{2}-\frac{p_1^3}{4}\right)+\frac{q^3p_1^3}{48}. \tag{4.40}$$

I. Now, (4.38) can be written as

$$\theta_1+1=\frac{qp_1}{4}. \tag{4.41}$$

Applying 'mod' on both sides of (4.41) and using Lemma 2.13.1, we have

$$|\theta_1+1|\leq\frac{q(2)}{4}, \tag{4.42}$$

it gives us

$$|\theta_1+1|\leq\frac{q}{2}, \tag{4.43}$$

which is required result.

II. From (4.41), it follows that

$$|\theta_1|\leq\frac{q|p_1|}{4}+1. \tag{4.44}$$

By using Lemma 2.13.1, (4.44) implies that

$$|\theta_1|\leq\frac{q}{2}+1, \tag{4.45}$$

which proofs the result.

III. By solving (4.39), we can easily obtain

$$2[(1+q)\theta_2-\theta_1^2+1]=\frac{q}{2}\left[p_2-\left(\frac{1}{2}-\frac{q}{4}\right)p_1^2\right]. \tag{4.46}$$

By taking "mod" on both sides of (4.46) and using Lemma 2.13.4, we will get

$$2|(1+q)\theta_2 - \theta_1^2 + 1| \leq q.$$

This implies that

$$|(1+q)\theta_2 - \theta_1^2 + 1| \leq \frac{q}{2}, \tag{4.47}$$

which proofs our result.

IV. BY using (4.38), we can write

$$\theta_1 = \frac{qp_1}{4} - 1. \tag{4.48}$$

Using (4.48) in (4.40) and then solving, we will get

$$(1+q)\theta_2 = \frac{q}{2}\left(\frac{p_2}{2} - p_1\right) + \frac{(q^2-q)p_1^2}{8}. \tag{4.49}$$

Applying the modulus operation to both sides of (4.49) and utilizing Lemma 2.13.1, we will have

$$(1+q)|\theta_2| \leq \frac{3q}{2} + \frac{q(q-1)}{2}. \tag{4.50}$$

This implies

$$|\theta_2| \leq \frac{3q}{2(1+q)} + \frac{q(q-1)}{2(1+q)}, \tag{4.51}$$

which is required resullt.

V. Using (4.40), we can write

$$2|(1+q+q^2)\theta_3 - (2+q)\theta_1\theta_2 + \theta_1^3 + 1| = \left|\frac{p_3 q}{2} - \frac{p_1 p_2}{2}\left(\frac{2q-q^2}{2}\right) + \frac{p_1^3}{8}\left(\frac{6q+q^3-6q^2}{6}\right)\right|. \tag{4.52}$$

By applying Lemma 2.13.5, we will get

$$2|(1+q+q^2)\theta_3 - (2+q)\theta_1\theta_2 + \theta_1^3 + 1| \leq \frac{q}{2}(2). \tag{4.53}$$

This implies that

$$|(1+q+q^2)\theta_3 - (2+q)\theta_1\theta_2 + \theta_1^3 + 1| \leq \frac{q}{2}, \tag{4.54}$$

which proves the theorem.

□

Taking the limit on q leads to 1^- in the above results, this gives the results proved in [52], as given in the corollary.

Corollary 4.4.1.1. [52] If $g \in G_e$. Then

I. $|\theta_1 + 1| \leq \frac{1}{2}$,

II. $|\theta_1| \leq \frac{3}{2}$,

III. $|2\theta_2 - \theta_1^2 + 1| \leq \frac{1}{2}$,

IV. $|\theta_2| \leq \frac{3}{4}$,

V. $|3\theta_3 - 3\theta_1\theta_2 + \theta_1^3 + 1| \leq \frac{1}{2}$.

4.5 Differential Subordination Results

This section will discuss differential subordination results for class $S^*_{b,q}$.

Theorem 4.5.1. Given that $g \in H$, $g(0) = 1$. If g satisfies the requirement for subordination,

$$\frac{(1+q)\eta D_q g(\eta)}{g(\eta)} + \frac{1+q\eta}{1-\eta} \prec 1 + q\eta, \eta \in M. \tag{4.55}$$

Then,

$$p(\eta) = \frac{(g(\eta))^2}{(1-\eta)^2} \prec e^{q\eta}, \eta \in M. \tag{4.56}$$

***Proof*.** Cosider (4.56), we can write it as

$$p(\eta) = \frac{(g(\eta))^2}{(1-\eta)^2} = e^{qw(\eta)}, \eta \in M, \tag{4.57}$$

where $w(\eta)$ is analytic and $w(0) = 1$ in M. Taking q-logarithmic differentiation of (4.57)

$$D_q log(p(\eta)) = D_q log\left(\frac{(g(\eta))^2}{(1-\eta)^2}\right) = D_q log\left(e^{qw(\eta)}\right). \tag{4.58}$$

This implies that

$$\left(\frac{lnq}{1-q}\right)\left(\frac{D_q p(\eta)}{p(\eta)}\right) = \left(\frac{lnq}{1-q}\right)\left(\frac{1}{\eta} + \frac{[2]_q g(\eta) D_q g(\eta)}{(g(\eta))^2} - \frac{[2]_q(1-\eta)(-1)}{(1-\eta)^2}\right) = \left(\frac{lnq}{1-q}\right)(qD_q w(\eta)). \tag{4.59}$$

$$\begin{aligned} \frac{D_q p(\eta)}{p(\eta)} &= \frac{1}{\eta} + \frac{(1+q)D_q g(\eta)}{g(\eta)} + \frac{1+q}{1-z} = qD_q w(\eta) \\ &= \frac{(1+q)D_q g(\eta)}{g(\eta)} + \frac{(1-\eta)+(1+q)\eta}{\eta(1-\eta)} = qD_q w(\eta) \\ &= \frac{(1+q)D_q g(\eta)}{g(\eta)} + \frac{1+q\eta}{\eta(1-\eta)} = qD_q w(\eta). \end{aligned} \tag{4.60}$$

Now, (4.60) can be written as

$$\frac{\eta D_q p(\eta)}{p(\eta)} = \frac{(1+q)\eta D_q g(\eta)}{g(\eta)} + \frac{1+q\eta}{1-\eta} = q\eta D_q w(\eta). \tag{4.61}$$

Now

$$\frac{(1+q)\eta D_q g(\eta)}{g(\eta)} + \frac{1+q\eta}{1-\eta} = q\eta D_q w(\eta). \tag{4.62}$$

On contrary, we assume that $\eta_0 \in M$, such that $|w(\eta_0)| = 1$, $\eta_0 D_q w(\eta_0) = kw(\eta_0)$, where $k \geq 1$. Suppose $w(z_0) = e^{\imath\theta}$, then by applying Lemma 2.13.8, we have

$$\begin{aligned} Re\left(\frac{(1+q)\eta D_q g(\eta)}{g(\eta)} + \frac{1+q\eta}{1-\eta}\right) &= kw(\eta_0) \\ &= ke^{\imath\theta} \\ &= k(cos\theta + \imath sin\theta). \end{aligned} \tag{4.63}$$

For $\theta = \pi$, we have

$$\begin{aligned} Re\left(\frac{(1+q)\eta D_q g(\eta)}{g(\eta)} + \frac{1+q\eta}{1-\eta}\right) &= k(-1+0) \\ &= -k < 0, \end{aligned} \tag{4.64}$$

which is contradiction to our given hypothesis, since $k \geq 1$.

By Lemma 2.13.6, it is clear that

$$Re\left(\frac{(1+q)\eta D_q g(\eta)}{g(\eta)} + \frac{1+q\eta}{1-\eta}\right) > 0, \tag{4.65}$$

which shows that

$$\frac{(1+q)\eta D_q g(\eta)}{g(\eta)} + \frac{1+q\eta}{1-\eta} \prec 1+q\eta, \eta \in M, \tag{4.66}$$

which proofs the theorem. □

Taking $q \to 1^-$ in the previous result gives the result as given in the corollary.

Corollary 4.5.1.1. [52] If $g \in G_e$. Then $\frac{2\eta g'(\eta)}{g(\eta)} + \frac{1+\eta}{1-\eta} \prec 1+\eta, \eta \in M$.

Theorem 4.5.2. Let $g(0) = 1$. If g satisfies the subordination condition,

$$\frac{(1+q)\eta D_q g(\eta)}{g(\eta)} + \frac{1+q\eta}{1-\eta} \prec e^{\eta} + \eta, \eta \in M. \tag{4.67}$$

Then,

$$p(\eta) = \eta\left(\frac{g(\eta)}{1-\eta}\right)^2\left(\int_0^{\eta}\left(\frac{g(t)}{1-t}\right)^2 dt\right)^{-1} \prec e^{q\eta}, \eta \in M. \tag{4.68}$$

Proof. We can write (4.68) as,

$$p(\eta)=\eta\left(\frac{g(\eta)}{1-\eta}\right)^2\left(\int_0^\eta\left(\frac{g(t)}{1-t}\right)^2dt\right)^{-1}=e^{qw(\eta)},\eta\in M,\tag{4.69}$$

where $w(\eta)$ is analytic and $w(0)=1$ in M, (4.69) becomes,

$$\left(\frac{lnq}{1-q}\right)\left(\frac{D_qp(\eta)}{p(\eta)}\right)=\left(\frac{lnq}{1-q}\right)\left(\frac{1}{\eta}+\frac{[2]_qg(\eta)D_qg(\eta)}{(g(\eta))^2}\right.$$
$$\left.-\frac{[2]_q(1-\eta)(-1)}{(1-\eta)^2}-\left(\frac{g(\eta)}{1-\eta}\right)^2\left(\int_0^\eta\left(\frac{g(t)}{1-t}\right)^2dt\right)^{-1}\right)=\left(\frac{lnq}{1-q}\right)(qD_qw(\eta)).\tag{4.70}$$

It implies that

$$\begin{aligned}\frac{D_qp(\eta)}{p(\eta)}&=\frac{1}{\eta}+\frac{[2]_qg(\eta)D_qg(\eta)}{(g(\eta))^2}-\frac{[2]_q(1-\eta)(-1)}{(1-\eta)^2}-\left(\frac{g(\eta)}{1-\eta}\right)^2\left(\int_0^\eta\left(\frac{g(t)}{1-t}\right)^2dt\right)^{-1}=qD_qw(\eta)\\&=\frac{1}{\eta}+\frac{(1+q)D_qg(\eta)}{g(\eta)}+\frac{1+q}{1-\eta}-\left(\frac{g(\eta)}{1-\eta}\right)^2\left(\int_0^\eta\left(\frac{g(t)}{1-t}\right)^2dt\right)^{-1}=qD_qw(\eta)\\&=\frac{(1-\eta)+(1+q)\eta}{\eta(1-\eta)}+\frac{(1+q)D_qg(\eta)}{g(\eta)}-\left(\frac{g(\eta)}{1-\eta}\right)^2\left(\int_0^\eta\left(\frac{g(t)}{1-t}\right)^2dt\right)^{-1}=qD_qw(\eta)\\&=\frac{1+q\eta}{\eta(1-\eta)}+\frac{(1+q)D_qg(\eta)}{g(\eta)}-\left(\frac{g(\eta)}{1-\eta}\right)^2\left(\int_0^\eta\left(\frac{g(t)}{1-t}\right)^2dt\right)^{-1}=qD_qw(\eta).\end{aligned}\tag{4.71}$$

Using (4.69) in (4.71), we have

$$\frac{D_qp(\eta)}{p(\eta)}=\frac{1+q\eta}{\eta(1-\eta)}+\frac{(1+q)D_qg(\eta)}{g(\eta)}-\frac{p(\eta)}{\eta}=qD_qw(\eta),\tag{4.72}$$

which leads to

$$\frac{\eta D_qp(\eta)}{p(\eta)}+p(\eta)=\frac{1+q\eta}{1-\eta}+\frac{(1+q)\eta D_qg(\eta)}{g(\eta)}=q\eta D_qw(\eta).\tag{4.73}$$

This implies that

$$\frac{1+q\eta}{1-\eta}+\frac{(1+q)\eta D_qg(\eta)}{g(\eta)}=q\eta D_qw(\eta).\tag{4.74}$$

On contrary, we assume the existance of a $\eta_0\in M$ such that $|w(\eta_o)|=1$ and let $w(\eta_0)=e^{\iota\theta}$. From (4.74), we have

$$\frac{1+q\eta}{1-\eta}+\frac{(1+q)\eta D_qg(\eta)}{g(\eta)}=q\eta D_qw(\eta_0).\tag{4.75}$$

Using lemma (2.13.8), we can write

$$
\begin{aligned}
Re\left(\frac{1+q\eta}{1-\eta}+\frac{(1+q)\eta D_q g(\eta)}{g(\eta)}\right) &= kw(\eta_0), k \geq 1 \\
&= ke^{\imath\theta} \\
&= k(cos\theta+\imath sin\theta).
\end{aligned} \tag{4.76}
$$

For $\theta = \pi$, we have

$$
\begin{aligned}
Re\left(\frac{(1+q)\eta D_q g(\eta)}{g(\eta)}+\frac{1+q\eta}{1-\eta}\right) &= k(-1+0) \\
&= -k < 0.
\end{aligned} \tag{4.77}
$$

This is contradiction to our given hypothesis, since $k \geq 1$.

By employing Lemma 2.13.6, we will get

$$
Re\left(\frac{(1+q)\eta D_q g(\eta)}{g(\eta)}+\frac{1+q\eta}{1-\eta}\right) > 0. \tag{4.78}
$$

Consequently, we can write

$$
\frac{(1+q)\eta D_q g(\eta)}{g(\eta)}+\frac{1+q\eta}{1-\eta} \prec e^{\eta}+\eta, \eta \in M, \tag{4.79}
$$

we get the needed result.

□

The result demonstrated in [52] is obtained by utilizing $q \to 1^-$ in the preceding result, as indicated by the subsequent corollary..

Corollary 4.5.2.1. [52] If $g \in G_e$. Then, $\frac{2\eta g'(\eta)}{g(\eta)}+\frac{1+\eta}{1-\eta} \prec e^{\eta}+\eta, \eta \in M$.

CHAPTER 5

CONCLUSION

This thesis focuses on the analysis of coefficients for Analytical, univalent, and normalized functions inside an open unit disc. The essential ideas of q-calculus and Geometric Function Theory are reviewed, with an emphasis on investigating how q-calculus can be applied to particular analytic functions. In relation to a boundary point, a new class of q-starlike functions is introduced using the q-difference operator.

Our research primarily focuses on the q-extension of the group of starlike functions with respect to boundary points. The work that is done by Lecko et al. [52] regarding the class G_e of star-like functions concerning boundary points subject to the exponential function is expanded. The previously described class's q-version is presented that is $S^*_{b,q}$ of q-starlike function associated to a boundary point. This class is defined by using q-derivative operator. Subordination technique is used to define this class.

A few intriguing characteristics of the functions that belong to our new class, such as the Fekete-Szegö problem, coefficient bounds, representation results, distortion bounds and differential subordination results are derived. The results of this thesis have demonstrated that the newly defined class is an important advancement over results proved by scholars in Geometric Function Theory. Compared to the previous class G_e, our new class is more extensive and developed, providing a refined foundation for studying complex functions. To validate the correctness of our results, limit q approaches to 1^- is used, our findings exhibited strong similarity to the established results for class G_e which showed that our new results are more advanced and refined as compared to the results proved in [52]. We hope that our contributions will significantly enhance the field of Geometric Function Theory, opening the path for future advances and insights in this area of study.

5.1 Future Work

Investigating q-starlike functionals concerning a boundary point. The focus of this thesis is related to the exponential function. Exploration of the results of our thesis for the classes of q-quasi convex functions and q-close-to-convex functions can be done. New boundary point classes can be examined using the subordination method and quantum calculus principles. comparing them with the classes discussed in our research. Further, It is possible to deduce the outcomes of the thesis for the classes for q-quasi convex functions, and q-close-to-convex functions.

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