NUMERICAL STUDY AND STABILITY ANALYSIS OF PARABOLIC EQUATION OF FRACTIONAL ORDER

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Candidate of **Master of Science in Mathematics (MS)** at the National University of Modern Languages do hereby declare that the thesis **Numerical study and stability analysis of parabolic equation of fractional order** submitted by me in partial fulfillment of MS degree is my original work, and has not been submitted or published earlier. I also solemnly declare that it shall not, in the future, be submitted by me for obtaining any other degree from this or any other university or institution. I also understand that if evidence of plagiarism is found in my thesis/dissertation at any stage, even after the award of a degree, the work may be cancelled and the degree revoked.

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ABSTRACT

Title: Numerical study and stability analysis of parabolic equation of fractional order

Nowadays the research community prefers fractional calculus over classical calculus, as fractional calculus (FC) has extensively explored the physical properties of fractional-order differentials and integrals. The main reason behind this is that classical calculus deals with more specific behavior, whereas fractional calculus deals with more generic behavior. In this framework, this thesis witnesses the numerical study of the fractional-order parabolic equations. In the modeled problem the fractional-order derivative is assumed in the Caputo sense due to its numerous advantages—the modeled time-fractional parabolic problem is discretized using the Crank-Nicolson method. The thesis also provides a comprehensive study of stability and convergence analysis of the proposed mathematical method. To demonstrate the accuracy as well as efficiency of the scheme, numerical problems with various fractional order derivatives are finally given and compared with the exact solutions. Furthermore, a comparative numerical study is done to demonstrate the efficiency of the proposed method. The thesis is finally summarised in the conclusion.

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LIST OF ABBREVIATIONS

TFDEs	Time-fractional Diffusion Equation
PDEs	Partial Differential Equation
FC	Fractional Calculus
1D	One-dimensional
FDM	Finite Difference Method
HCT	Heat Conduction Toolbox
FHCT	Fractional Heat Conduction Toolbox
HPM	Homotopy perturbation method
CIIF	Compact implicit integration factor
2D	Two-dimensional
A.E	Absolute Error

LIST OF SYMBOLS

D	Differentiation operator
α	Fractional Parameter
M,N	Mesh Parameter
$f(x,t,u)$	Source Function
$D_t^\alpha f(t)$	Caputo Fractional Derivative
Δt	Time Step
Δx^2	Space Step
$\phi(x,t)$	Source Function in 1D

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Regarding this work, I was impacted both emotionally and intellectually by several people. I am grateful to Allah for having them because they have supported me in any cause. Before anything else, I would like to express my gratitude to **Dr. Muhammad Usman**, my research supervisor, for his kindness, patient guidance, and plenty of ideas that have helped me with our study and thesis. My effort was significantly supported by my supervisor's great knowledge, experience, and proficiency in fractional calculus, particularly with fractional order PDEs. Without his help and contribution, this would not have been possible.

I would like to express my gratitude to the Department of Mathematics at the National University of Modern Languages in Islamabad, especially to the honourable department head **Dr. Sadia Riaz**, for her support and guidance throughout my thesis.

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DEDICATION

This thesis work is dedicated to my parents especially my father, my spouse and my teachers throughout my education career who have not only loved me unconditionally but whose good examples have taught me to work hard for the things that I aspire to achieve.

CHAPTER 1

INTRODUCTION

1.1 Overview

In mathematics, any order's derivative, whether complex or real, is known as a fractional derivative. Fractional calculus is the study of the many real number or complex number definitions of the differentiation operator D and the integration operator J, as well as the creation of a calculus for these operators that extends in addition to the traditional one. The fractional derivative first appeared in a letter written by Leibniz to L'Hopital in 1695 [1] as mentioned in Figure 1.1 [2]. Leibniz wrote a letter to one of the Bernoulli's brothers around the same time in which he described how the binomial theorem and the Leibniz rule for the fractional derivative of a product of two functions are identical. N.H Abel introduced fractional calculus in one of his early papers [2-3]. Many mathematical models arising in physics have been constructed in recent years employing the theory of fractional derivative [1].

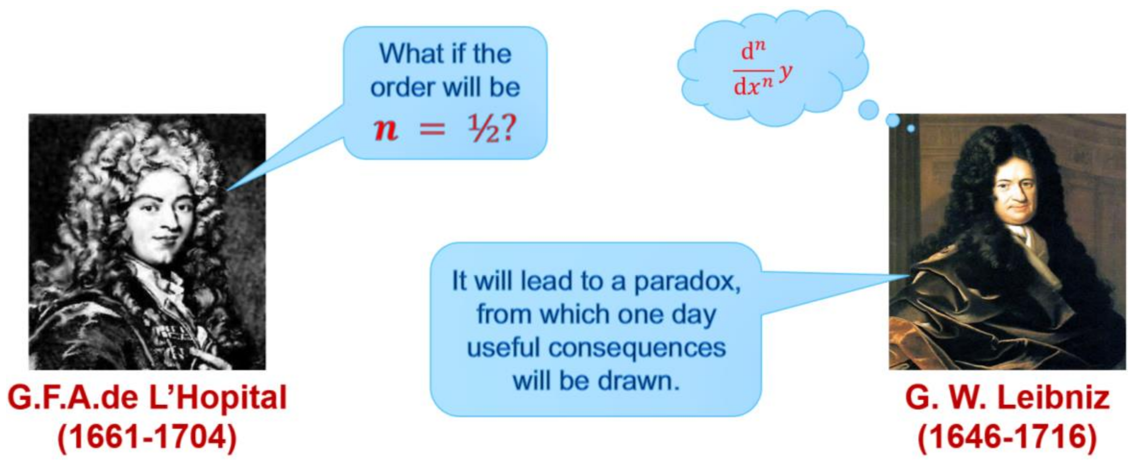


Figure 1.1 Conversation between Hospital and Leibniz on the order of derivative

Fractional calculus (FC) has extensively explored the physical properties of fractional-order differentials and integrals. Classical calculus deals with more specific behavior, whereas fractional calculus deals with more generic behavior as portrayed in Figure 1.2. Numerous researchers made valuable contributions in this field as shown in Figure 1.3. An efficient method for illustrating the memory and heredity characteristics of different materials and processes is to use fractional derivatives. When compared to conventional integer-order derivatives, this is their main advantage.

Figure1.2 A generic concept between classical and fractional derivatives

Biological population models, electromagnetics, viscoelasticity, fluid mechanics, electrochemistry, optics, and signals processing are among the technological and scientific fields that require fractional calculus [3]. Fractional differential equations are the most effective tool for modeling physical and technical processes. For appropriate modelling of systems that need accurate damping modelling, the fractional derivative approach can be applied. These disciplines have recently presented a variety of numerical and analytical techniques as well as their applications to new problems.

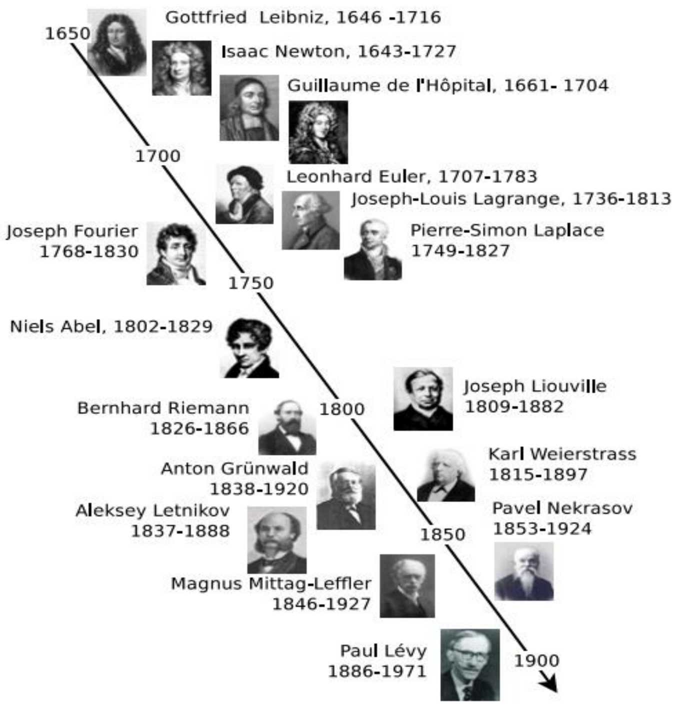


Figure 1.3 Author’s contribution to fractional calculus

Mathematical modelling of real-world problems frequently leads to fractional differential equations and other challenges involving specific mathematical physics functions, as well as their extensions and modifications in one or more variables. In many other models, fractional order partial differential equations (PDEs) likewise regulate most physical processes, such as those associated with electricity, fluid dynamics, quantum physics, ecological systems, and many more. Consider two-dimensional problems as

$$a\frac{\partial^2 u}{\partial x^2}+b\frac{\partial^2 u}{\partial x\partial y}+c\frac{\partial^2 u}{\partial y^2}+d\frac{\partial u}{\partial x}+e\frac{\partial u}{\partial y}+fu+g(x,y)=0. \tag{1.1}$$

In the above equation if $b^2-4ac=0$ then the Eq. (1.1) is said to be a parabolic equation. It's necessary to keep in mind that many time-dependent processes, such as heat conduction and particle diffusion can be explained by the parabolic problem (1.1). Here, we consider the very famous parabolic equation (heat equation) as,

$$\frac{\partial u}{\partial t}=k\Delta u(x,t)+f(x,t,u), \text{where } \Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}. \tag{1.2}$$

where k is shown by the material's thermal diffusivity, which processes the rod's ability for heat conduction. Heat equation (1.2) is an important PDE that describes the heat transfer in a certain area over time. The concept of a parabolic PDE can be applied in several contexts. In this case, the manner in which heat flows through a material body is determined by the three-dimensional heat equation. The distribution of heat, or temperature variation, over time in a given region is defined by the heat equation. Figure 1.4 deliberate some applications of the heat equation.

(a) **(b)** **(c)**

Figure 1.4(a-c). Application of heat equation in daily life

In above Figure 1.4(a), the heat is transferred from one end of the spoon to the other end when it is put into hot coffee or boiled water. In Figure 1.4(b) an iron rod is placed on fire and heat is transferred to the hand of the person holding the rod from another end. In Figure 1.4(c) there is cattle in which water is boiling fire is set under it which provides radiation when water becomes hot it conducted the heat towards its upper opening and then vapors evaporate in the air which is the convection of heat transfer. Along with that FC has another application in the biological field. A dynamic system can be used to approximate the hereditary characteristics and efficacy (efficiency, usefulness) of memory, which are crucial components of many biological functions, by utilizing fractional-order derivatives.

1.2 Preliminaries

We will look over the basic definition of fractional derivative, which we used in our thesis, in this section. Furthermore, we have to address the finite difference scheme that was used for the result.

1.2.1 Definition of Caputo Fractional Derivatives

Michele Caputo [4] presented the Caputo fractional derivative in 1967 for calculating fractional derivatives. Unlike the fractional derivative of Riemann-Liouville, the fractional order beginning conditions do not need to be defined for solving differential equations using Caputo's definition. Here is an illustration of Caputo's definition, where once more $n = \lceil \alpha \rceil$

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^n(\tau)}{(t-\tau)^{\alpha+1-n}} \mathrm{d}\tau,$$

$$f(x,0) = f_0(x), \qquad for\ x \in \Omega$$

$$f(x,t) = 0, \qquad for\ (x,t) \in \partial\Omega \times J$$

1.2.2 Finite Difference scheme

The finite difference scheme used in this thesis is the Crank-Nicolson scheme. The Crank–Nicolson approach, a finite difference approach, can be employed to mathematically solve the heat equation and associated partial differential equations using numerical analysis. This is a second-order temporal approach. It is implicit in time, provides numerical stability, and may be written as an implicit Runge–Kutta algorithm. John Crank and Phyllis Nicolson developed this method in the middle of the 20th century[5].

One can show the unconditional stability of the Crank–Nicolson method for diffusion equations, as well as many other physical equations. Still, if the ratio of time step Δt is small enough, the approximate solutions may include (decaying) spurious oscillations. According to Von Neumann stability analysis, Δt times the thermal diffusivity to the square of space step, or Δx^2, is considerable, usually greater than 1/2.

Assume, for instance, that the partial differential equation in one dimension is

$$\frac{\partial u}{\partial t} = F\left(u, x, t, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}\right).$$

Considering $u(p\Delta x, q\Delta t) = u_{p,q}$ and $F_{p,q} = F$ evaluated for p, q and $u_{p,q}$. The Crank-Nicolson methods for the parabolic equation are the result of combining the forward Euler method at q with the backward Euler method at q+1. It should be noted, though, that the technique is not just the average of those two approaches because the backward Euler equation implicitly depends on the solution:

$$\frac{u_{p,q+1} - u_{p,q}}{\Delta t} = F_{p,q}\left(u, x, t, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}\right). \qquad \text{(Forward Euler)}$$

$$\frac{u_{p,q+1} - u_{p,q}}{\Delta t} = F_{p,q+1}\left(u, x, t, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}\right). \qquad \text{(Backward Euler)}$$

$$\frac{u_{p,q+1} - u_{p,q}}{\Delta t} = \frac{1}{2}\left[F_{p,q+1}\left(u, x, t, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}\right) + F_{p,q}\left(u, x, t, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}\right)\right]. \qquad \text{(Crank–Nicolson)}$$

Keep in mind that this is an implicit method that requires solving an algebraic system of equations to determine the "next" value of u in time. If the partial differential equation is nonlinear, linearization is achievable, however moving in time will require solving a set of nonlinear algebraic equations. This is because discretization will likewise be nonlinear if it is nonlinear. Many algebraic problems, particularly those involving linear diffusion, can be effectively addressed using the tridiagonal matrix algorithm. This method provides a quick and direct solution, in contrast to the typical method for a complete matrix, where N is the size of the matrix.

1.3 Problem Statement

Diffusion problems are the first numerical solutions of partial differential equations that we explore. These are problems with the movement or transfer of particles (molecules, ions, etc.) from areas of higher concentration to areas of lower concentration, to put it literally. In this proposal, we will deliberate the subsequent problem as,

$${}_{0}^{c}D_{t}^{\alpha}\, u(x,t) = \Delta u(x,t) + f\big(x, t, u(x,t)\big), 0 < \alpha \le 1, t > 0, a < x < b, \qquad (1.3)$$

along with the following initial and boundary conditions

$$u(x,0) = g(x), a \leq x \leq b,$$

and
$$u(x,t) = \emptyset(x,t), x \in \partial\Omega \times [a,b].$$

In the above, Ω, Δ and ${}_{0}^{C}D_t^{\alpha}$ are given as,

$$\Omega = \{x | a \leq x \leq b\}, \Delta = \frac{\partial^2}{\partial x^2},$$

and

$${}_{0}^{C}D_t^{\alpha}u(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{1}{(t-x)^{\alpha-n+1}} u^{(n)}(x) \mathrm{d}x, n-1 < \alpha \leq n, n \in \mathbb{N},$$

Correspondingly. A numerical method will be presented to examine the impact of fractional and mesh parameters on the solutions, as well as the method's stability, convergence, and error bounds.

1.4 Research Questions

This research thesis will address the following questions:

1. As already mentioned, there are numerous applications for the diffusion of the fractional order equation in science and engineering. Therefore, its precise numerical solution is very important. In this context, the first question is how we can construct an appropriate mathematical formulation for the numerical investigation of parabolic problems.
2. Next, what will be the physical behavior of the proposed fractional-order model under different circumstances, especially against the mesh parameters M, N and fractional parameter α?
3. Developing or modifying a method is not a big deal in the research community but theoretical study of the developed or modified method is significant. Therefore, another aim is to investigate the will be error bound, of the proposed discretized scheme theoretically and numerically.
4. Investigate the proposed discretized scheme's theoretical and numerical convergence and stability.

1.5 Aim of Research

This research thesis will play a significant role in applied and computational mathematics because:

1. This research is important in the modeling and simulation of parabolic equations that arise in heat transfer phenomena.

2. This research will also report the heat transfer behavior as varying α.
3. This research will significantly improve computational mathematics by providing a comprehensive numerical technique including error bound, convergence, and stability analysis.
4. The behavior of a more real fractional order model is investigated as well by the numerical scheme.

1.6 Research Objectives

Considering the research questions and research gaps mentioned in the above section, the main objectives of the thesis are:

1. Appropriate mathematical modeling of parabolic problems using the fractional calculus in the space.
2. To examine the behavior of parabolic problems besides the mesh parameters and fractional parameters.
3. A numerical analysis of how boundary conditions influence the heat problem.
4. A comprehensive examination of the proposed numerical technique's error bound.
5. Investigation of convergence and stability of the numerical method proposed.

1.7 Thesis Organization

In this thesis, we applied the finite difference technique to a variety of problems. The proposed algorithm is very efficient and understandable. Computational effort and subsequent numerical data fully support the accuracy of the proposed strategy. The following is the structure of the assignment:

There are four sections in the first chapter. Basic definitions of TFDEs are given in the first section. Various preliminary concepts of fractional calculus are addressed in the second section. The third section covers the definition of Caputo's fractional derivatives. The fourth section addresses about the time-fractional diffusion equation.

A brief review of the literature on fractional parabolic differential equations is included in the second chapter. The diffusion equation for time fractional order is the primary focus of the study. As a result, the fractional order diffusion equation in one and two dimensions has been studied analytically. An equation evaluation of the Caputo fractional-order derivations is also included in this paper. The differential Crank-Nicolson, implicit, and explicit finite difference methods have all been used to analyze the parabolic equation. This thesis also describes the potential advantages of using fractional order diffusion in situations from the real world.

Chapter 3 addresses the importance of one-dimensional parabolic problems and their analytical solution. This chapter covers mathematical modeling, stability analysis, and convergence analysis. Numerical solutions for the fractional-order diffusion problems are produced using the Tec plot software.

Two-dimensional parabolic problems and their analytical analysis are addressed in Chapter 4. We go over the mathematical modeling, stability analysis, and convergence analysis in this chapter.

Chapter 5 includes an analysis of the performance of the numerical results. In Chapters 3 and 4, we proposed a time-fractional parabolic partial differential equation, which this chapter will examine and demonstrate using simulation Maple software. There are two sections to this chapter. Results analysis is included, which measures the performance of TFDEs according to certain parameters and compares it with the exact solution.

Chapter 6 includes the conclusion of this thesis, which will provide both the plans and an overview of the contributions.

CHAPTER 2

LITERATURE REVIEW

2.1 Overview

Over the previous few decades, in comparison to the application of classical calculus, the interests in fractional calculus have increased significantly from researchers due to its various advantages and the ability to reveal considerable information about complex systems. Because fractional equations must be used to model a wide variety of real-world problems, the significance of modelling using fractional differential equations has recently come to light. A close connection exists between fractional calculus and the dynamics of challenging problems in everyday life. Because they are non-local, fractional operators provide a more thorough and organized description of several natural phenomena. Analysing fractions is important since it allows one to observe variations in results between two integers. The non-local property and memory impact of the fractional operators are the main advantages of PDE in mathematical modelling. Several mathematical models are accurately defined by differential equations of fractional order. Modelling with fractional equations has diverse applications, including in the areas of signal processing and image theory, as well as in the disciplines of economics, biology, mechanics, geology, heat transport, chemistry, biology, and physics. Many researchers have focused on developing different numerical methods to find an approximation of a solution to fractional models because they frequently lack an analytical solution. The fractional-order differential equation is a hot and challenging topic among researchers in this field.

Liu *et al.* [6] considered the space-time fractional diffusion equation's Cauchy problem, which may be derived from the conventional diffusion equation by substituting a Caputo or Riemann-Liouville derivative for the second-order space derivative starting with its Fourier-Laplace representation, the fundamental solution to the Cauchy problem is examined in terms of its scaling and similarity features. The Green function's explicit expression is derived by the authors. They go on to describe the similarity property by talking about the space-time fractional diffusion equation's scale-invariance. Zhuang *et al.* [7] have described approximation via implicit finite differences for the two-dimensional time Fractional diffusion problem within a bounded domain. The implicit difference approximation's unconditional stability and convergence have been demonstrated by the authors. Fractional differential equation solutions can be accomplished using this approach. At last, a numerical example is included by the authors. Their theoretical analysis and the numerical result correspond extremely well. Chen *et al.* [8] considered a fractional a partial differential equation that explains sub diffusion. The authors presented a method for approximating implicit differences in the solution of a fractional partial differential equation is described. They used Fourier series method to prove the stability and convergence of proposed method. When the partial time fractional derivative has been defined in the sense of Caputo, the linear time-fractional diffusion problem on a finite slab can be numerically solved using a fully implicit unconditionally stable finite difference approach by Murio [9]. He described methods for resolving the linear algebraic equations in the triangle system merely once for each time step in order to approach the first Volterra integral formulation of the derivative of Caputo. He also demonstrates how the first-order approach of time makes it easier to model sub-diffusion processes, especially for modest orders of fractional differentiation. Francesco *et al.* [10] studied the generalized Gaussian diffusion partial differential equation through both the Caputo (C) and Riemann-Liouville senses of the distributed order's time-fractional derivative between 0 and 1. They give the basic solution expressed in terms of an integral of Laplace type for a general distribution of time orders, which is still a probability density. Li *et al.*[11] demonstrated an invaluable numerical algorithm is the generalized technique of Adams, Basforth, and Moulton, sometimes known as " the fractional Adams approach", for solving a fractional ordinary differential equation. The primary focus of this research is to examine the error assessments of the fractional Adams technique for solving fractional differential equations in various scenarios. In accordance with the theoretical study, numerical simulations are also given. Chuanju Xu *et al.* [12] investigated the initial boundary value problems of the space-time fractional diffusion equation and their numerical solution. The definitions of the Caputo and Riemann liouville fractional derivative are investigated simultaneously. Furthermore, the authors develop an effective spectral approach for numerical approximations based on the suggested weak formulation of the weak solution. Ultimately, the weak formulation allows us to build a numerically effective space-time fractional heat problem's solution. Luchko [13] discuss and formulate maximal principle for the distributed-order, multi-term diffusion equations, in addition to other generalized time-fractional diffusion equations. Luchko [14] considered initial-boundary-value problems that are bounded open in n-dimensional domains for the equation of time-fractional diffusion. Particularly, the equation of time-fractional diffusion benefits from an extension of the maximum principle, which is widely recognized for PDEs of the parabolic and elliptic kinds. The maximal concept is then applied to demonstrate the uniqueness of the solution of initial boundary value problems for time-fractional diffusion equations. Finally, the smoothness and asymptotic of this solution are examined for some particular usage of a source function by the author. Ashralyev *et al*. [15] proposed Stable difference methods with first and second-order accuracy for the Neumann boundary conditions on the fractional parabolic equation. Regarding the solution of several distinct schemes, the authors of this research derive estimations of almost coercive stability. The method is demonstrated through numerical examples. Priya *et al*. [16] proposed a numerical method with higher order considering the Neumann and Dirichlet boundary conditions for the fractional heat equation. For the spatial variable, they propose a fourth-order compact finite difference technique, which converts the fractional heat equation into a collection of regular fractional differential equations that can be expressed integrally. Additionally, a modified trapezoidal rule is applied to create a difference equation by converting the integral equation. In order to confirm the precision and effectiveness of the suggested strategy, numerical results are provided. Wang *et al*. [17] studied Generalized compact finite difference techniques for fractional diffusion-wave equation and anomalous fractional sub-diffusion equation. The order of fractional derivatives in the equations determines the maximum temporal precision that previously presented schemes can attain, which is typically less than two. Abdullah *et al*. [18] proposed an approach for the Solution of time-fractional sub diffusion problems numerically that combines the continuous conforming space-based finite element approach with a time-stepping discontinuous Petrov-Galerkin method. They also proved the stability and existence of approximations, as well as the extraction of error estimates. Rebelo *et al*. [19] presented an innovative numerical strategy for addressing the fractional sub-diffusion problem. This novel approach combines the method of lines with a non-polynomial collocation method for fractional ordinary differential equations has been proposed lately. The authors finally conclude that Similar to the case of fractional differential equations, the method's convergence order remains constant regardless of the time derivative's order and does not diminish while addressing specific non smooth solutions. Zecova *et al*. [20] developed a heat conduction model based on fractional derivatives. In this study, two library functions that have been built in MATLAB are used: First, there is the Fractional Heat Conduction Toolbox (FHCT) and the Heat Conduction Toolbox (HCT). It also explains how fractional and integer-order derivatives are used in a one-dimensional heat conduction mathematical model. The paper explains the numerical and analytical methods for resolving heat conduction models. Numerical approaches are related to the finite difference approach through the applicability of the fractional time derivative to the Grünwald-Letnikov concept. The article's conclusion includes a collection of simulation examples that make use of toolboxes. The examples show the various applications for these libraries, which permit the modelling of diffusion processes in addition to heat conduction processes. Alikhanov [21] constructed the Caputo fractional derivative's new difference analogue, defined as the L2-1σ formula. He studies this difference operator's basic properties and apply it to create a number of difference schemes that roughly represent the second and fourth orders in space and the second order in time for the time Equation of fractional diffusion with variable coefficients. The author provide evidence that the suggested schemes are stable and that they converge in the grid L2-norm at a rate proportional to the order of the approximation error. The numerical computations performed for a few test problems validate the given results. Kurt *et al.* [22] discuss the fractional diffusion equations' time- and space-dependent solution. In this paper, the time-fractional heat equation is solved using a conformable fractional derivative. The conformable Fourier transform, which is similar to the conformable Laplace transform in fractional calculus, is used for solving the fractional heat equation in space. This article can help researchers realize that, in the appropriate situations, specific definitions can be quite helpful. Stokes *et al.* [23] expressed the resolution of an order $0 < \alpha < 1$ Caputo time fractional diffusion equation in terms of an integer-order related diffusion equation's solution. The authors explain how there is a temporal map that is linear between these answers, which speeds up the computation of the fractional order problem's solution. The authors' effective approach to the least squares fitting of a fractional advection-diffusion model for the current in a time-of-flight experiment resulted in a computational performance boost of one to three orders of magnitude for practical issue sizes. Sweilam *et al.* [24] recommended utilizing numerical methods to solve the diffusion problem of fractional order. Third-order shifted Chebyshev polynomials were employed by the authors. The Caputo sense is used to express fractional derivatives. The authors also condensed these differential equations to an algebraic series of problems accessible to numerical resolution using the finite difference approach and the Chebyshev collocation technique. The proposed method's results were also contrasted with those of other numerical approaches by the authors. They demonstrate from this comparison that the suggested method is a straightforward and efficient numerical method. Singh *et al.* [25] construct a higher spatial-temporal dimension generalization fractional solution. The findings of this article examined the implications in various limiting cases after obtaining fractional solutions to the heat equations. Additionally, they look forward to potential uses of physical system solutions for fractional heat flow. Yanmin Zhao *et al.* [26] developed two fully-discrete approximation approaches with unconditional stability for the problem of time-fractional diffusion with Caputo derivative of order $0 < \alpha < 1$, combining established mixed finite element methods with the classical L1 time stepping method on the basis of spatial nonconforming and conforming. Ghazanfari *et al.* [27] solved the stochastic generalized fractional diffusion issue using the finite difference method. The writers of this paper have examined the requirements needed for the numerical solution's mean square convergence. They also offered numerical examples to show the effectiveness and application of the approach. Makhtomi [28] successfully developed the Homotopy perturbation method (HPM) to solve the one-dimensional thin rod heat diffusion problem between specified initial and boundary conditions. A comprehensive comparative analysis of the rod has been conducted using the PDE system's the finite difference technique and the Homotopy perturbation method. He came to the conclusion that, when it comes to solving partial differential equations, the Homotopy perturbation method (HPM) is a more effective numerical technique than the finite difference approach. Furthermore, as solutions come from HPM problems it is convenient to represent them using various functions, leading to the conclusion that a strong numerical method for solving partial differential equations is Homotopy perturbation. Fawang *et al.* [29] proposed that the time fractional order diffusion equation is the counterpart of the classical diffusion equation. to the spectral signal's peak-preserving smoothing. There were two implementation strategies offered: both the implicit and explicit difference schemes. Designing the diffusion function by using the under-processing spectrum as a reference signal, hence because the peak's diffusion is weaker, peak-preserving smoothing can be accomplished. The authors conclude that the results show that time fractional order diffusion filtering is superior to classical diffusion filtering and that it smoothest more effectively than classical smoothing methods. Wen Chen *et al.* [30] presented the real world applications of fractional calculus. The main objective of this review article is to provide some brief summaries written by renowned fractional calculus researchers. Attaullah *et al.* [31] developed the solution to a two-dimensional heat equation with a fractional order external source term. In this article, the authors provide some examples to highlight the results. They also demonstrate the convergence of the derived series form solution to the exact solution. Anley *et al.* [32] used a fractional Crank-Nicolson scheme to solve the limit approach using the right-shifted Grunwald-Letnikov approximation, one elaborates the Fractional convection-diffusion equation with coefficients in space variables and the one-dimensional space fractional diffusion problem. The authors of this paper give a few numerical examples that show how the Crank-Nicolson approach can be employed to resolve the problem of spatial fractional diffusion and convection even over a wide variety of time scales. For the purpose of solving the 1D fractional diffusion equations with periodic boundary conditions Ali Demir *et al.* [33] established the analytic solution by implementing well-known separation of variables method. The authors Investigated every possibility provides the Sturm-Liouville problem's obtained eigenvalues. In the second stage, the matching Eigen functions are found. The solution's Fourier series is formed with reference to the Mittag-Leffler function using eigenvalues and Eigen functions, and the coefficients are calculated by weighing the beginning condition and L^2 inner product in the last stage. To resolve the fractional parabolic partial differential equation Sunarto *et al.* [34] find the approximation half sweep and preconditioned relaxation solutions. The authors of this paper demonstrate that, in comparison to other methods, the SOR with half-sweep preconditioning (HSPSOR) iterative method requires the least amount of computation time and iterations. Additionally, they discovered that the SOR with half-sweep preconditioning (HSPSOR) caused somewhat larger overall maximum errors when employed instead of the time-fractional diffusion equation when resolving the space fractional diffusion problem. A thorough analysis of this method error can be done in the future. This article demonstrates how half-sweep preconditioned SOR (HSPSOR) can simplify computations when solving heat equations with finite space and time dimensions. Zhang *et al.* [35] developed an efficient finite difference approach to solve fractional reaction-diffusion problems in two dimensions. The authors of this study created a rigid nonlinear ordinary differential equation system by discretizing space using the weighted shifting Grunwald difference. The ordinary differential equations in this system is resolved via the efficient compact implicit integration factor (CIIF) technique. The authors demonstrated the second-order convergent behaviour and discrete L_2 −norm stability of the CIIF system. Furthermore, they included numerical examples to demonstrate the method's efficiency, accuracy, and stability. Lenzi *et al.* [36] have suggested a finite difference-based convergent numerical approach for solving a series of differential equations with fractional order with nonlinear advection-diffusion, which are used to describe phenomena like anomalous diffusion as well as porous media. About the anomalous diffusion, it is hoped that the results presented in this article will help solve fractional differential equations involving nonlinear advection-diffusion. Hosseini *et al.*[37] have investigated a new technique for solving time fractional diffusion equation numerically. The authors represented fractional derivative term in Lagrange operational sense and use Legendre orthogonal polynomials for temporal direction of considered model. The authors applied nonlocal peri-dynamic differential operator on the considered model and finally authors find stability and convergence numerically. Eftekhari *et al.*[38] discussed Jacobi wavelets for distributed order linear and nonlinear time-fractional diffusion equations. For the Riemann–Liouville fractional integral operator, the authors derive the Jacobi wavelet operational vector. A theoretical investigation by the authors is conducted into the specified method's convergence and certain error bounds. The suggested method's effectiveness, applicability, and dependability are demonstrated by numerical examples. Jafari *et al.*[39] studied time-fractional diffusion equations, such as the Caputo sense's time-fractional advection–diffusion equations (TF–ADEs) and time-fractional Kolmogorov equations (TF–KEs). In order to solve the TF–KEs and the TF–ADEs, the authors developed operational matrices (OMs) utilising the Hosoya polynomial (HP) as the basis function. Nadeem *et al.*[40] have outlined the Elzaki Homotopy perturbation transform method (EHPTS) for analysing the multi-dimensional fractional diffusion equation's approximate solution. The authors come to the conclusion that HPS generates iterations in the form of convergence series that get closer to the exact answer.

2.2 Research Gap and Conceptual Framework

The following research gap was identified following the above-mentioned inclusive literature survey, after the review of the literature, we observed that there is a lack of research on the parabolic equation of fractional order. In this framework, the current study will address the comprehensive study regarding the time-fractional diffusion model's numerical solution via the Crank-Nicholson method. This thesis also emphasizes the theoretical study of the time-fractional diffusion model.

Firstly, I would construct an appropriate mathematical formulation for the numerical investigation of parabolic problems. Then, I would investigate the theoretical and numerical convergence and stability of the proposed discretized system. This research is important in the modeling and simulation of parabolic equations that arise in heat transfer phenomena. This research will also report the heat transfer behavior as varying α. This study will significantly improve computational mathematics by providing an extensive numerical scheme including error bound, convergence, and stability analysis. A more physical fractional order model's behavior is also investigated via the numerical scheme.

CHAPTER 3

MODIFICATION OF ONE-DIMENSIONAL PARABOLIC PROBLEM

3.1 Introduction

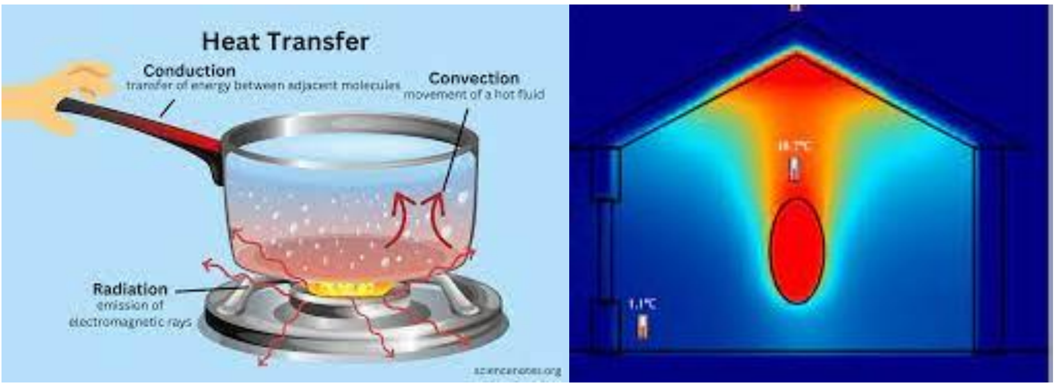
A specific type of partial differential equation that arises in science and mathematics is the heat equation. Another name for the heat equation solutions is calorie functions. In 1822, Joseph Fourier proposed the heat equation theory to model the diffusion of a quantity—heat, for example—through a certain space.

In pure mathematics, one of the most discussed topics is the heat equation, the traditional parabolic partial differential equation, as understanding it is seen to be essential for understanding partial differential equations in general. Many geometric applications result from the consideration of the heat equation on Riemannian manifolds as well. The work of Ike Pleijel and Subbaramiah Minakshisundaram has resulted in a close link between spectral geometry and the heat equation. In 1964, James Eells and Joseph Sampson used differential geometry to develop a groundbreaking nonlinear version of the heat equation. This version was used as the foundation for Grigori Perelman's 2003 proof of the Poincare conjecture and Richard Hamilton's 1982 development of the Ricci flow [41].

Furthermore, the heat equation and its modifications are essential to many fields of applied mathematics and research. The Fokker-Planck equation in probability theory links the study of random walks and Brownian motion to the heat equation. The Schrödinger equation in quantum mechanics can be viewed as a heat equation in imaginary time, while the Black-Scholes equation in financial mathematics is a small variation of the heat equation. The heat equation can occasionally be used in image analysis to pinpoint edges and rectify pixelation. Since the advent of "artificial viscosity" techniques by Richtmyer Robert and John von Neumann, respectively, heat equation solutions have proven helpful in the mathematical characterization of hydrodynamic shocks.

$$\frac{\partial u}{\partial t} = k\Delta u(x,t) + f(x,t,u). \tag{3.1}$$

Heat equation (3.1) is an important PDE that describes the heat transfer in a certain area over time. The distribution of heat, or temperature variation, over time in a given region is described by the heat equation. Figure 3.1 deliberate some applications of the heat equation.



(a) **(b)**

Figure 3.1(a,b). Heat problem arises in different physical phenomena

The above Figure 3.1 (a) illustrates the heat transfer of boiling water in a pan. In Figure 3.1 (b) a room heater is placed in the center of the room and the heat is transferred from one point to another. Along with that FC has another application in Biological Field. A dynamic system can be used to approximate the hereditary characteristics and efficacy (efficiency, usefulness) of memory, which are crucial components of many biological functions, by utilizing fractional-order derivatives.

There are many methods to solve the 1D heat equation. Fourier series can be used to solve the heat equation. Joseph Fourier first presented this technique in his 1822 dissertation entitled as “Theorie analytique de la chaleur”. Heat conduction in a rod could be modeled using this technique. Generalizing the solution technique is another approach to solving the heat equation. Many more kinds of equations can be solved with a great deal of flexibility using the method discussed above. The objective is to express the operator u_{xx} in terms of its eigen functions when it has zero boundary conditions. Eventually, this leads to a fundamental concept in the theory of self-adjoint operators that are linear. The solution to the heat equation that depicts the initial state of an initial point source of heat at a specific location is known as a basic solution, also often called a heat kernel. Over certain domains, they can be utilized to construct a general solution for the heat equation. Numerous basic one-dimensional solutions for Green’s function are presented for the 1D heat equation. The spatial domain of a few of them is $(-\infty,\infty)$. In others, both Dirichlet and Neumann boundary conditions are feasible, and the interval is semi-infinite $(0, \infty)$. Another difference is that some of these are capable of solving the in homogeneous equation, $u_t = ku_{xx} + f$ where f is some given function of x and t.

3.2 Mathematical Methodology

We consider the following 1D-TFDE in this section, of the form

$$ {}_0^C D_t^\alpha u(x,t) = \frac{\partial^2 u}{\partial x^2} + \phi(x,t), a < x < b\ , t > 0, \tag{3.2}$$

with the initial and boundary conditions given as

$$\begin{gathered} u(a,t) = f_1(t), u(b,t) = f_2(t), t > 0, \\ u(x,0) = g(x), a \le x \le b. \end{gathered} \tag{3.3}$$

We initially discuss the Second-order space derivative discretization before introducing the Crank Nicolson scheme, which is illustrated below.

$$\frac{\partial^2 u\left(x_p, t_{\frac{q+1}{2}}\right)}{\partial x^2} \cong \frac{1}{2}\left(\frac{u_{p+1,q+1} - 2u_{p,q+1} + u_{p-1,q+1}}{h^2} + \frac{u_{p+1,q} - 2u_{p,q} - u_{p-1,q}}{h^2}\right) + \mathrm{O}(h^2).$$

Also, the time-fractional derivative discretized form is

$$ {}_0^C D_t^\alpha u\left(x_p, t_{\frac{q+1}{2}}\right) \cong \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[u_{p,q+1} - u_{p,q} + \sum_{j=1}^{q}\left(u_{p,q+1-j} - u_{p,q-j}\right)b_j^\alpha\right] + \mathrm{O}(\tau^{2-\alpha}),$$

where $b_j^\alpha = (j+1)^{1-\alpha} - j^{1-\alpha}$. Substituting the discretized form mentioned above for the temporal and spatial derivatives, the comprehensive Crank-Nicolson scheme can be expressed by employing the time-fractional diffusion model. as follows:

$$\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[u_{p,q+1} - u_{p,q} + \sum_{j=1}^{q}\left(u_{p,q+1-j} - u_{p,q-j}\right)b_j^\alpha\right] + \mathrm{O}(\tau^{2-\alpha}) =$$

$$\frac{1}{2h^2}\left(u_{p+1,q+1} - 2u_{p,q+1} + u_{p-1,q+1} + u_{p+1,q} - 2u_{p,q} - u_{p-1,q}\right) + \mathrm{O}(h^2)$$

$$+\frac{1}{2}\phi\left(x_p, t_q\right) + \frac{1}{2}\phi\left(x_p, t_{q+1}\right).$$

After simplification, we obtained the following form

$$\begin{gathered} -\frac{r}{2}u_{p-1,q+1} + (1+r)u_{p,q+1} - \frac{r}{2}u_{p+1,q+1} = \frac{r}{2}u_{p-1,q} + (1-r)u_{p,q} + \frac{r}{2}u_{p+1,q} \\ -\sum_{j=1}^{q}\left(u_{p,q+1-j} - u_{p,q-j}\right)b_j^\alpha + \frac{\Gamma(2-\alpha)}{2}\Delta t^\alpha \phi_{p,q+1} + \frac{\Gamma(2-\alpha)}{2}\Delta t^\alpha \phi_{p,q}, \end{gathered} \tag{3.4}$$

for $1 \le p \le M-1, 0 \le q \le N-1$. The boundary condition reduces to as:

$$u_{p,0} = f_p, 0 \le p \le M,$$

$$u_{0,q} = (f_1)_q, u_{M,q} = (f_2)_q, q > 0,$$

where

$$\Delta x=\frac{b-a}{M},\Delta t=\frac{T}{N},n=,x_p=a+p\Delta x,t_q=q\Delta t,r=\frac{\Gamma(2-\alpha)\Delta t^{\alpha}}{\Delta x^2},$$

$$u_{p,q}=u(x_p,t_q),\phi_{p,q}=\phi(x_p,t_q),f_p=f(x_p),(f_1)_q=f_1(t_q),(f_2)_q=f_2(t_q).$$

It can be rewritten as in the matrix form:

for $q=0$

$$\mathbb{A}\vec{\mathbf{U}}^1=\mathbb{B}\vec{\mathbf{U}}^0+\frac{1}{2}\left(\vec{\mathbf{f}}^0+\vec{\mathbf{f}}^1\right)+\vec{\mathbf{b}}^p,$$

for $q=1$

$$\mathbb{A}\vec{\mathbf{U}}^2=\mathbb{B}\vec{\mathbf{U}}^1-b_1\vec{\mathbf{U}}^1+b_1\vec{\mathbf{U}}^0+\frac{1}{2}\left(\vec{\mathbf{f}}^1+\vec{\mathbf{f}}^2\right)+\vec{\mathbf{b}}^p,$$

for $q\geq 2$

$$\mathbb{A}\vec{\mathbf{U}}^{q+1}=\mathbb{B}\vec{\mathbf{U}}^q-b_1\vec{\mathbf{U}}^q+\sum_{j=2}^{q}\left(b_{j-1}-b_j\right)\vec{\mathbf{U}}^{q-j+1}+b_q\vec{\mathbf{U}}^0+\frac{1}{2}\left(\vec{\mathbf{f}}^q+\vec{\mathbf{f}}^{q+1}\right)+\vec{\mathbf{b}}^q,$$

where

$$\mathbb{A}=\begin{bmatrix}1+r & -\frac{r}{2} & 0 & \cdots & 0 & 0\\ -\frac{r}{2} & 1+r & -\frac{r}{2} & \cdots & 0 & 0\\ 0 & -\frac{r}{2} & 1+r & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 1+r & -\frac{r}{2}\\ 0 & 0 & 0 & \cdots & -\frac{r}{2} & 1+r\end{bmatrix},\vec{\mathbf{U}}^q=\begin{bmatrix}u_{1,q}\\ u_{2,q}\\ u_{3,q}\\ u_{4,q}\\ \vdots\\ u_{M-2,q}\\ u_{M-1,q}\end{bmatrix},\vec{\mathbf{b}}^q=\frac{r}{2}\begin{bmatrix}u_{0,q}+u_{0,q+1}\\ 0\\ 0\\ \vdots\\ 0\\ u_{M,q}+u_{M,q+1}\end{bmatrix},$$

$$\vec{\mathbf{f}}^p=\Gamma(2-\alpha)\Delta t^{\alpha}\begin{bmatrix}\phi_{1,q}\\ \phi_{2,q}\\ \phi_{3,q}\\ \vdots\\ \phi_{M-2,q}\\ \phi_{M-1,q}\end{bmatrix},\mathbb{B}=\begin{bmatrix}2-r & \frac{r}{2} & 0 & \cdots & 0 & 0\\ \frac{r}{2} & 2-r & \frac{r}{2} & \cdots & 0 & 0\\ 0 & \frac{r}{2} & 2-r & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 2-r & \frac{r}{2}\\ 0 & 0 & 0 & \cdots & \frac{r}{2} & 2-r\end{bmatrix},\vec{\mathbf{U}}^0=\begin{bmatrix}f_1\\ f_2\\ f_3\\ \vdots\\ f_{M-2}\\ f_{M-1}\end{bmatrix}.$$

Algorithm:

$a,b,T,M,N\in N,\ \alpha,\phi$, ICs and BCs	(Input data)
for p **from** 0 **by** 1 **while** $p\leq M$ **do**;	
$x_p=a+\Delta xp$, and *ICs*	(Discretization of spatial domain and ICs)
end do:	
for q **from** 0 **by** 1 **while** $q\leq N$ **do**;	
$t_q=q\Delta t$, and *BCs*	(Discretization of time domain and BCs)

end do:
for p **from** 0 **by** 1 **while** $p \leq M$ **do**;
for q **from** 0 **by** 1 **while** $q \leq N$ **do**;
$\phi_{p,q} = \phi(x_p, t_q)$, (Discretization source term)
end do:
end do:
$\mathbb{A}, \mathbb{B}, \vec{\boldsymbol{f}}^q, \vec{\boldsymbol{U}}^{-1}, \vec{\boldsymbol{U}}^0, \vec{\boldsymbol{b}}^q$ (Evaluation of required matrices and vectors)
for q **from** 0 **by** 1 **while** $q \leq N-1$ **do**;
if $q = 0$ **then**
$\mathbb{A}\vec{\mathbf{U}}^1 = \mathbb{B}\vec{\mathbf{U}}^0 + \frac{1}{2}(\vec{\mathbf{f}}^0 + \vec{\mathbf{f}}^1) + \vec{\mathbf{b}}^p$
else if $q = 1$ **then**,
$\mathbb{A}\vec{\mathbf{U}}^2 = \mathbb{B}\vec{\mathbf{U}}^1 - b_1\vec{\mathbf{U}}^1 + b_1\vec{\mathbf{U}}^0 + \frac{1}{2}(\vec{\mathbf{f}}^1 + \vec{\mathbf{f}}^2) + \vec{\mathbf{b}}^p$
else,
$\mathbb{A}\vec{\mathbf{U}}^{q+1} = \mathbb{B}\vec{\mathbf{U}}^q - b_1\vec{\mathbf{U}}^q + \sum_{j=2}^{q}(b_{j-1} - b_j)\vec{\mathbf{U}}^{q-j+1} + b_q\vec{\mathbf{U}}^0 + \frac{1}{2}(\vec{\mathbf{f}}^q + \vec{\mathbf{f}}^{q+1}) + \vec{\mathbf{b}}^q$
end if:
end do:

3.3 Stability Analysis

This section is dedicated to analyzing the stability analysis of the proposed method using the Crank-Nicolson logic. Let $L_{p,q} = u_{p,q} - U_{p,q}$ where $U_{p,q}$ represents the approximate solution at mesh point (x_p, t_q) ,$(p = 1,2, \dots \dots \dots, M-1; q = 1,2 \dots \dots \dots, N)$ and

$$L^q = [L_{1,q}, L_{2,q}, \dots \dots \dots, L_{M-1,q}]$$

The expression of $L^q (q = 1,2, \dots \dots, N)$ is defined as follow:

$$L^q(x) = \begin{cases} L_{p,q}, & \text{if } x_p - \frac{h}{2} < x \leq x_p + \frac{h}{2}, \\ 0, & \text{if } 0 \leq x \leq \frac{h}{2} \text{ or } p - \frac{h}{2} < x \leq p, \end{cases}$$

Then, $L^q(x)$ could be expanded in a Fourier series as given below:

$$L^q(x) = \sum_{m=-\infty}^{\infty} \delta_k(m) \mathrm{e}^{i2\pi mx/p} \quad , (q = 1,2, \dots \dots \dots, N),$$

where

$$\delta_k(m) = \frac{1}{p}\int_0^p L^q(x)\, e^{-2\pi mx/p} dx.$$

It has already been proved by Chen *et al.* [8]

$$\|L^q\|_2^2 = \sum_{m=-\infty}^{\infty} |\delta_k(m)|^2.$$

Proposition 3.1. The coefficients $r_{p,q,}$ and $d_j^{p,q}$ have the following properties

1. $r_{p,q} > 0 \quad, 0 < b_j^{p,q} < b_{j-1}^{p,q} \leq 1 \ \forall\, p = 1,2, \dots \dots \dots, M \ \forall \quad j,k = 1,2, \dots \dots \dots, N,$
2. $0 < d_j^{p,q} < 1 \ \ \sum_{j=0}^{k-1} d_{j+1}^{p,q+1} = 1 - b_q^{p,q+1}$,

Since $d_1^{p,q+1} = 1 - b_1^{p,q+1}$, the round-off error corresponding to equation (3.4) can be stated as follows

$$\begin{aligned} &\left(1 + 2r_{p,q+1}\right)L_{p,q+1} - r_{p,q+1}L_{p+1,q+1} - r_{p,q+1}L_{p-1,q+1} \\ &= r_{p,q+1}L_{p+1,q} + \left(1 - 2r_{p,q+1} - b_1^{p,q+1}\right)L_{p,q} + r_{p,q+1}L_{p-1,q} \\ &\quad + \sum_{j=1}^{q-1} d_{j+1}^{p,q+1} L_{p,q-j} + b_q^{p,q+1} L_{p,0}\, , q \geq 1. \end{aligned} \tag{3.5}$$

Assuming the following, the homogeneous equation of (3.4) gives the following solution.

$$L_{p,q} = \delta_q e^{i\beta p h}.$$

Substituting this in the above equation gives us

$$\begin{aligned} &\left(1 + 2r_{p,q+1}\right)\delta_{q+1}\mathrm{e}^{i\beta p h} - r_{p,q+1}\delta_{q+1}\mathrm{e}^{i\beta(p+1)h} - r_{p,q+1}\delta_{q+1}\mathrm{e}^{i\beta(p-1)h} \\ &= r_{p,q+1}\delta_q\mathrm{e}^{i\beta p h} + \left(1 - 2r_{p,q+1} - b_1^{p,q+1}\right)\delta_q\mathrm{e}^{i\beta p h} + r_{p,q+1}\mathrm{e}^{i\beta(p-1)h} \\ &\quad + \sum_{j=1}^{q-1} d_{j+1}^{p,q+1}\delta_{q-j}\mathrm{e}^{i\beta p h} + b_q^{p,q+1}\,\delta_0\mathrm{e}^{i\beta p h}\,. \end{aligned} \tag{3.6}$$

Dividing the equation by $\mathrm{e}^{i\beta p h}$ yields the following form:

$$\begin{aligned} &\left(1 + 2r_{p,q+1}\right)\delta_{q+1} - r_{p,q+1}\delta_{q+1}\mathrm{e}^{i\beta h} - r_{p,q+1}\delta_{q+1}\mathrm{e}^{-i\beta h} \\ &= r_{p,q+1}\delta_q\mathrm{e}^{i\beta h} + \left(1 - 2r_{p,q+1} - b_1^{p,q+1}\right)\delta_q + r_{p,q+1}\mathrm{e}^{-i\beta h} + \sum_{j=1}^{q-1} d_{j+1}^{p,q+1}\delta_{q-j} + b_q^{p,q+1}\,\delta_0\,, \end{aligned}$$

or

$$\begin{aligned} &-r_{p,q+1}\delta_{q+1}\left(\mathrm{e}^{i\beta h} + \mathrm{e}^{-i\beta h} - 2\right) + \delta_{q+1} = r_{p,q+1}\delta_q\left(\mathrm{e}^{i\beta h} + \mathrm{e}^{-i\beta h} - 2\right) - b_1^{p,q+1}\delta_q + \delta_q \\ &\quad + \sum_{j=1}^{q-1} d_{j+1}^{p,q+1}\delta_{q-j} + b_q^{p,q+1}\,\delta_0. \end{aligned}$$

The above expression can be rewritten as:

$$\begin{aligned} &-r_{p,q+1}\delta_{q+1}\left(2\frac{\mathrm{e}^{i\beta h} + \mathrm{e}^{-i\beta h}}{2} - 2\right) + \delta_{q+1} = r_{p,q+1}\delta_q\left(2\frac{\mathrm{e}^{i\beta h} + \mathrm{e}^{-i\beta h}}{2} - 2\right) - b_1^{p,q+1}\delta_q \\ &\quad + \delta_q + \sum_{j=1}^{q-1} d_{j+1}^{p,q+1}\delta_{q-j} + b_q^{p,q+1}\,\delta_0. \end{aligned}$$

Using the relation $\left(e^{i\beta h}+e^{-i\beta h}\right)/2=\cos\beta h$ in the above expression which then reduces to as:

$$-2r_{p,q+1}\delta_{q+1}(\cos\beta h-1)+\delta_{q+1}=2r_{p,q+1}\delta_q(\cos\beta h-1)-b_1^{p,q+1}\delta_q+\delta_q$$
$$+\sum_{j=1}^{q-1} d_{j+1}^{p,q+1}\delta_{q-j}+b_q^{p,q+1}\,\delta_0,$$

or

$$-2r_{p,q+1}\delta_{q+1}\left(-2\sin^2\frac{\beta h}{2}\right)+\delta_{q+1}=2r_{p,q+1}\delta_q\left(-2\sin^2\frac{\beta h}{2}\right)-b_1^{p,q+1}\delta_q+\delta_q$$
$$+\sum_{j=1}^{q-1} d_{j+1}^{p,q+1}\delta_{q-j}+b_q^{p,q+1}\,\delta_0.$$

This implies that

$$\begin{aligned}\left(1+4r_{p,q+1}\sin^2\frac{\beta h}{2}\right)\delta_{q+1} &= \left(1-4r_{p,q+1}\sin^2\frac{\beta h}{2}\right)\delta_q-b_1^{p,q+1}\delta_q\\ &+\sum_{j=1}^{q-1} d_{j+1}^{p,q+1}\delta_{q-j}+b_q^{p,q+1}\,\delta_0.\end{aligned}\tag{3.7}$$

For $q=0$ in equation (3.7) we have

$$\left(1+4r_{p,q+1}\sin^2\frac{\beta h}{2}\right)\delta_1=\left(1-4r_{p,q+1}\sin^2\frac{\beta h}{2}\right)\delta_0-b_1^{p,q+1}\delta_0+\sum_{j=1}^{0-1} d_{j+1}^{p,1}\delta_{-j}+b_q^{p,1}\,\delta_0.$$

For $q=1,2,3,\dots,N-1$ in equation (3.7) we have

$$\left(1+4r_{p,q+1}\sin^2\frac{\beta h}{2}\right)\delta_{q+1}=\left(1-b_1^{p,q+1}-4r_{p,q+1}\sin^2\frac{\beta h}{2}\right)\delta_q$$
$$+\sum_{j=1}^{q-1} d_{j+1}^{p,q+1}\delta_{q-j}+b_q^{p,q+1}\,\delta_0.$$

The above equation can be rewritten as

$$\delta_{q+1}=\frac{\left(1-b_1^{p,q+1}-4r_{p,q+1}\sin^2\frac{\beta h}{2}\right)\delta_q+\sum_{j=1}^{q-1} d_{j+1}^{p,q+1}\delta_{q-j}+b_q^{p,q+1}\,\delta_0}{\left(1+4r_{p,q+1}\sin^2\frac{\beta h}{2}\right)}.\tag{3.8}$$

Lemma 3.1. Suppose that δ_q for $q=1,2,\dots,N-1$ is the solution of eqn. (3.8) Then we have

$$\left|\delta_q\right|\le|\delta_0|,q=1,2,\dots,N-1.$$

Proof. Using mathematical induction, we prove that for $p=1,2,\dots,M-1$, when $q=0$, we get

$$\delta_1 = \frac{1 - 4r_{p,q+1}\sin^2\frac{\beta h}{2}}{1 + 4r_{p,q+1}\sin^2\frac{\beta h}{2}}\delta_0.$$

Noticing that $r_{p,1} > 0$, we get

$$|\delta_1| \le |\delta_0|.$$

Next, we let $|\delta_n| \le |\delta_0|$ for $n = 2,3,\dots,q$,

$$\left|\delta_{q+1}\right| = \frac{\left|\left(1 - b_1^{p,q+1} - 4r_{p,q+1}\sin^2\frac{\beta h}{2}\right)\delta_q + \sum_{j=1}^{q-1} d_{j+1}^{p,q+1}\delta_{q-j} + b_q^{p,q+1}\,\delta_0\right|}{1 + 4r_{p,q+1}\sin^2\frac{\beta h}{2}}.$$

or

$$\left|\delta_{q+1}\right| = \frac{\left|-4r_{p,q+1}\sin^2\frac{\beta h}{2}\delta_q + \sum_{j=0}^{q-1} d_{j+1}^{p,q+1}\delta_{q-j} + b_q^{p,q+1}\,\delta_0\right|}{1 + 4r_{p,q+1}\sin^2\frac{\beta h}{2}},$$

since $d_1^{p,q+1} = 1 - b_1^{p,q+1}$, therefore

$$\left|\delta_{q+1}\right| \le \frac{-4r_{p,q+1}\sin^2\frac{\beta h}{2}\left|\delta_q\right| + \sum_{j=0}^{q-1} d_{j+1}^{p,q+1}\left|\delta_{q-j}\right| + b_q^{p,q+1}\,|\delta_0|}{1 + 4r_{p,q+1}\sin^2\frac{\beta h}{2}},$$

$$= \frac{1 + 4r_{p,q+1}\sin^2\frac{\beta h}{2}}{1 + 4r_{p,q+1}\sin^2\frac{\beta h}{2}}|\delta_0| \le |\delta_0|,$$

or

$$\left|\delta_{q+1}\right| \le |\delta_0|.$$

It is proved that the scheme is unconditionally stable for the diffusion problem.

3.4 Convergence Analysis

We assume that $u(x_p, t_q)$, for $p = 1,2,\dots,M-1, q = 1,2,\dots,N$ is the exact solution of equation (3.2) at the mesh point (x_p, t_q).

We define $e_{p,q} = u_{p,q} - U_{p,q}$ and $e_q = (e_{1,q}, e_{2,q}, \dots\dots\dots, e_{M-1,q})$. Since $e_{0,0} = 0$ we obtain the following relations for the Crank-Nicolson scheme

$$\begin{cases} (1 + 2r_{p,1})e_{p,1} - r_{p,1}e_{p+1,1} - r_{p,1}e_{p-1,1} = R_{p,1}\,, & q = 0, \\ (1 + 2r_{p,q+1})e_{p,q+1} - r_{p,q+1}e_{p+1,q+1} - r_{p,q+1}e_{p-1,q+1} = \sum_{j=1}^{q-1} d_{j+1}^{p,q+1}e_{p,q-j} \;\; + R_{p,q+1}, & q > 0. \end{cases}$$

In the above $R_{p,q+1}$ has the following expression:

$$R_{p,q+1} \;\; = u_{p,q+1} - \sum_{j=0}^{q-1} d_{j+1}^{p,q+1}u_{p,q-j} \;\; - b_q^{p,q+1}u_{p,0} - r_{p,q+1}\left(u_{p+1,q+1} - 2u_{p,q+1} + u_{p-1,q+1}\right).$$

From the basic discretized form we get

$$\begin{cases} \dfrac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left(u_{p,q+1} - \sum_{j=0}^{q-1} d_{j+1}^{p,q+1} u_{p,q-j} \ - b_q^{p,q+1} u_{p,0}\right) = \dfrac{\partial^\alpha u_{p,q+1}}{\partial t^\alpha} + C_1 \tau, \\ \dfrac{u_{p+1,q+1} - 2u_{p,q+1} + u_{p-1,q+1}}{h^2} = \dfrac{\partial^2 u_{p,q+1}}{\partial x^2} + C_2 h^2. \end{cases}$$

Then

$$R_{p,q+1} \ = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left(\frac{\partial^\alpha u_{p,q+1}}{\partial t^\alpha} - \frac{\partial^2 u_{p,q+1}}{\partial x^2}\right) + C_1 \tau^{1+\alpha} + C_2 \tau^\alpha h^2,$$

which implies that

$$R_{p,q+1} \ \leq C(\tau^{1+\alpha} + \tau^\alpha h^2),$$

where C_1, C_2, C are constants. The detailed error analysis on the above scheme considering Caputo-type fractional derivative can refer to work by Li and Tao [42].

Lemma 3.2. The expression

$$\|e^{q+1}\|_\infty \leq C\left(b_q^{p,q+1}\right)^{-1}(\tau^{1+\alpha} + \tau^\alpha h^2),$$

hold for $q = 0,1,\dots,N-1$, where $\|e^q\|_\infty = \max_{1\leq p\leq M-1} |e_{p,q}|$, C is a constant and

$$\alpha^{q+1} = \begin{cases} \min_{1\leq p\leq M-1} \alpha_p^{q+1}, & \text{if } \tau \leq 1, \\ \max_{1\leq p\leq M-1} \alpha_p^{q+1}, & \text{if } \tau > 1. \end{cases}$$

Proof. We also prove it by performing mathematical induction. For $q = 0$ we have

$$|e_{p,1}| \leq (1 + 2r_{p,1})|e_{p,1}| - r_{p,1}|e_{p+1,1}| - r_{p,1}|e_{p-1,1}|,$$

$$\leq |(1 + 2r_{p,1})e_{p,1} - r_{p,1}e_{p+1,1} - r_{p,1}e_{p-1,1}| = |R_{p,1}| \leq C\left(b_0^{p,1}(\tau^{1+\alpha^1} + \tau^{\alpha^1} h^2)\right).$$

Suppose that $\|e^{j+1}\|_\infty \leq C\left(b_q^{p,q+1}\right)^{-1}\left(\tau^{1+\alpha^{j+1}} + \tau^{\alpha^{j+1}} h^2\right)$, $j = 1,2,\dots,N-2$, then

$$|e_{p,q+1}| \leq (1 + 2r_{p,q+1})|e_{p,q+1}| - r_{p,q+1}|e_{p+1,q+1}| - r_{p,q+1}|e_{p-1,q+1}|,$$

$$\leq |(1 + 2r_{p,q+1})e_{p,q+1} - r_{p,q+1}e_{p+1,q+1} - r_{p,q+1}e_{p-1,q+1}|,$$

$$= \left|\sum_{j=0}^{q-1} d_{j+1}^{p,q+1} e_{p,q-j} \ + R_{p,q+1}\right| = \sum_{j=0}^{q-1} d_{j+1}^{p,q+1} |e_{p,q-j}| \ + R_{p,q+1},$$

$$\leq \sum_{j=0}^{q-1} d_{j+1}^{p,q+1} \left\|e^{q-j}\right\|_\infty + C(\tau^{1+\alpha} + \tau^\alpha h^2),$$

$$\leq \left(b_0^{p,q+1} - b_q^{p,q+1} + b_q^{p,q+1}\right)\left(b_q^{p,q+1}\right)^{-1} \times C(\tau^{1+\alpha} + \tau^\alpha h^2)$$

$$\leq C\left(b_q^{p,q+1}\right)^{-1}(\tau^{1+\alpha} + \tau^\alpha h^2),$$

which completes the proof. In addition, we can obtain

$$\lim_{q \to -\infty} \frac{b_q^{p,q}}{q^{\alpha_{p,q}}} = \frac{1}{1-\alpha}.$$

Hence we get the following result

$$\|e^q\|_\infty \le Cq^\alpha(\tau^{1+\alpha} + \tau^\alpha h^2).$$

Since $q^\tau < T$, the following results can be obtained.

Theorem 3.1. Crank Nicolson scheme convergence is established, and a positive constant C exists such that

$$|u_{p,q} - U_{p,q}| \le C(\tau + h^2), p = 1, 2, \dots, M-1, q = 1, 2, \dots, N.$$

It is noteworthy to mention that the Crank-Nicolson scheme's solvability can also be proved Chen *et al.* [8] so the proof is omitted.

CHAPTER 4

MODIFICATION OF TWO-DIMENSIONAL PARABOLIC PROBLEM

4.1 Introduction

A partial differential equation that demonstrates the temperature distribution in a two-dimensional region changes over time is the two-dimensional heat equation. It is a basic equation in the study of heat transport and has uses in many different branches of science and engineering. A partial differential equation that controls thermal conduction-based heat transfer through a material is known as the equation for two-dimensional heat conduction. The form of the equation is

$$\frac{\partial u}{\partial t} = \alpha\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right).$$

The 2D heat equation has many applications in the field of engineering, Electronics, Environmental science, Geophysics, Climate science and manufacturing processes, supporting the development of systems and optimization as well as expanding scientific knowledge across a wide range of disciplines.

4.2 Mathematical Method

We consider the following 2D-TFDE in this section, of the form

$$ {}_0^C D_t^{\alpha} u(x,y,t) = \frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2} + f(x,y,t). \tag{4.1}$$

With the initial and boundary conditions are as follows:

$$\begin{gathered} u(x,y,0) = g(x,y)\ \ ,(x,y) \in \Omega, \\ u(a,y,t) = \phi_1(y,t), u(b,y,t) = \phi_2(y,t), 0 < t < T, \\ u(x,c,t) = \phi_3(x,t), u(x,d,t) = \phi_4(x,t), 0 \le t \le T, \end{gathered} \tag{4.2}$$

where $\Omega = \{(x,y)|a \le x \le b, c \le y \le d\}$. The fractional derivative ${}_0^C D_t^{\alpha} u(x,y,t)$ in (4.1) represents the Caputo fractional derivative of order α.

$$ {}_0^C D_t^{\alpha} u(x,y,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,y,t)}{\partial \tau} \frac{\mathrm{d}\tau}{(t-\tau)^{\alpha}}, 0 < \alpha < 1. \tag{4.3} $$

Defining $t_q = q\tau, q = 0,1,2,\ldots,N, x_p = p\Delta x, p = 0,1,2,\ldots,M_1, y_j = j\Delta y, j = 0,1,2,\ldots,M_2$, where $\tau = t/N, \Delta x = (a-b)/M_1, \Delta y = (c-d)/M_2$ are the time and space steps, respectively. Let the numerical approximation of

$$ u(x_p, y_j, t_q) = u_{p,j}^q, f(x_p, y_j, t_q) = f_{p,j}^q, \varphi(x_p, y_j) = \varphi_{p,j}. $$

Applying the Crank-Nicholson scheme to the problem (4.1)-(4.2) and proceed as before as in the previous chapter, we obtained the following form

$$ \begin{gathered} \frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)} \left[u_{p,j}^{q+1} - u_{p,j}^q + \sum_{s=1}^{q} b_s \left(u_p^{q+1-s} - u_p^{q-s} \right) \right] = \frac{u_{p+1,j}^{q+1} - 2u_{p,j}^{q+1} + u_{p-1,j}^{q+1}}{2\Delta x^2} \\ + \frac{u_{p+1,j}^{q} - 2u_{p,j}^{q} + u_{p-1,j}^{q}}{2\Delta x^2} + \frac{u_{p,j+1}^{q+1} - 2u_{p,j}^{q+1} + u_{p,j-1}^{q+1}}{2\Delta y^2} + \frac{u_{p,j+1}^{q} - 2u_{p,j}^{q} + u_{p,j-1}^{q}}{2\Delta y^2} \\ + \frac{1}{2} f_{p,j}^q + \frac{1}{2} f_{p,j}^{q+1}, \end{gathered} \tag{4.4} $$

where $b_s = (1+s)^{1-\alpha} - s^{1-\alpha}$. For $p = 1,2,\ldots,M_1-1, j = 1,2,\ldots,M_2-1, q = 0,1,2,\ldots,N-1$. The initial and boundary conditions take the following form:

$$ \begin{gathered} u_{p,j}^0 = g_{p,j}, p = 0,1,2,\ldots,M_1, j = 0,1,2,\ldots,M_2, \\ u_{0,j}^q = \phi_{1,j}^q, u_{M_1,j}^q = \phi_{2,j}^q,, 1 < q \le N, 0 \le j, \le M_2, \\ u_{0,p}^q = \phi_{3,p}^q, u_{M_2,p}^q = \phi_{4,p}^q,, 1 < q \le N, 1 \le p, \le M_1 - 1. \end{gathered} \tag{4.5} $$

After simplification, we obtained the following form:

$$ \begin{gathered} -r_1 u_{p+1,j}^{q+1} - r_2 u_{p,j+1}^{q+1} + (1 + 2r_1 + 2r_2) u_{p,j}^{q+1} - r_1 u_{p-1,j}^{q+1} - r_2 u_{p,j-1}^{q+1} = r_1 u_{p+1,j}^q \\ + r_2 u_{p,j+1}^q + (1 - 2r_1 - 2r_2) u_{p,j}^q + r_1 u_{p-1,j}^q + r_2 u_{p,j-1}^q + r_3 \left(f_{p,j}^q + f_{p,j}^{q+1} \right) \\ - \sum_{s=1}^{q-1} u_{p,j}^{q-s} (b_{s-1} - b_s), \end{gathered} \tag{4.6} $$

where

$$ r_1 = \frac{\Gamma(2-\alpha)}{2\Delta x^2 \Delta t^{-\alpha}}, r_2 = \frac{\Gamma(2-\alpha)}{2\Delta y^2 \Delta t^{-\alpha}}, r_3 = \frac{\Gamma(2-\alpha)}{2\Delta t^{-\alpha}}. $$

The previously mentioned problem can be transformed into the following matrix form by carrying out the steps outlined in the preceding chapter:

$$ \mathbb{A} \vec{\mathbf{V}}^1 = \mathbb{B} \vec{\mathbf{V}}^0 + r_3 \left(\mathbf{f}^0 + \mathbf{f}^1 \right) + \mathbf{b}, $$

$$\mathbb{A}\vec{\mathbf{V}}^{q+1} = \mathbb{B}\vec{\mathbf{V}}^{q} + b_k\vec{\mathbf{V}}^{0} + \sum_{k=0}^{q-1}(b_k - b_{k+1})\,\vec{\mathbf{V}}^{q-k} + r_3\left(\mathbf{f}^{q} + \mathbf{f}^{q+1}\right) + \mathbf{b},$$

where the square matrix $\mathbb{A}$ and $\mathbb{B}$ are the block tridiagonal matrices and vector $\vec{\mathbf{e}}$ computed using the boundary conditions and source term, for $q = 0$ the vector $\vec{\mathbf{V}}^{0}$ can be obtained using the initial condition.

Algorithm:

$a, b, T, M_1, M_2, N \in N, \alpha, \phi$, ICs and BCs (Input data)
for p **from** 0 **by** 1 **while** $p \leq M$ **do**;
  $x_p = a + \Delta xp$ (Discretization of the spatial domain)
  $y_j = c + \Delta yj$, and *ICs* (Discretization of spatial domain and ICs)
end do:
for q **from** 0 **by** 1 **while** $q \leq N$ **do**;
  $t_q = q\Delta t$, and *BCs* (Discretization of time domain and BCs)
end do:
for p **from** 0 **by** 1 **while** $p \leq M_1$ **do**;
for p **from** 0 **by** 1 **while** $p \leq M_2$ **do**;
for q **from** 0 **by** 1 **while** $q \leq N$ **do**;
  $\phi_{p,j}^{q} = \phi\left(x_p, y_j, t_q\right)$, (Discretization source term)
end do:
end do:
end do:
  $\mathbb{A}, \mathbb{B}, \vec{\mathbf{f}}^{q}, \vec{\boldsymbol{V}}^{0}, \vec{\boldsymbol{b}}^{q}$ (Evaluation of required matrices and vectors)
for q **from** 0 **by** 1 **while** $q \leq N - 1$ **do**;
 if $q = 0$ **then**
  $\mathbb{A}\vec{\mathbf{V}}^{1} = \mathbb{B}\vec{\mathbf{V}}^{0} + r_3\left(\mathbf{f}^{0} + \mathbf{f}^{1}\right) + \mathbf{b},$
 else,
  $\mathbb{A}\vec{\mathbf{V}}^{q+1} = \mathbb{B}\vec{\mathbf{V}}^{q} + b_k\vec{\mathbf{V}}^{0} + \sum_{k=0}^{q-1}(b_k - b_{k+1})\,\vec{\mathbf{V}}^{q-k} + r_3\left(\mathbf{f}^{q} + \mathbf{f}^{q+1}\right) + \mathbf{b},$
end if:
end do:

Lemma 4.1: In $b_s = (1+s)^{1-\alpha} - s^{1-\alpha}$ the coefficients $b_s (s = 0,1,2,\dots)$ satisfy:

1. $b_s > b_{s+1}, s = 0,1,2, \dots$.
2. $b_0 = 1, b_s > 0, s = 0,1,2, \dots$.

4.3 Stability and Convergence Analysis

This section is dedicated to analyzing the stability analysis of the proposed method using the Crank-Nicolson logic. We suppose that $U_{p,j}^{q}$, $(p = 0,1,2,\dots,M_1, j = 0,1,2,\dots,M_2, q = 0,1,2,\dots,N)$ is the approximate solution of (4.5) and (4.6), the error

$\varepsilon_{p,j}^{q} = U_{p,j}^{q} - u_{p,j}^{q}$, For $p = 0,1,2,\dots,M_1, j = 0,1,2,\dots,M_2, q = 0,1,2,\dots,N,$

satisfies the following equations

$$-r_1\left(\varepsilon_{p+1,j}^1+\varepsilon_{p-1,j}^1\right)+(1+2r_1+2r_2)2\varepsilon_{p,j}^1-r_2\left(\varepsilon_{p,j+1}^1+\varepsilon_{p,j-1}^1\right)$$
$$=r_1\left(\varepsilon_{p+1,j}^0+\varepsilon_{p-1,j}^0\right)+r_2\left(\varepsilon_{p,j+1}^0+\varepsilon_{p,j-1}^0\right)+(1-2r_1-2r_2)\varepsilon_{p,j}^0 \tag{4.7}$$

$$-r_1\left(\varepsilon_{p+1,j}^{q+1}+\varepsilon_{p-1,j}^{q+1}\right)+(1+2r_1+2r_2)2\varepsilon_{p,j}^{q+1}-r_2\left(\varepsilon_{p,j+1}^{q+1}+\varepsilon_{p,j-1}^{q+1}\right)$$
$$=r_1\left(\varepsilon_{p+1,j}^{q}+\varepsilon_{p-1,j}^{q}\right)+r_2\left(\varepsilon_{p,j+1}^{q}+\varepsilon_{p,j-1}^{q}\right)+(1-2r_1-2r_2)\varepsilon_{p,j}^{q}$$
$$-\sum_{s=1}^{q-1}\varepsilon_{p,j}^{q-s}(b_{s-1}-b_s). \tag{4.8}$$

It can be written as:

$$\begin{cases} A\vec{\boldsymbol{E}}^1=B\vec{\boldsymbol{E}}^0, \\ A\vec{\boldsymbol{E}}^{q+1}=(b_0-b_1)\vec{\boldsymbol{E}}^q+(b_1-b_2)\vec{\boldsymbol{E}}^{q-1}+\cdots+\left(b_{q-1}-b_q\right)\vec{\boldsymbol{E}}^1+Bb_q\vec{\boldsymbol{E}}^0, \\ \vec{\boldsymbol{E}}^0\ , \end{cases} \tag{4.9}$$

where,

$$\vec{\boldsymbol{E}}^q=\begin{bmatrix}\vec{\boldsymbol{E}}_1^q\\ \vec{\boldsymbol{E}}_2^q\\ \vdots\\ \vec{\boldsymbol{E}}_{l-1}^q\end{bmatrix}\text{with }\vec{\boldsymbol{E}}_p^q=\begin{bmatrix}\varepsilon_{p,1}^q\\ \varepsilon_{p,2}^q\\ \vdots\\ \varepsilon_{p,m-1}^q\end{bmatrix},p=1,2,\dots,M_1-1. \tag{4.10}$$

Hence, the following result can be proved using mathematical induction.

Theorem 4.1. Prove that

$$\left\|\vec{\boldsymbol{E}}^q\right\|_\infty\le\left\|\vec{\boldsymbol{E}}^0\right\|_\infty,q=1,2,3,\cdots.$$

Proof. For $q=1$,

$$-r_1\left(\varepsilon_{p+1,j}^1+\varepsilon_{p-1,j}^1\right)+(1+2r_1+2r_2)\varepsilon_{p,j}^1-r_2\left(\varepsilon_{p,j+1}^1+\varepsilon_{p,j-1}^1\right)=r_1\left(\varepsilon_{p+1,j}^0+\varepsilon_{p-1,j}^0\right)$$
$$+r_2\left(\varepsilon_{p,j+1}^0+\varepsilon_{p,j-1}^0\right)+(1-2r_1-2r_2)\varepsilon_{p,j}^0.$$

Suppose $\left|\varepsilon_{i,k}^1\right|=\max\limits_{1\le p\le M_1-1;1\le j\le M_2-1}\left|\varepsilon_{p,j}^1\right|$ we have

$$\begin{aligned}\left|\varepsilon_{i,k}^1\right|&=-r_1\left(\left|\varepsilon_{i,k}^1\right|+\left|\varepsilon_{i,k}^1\right|\right)+(1+2r_1+2r_2)\left|\varepsilon_{i,k}^1\right|-r_2\left(\left|\varepsilon_{i,k}^1\right|+\left|\varepsilon_{i,k}^1\right|\right),\\ &\le -r_1\left(\left|\varepsilon_{i+1,k}^1\right|+\left|\varepsilon_{i-1,k}^1\right|\right)+(1+2r_1+2r_2)\left|\varepsilon_{i,k}^1\right|-r_2\left(\left|\varepsilon_{i,k+1}^1\right|+\left|\varepsilon_{i,k-1}^1\right|\right),\\ &\le\left|-r_1\left(\varepsilon_{i+1,k}^1+\varepsilon_{i-1,k}^1\right)+(1+2r_1+2r_2)\varepsilon_{i,k}^1-r_2\left(\varepsilon_{i,k+1}^1+\varepsilon_{i,k-1}^1\right)\right|,\\ &=\left|r_1\left(\varepsilon_{p+1,j}^0+\varepsilon_{p-1,j}^0\right)+r_2\left(\varepsilon_{p,j+1}^0+\varepsilon_{p,j-1}^0\right)+(1-2r_1-2r_2)\varepsilon_{p,j}^0\right|\le\left\|\vec{\boldsymbol{E}}^0\right\|_\infty.\end{aligned} \tag{4.11}$$

Also, $\left\|\vec{\boldsymbol{E}}^1\right\|_\infty\le\left\|\vec{\boldsymbol{E}}^0\right\|_\infty$. Suppose that $\left\|\vec{\boldsymbol{E}}^s\right\|_\infty\le\left\|\vec{\boldsymbol{E}}^0\right\|_\infty,s=1,2,\dots,q$. Assume that $\left|\varepsilon_{i,k}^{q+1}\right|=\max\limits_{1\le p\le l-1;1\le j\le m-1}\left|\varepsilon_{p,j}^{q+1}\right|$.

$$\begin{aligned}\left|\varepsilon_{i,k}^{q+1}\right|&=-r_1\left(\left|\varepsilon_{i,k}^{q+1}\right|+\left|\varepsilon_{i,k}^{q+1}\right|\right)+(1+2r_1+2r_2)\left|\varepsilon_{i,k}^{q+1}\right|-r_2\left(\left|\varepsilon_{i,k}^{q+1}\right|+\left|\varepsilon_{i,k}^{q+1}\right|\right),\\ &\le -r_1\left(\left|\varepsilon_{i+1,k}^{q+1}\right|+\left|\varepsilon_{i-1,k}^{q+1}\right|\right)+(1+2r_1+2r_2)\left|\varepsilon_{i,k}^{q+1}\right|-r_2\left(\left|\varepsilon_{i,k+1}^{q+1}\right|+\left|\varepsilon_{i,k-1}^{q+1}\right|\right),\end{aligned}$$

$$\leq \left|-r_1\left(\varepsilon_{i+1,k}^{q+1}+\varepsilon_{i-1,k}^{q+1}\right)+(1+2r_1+2r_2)2\varepsilon_{i,k}^{q+1}-r_2\left(\varepsilon_{i,k+1}^{q+1}+\varepsilon_{i,k-1}^{q+1}\right)\right|,$$

$$=\left|r_1\left(\varepsilon_{p+1,j}^{q}+\varepsilon_{p-1,j}^{q}\right)+r_2\left(\varepsilon_{p,j+1}^{q}+\varepsilon_{p,j-1}^{q}\right)+(1-2r_1-2r_2)\varepsilon_{p,j}^{q}-\sum_{s=1}^{q-1}\varepsilon_{p,j}^{q-s}(b_{s-1}-b_s)\right|,$$

$$\leq (b_0-b_1)\left|\varepsilon_{i,k}^{q}\right|+\sum_{s=1}^{q-1}(b_s-b_{s+1})\left|\varepsilon_{i,k}^{q-s}\right|+Bb_q\left|\varepsilon_{i,k}^{0}\right|,$$

$$\leq (b_0-b_1)\left\|\vec{\boldsymbol{E}}^{q}\right\|_\infty+\sum_{s=1}^{q-1}(b_s-b_{s+1})\left\|\vec{\boldsymbol{E}}^{q-s}\right\|_\infty+Bb_q\|\boldsymbol{E}^0\|_\infty,$$

$$\leq \left\{(b_0-b_1)+\sum_{s=1}^{q-1}(b_s-b_{s+1})+Bb_q\right\}\left\|\boldsymbol{E}^{\mathbf{0}}\right\|_\infty=\left\|\boldsymbol{E}^{\mathbf{0}}\right\|_\infty. \tag{4.12}$$

also, $\left\|\vec{\boldsymbol{E}}^{q+1}\right\|_\infty \leq \left\|\vec{\boldsymbol{E}}^{0}\right\|_\infty$. Hence the difference approximation is unconditionally stable.

Now, suppose that $u(x_p,y_j,t_q), p=0,1,2,\dots,M_1; j=0,1,2,\dots,M_2-1; q=0,1,2,\dots,N$ be the exact solution to the partial differential equation with fractional order (4.1) at mesh point (x_p,y_j,t_q). Defining $\eta_{p,j}^{q}=u(x_p,y_j,t_q)-u_{p,j}^{q}$ and $\boldsymbol{e}^q=\left(\boldsymbol{e}_1^q,\boldsymbol{e}_2^q,\dots,\boldsymbol{e}_{m-1}^q\right)$. Using $\boldsymbol{e}^0=0$, where

$$\boldsymbol{e}_p^q=\begin{bmatrix}\eta_{p,1}^{q}\\ \eta_{p,2}^{q}\\ \vdots\\ \eta_{p,M_2-1}^{q}\end{bmatrix}, p=1,2,\dots,M_1-1.$$

Substituting in the above yields the following result:

$$-r_1\left(\eta_{p+1,j}^{1}+\eta_{p-1,j}^{1}\right)+(1+2r_1+2r_2)2\eta_{p,j}^{1}-r_2\left(\eta_{p,j+1}^{1}+\eta_{p,j-1}^{1}\right)=R_{p,j}^{0}, \tag{4.13}$$

$$-r_1\left(\eta_{p+1,j}^{q+1}+\eta_{p-1,j}^{q+1}\right)+(1+2r_1+2r_2)2\eta_{p,j}^{q+1}-r_2\left(\eta_{p,j+1}^{q+1}+\eta_{p,j-1}^{q+1}\right)$$
$$=R_{p,j}^{q}+\sum_{s=1}^{q-1}(b_{s-1}-b_s)\,\eta_{p}^{q-s}, \tag{4.14}$$

where

$$R_{p,j}^{q}=r_1\left(u_{p+1,j}^{q}+u_{p-1,j}^{q}\right)+r_2\left(u_{p,j+1}^{q}+u_{p,j-1}^{q}\right)+(1-2r_1-2r_2)u_{p,j}^{q}$$
$$R_{p,j}^{q+1}=r_1\left(u_{p+1,j}^{q+1}+u_{p-1,j}^{q+1}\right)+r_2\left(u_{p,j+1}^{q+1}+u_{p,j-1}^{q+1}\right)+(1+2r_1+2r_2)u_{p,j}^{q+1}$$
$$+\sum_{s=1}^{q-1}(b_{s-1}-b_s)\,u_{p}^{q-s}.$$

Since that

$$\frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}\sum_{s=0}^{q} b_s\left[u\left(x_p,y_j,t_{q+1-s}\right)-u\left(x_p,y_j,t_{q-s}\right)\right]=\frac{\partial^\alpha}{\partial t^\alpha}u\left(x_p,y_j,t_{q+1}\right)+\mathrm{O}(\Delta t),$$

$$\frac{u\left(x_{p+1},y_j,t_{q+1}\right)-2u\left(x_p,y_j,t_{q+1}\right)+u\left(x_{p-1},y_j,t_{q+1}\right)}{\Delta x^2}=\frac{\partial^2 u\left(x_p,y_j,t_{q+1}\right)}{\partial x^2}+\mathrm{O}(\Delta x^2),$$

$$\frac{u\left(x_p,y_{j+1},t_{q+1}\right)-2u\left(x_p,y_j,t_{q+1}\right)+u\left(x_p,y_{j-1},t_{q+1}\right)}{\Delta y^2}=\frac{\partial^2 u\left(x_p,y_j,t_{q+1}\right)}{\partial y^2}+\mathrm{O}(\Delta y^2).$$

Hence, we have

$$R_{p,j}^{q+1}=\mathrm{O}(\tau^{1+\alpha}+\tau^\alpha(\Delta x)^2+\tau^\alpha(\Delta y)^2),$$

also

$$\left|R_{p,j}^{q+1}\right|\le C(\tau^{1+\alpha}+\tau^\alpha(\Delta x)^2+\tau^\alpha(\Delta y)^2),$$

for $p=1,2,\dots,M_1-1, j=1,2,\dots,M_2-1, q=0,1,2,\dots,N$, where C is a constant.

Theorem 4.2. Show that

$$\|e^q\|_\infty\le Cb_{q-1}^{-1}(\tau^{1+\alpha}+\tau^\alpha(\Delta x)^2+\tau^\alpha(\Delta y)^2), q=1,2,\dots,N,$$

where $\left|\varepsilon_{i,k}^q\right|=\max\limits_{1\le p\le l-1;1\le j\le m-1}\left|\varepsilon_{p,j}^q\right|$ and C is a constant.

Proof. By means of the mathematical induction for this when $q=1$, let

$$\|\boldsymbol{e}^q\|_\infty=\left|\varepsilon_{i,k}^1\right|=\max_{1\le p\le l-1;1\le j\le m-1}\left|\varepsilon_{p,j}^1\right|.$$

We have

$$\begin{aligned}\left|\varepsilon_{i,k}^1\right|&=-r_1\left(\left|\varepsilon_{i,k}^1\right|+\left|\varepsilon_{i,k}^1\right|\right)+(1+2r_1+2r_2)\left|\varepsilon_{i,k}^1\right|-r_2\left(\left|\varepsilon_{i,k}^1\right|+\left|\varepsilon_{i,k}^1\right|\right)\\ &\le -r_1\left(\left|\varepsilon_{i+1,k}^1\right|+\left|\varepsilon_{i-1,k}^1\right|\right)+(1+2r_1+2r_2)\left|\varepsilon_{i,k}^1\right|-r_2\left(\left|\varepsilon_{i,k+1}^1\right|+\left|\varepsilon_{i,k-1}^1\right|\right)=\left|R_{i,k}^1\right|\\ &\le Cb_0^{-1}(\tau^{1+\alpha}+\tau^\alpha(\Delta x)^2+\tau^\alpha(\Delta y)^2).\end{aligned}$$

Suppose that $\|e^s\|_\infty\le Cb_{s-1}^{-1}(\tau^{1+\alpha}+\tau^\alpha(\Delta x)^2+\tau^\alpha(\Delta y)^2), s=0,1,2,\dots,q-1$ and $\left|\varepsilon_{i,k}^{q+1}\right|=\max\limits_{1\le p\le l-1;1\le j\le m-1}\left|\varepsilon_{p,j}^{q+1}\right|$. Note that $b_s^{-1}\le b_q^{-1}, s=0,1,\dots,q$. We have

$$\begin{aligned}\left|\varepsilon_{i,k}^{q+1}\right|&\le -r_1\left(\left|\varepsilon_{i+1,k}^{q+1}\right|+\left|\varepsilon_{i-1,k}^{q+1}\right|\right)+(1+2r_1+2r_2)\left|\varepsilon_{i,k}^{q+1}\right|-r_2\left(\left|\varepsilon_{i,k+1}^{q+1}\right|+\left|\varepsilon_{i,k-1}^{q+1}\right|\right),\\ &\le\left|-r_1\left(\varepsilon_{i+1,k}^{q+1}+\varepsilon_{i-1,k}^{q+1}\right)+(1+2r_1+2r_2)2\varepsilon_{i,k}^{q+1}-r_2\left(\varepsilon_{i,k+1}^{q+1}+\varepsilon_{i,k-1}^{q+1}\right)\right|,\\ &=\left|\sum_{s=1}^{q-1}(b_s-b_{s+1})\varepsilon_{i,k}^{q-s}+R_{i,k}^q\right|\le\sum_{s=1}^{q-1}(b_s-b_{s+1})\left|\varepsilon_{i,k}^{q-s}\right|+\left|R_{i,k}^q\right|,\\ &\le\sum_{s=1}^{q-1}(b_s-b_{s+1})\left|\varepsilon_{i,k}^{q-s}\right|+C(\tau^{1+\alpha}+\tau^\alpha(\Delta x)^2+\tau^\alpha(\Delta y)^2),\\ &\le\sum_{s=1}^{q-1}(b_s-b_{s+1})\left\|e_{i,k}^{q-s}\right\|_\infty+C(\tau^{1+\alpha}+\tau^\alpha(\Delta x)^2+\tau^\alpha(\Delta y)^2),\end{aligned}$$

$$\leq \sum_{s=1}^{q-1}(b_s - b_{s+1})\,\|e^{q-s}\|_\infty + \ C(\tau^{1+\alpha} + \tau^\alpha(\Delta x)^2 + \tau^\alpha(\Delta y)^2),$$

$$\leq \left[\sum_{s=1}^{q-1}(b_s - b_{s+1}) + b_q\right] b_q^{-1} C(\tau^{1+\alpha} + \tau^\alpha(\Delta x)^2 + \tau^\alpha(\Delta y)^2),$$

$$= b_q^{-1} C(\tau^{1+\alpha} + \tau^\alpha(\Delta x)^2 + \tau^\alpha(\Delta y)^2).$$

Because

$$\lim_{q\to\infty}\frac{b_q^{-1}}{q^\alpha} = \lim_{q\to\infty}\frac{q^{-\alpha}}{(q+1)^{1-\alpha} - q^{1-\alpha}},$$

$$= \lim_{q\to\infty}\frac{q^{-\alpha}}{\left(1+\frac{1}{q}\right)^{1-\alpha} - 1},$$

$$= \lim_{q\to\infty}\frac{q^{-\alpha}}{(1-\alpha)q^{-1}},$$

$$= \frac{1}{1-\alpha}.$$

Hence, there is a constant C,

$$\|e^q\|_\infty = Cq^\alpha(\tau^{1+\alpha} + \tau^\alpha(\Delta x)^2 + \tau^\alpha(\Delta y)^2).$$

If $k\Delta t \leq T$ is finite, then the theorem is obtained.

Theorem 4.3. Suppose $u_{p,j}^q$ be the approximate solution of $u(x_p, y_j, t_q)$ determined by using difference scheme (4.4) and (4.5). In that case, the constant C is positive, such that

$$\left|u_{p,j}^q - u\big(x_p, y_j, t_q\big)\right| \leq C(\tau^\alpha + (\Delta x)^2 + (\Delta y)^2),$$

$$p = 1,2,\dots,M_1 - 1, j = 1,2,\dots,M_2 - 1, q = 0,1,2,\dots,N.$$

CHAPTER 5

PERFORMANCE EVALUATION

5.1 Results and Discussion

To evaluate the parabolic partial differential equation behavior under the influence of temporal fractional factors and convergence control parameters presented in Chapters 3 and 4, we developed mathematical code in Maple and used Tec plot software to show the simulation results. This section is divided into two main sections. The analysis of results, which compares the exact solution with various parameters used to evaluate the effectiveness of TFDEs, is shown below. Many challenges have been resolved to evaluate the effectiveness of the suggested method. The same results as well as a few new ones have been obtained in this chapter.

5.2 Graphical Solutions of 1D Parabolic Problem

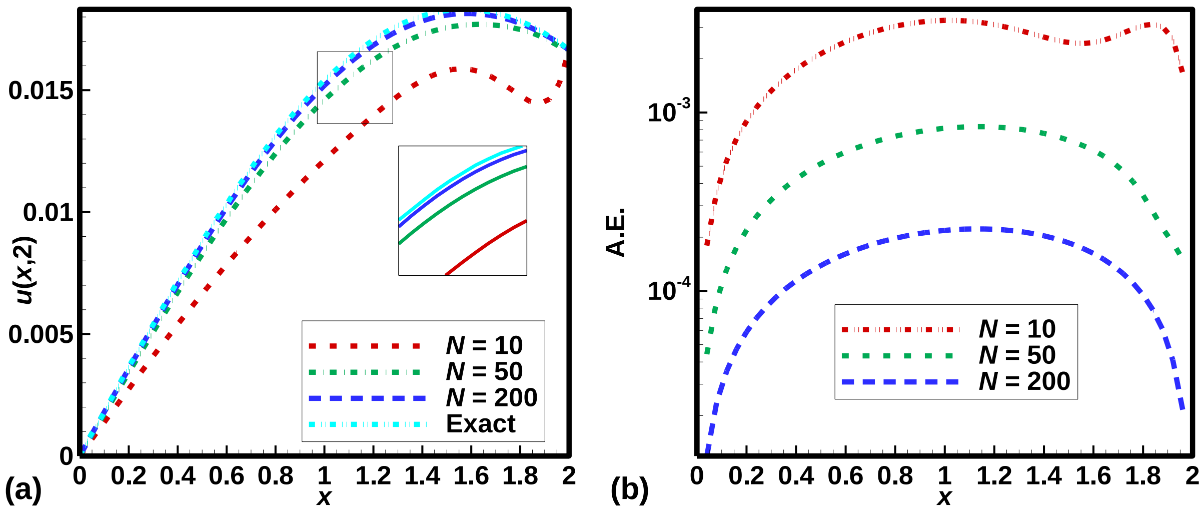


Figure 5.1. 2D graphical behavior of **(a)** approximate and exact solution **(b)** absolute error against N the time-fractional problem when $u(x,t) = \sin(x)\,\mathrm{e}^{-2t}, 0 \leq x \leq 2, 0 < t \leq 2$

For different values of N, an exact and numerical solution to the first problem when $u(x,t) = \sin(x)\,\mathrm{e}^{-2t}, 0 \le x \le 2, 0 < t \le 2$ is obtained in Figure 5.1. In this figure, two-dimensional representations of exact and approximate solutions, and absolute error behavior are obtained against the different mesh sizes i.e., $\Delta t = 2/N$. Figure 5.1 (a) shows that as the mesh size decreases the approximate solution gets closer to the exact solution gradually which untimely shows the proposed method's convergence against decreases in the mesh size in the temporal direction. Similar phenomena have been obtained in Figure 5.1 (b) where the absolute difference in accuracy between the exact and approximate solutions decreases when $N \to \infty$. The outcomes are the evidence that the higher value of N has more accuracy and close solution to the exact solution.

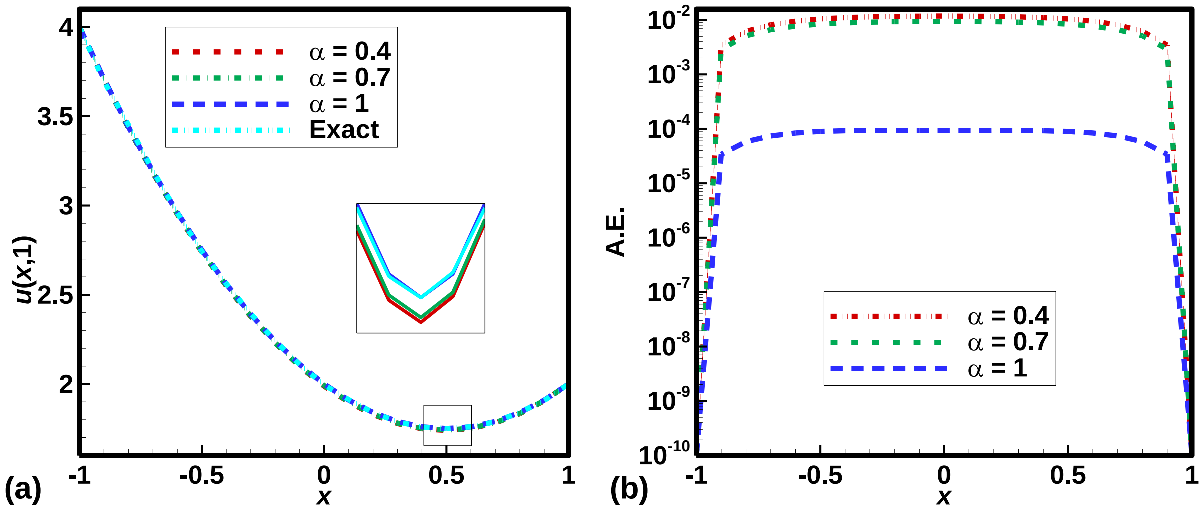


Figure 5.2. 2D graphical behavior of **(a)** approximate and exact solution **(b)** absolute error against the parameter α the time-fractional problem when $u(x,t) = x^2t^3 + 1 - x + t, -1 \le x \le 1, 0 < t \le 1$

An exact numerical solution to the first problem for various values of α when $u(x,t) = x^2t^3 + 1 - x + t, -1 \le x \le 1, 0 < t \le 1$ is obtained in Figure 5.2. In this figure, two-dimensional representations of exact and approximate solutions, and absolute error behavior are obtained against the different values of α. As the fractional order rises, Figure 5.2(a) demonstrates how the approximation solution progressively gets closer to the exact solution, demonstrating the suggested method's convergence against fractional order increases. Figure 5.2 (b) displays a similar phenomenon, where in there is a decrease in the absolute error between the exact and approximation solutions as α approaches 1. The results show that a higher value of α corresponds to a more accurate and nearly precise solution.

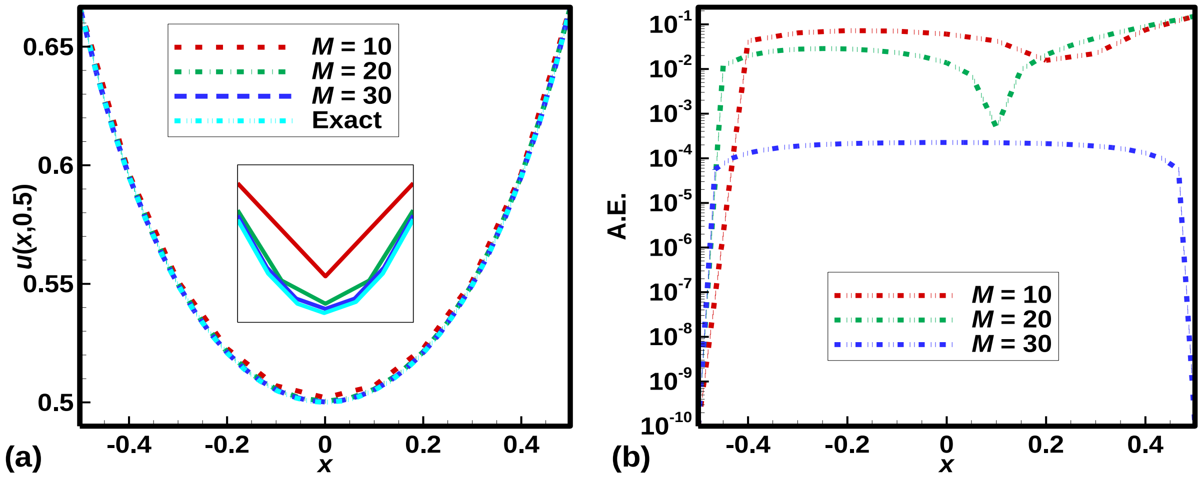


Figure 5.3. 2D graphical behavior of **(a)** approximate and exact solution **(b)** absolute error against the parameter M the time-fractional problem when $u(x,t) = t/(2 - x^2)$ $-0.5 \leq x \leq 0.5, 0 < t \leq 0.5$

Figure 5.3 provides an accurate and numerical solution to the first problem for a variety of values of M where $u(x,t) = t/(2 - x^2)$ $-0.5 \leq x \leq 0.5, 0 < t \leq 0.5$. This figure shows the absolute error behavior and two-dimensional representations of the exact and approximation solutions versus the various mesh sizes, i.e. $\Delta x = 1/M$. As the mesh size decreases, the approximate solution quickly approaches the precise solution, as illustrated in Figure 5.3 (a). The convergence of the proposed approach against changes in the mesh size in the spatial direction is depicted in this picture. A similar effect is shown in Figure 5.3 (b), where the absolute error between the precise and approximation solutions decreases as M gets closer to ∞. The results show that a solution that is closer to the exact solution and has a higher value of M is more accurate.

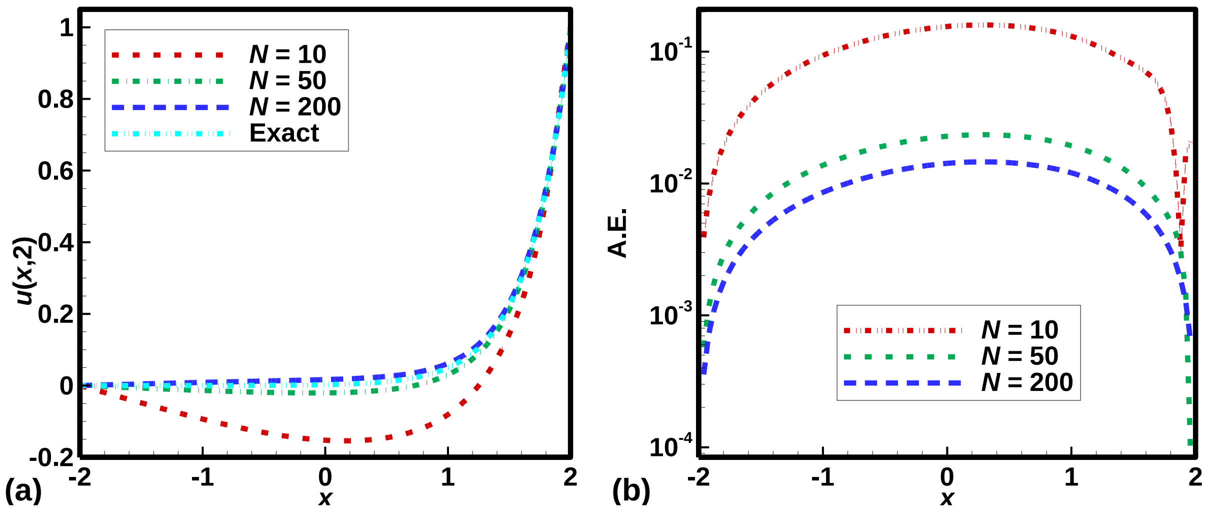


Figure 5.4. 2D graphical behavior of **(a)** approximate and exact solution **(b)** absolute error against the parameter N the time-fractional problem when $u(x,t) = \mathrm{e}^{3(x-t)}, -2 \leq x \leq 2, 0 < t \leq 2$

Figure 5.4 provides an exact numerical solution for $u(x,t) = e^{3(x-t)}$ for a range of N values where $-2 \le x \le 2, 0 < t \le 2$. Two-dimensional representations of the exact and approximate solutions, as well as the absolute error behavior, are derived against the various mesh sizes, i.e. $\Delta t = 2/N$, in this image. The approximate solution rapidly approaches the precise solution as the mesh size drops, as illustrated in Figure 5.4(a). This illustrates the timely convergence of the proposed method against mesh size reductions in the temporal direction. Figure 5.4 (b) displays a similar phenomenon, wherein $N \to \infty$ leads to a reduction in the absolute error between the approximation and perfect solutions. The results show that a larger number of N has greater precision and a solution that is closer to the exact solution.

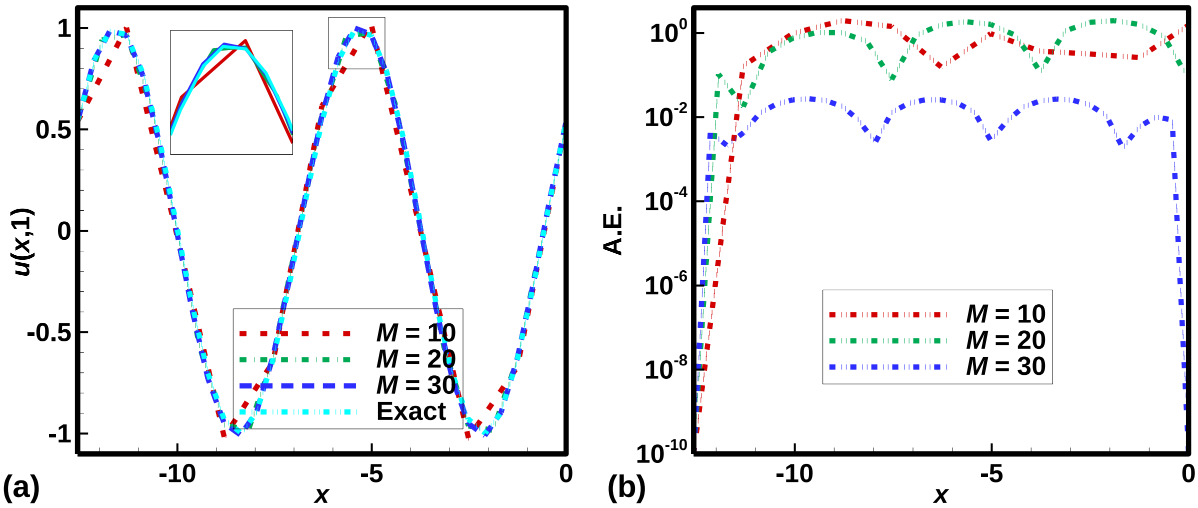


Figure 5.5. 2D graphical behavior of **(a)** approximate and exact solution **(b)** absolute error against the parameter M the time-fractional problem when $u(x,t) = \cos(x-t), -4\pi \le x \le 0, 0 < t \le 1$

Figure 5.5 illustrates a precise and numerical solution to the first problem for a variety of values of M where $u(x,t) = \cos(x-t), -4\pi \le x \le 0, 0 < t \le 1$. In this figure, absolute error behavior and two-dimensional representations of precise and approximate solutions are produced against various mesh sizes, i.e., $\Delta x = 4\pi/M$. Figure 5.5(a) shows how the approximate solution gradually becomes closer to the exact solution as the mesh size decreases. The convergence of the suggested approach against changes in mesh size in the spatial direction is shown in this image. A similar effect is shown in Figure 5.5 (b), where the absolute error between the precise and approximation solutions decreases as M gets closer to ∞. The results show that a solution that is closer to the exact solution and has a higher value of M is more accurate.

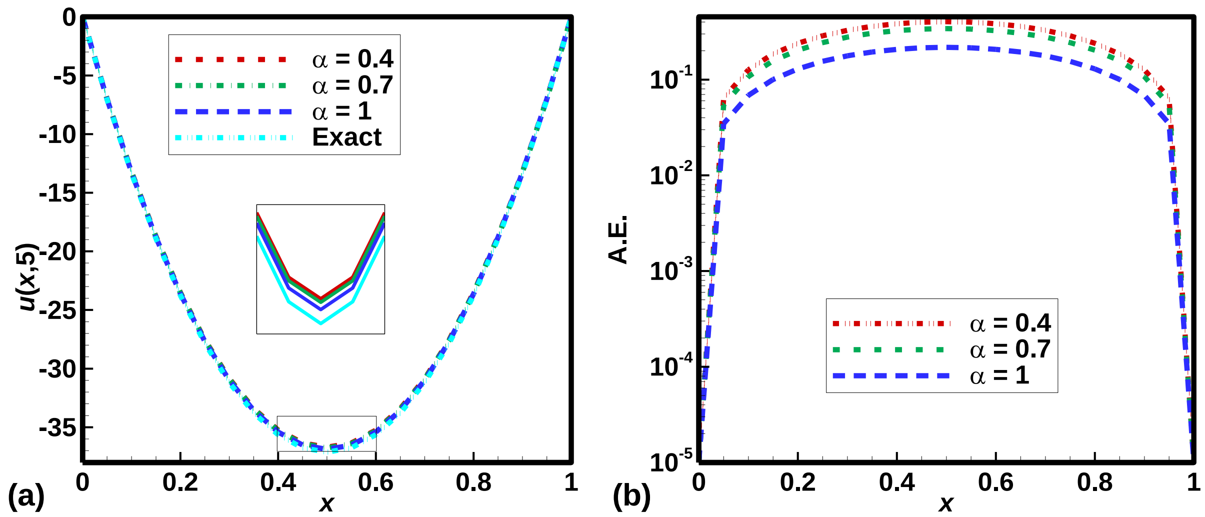


Figure 5.6. 2D graphical behavior of **(a)** approximate and exact solution **(b)** absolute error against the parameter α the time-fractional problem when $u(x,t) = x(x-1)\mathrm{e}^t, 0 \leq x \leq 1, 0 < t \leq 5$

A precise and numerical solution to the first problem for various values of α when $u(x,t) = x(x-1)\mathrm{e}^t, 0 \leq x \leq 1, 0 < t \leq 5$ is obtained in Figure 5.6. This illustration shows two-dimensional representations of precise and approximate solutions, as well as absolute error behavior for different α values. Figure 5.6(a) shows how the approximation solution gradually approaches the precise solution as the fractional order increases, proving the proposed method's convergence against fractional order increases. Figure 5.6 (b) shows a decrease in absolute error between exact and approximation solutions as α approaches one. Higher α values result in more accurate and near-exact solutions.

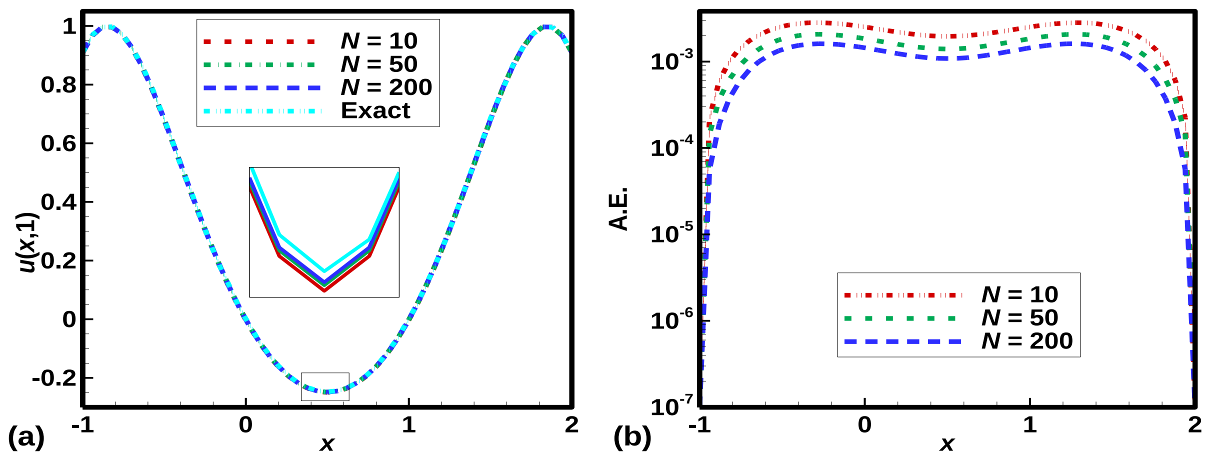


Figure 5.7. 2D graphical behavior of **(a)** approximate and exact solution **(b)** absolute error against the parameter N the time-fractional problem when $u(x,t) = \sqrt{t}\sin(x^2 - x), -1 \leq x \leq 2, 0 < t \leq 1$

Figure 5.7 shows a perfect numerical solution to the first problem for different N values where $u(x,t) = \sqrt{t}\sin(x^2 - x), -1 \leq x \leq 2, 0 < t \leq 1$. In this figure, two-dimensional representations of exact and approximate solutions, and absolute error behavior are obtained against the different mesh sizes i.e., $\Delta t = 1/N$. Figure 5.7 (a) shows that as the mesh size decreases the approximate solution gets closer to the exact solution gradually which untimely shows the convergence of the proposed method against decreases in the mesh size in the temporal direction. Similar phenomena have been obtained in Figure 5.7 (b) Where the absolute error of the approximate solutions and the exact solutions reduces when $N \to \infty$. The outcomes are the evidence that the higher value of N has more accuracy and close solution to the exact solution.

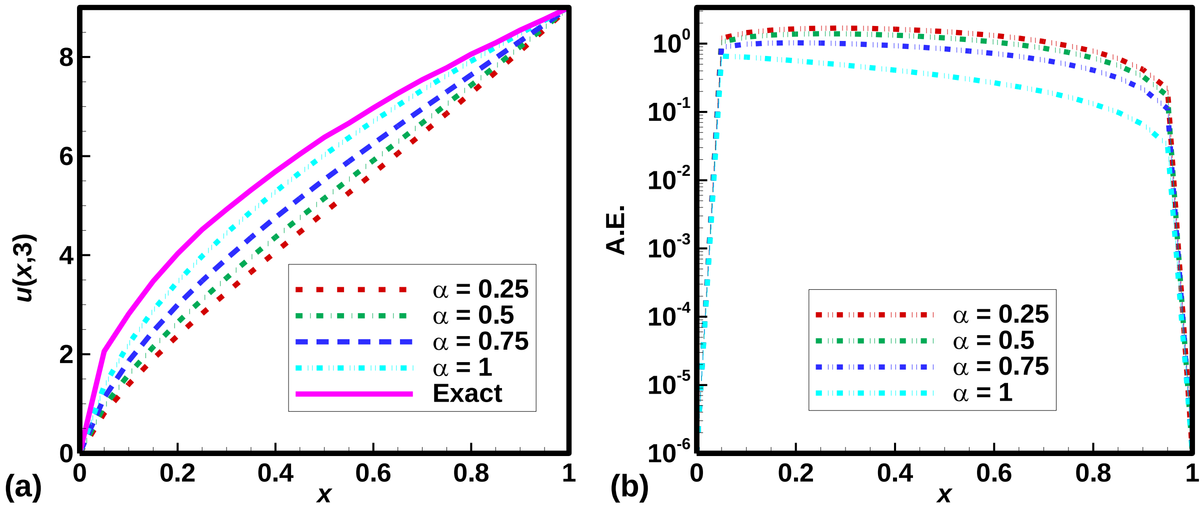


Figure 5.8. 2D graphical behavior of **(a)** approximate and exact solution **(b)** absolute error against the parameter α the time-fractional problem when $u(x,t) = \sqrt{x}t^{1+\alpha}, 0 \leq x \leq 1, 0 < t \leq 3$

For various values of α, an exact and numerical solution to the first problem when $u(x,t) = \sqrt{x}t^{1+\alpha}, 0 \leq x \leq 1, 0 < t \leq 3$ is obtained in Figure 5.8. In this figure, two-dimensional representations of exact and approximate solutions, and absolute error behavior are obtained against the different values of α. It is evident from Figure 5.8(a) that the proposed method converges against increases in fractional order, with the approximate solution progressively approaching the exact solution as the fractional order increases. Figure 5.8 (b) displays a similar phenomenon, wherein there is a decrease in the absolute error between the exact and approximation solutions as α approaches 1. The results show that a higher value of α corresponds to a more accurate and nearly precise solution.

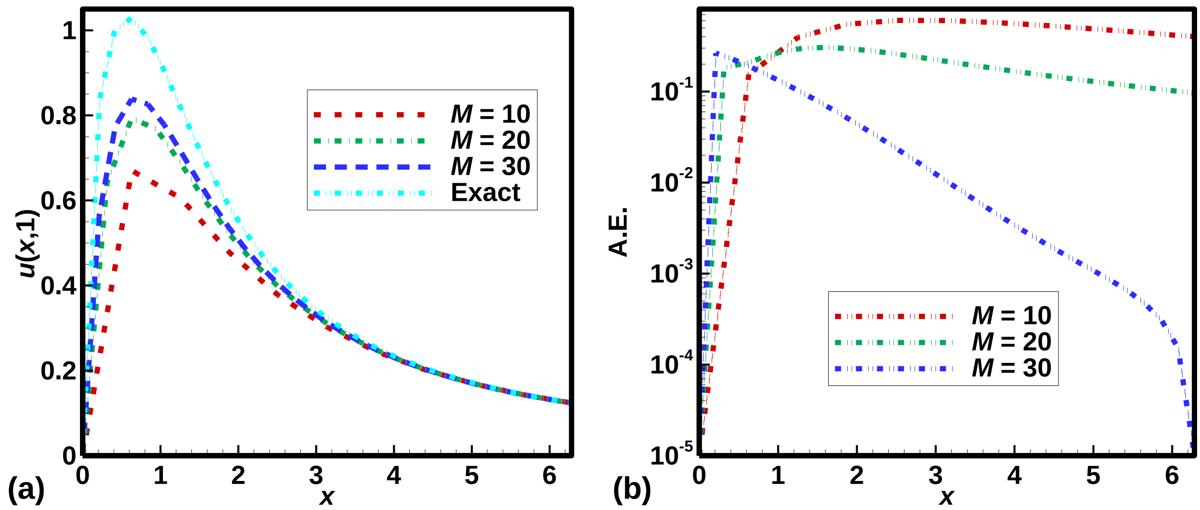
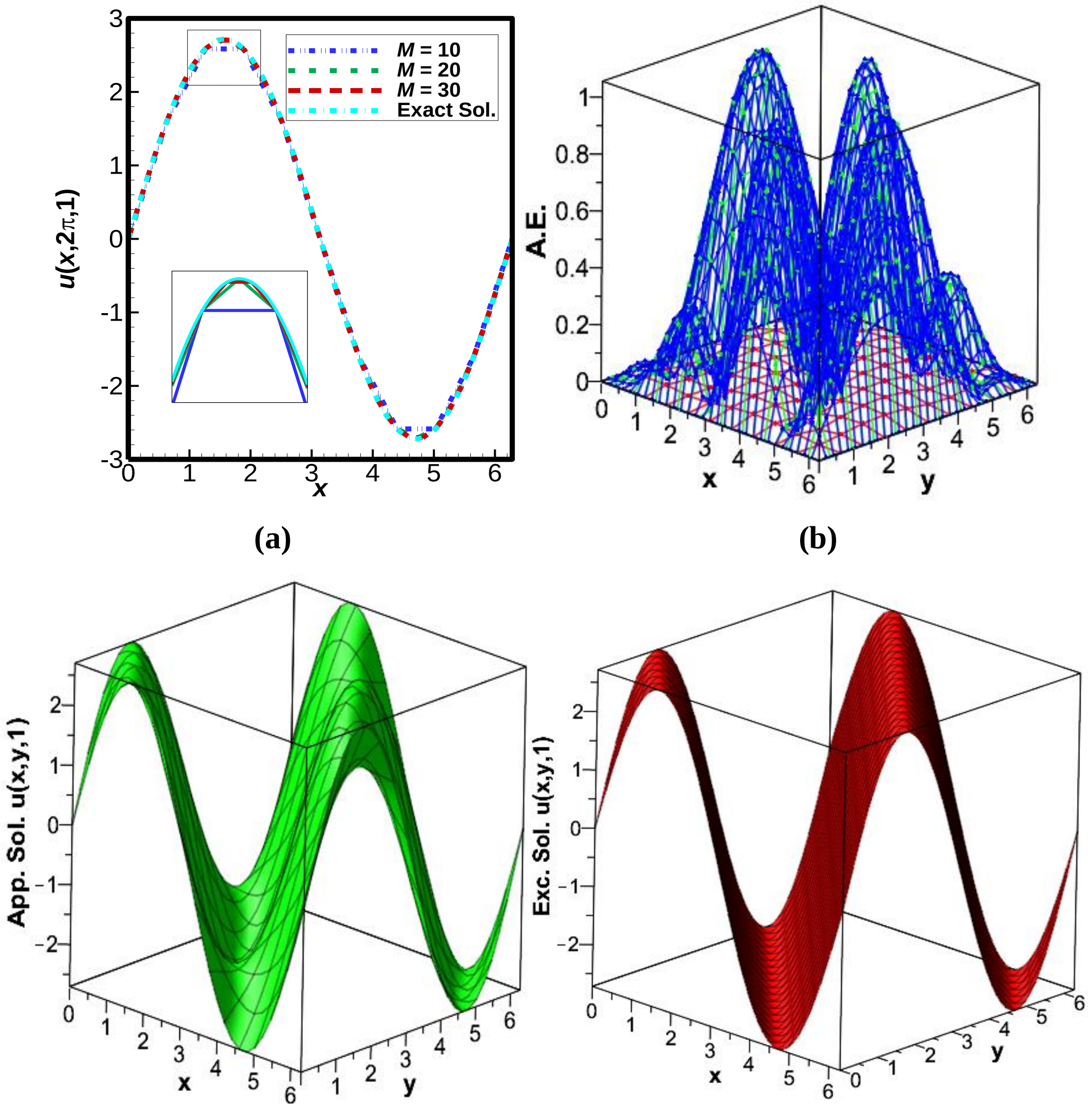


Figure 5.9. 2D graphical behavior of **(a)** approximate and exact solution **(b)** absolute error against the parameter M the time-fractional problem when $u(x,t) = (t-1)\tanh\left(\sqrt{x}/(x^2-1)\right), 0 \leq x \leq 2\pi, 0 < t \leq 1$

An exact and numerical solution to the first problem for a range of values of M when $u(x,t) = (t-1)\tanh\left(\sqrt{x}/(x^2-1)\right), 0 \leq x \leq 2\pi, 0 < t \leq 1$ is obtained in Figure 5.9. In this figure, two-dimensional representations of exact and approximate solutions, and absolute error behavior are obtained against the different mesh sizes i.e., $\Delta x = 2\pi/M$. Figure 5.9(a) illustrates how, as the mesh size reduces, the approximate solution progressively approaches the exact solution. This picture illustrates the proposed method's convergence against mesh size decreases in the spatial direction. Figure 5.9 (b) displays a similar phenomenon, wherein there is a decrease in the absolute error between the exact and approximation solutions as $M \to \infty$. The findings indicate that a solution with a greater value of M is more accurate and approaches the precise solution more closely.

5.3 Graphical Solutions of 2D Parabolic Problem



(a) **(b)**

(c) **(d)**

Figure 5.10. Three-dimensional behavior of **(a)** exact and approximate solution against $M = M_1 = M_2$ **(b)** absolute error as varying $M = M_1 = M_2$ **(c)** approximate solution in space when $M = 30$ and **(d)** exact solution for 2D parabolic problem of fractional order when $u(x, y, t) = \sin(x + y)\,\mathrm{e}^t, 0 < x < 2, 0 < y < 2, 0 < t < 1$

Figure 5.10 Graphical illustration of (a) Displays the numerical solution's 2D behavior against M (b) demonstrates the approximate solution in space for $M = 30$ in three dimensions (c) and the exact solution for the 2D parabolic problem of fractional order when $u(x, y, t) = \sin(x + y)\,\mathrm{e}^t$.The behaviour of the absolute error against M varies in three dimensions. It has been observed that there is little difference between the exact and approximate replies. The graphs show the degree of relationship between the exact results and the produced results. The graphs have demonstrated the applicability of the existing methodology.

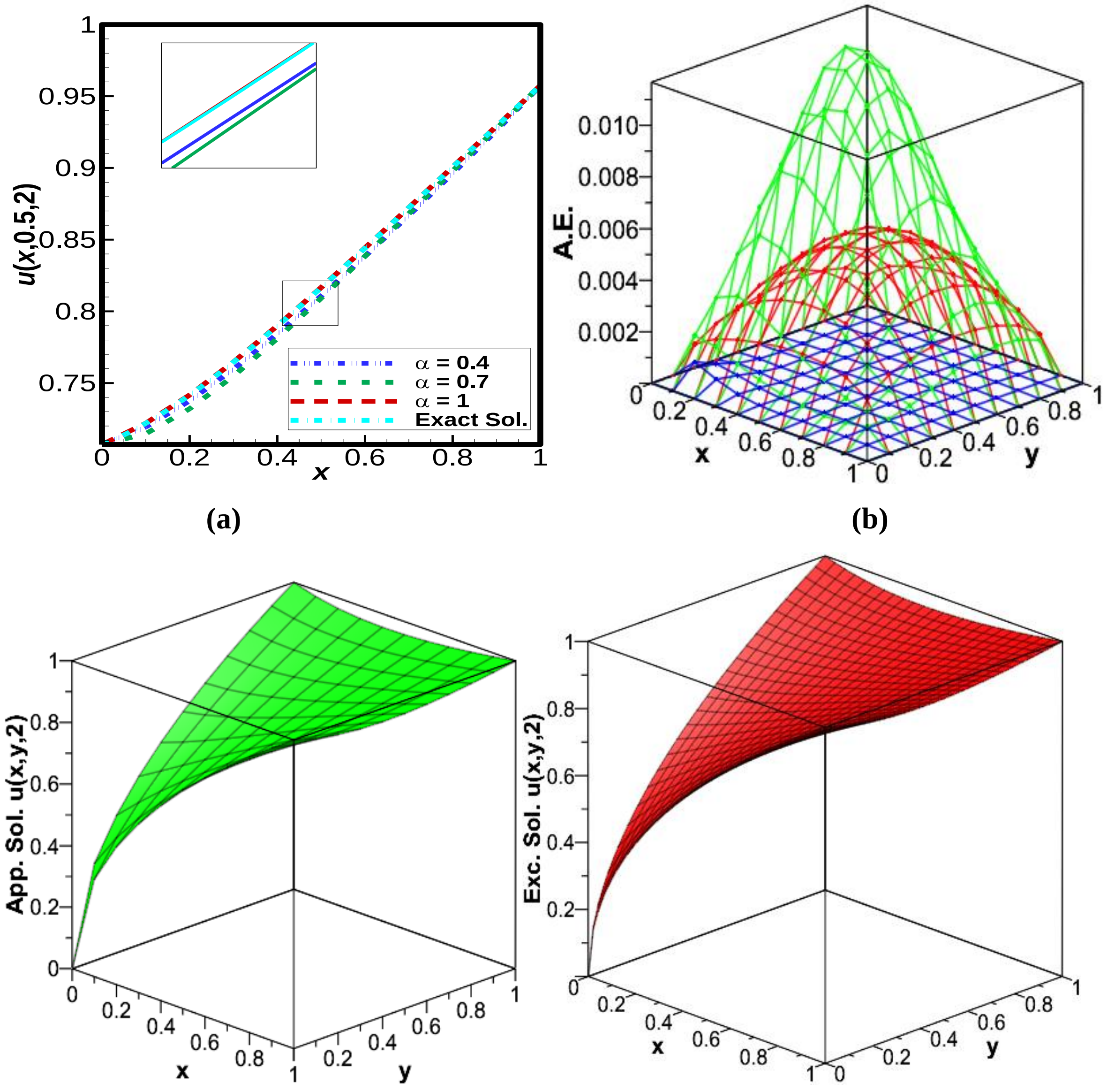
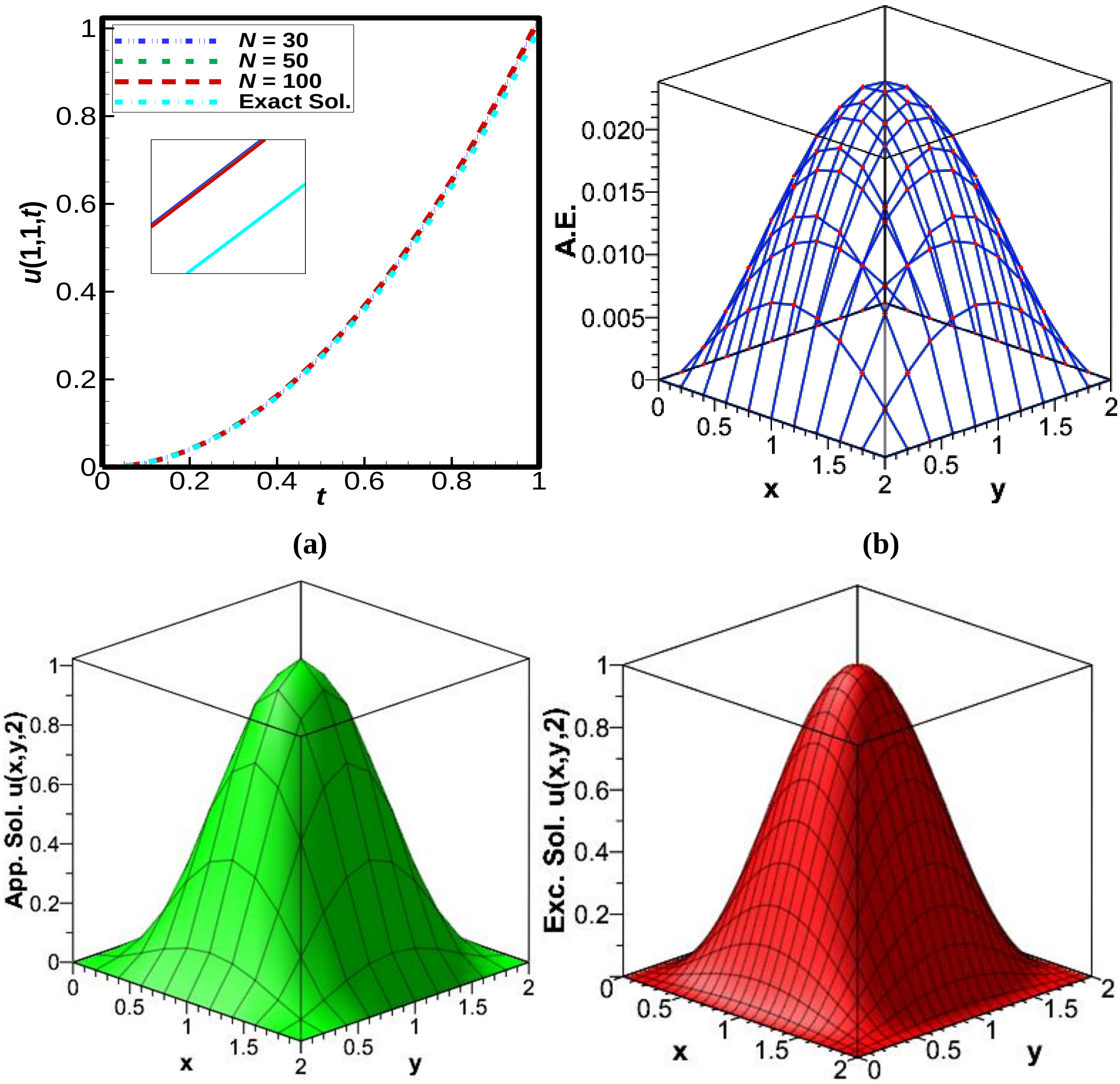


Figure 5.11. Behavior in 3D of **(a)** the exact and approximate solution against α **(b)** absolute error as varying α **(c)** approximate solution in space when $\alpha = 1$ and **(d)** exact solution for 2D parabolic problem of fractional order when $u(x, y, t) = \sqrt{x/(1+y) + y/(1+x)}(1 - t)^2, 0 < x < 1, 0 < y < 1, 0 < t < 2$

Figure 5.11 shows a graphic representation of (a) Demonstrate the numerical solution's 2D behavior in relation to α (b). Demonstrate the 3D variation in the behavior of the absolute error against α (c). In three dimensions, represent the approximate answer (where $\alpha = 1$). (d) Express the precise solution to the fractional order 2D parabolic problem $u(x, y, t) =$

$\sqrt{x/(1+y)+y/(1+x)}(1-t)^2$. It is observed that there is a close relationship between the exact and CNM solutions. The graphs have confirmed the applicability of the existing process and shown how exact and near the obtained results are.



(a) **(b)**

(c) **(d)**

Figure 5.12. 3D representation of **(a)** the exact and approximate solution beside N **(b)** absolute error as changing N **(c)** approximate solution in space when $N = 100$ and **(d)** exact solution for the 2D parabolic problem of fractional order when $u(x,y,t) = x^2y^2(x-2)^2(y-2)^2t^2, 0 < x < 2, 0 < y < 2, 0 < t < 1$

Figure 5.12 Graphical representation of (a) Display the 2D behavior of the numerical solution against N (b) Display the 3D change of the absolute error against changing N (c) Offer a rough spatial solution for $N = 100$ and (d). Provide an exact solution to the fractional order 2D parabolic problem $u(x,y,t) = x^2y^2(x-2)^2(y-2)^2t^2$. It is evident from the 2D figure that as N increases, the approximate solution approaches the precise solution, and as N approaches ∞, the absolute error diminishes. It is observed that the estimated and exact solutions are very similar. The graphs demonstrate how the current technique can be used and how exact and close the results are.

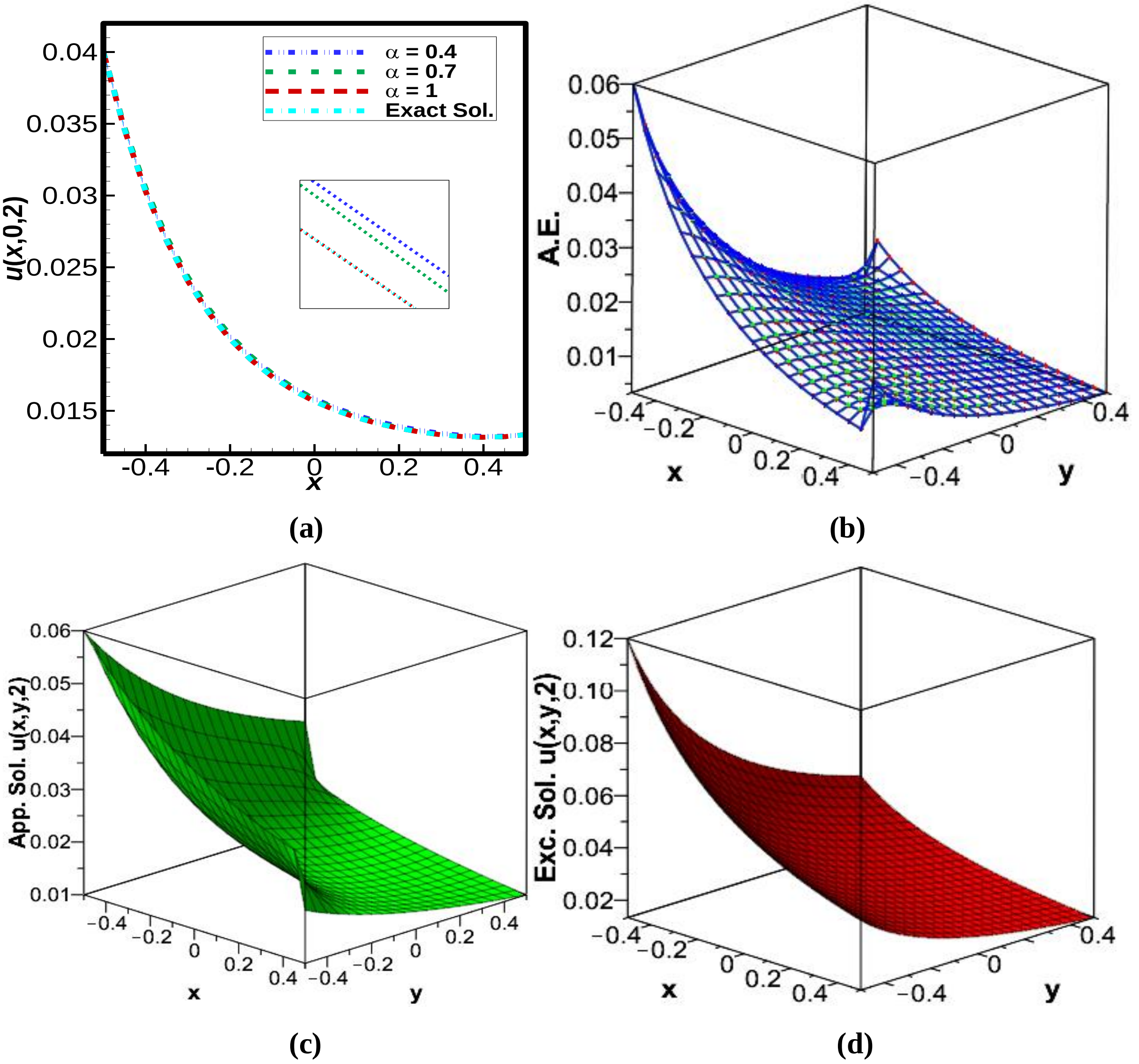


Figure 5.13. Representation in three-dimensional of **(a)** the exact and approximate solution beside α **(b)** absolute error as changing α **(c)** approximate solution in space when $\alpha = 1$ and **(d)** exact solution for the 2D parabolic problem of fractional order when $u(x, y, t) = (1 + t)/(1 + x)(1 + y), -0.5 < x < 0.5, 0.5 < y < -0.5, 0 < t < 5$

The three-dimensional representation in Figure 5.13 shows (a) the exact and approximate solution next to α, (b) the absolute error expressed as a change in α, (c) the approximate solution in space when $\alpha = 1$, and (d) the precise solution for the 2D parabolic problem of fractional order when $u(x, y, t) = (1 + t)/(1 + x)(1 + y)$. We find that the CNM and exact solutions are in close proximity to one other. The graphs have validated the applicability of the existing technique and shown how exact and accurate the findings produced are.

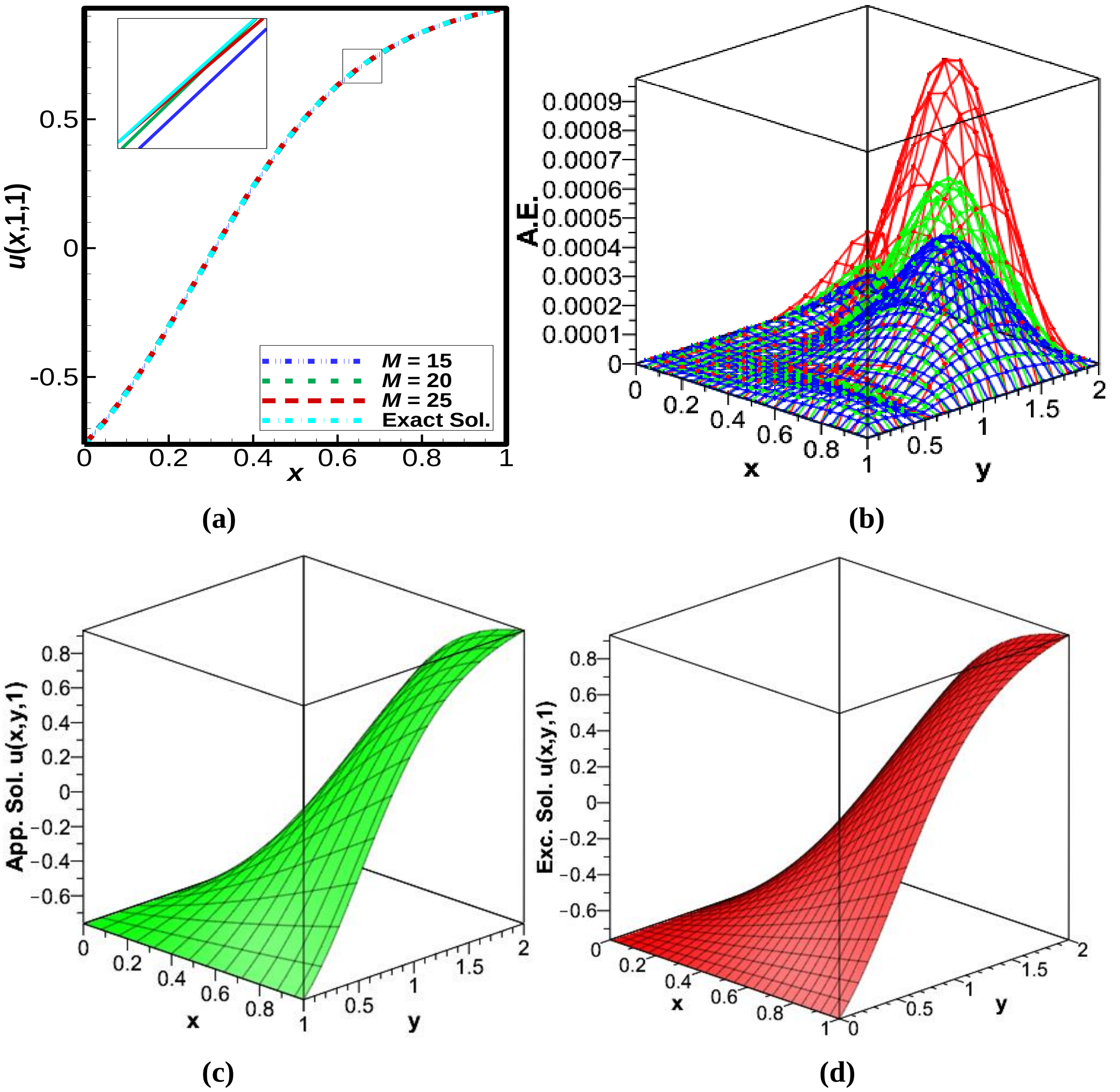


Figure 5.14. Illustration in 3D of **(a)** the exact and approximate solution beside $M = M_1 = M_2$ **(b)** absolute error as changing $M = M_1 = M_2$ **(c)** approximate solution in space when $M = 25$ and **(d)** exact solution for 2D parabolic problem of fractional order when $u(x, y, t) = t^{3/2} \tanh(xy - 1/(1 + xy)), 0 < x < 1, 0 < y < 2, 0 < t < 1$

Figure 5.14 shows an illustration in three dimensions of (a) Display the precise and approximate solutions, in addition to $M = M_1 = M_2$ (b) Display the absolute error as it changes $M = M_1 = M_2$ (c) Display the approximate solution in space for $M = 25$. (d) Showcase the exact solution to the two-dimensional parabolic problem of fractional order when $u(x, y, t) = t^{3/2} \tanh(xy - 1/(1 + xy))$. The 2D figure makes it clear that the approximate solution gets closer to the precise answer as N increases, while the absolute error gets smaller as N approaches ∞. The CNM and exact solutions are found to be relatively near to one another. The graphs highlight the current technique's applicability and highlight how precise and near the findings are that are generated.

5.4 Summary

In this chapter, we have included a graphical solution and compared the exact and approximated solution through graph representation. The 1D graphical solutions show that when mesh size decreases the proposed method shows the convergence to the exact solution for both temporal and spatial direction. It also shows that in 1D when $\alpha \to 1$ the proposed method converges to the exact method. The 2D graphical solutions show that the approximate and exact solutions are in close connection to one another. The graphs of 2D demonstrate the applicability of the proposed method. The graphs also show how close the exact and proposed method is.

CHAPTER 6

CONCLUSION AND FUTURE WORK

6.1 Summary of Current Work

In this thesis, we have included a brief review of TFDE and the preliminaries of fractional calculus. For the one-dimensional temporal fractional diffusion problem, we have proposed a finite difference approach. We use the scheme of Crank-Nicolson to discretize the TFDE for one dimension. We have studied the physical behavior of the proposed fractional-order model under different circumstances, especially against the mesh parameters $M, N,$ and fractional parameter α. The proposed 1D system has been proved to be unconditionally stable and convergent. We have compared the exact and approximate solution graphically. Graphically it has been proved that exact and approximate solutions are in close connection to one another in 1D. For the two-dimensional temporal fractional diffusion problem, we have proposed a finite difference approach. For two dimensions, we discretize the TFDE using the Crank-Nicolson approach. We have examined the suggested fractional-order model's physical behavior in several scenarios, particularly in relation to the mesh parameters $M, N,$ and the fractional parameter α. The suggested 2D system has been shown to be convergent and unconditionally stable. A graphic comparison of the exact and approximate solutions has been made. It has been demonstrated graphically that in 1D, exact and approximate solutions have a strong connection to one another. The proposed method is very clear and efficient. The accuracy of the proposed technique is totally supported by computational effort and the resulting numerical data. The proposed method is also supported by the graphical representation in both 1D and 2D.

6.2 Future Directions

It is important to keep in mind that, even though in this thesis, we simply took the time fractional diffusion equation under consideration. Perhaps it could be possible to apply the suggested approach to the fractional space diffusion problem. In future, the researchers can take the fractional order to the space and solve it and can possibly find the stability and convergence for it.

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