A NOVEL STUDY OF Q-RUNG ORTHOPAIR INTERVAL VALUED FUZZY SOFT EXPERT SETS

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Candidate of **Master of Science in Mathematics (MS Math)** at the National University of Modern Languages, affirm that the thesis titled**A Novel Study of Q-Rung Orthopair Interval Valued Fuzzy Soft Expert Sets** submitted by me in partial fulfillment of MS Math degree, is entirely my own work and has not been previously submitted or published. I further declare that I will not submit this work for any other degree at this university or any other institution in the future. I acknowledge that any instance of plagiarism discovered in my thesis/dissertation, even after the degree is conferred, may lead to the annulment of the work and the revocation of the degree.

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ABSTRACT

Title: A Novel Study of Q-Rung Orthopair Interval Valued Fuzzy Soft Expert Sets

The research introduces Q-Rung Orthopair Interval Valued Fuzzy Soft Expert Sets, which serve as a robust tool for decision-making. By integrating Interval-Valued Intuitionistic Fuzzy Soft Expert Sets and Q-Rung Orthopair Fuzzy Sets, this approach effectively handles multiple-criteria decision-making problems. The primary objectives of this study involve creating a mathematical framework, introducing new aggregation operators and improving the overall decision-making process. To achieve these goals, the research explores the structure of Q-Rung Orthopair Interval Valued Fuzzy Soft Expert Sets and the integration of aggregation operators. The methodology encompasses the development of a theoretical framework, the Definition of algorithms and the evaluation of their performance. Ultimately, this study contributes to enhancing decision-making efficiency and broadening the application of these techniques across various domains.

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FIGURE NO.	TITLE	PAGE

LIST OF ABBREVIATIONS

AOs	-	Aggregation operators
DMPs	-	Decision-making problems
FSs	-	Fuzzy Sets
IFSs	-	Intuitionistic fuzzy sets
IFSS	-	Intuitionistic Fuzzy Soft Set
IFWAO	-	Intuitionistic fuzzy weighted averaging operator
IFWGO	-	Intuitionistic fuzzy weighted geometric operator
IVFSs	-	Interval valued fuzzy sets
IVFSSs	-	Interval valued Fuzzy soft sets
IVIFSs	-	Interval-valued Intuitionistic Fuzzy Sets
IVIFSESs	-	Interval Valued Intuitionistic Fuzzy Soft Expert Sets
IVIFSEWAO	-	Interval Valued Intuitionistic Fuzzy Soft Expert Weighted Averaging Operator
IVIFSEWGO	-	Interval Valued Intuitionistic Fuzzy Soft Expert Weighted Geometric Operator
IVPFSEWAO	-	Interval Valued Pythagorean Fuzzy Soft Expert Weighted Averaging Operator
IVPFSEWGO	-	Interval Valued Pythagorean Fuzzy Soft Expert Weighted Geometric Operator
MCDMPs	-	Multiple-criteria decision-making problems
Q-ROFN	-	Q-rung Orthopair Fuzzy number
Q-ROFSs	-	Q-Rung Orthopair Fuzzy Sets
Q-ROFWAO		Q-Rung orthopair fuzzy weighted averaging operator
Q-ROFWG	-	Q-Rung orthopair fuzzy weighted geometric operator
Q-ROIVFSEAMO	-	Q-Rung Orthopair Interval Valued Fuzzy Soft Expert Arithmetic mean operator
Q-ROIVFSEFWAO	-	Q-Rung Orthopair Interval Valued Fuzzy Soft Expert fusion weighted averaging operator
Q-ROIVFSEFWGO	-	Q-Rung Orthopair Interval Valued Fuzzy Soft Expert fusion weighted geometric operator

Q-ROIVFSEGFWAO	-	Q-Rung Orthopair Interval Valued Fuzzy Soft Expert generalized fusion weighted averaging operator
Q-ROIVFSEGMO	-	Q-Rung Orthopair Interval Valued Fuzzy Soft Expert geometric mean operator
Q-ROIVFSEGOWAO	-	Q-Rung Orthopair Interval Valued Fuzzy Soft Expert generalized ordered weighted averaging operator
Q-ROIVFSEOs	-	Q-rung orthopair interval valued fuzzy soft expert operators
Q-ROIVFSEOWAO	-	Q-Rung Orthopair Interval Valued Fuzzy Soft Expert ordered weighted averaging operator
Q-ROIVFSEOWGO	-	Q-Rung Orthopair Interval Valued Fuzzy Soft Expert ordered weighted geometric operator
Q-ROIVFSEWAO	-	Q-Rung Orthopair Interval Valued Fuzzy Soft Expert Weighted Averaging Operator
Q-ROIVFSEWGO	-	Q-Rung Orthopair Interval Valued Fuzzy Soft Expert Weighted Geometric Operator
SESs	-	Soft Expert Sets
SSs	-	Soft sets

LIST OF SYMBOLS

$\breve{s}$	-	Any element of universe set S
θ	-	A function defined on S such that $\theta: S \to [0,1]$, which assigns a membership value to each element
I^S	-	Collection of all fuzzy sets
$\vee$	-	Join
$\wedge$	-	Meet
$\boldsymbol{\Delta}_0$	-	Null fuzzy subset or empty fuzzy set
$\boldsymbol{\Delta}_1$	-	Whole fuzzy set of S
$\theta^-(\breve{s})$, $\theta^+(\breve{s})$	-	Lower and upper degree of membership, respectively
$\underline{I}$ ([0, 1])	-	The set of all closed sub-intervals of [0, 1]
$\theta_A(\breve{s})$, $\varphi_A(\breve{s})$	-	Values of degree of membership and non-membership, respectively, of an element $\breve{s} \in A \subseteq S$
$\breve{p}$	-	Parameter belonging to set of parameter $\breve{P}$
$\alpha(\breve{p})$	-	Intuitionistic fuzzy subset of S for a parameter $\breve{p} \in A$
$\pi_A(\breve{s})$	-	Interval valued hesitancy degree or intuitionistic fuzzy index
(j, A)	-	Soft set
$S^{\breve{P}}$	-	Power set of S
$\emptyset_A$	-	Relative null soft set
W_A	-	Relative whole soft set
$\tilde{e}$	-	Expert belonging to set of expert $\tilde{E}$
O	-	Opinion Set
F(S)	-	Collection of all intuitionistic fuzzy sets
$U_I(S)$	-	Set of IVIFSs on the universe set S
α	-	Q-rung Orthopair Fuzzy number (Q-ROFN) $\alpha = <\theta_A(\breve{s}), \varphi_A(\breve{s})> = <\theta_A, \varphi_A>$.
§	-	Score function
Â	-	Accuracy function

w	-	Weight vector
δ	-	Q-Rung orthopair fuzzy weighted averaging operator.
γ	-	Q-Rung orthopair fuzzy weighted geometric operator
F	-	Set of all Q-ROFNs
$\subset$	-	Contained
$\overline{\omega}$	-	Q-ROIVFSEWAO
$\underline{\omega}$	-	Q-ROIVFSEWGO
$\underline{\omega}_G$	-	Q-ROIVFSEGMO
$\boldsymbol{W}_i$	-	Position weight vector
$\overline{\omega_o}$	-	Q-ROIVFSEOWAO
$\overline{\omega_f}$	-	Q-ROIVFSEFWAO
$G\overline{\omega_o}$	-	Q-ROIVFSEGOWAO
$G\overline{\omega_f}$	-	Q-ROIVFSEGFWAO
$\underline{\omega_o}$	-	Q-ROIVFSEOWGO
$\underline{\omega_f}$	-	Q-ROIVFSEFWGO

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DEDICATION

I dedicate this thesis to my parents, family and the teachers who have been a constant presence throughout my educational journey. Their unconditional love and exemplary guidance have not only provided unwavering support but have also imparted valuable lessons on the importance of hard work in striving for my goals.

CHAPTER 1

INTRODUCTION

1.1 Introduction

The paradigm of Fuzzy Sets (FSs), put forward by Lotfi Zadeh [1] in 1965, extends classical sets by allowing partial membership. Intuitionistic fuzzy sets (IFSs); proposed by Krassimir Atanassov [2] in 1983, further expand FSs by incorporating not only membership but also non-membership degrees for elements. Later, Interval-valued Intuitionistic Fuzzy Sets (IVIFSs) were proposed by Krassimir Atanassov [3] in 1989, allowing membership and non-membership degrees to be expressed as ranges. Soft sets (SSs), introduced by Molodtsov [4] in 1999, generalize FSs and include uncertainty and vagueness in data representation by including parameters in study. Soft Expert Sets (SESs), proposed by Alkhazaleh and Salleh [5] in 2011, incorporate expertise from multiple experts. Yager [6] suggested Q-Rung Orthopair Fuzzy Sets (Q-ROFSs) in 2016 as an extension of IFSs for handling uncertainty in decision-making. Interval Valued Intuitionistic Fuzzy Soft Expert Sets (IVIFSESs), introduced by Afshan Qayyum [7] in 2017, combine IVIFSs and SESs for uncertain data representation involving multiple experts.

The research focuses on developing a decision-making algorithm using a generalized structure called Q-Rung Orthopair Interval Valued Fuzzy Soft Expert Sets (Q-ROIVFSESs) to address multiple-criteria decision-making problems (MCDMPs). The study aims to explore different algorithms to support this theory and define new operations on Q-ROIVFSESs, including aggregation operators, to enhance the effectiveness of the process of decision-making. The research seeks to provide an efficient and effective approach to decision-making, incorporating uncertainty and expertise in a unified framework.

1.2 Motivation

The motivation behind this research is to address the limitations of existing decision-making frameworks in dealing with uncertain or vague criteria. The proposed Q-ROIVFSESs framework aims to provide a more robust and flexible approach to decision-making that can be effectively applied to real-world decision-making problems (DMPs).

IVIFSESs combine SESs and IVIFSs and it expresses the expert's opinion as an interval-valued intuitionistic fuzzy sets. However, in many real world scenarios the expert's opinion may not be confined as IVIFSs. Due to uncertainty and vagueness in past domains, there occurs a need of expressing expert's opinion as Q-ROFSs. So an integration of IVIFSESs and Q-ROFSs may result a better mathematical structure to handle uncertainties in decision making process in a fruitful way.

Overall, the motivation behind this research is to contribute to the field of mathematics and improve decision-making processes in real-world scenarios.

1.2.1 Existing Structures:

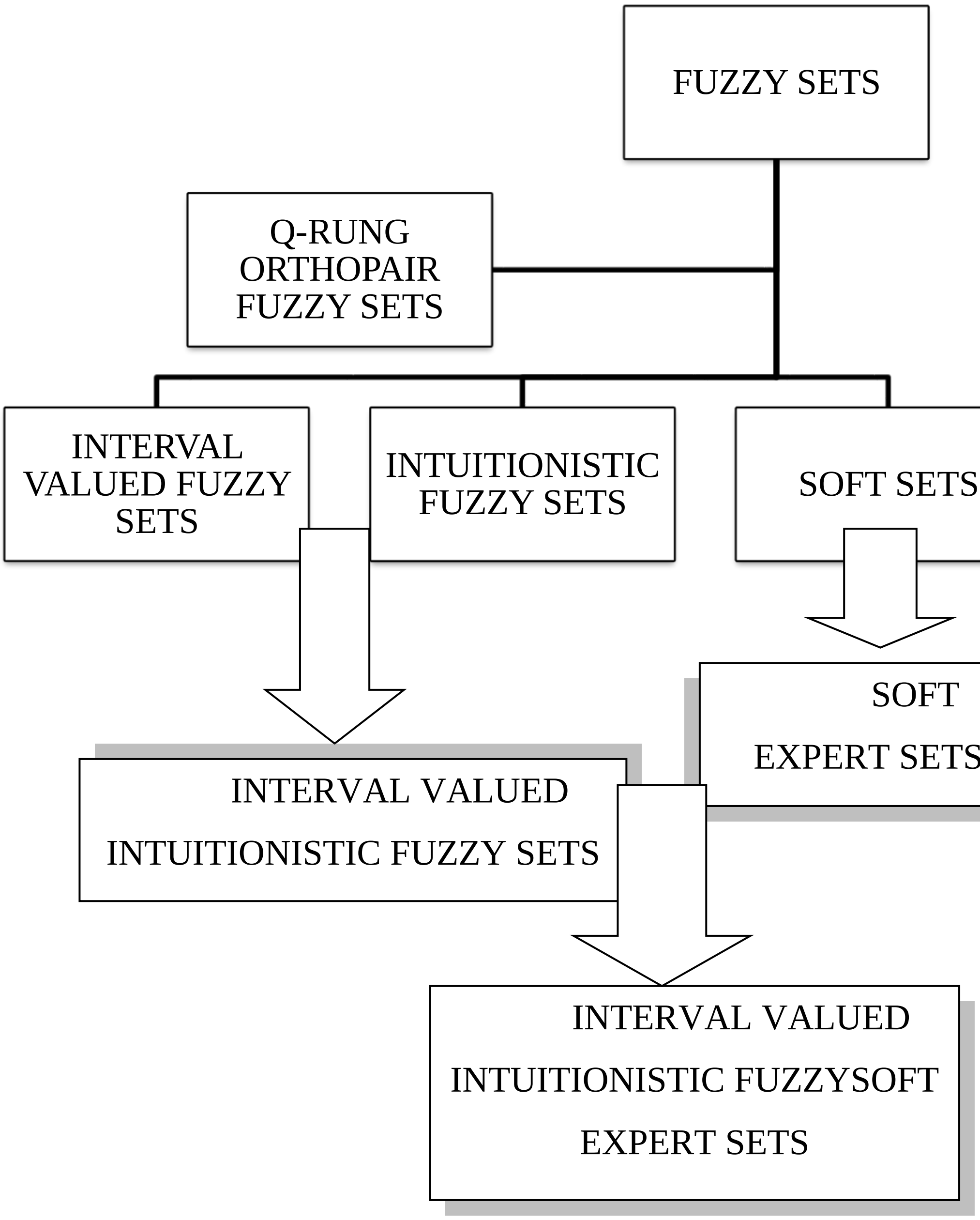

Figure 1.1: Fuzzy Sets Existing Structures Diagram

1.2.2 Proposed Structure:

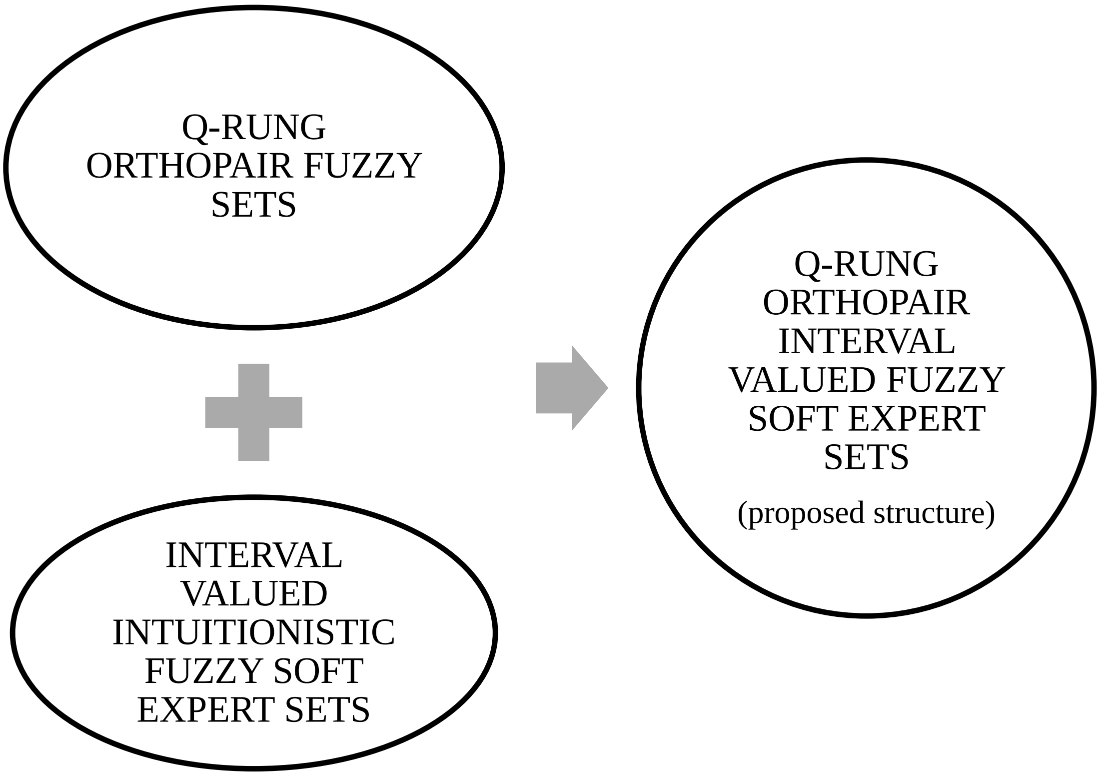

Figure 1.2: Proposed Structure

1.3 Problem Background

In current decision-making models, IVIFSESs are utilized to represent expert opinions by combining SESs and IVIFSs. However, these models may not fully capture the true nature of expert opinions in many real-world scenarios, where opinions can be better represented using Q-ROFSs due to uncertainties and vagueness in past domains. Thus, the integration of IVIFSESs and Q-ROFSs is proposed as a more effective mathematical structure for handling uncertainties in the decision-making process.

1.4 Problem Statement

Effective decision making by generalizing IVIFSESs and combining with Q-ROFSs. The target of research is to form and apply a better decision-making framework that combines IVIFSESs and Q-ROFSs to form Q-ROIVFSESs, which are useful in DMPs with uncertain or vague criteria. The research will evaluate the advantages of Q-ROIVFSESs, develop algorithms and operations to support the implementation of Q-ROIVFSESs in decision-making problems and identify the strengths and limitations of different algorithms. The research will also identify new operations on Q-ROIVFSESs that can enhance the effectiveness of aggregation operators in dealing with MCDMPs. The overall aim of the research is to build a new decision-making framework which can be effectively applied to real world DMPs.

1.5 Research Questions

i. What is the structure and mathematical framework of Q-Rung Orthopair Interval Valued Intuitionistic Fuzzy Soft Expert Sets (Q-ROIVFSESs), which combines IVIFSESs and Q-ROFSs to create a more powerful tool for decision-making?
ii. How can novel aggregation operators be defined for Q-ROIVFSESs and integrated into Q-ROIVFSES-based algorithms to enhance the effectiveness of multiple-criteria decision-making?

1.6 Aim of the Research

The research is driven by the aim to address the limitations of existing decision-making frameworks when faced with uncertain or ambiguous criteria. To achieve this, the proposed Q-ROIVFSESs framework is designed to offer a stronger and adaptable method for making decisions in real-world scenarios and addressing DMPs.

1.7 Research Objectives

i. To investigate the structure and mathematical framework of Q-Rung Orthopair Interval Valued Intuitionistic Fuzzy Soft Expert Sets (Q-ROIVFSESs) by combining IVIFSESs defined by Qayyum [7] and Q-ROFSs introduced by Yager [6].

ii. To put forward novel operations for Q-ROIVFSESs to define aggregation operators and to develop Q-ROIVFSES-based algorithms to support our MCDMP theory.

1.8 Method of Study

i. **Develop Theoretical Framework**

Identify comprehensively and understand the existing research on fuzzy set theory, DMPs and related topics. Analyze the literature to develop a theoretical framework that includes definitions of key terms and concepts related to Q-ROIVFSESs.

ii. **Define New Algorithms and Operations**

Use the results of the analysis to develop new algorithms and operations related to Q-ROIVFSESs. Evaluate the effectiveness of these algorithms and operations in multiple-criteria decision-making scenarios.

iii. **Discuss Findings**

The analysis and evaluation of effectiveness of proposed algorithms and operations would be used to address the research questions related to the development and application of Q-ROIVFSESs in decision-making. Discuss the findings and make conclusions about the significance and practical implications of the research

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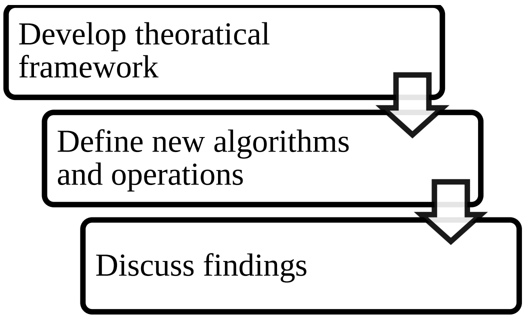

Figure 1.3: Fuzzy Sets Contribution Diagram

1.9 Significance of Research Work

i. To develop an innovative decision-making method that combines IVIFSESs and Q-ROFSs.
ii. To improve the efficiency and effectiveness of MCDMPs.
iii. This research can benefit various fields such as engineering, finance, healthcare and many others where complex decision-making is involved.
iv. The development of new algorithms and aggregation operators can provide decision-makers with better tools to address challenges in complex decision-making processes.
v. Ultimately, this research has the potential to contribute to advancements in decision-making theory and practice.

1.10 Thesis Organization

Starting from chapter 1, this chapter provides an introduction to the research and a brief overview of proposed mathematical Q-ROIVFSESs framework. The framework aims to address the limitations of existing decision-making models by combining IVIFSESs and Q-ROFSs. The chapter begins by discussing the motivation behind the research and the problem background that led to the development of the Q-ROIVFSESs framework. The research questions that the framework seeks to answer and the aim of the research are also introduced. Finally, the chapter discusses the potential applications of the Q-ROIVFSESs framework in real-world decision-making scenarios and the contributions that this research makes to the field of decision-making.

Chapter 2 named literature review explores mathematical concepts for decision-making under uncertainty. It discusses classical set theory limitations, introduces FSs and extensions like IFSs and IVIFSs. SESs and Q-ROFSs are presented. The review identifies research gaps, including integrating Q-ROFSs with IVIFSESs. Overall, it offers valuable insights for enhancing decision-making processes in diverse domains.

Overall, chapter 3 covers various fuzzy set extensions, their operations and combinations with weight vectors. These structures include Fuzzy Set (A FS is a set where each element is assigned a membership value between 0 and 1, indicating the degree of membership), Complement of FS, Null FS and Whole FS, IVFS and its complement, IFS, IFSS, IVIFS, SS, SES, IVIFSES,Q-ROFS and its operation, Q-Rung Orthopair Fuzzy Weighted Averaging Operator and Q-Rung Orthopair Fuzzy Weighted Geometric Operator.

In chapter 4, Q-Rung Orthopair Interval Valued Intuitionistic Fuzzy Soft Expert Sets (Q-ROIVFSESs) are explored by combining ideas from Qayyum [7] and Yager [6]. The focus is on defining their structure, containment, addition, multiplication, power and scalar multiplication. The chapter is divided into two sections: the first covers basic operations and the second explores properties like commutativity and inverses in Q-ROIVFSESs.

Chapter 5 builds on Q-ROIVFSESs rules, introducing and analyzing aggregation operators in three sections. It covers Q-Rung Orthopair Interval Valued Fuzzy Soft Expert Weighted Averaging and Geometric Operators, along with theorems on attributes like idempotency and boundedness. Some additional operators are discussed. The chapter concludes with decision analysis using an algorithm on real-world scenarios. Examples are discussed which compare aggregation operators effectiveness and result consistency, analyzes the flexibility of parameter Q in a practical decision-making scenario. Finally, Q-ROIVFSEFWAO is compared with IVIFSEFWAO, concluding the chapter concisely.

The concluding chapter 6 encapsulates the essence of the entire research journey, providing a detailed overview of the primary objectives, accomplishments and potential future directions. It commences with a concise introduction, underscoring the central focus of the research and the envisaged goals.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

The following chapter literature review provides a brief exploration of various mathematical concepts and methodologies related to decision-making and handling uncertainties. It covers several key areas, including classical set theory, limitations of conventional approaches, the evolution of fuzzy numbers and the emergence of fuzzy sets theory. Additionally, the review delves into interval valued fuzzy sets (IVFSs), IFSs and their combination into IVIFSs. The introduction of soft set theory and its extensions, such as fuzzy SSs and SESs, is discussed, highlighting their applications in managing uncertainty. The review introduces the concept of Q-ROFSs, a generalization of IFSs, to handle uncertainty in information. Aggregation operators (AOs), which are fundamental to knowledge-based systems, decision-making and pattern recognition, are also explored.

The review identifies research gaps and proposes potential research directions for improving decision-making models. It emphasizes the need for alternative mathematical structures that can better handle uncertainties in real-world scenarios. The integration of Q-ROFSs and IVIFSESs is proposed as a bridge to existing gaps to develop a more effective decision-making framework.

2.2 Classical set Theory

The classical set theory plays a fundamental role in Mathematics. A crisp set, also known as a classical set, is a well-defined collection of elements where each element either belongs to the set or does not. In other words, a crisp set is characterized by its sharp boundaries and every element is unambiguously either a member or not a member of the set.

2.2.1 Limitations of Classical Set Theory

The classical set theory, although valuable, is limited to conventional methods of modeling and computing. However, in numerous real-world domains such as field of economics, practical world engineering, studies related to environmental sciences, research in medical sciences and social sciences, problems involve vague and uncertain information sets. This renders traditional approaches inadequate in addressing such complexities.

2.2.2 Fields of Application

Mathematicians possess crucial critical thinking skills and problem-solving approaches, enabling them to tackle a wide array of challenges in field of economics, commerce, social science and arts. In contemporary global decision-making processes, mathematical models, simulations and interpretations are increasingly utilized, especially as various fields like business; politics and management adopt more quantitative methodologies.

2.3 Fuzzy Set Theory

2.3.1 Evolution of Fuzzy Numbers

Mathematics plays a significant role in social sciences realm, especially economics, where mathematical structures and tools are utilize to formulate and analyze models for intricate interactions within an economic system. To assess uncertainty in future performance for various problems, stochastic methods are frequently employed. While the probabilistic approach has been widely used, it may encounter difficult problems. In response to uncertainty, fuzzy numbers have emerged as an important theoretical and practical tool. They offer valuable means to handle uncertainty in various applications.

2.3.2 The word “FUZZY”

"Fuzzy" refers to a concept or property that lacks a precise or well-defined boundary, making it uncertain or vague. In the realm of fuzzy logic and fuzzy sets, the concept of "fuzziness" allows for a degree of membership between 0 and 1, representing the extent to which an element belongs to a set. This is in contrast to the crisp set, where membership is strictly binary (either 0 or 1).

2.3.3 Fuzzy Set Theory

FS Theory, first proposed by Lotfi A. Zadeh [1] in 1965, has emerged to be a powerful tool in dealing with uncertain and imprecise information. This theory provides a flexible framework for modeling and handling uncertain data, which are pervasive in various areas of practical world, including mathematics, information science, and engineering and also in decision making.

2.3.4 Interval Valued Fuzzy Sets

IVFSs by Zadeh [8] are an extension of FSs that allow for a more flexible representation of uncertainty. In a standard fuzzy set, an element can belong to the set to some degree between 0 and 1. However, in an IVFS, the membership degree of an element is represented not as a single value but as an interval. Interval valued fuzzy sets are discussed in detail by Turken [9, 10, 11]. Uncertainty in information and fuzzy sets are discussed by Klir [12].

2.3.5 Intuitionistic Fuzzy Sets

IFSs were defined by Krassimir Atanassov [13] in the year 1983 as an extension of FSs to represent uncertainty not just membership but also by non-membership degrees of members in a set. In IFS, each element has a membership degree, a non-membership degree and an indeterminacy degree, which represents the degree to which the membership and non-membership degrees are not complementary. IFSs have been applied in various areas, such as DMPs, image processing and data mining.

2.3.6 Interval valued Intuitionistic Fuzzy Sets

Atanassov [3] defined IVIFSs that combine IVFSs and IFSs.

2.3.7 Soft Sets

Molodstov [4] introduced a mathematical theory known as soft set theory in 1999, dealing with uncertainties and it gained attention due to its parameter-rich nature. SSs theory has found applications in various real life fields [14]. He explored several extensions and applications of soft set theory [15]. Fuzzy SSs, also called vague soft set were also introduced with their properties defined in [16].

2.3.8 Combination of Interval Valued Fuzzy sets and Soft Sets

Yang et al. [17] combines SSs and IVFSs and discussed them. Interval valued Fuzzy soft sets (IVFSSs) were discussed by Jiang et al. [18] along with their properties.

2.3.9 Soft Expert Sets

Alkhazaleh and Salleh [19] introduced the concept of SESs in the realm of decision-making analysis. This structure can be viewed as an extension of SSs, incorporating experts and their opinions to enhance the manageability of decision analysis.

SESs are a useful tool for gathering and synthesizing expert opinions in a structured and systematic way. One of the advantages of SESs is that they allow experts to provide their opinions for each parameter separately, which can help to identify areas of agreement and disagreement among experts.

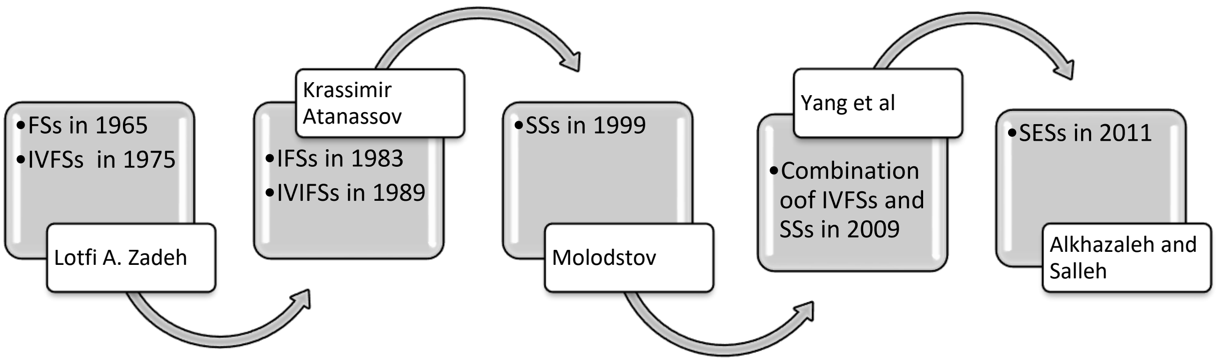

Figure 2.1: Contribution in the Field of Fuzzy Set Theory

2.3.10 Interval Valued Intuitionistic Fuzzy Soft Expert Sets

IVIFSESs were introduced by Qayyum [7] in 2017 as a generalization of IVIFSs and SESs. They allow for the representation of uncertain and vague data in the presence of multiple experts. IVIFSESs are characterized by two interval-valued functions, namely the membership and non-membership function.

2.3.11 Q-Rung Orthopair Fuzzy Sets

Q-ROFSs were introduced by Yager [6] in the year 2016 as a generalization of IFSs to handle uncertainty in information. Q-ROFS allows for the representation of both membership and non-membership degrees, as well as a degree of hesitancy in the form of a Q-number. The membership and non-membership degrees in Q-ROFS are orthogonal to each other and orthogonality between them is controlled by a parameter Q, which determines the degree of separation between the two degrees. When Q = 1, the membership and non-membership degrees are completely orthogonal to each other and Q-ROFSs reduces to IFSs.

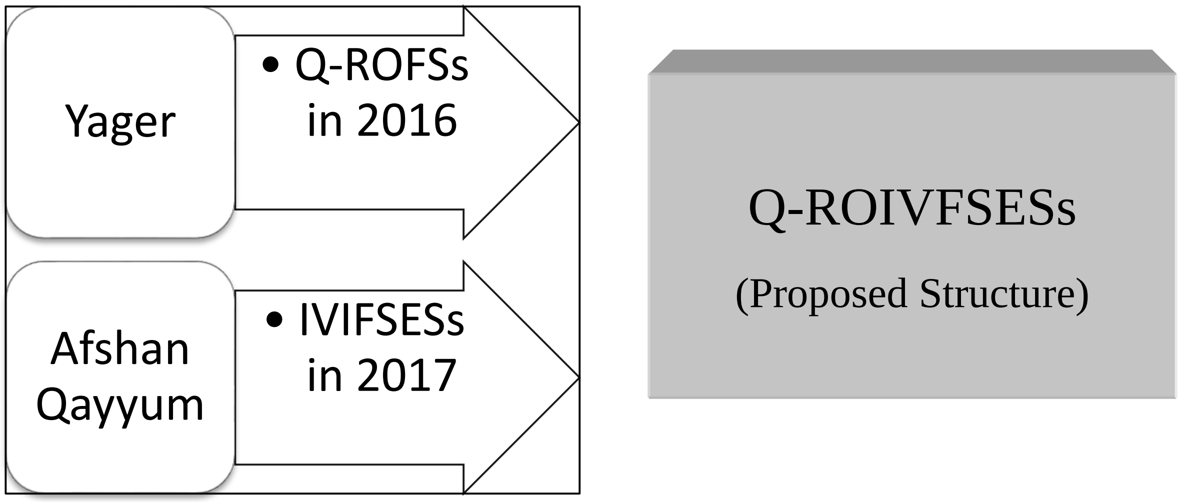

Figure 2.2: Elements of Proposed Structure

2.4 Aggregation

In various fields of human knowledge, there is a growing demand for information combination strategies. Aggregation serves as a fundamental focus for all types of information based systems, whether its image processing, pattern recognition or DMPs. The primary target of aggregation is to reach a conclusion or make a decision by synchronously utilizing diverse pieces of information from multiple sources.

2.4.1 Mutli Criteria Decision making

Among various research groups seeking solutions, we can pinpoint the several communities like data mining, sensor fusion, decision making and many others. These groups employ methodologies to achieve intelligent aggregation. Each of these groups employs unique methodologies to achieve intelligent aggregation. These methods encompass the utilization of rules, neural networks, fusion-specific techniques, probability theory, fuzzy set theory and more. It's important to highlight that all these methodologies depend on different type of aggregation operators.

2.4.2 Aggregation operators

Dombi introduced the aggregated operator in [20], while [21, 22,23] discusses fuzzy multi-criteria decision making. Xu [24] presented methods to combine IVIFSs information for decision-making purposes. He pioneered the ordered weighted geometric averaging operator and the ordered weighted averaging operator was introduced by Yager [25]. In 1988, a family of aggregation operators was suggested Yager [26] which has applications in diverse fields. Moreover, Yager [27] generalized the ordered weighted geometric averaging operator by combining it with the generalized mean operator, creating a class of operators termed the generalized ordered weighted averaging operators [28]. Decision-making problems have also been addressed using soft set theory [29, 30, 31]. P Liu, Peng Wang [32] discussed some Q-Rung Orthopair fuzzy aggregation operators which includes Q-Rung Orthopair Fuzzy weighted averaging and geometric operator. They also discussed how these operators are utilized in Multiple-Attribute Decision Making.

2.5 Research Gap and Directions

i) Existing decision-making models utilizing IVIFSESs may not fully capture the true nature of expert opinions in real-world scenarios. There is a need to explore alternative mathematical structures that can better handle uncertainties and vagueness.

ii) The integration of Q-ROFSs with IVIFSESs presents an opportunity to develop a more effective decision-making framework that combines the benefits of both approaches.

iii) Develop a novel Q-ROIVFSESs framework that integrates the concepts of IVIFSESs and Q-ROFSs. This framework should be designed to effectively handle uncertainties and vagueness in decision-making problems.

iv) Apply Q-rung orthopair interval valued fuzzy soft expert operators(Q-ROIVFSEOs) on practical applications to validate the effectiveness of the proposed Q-ROIVFSESs framework in real-world DMPs. Compare the results with existing decision-making models to demonstrate the improvements in handling uncertainties.

v) Comparative studies can be conducted to analyze the strengths and weaknesses of IVIFSESs and Q-ROIVFSESs in different decision-making scenarios. Identify the specific scenarios where one approach outperforms the other and vice versa.

vi) Contribute to the field of mathematics by developing a more robust mathematical structure for decision-making that can be applied to a wide range of real-world scenarios.

CHAPTER 3

PRELIMINARIES

3.1 Introduction

In this chapter, the focus is on an array of fuzzy set extensions, encompassing their operations and combinations with weight vectors. The first section elaborates on diverse definitions and concepts pertinent to various types of FS, including the conventional fuzzy set and its complement, IVFS, IFS, Intuitionistic Fuzzy Soft Set (IFSS) and SES.

Sections 3.2 and 3.3 are devoted to a comprehensive exploration of two specific topics. The first section focuses on IVIFSESs, wherein their properties will be thoroughly examined. The second section centers on Q-ROFSs, with a particular emphasis on exploring their respective operations. The aim is to provide an in-depth understanding of these structures and their implications in the context of fuzzy set theory.

3.2 Basic Definition

3.2.1 Fuzzy Set [1]

Consider a universe of discourse S where S = $\{\check{s}_j\colon 1 \leq j \leq n\ \}$ represents a set of elements, then A⊆ S is called FS if A = $\{\langle\check{s},\ \theta(\check{s})\rangle\colon\ \check{s}$∈ S &$\theta(\check{s})$∈ [0, 1]} where $\theta$is a function defined on S such that $\theta\colon S \rightarrow [0,1]$ such that $\theta(\check{s}) \in [0,1]$The function θ assigns a membership value to each element in the set S. $\theta(\check{s})$ lies in the range [0, 1], where 0 indicates complete non-membership and 1 indicates complete membership. Let the collection of all FSs be symbolized by I^S.

3.2.2 Join and Meet [1]

For some θand θ′∈ I^S, the meet and join of θand θ′ symbolized as ⋀ and ⋁ respectively, are defined as $(\theta \wedge \theta')(\check{s})$ = inf {θ($\check{s}$),θ′($\check{s}$)} and $(\theta \vee \theta')(\check{s})$ = sup {θ($\check{s}$),θ′($\check{s}$)}, respectively.

3.2.3 Complement of Fuzzy Set [1]

Consider a universe of discourse S where S = $\{\check{s}_j\colon 1 \leq j \leq n\}$ represents a set of elements and A⊆ S is called FS then complement of A, symbolized as A^c and is defined as $A^c = \{\langle \check{s}, 1 - \theta(\check{s})\rangle : \check{s} \in S \,\&\, \theta(\check{s}) \in [0, 1]\}$ where $\theta$is a function defined on S such that $\theta\colon S \to [0,1]$ such that $\theta(\check{s}) \in [0,1]$The function θ assigns a membership value to each element in the set S. $\theta(\check{s})$ lies in the range [0, 1], where 0 indicates complete non-membership and 1 indicates complete membership.

3.2.4 The null fuzzy subset or empty fuzzy set (denoted as $\boldsymbol{\Delta_0}$) [1]

This fuzzy subset maps every element of S onto 0. In other words, for all $\check{s}$∈ S, the membership value $\boldsymbol{\Delta}_0(\check{s})$ is equal to 0. This implies that no element belongs to the fuzzy subset $\boldsymbol{\Delta}_0$, as all elements have a membership degree of 0. Mathematically, $\boldsymbol{\Delta}_0$: S → [0,1] such that $\boldsymbol{\Delta}_0(\check{s}) = 0$, ∀$\check{s}$∈ S.

3.2.5 The whole fuzzy set of S (denoted as $\boldsymbol{\Delta_1}$) [1]

This fuzzy subset maps every element of S onto 1. In other words, for all $\check{s}$∈ S, the membership value $\boldsymbol{\Delta}_1(\check{s})$ is equal to 1. This implies that all elements of S completely belong to the fuzzy subset $\boldsymbol{\Delta}_1$, as all elements have a membership degree of 1. Mathematically, $\boldsymbol{\Delta}_1$: S → [0,1] such that $\boldsymbol{\Delta}_1(\check{s}) = 1$, ∀ $\check{s}$∈ S.

3.2.6 Interval-Valued Fuzzy Set [3]

Let S = $\{\check{s}_j : 1 \leq j \leq n\}$ be a set of elements of universe and $B \subseteq$ S is defined to be interval-valued fuzzy set denoted as

$$B = \{\langle \check{s}, [\theta^-(\check{s}), \theta^+(\check{s})]\rangle : \check{s} \in S \ \& \theta^-(\check{s}), \theta^+(\check{s}) \in [0, 1] \text{ and } \theta^-(\check{s}) \leq \theta^+(\check{s})\},$$

where $\theta^-(\check{s})$, $\theta^+(\check{s})$ represent the lower and upper degree of membership, respectively, for each element $\check{s} \in$ S. And θ : S → $\underline{I}$([0, 1]) is a mapping, where $\underline{I}$ ([0, 1]) denotes the set of all closed sub-intervals of [0, 1]. For each $\check{s} \in$ S, the membership degree of an element $\check{s}$ to B is represented as $\theta(\check{s}) = [\theta^-(\check{s}), \theta^+(\check{s})]$, where θ^-: S → [0,1] and θ^+ : S → [0,1] are fuzzy sets in S, known as the lower and upper fuzzy set in S, respectively.

3.2.7 Complement of an Interval-Valued Fuzzy Set [3]

For any IVFS B, the complement of B is denoted as B^C and is characterized as

$$B^C = \{\langle \check{s}, [1-\theta^+(\check{s}), 1-\theta^-(\check{s})]\rangle : \check{s} \in S \ \& \theta^-(\check{s}), \theta^+(\check{s}) \in [0, 1] \text{ and } \theta^-(\check{s}) \leq \theta^+(\check{s})\}.$$

This represents the IVFS where the lower membership degrees are the complements of the upper membership values in θ and vice versa.

3.2.8 Intuitionistic Fuzzy Set on Universal Set S [13]

Let S = $\{\check{s}_j : 1 \leq j \leq n\}$ be a set of elements of universeand θ_A : S → [0,1] , φ_A : S → [0,1] such that $\check{s} \mapsto \theta_A(\check{s})$, $\check{s} \mapsto \varphi_A(\check{s})$ be two mappings on S satisfying $\theta_A(\check{s}) + \varphi_A(\check{s}) \leq 1 \ \forall \ \check{s} \in$ S. An intuitionistic fuzzy set on S is delineated as A = $\{<\check{s}, \theta_A(\check{s}), \varphi_A(\check{s})>: \check{s} \in S$. $\theta_A(\check{s})$, $\varphi_A(\check{s})$ are the values of membership and non membership degree, respectively, of an element $\check{s} \in A \subseteq$ S. These degrees indicate the extent to which an element belongs or does not belong to the intuitionistic fuzzy set A.

3.2.9 Intuitionistic Fuzzy Soft Set over S [18]

Let the set of universe of discourse be S, P be a set of parameters and A ⊆ P. A pair (α, A) is referred to be an intuitionistic fuzzy soft set over S, where α: $A \rightarrow$ F(S) represents a mapping defined on A. Here F(S) represents the collection of all IFSs on S. $\alpha(\check{p})$ is an intuitionistic fuzzy subset of S for a parameter $\check{p} \in A$, denoted as

$$\alpha(\check{p}) = \{<\check{s}, \theta_{\alpha(\check{p})}(\check{s}), \varphi_{\alpha(\check{p})}(\check{s})>: \check{s} \in S\}.$$

In this definition, $\theta_{\alpha(\check{p})}(\check{s})$ and $\varphi_{\alpha(\check{p})}(\check{s})$ are the membership and non-membership degree, respectively, corresponding to the IFSS $\alpha(\check{p})$ over S.

Degenerate Case [18]

If for all $\check{s} \in$ S, $\theta_{\check{p}}(\check{s}) = 1 - \varphi_{\check{p}}(\check{s})$, then $\alpha(\check{p})$ will become standard FS. Consequently, the pair (α, A) will degenerate to a traditional FSS.

3.2.10 Interval-Valued Intuitionistic Fuzzy Set [3]

Let S be a set representing the universe. Consider two interval valued mappings: $\theta_A : S \rightarrow \underline{I}[0,1]$, $\varphi_A : S \rightarrow \underline{I}[0,1]$ such that $s \mapsto \theta_A(\check{s})$, $s \mapsto \varphi_A(\check{s})$ and $0 \le \sup(\theta_A(\check{s})) + \sup(\varphi_A(\check{s})) \le 1$. The IVIFS is defined as A = $\{<\check{s}, \theta_A(\check{s}), \varphi_A(\check{s})>: \check{s} \in S\}$, where $\theta_A(\check{s})$, $\varphi_A(\check{s})$ represents the membership and non-membership degrees in form of interval values, respectively, for an element $\check{s}$ in the IVIFS 'A'.

Hence A can be denoted as A = $\{<\check{s}, [\theta_A^-(\check{s}), \theta_A^+(\check{s})], [\varphi_A^-(\check{s}), \varphi_A^+(\check{s})]>: \check{s} \in S\}$, where $\theta_A^-(\check{s}), \theta_A^+(\check{s}), \varphi_A^-(\check{s}), \varphi_A^+(\check{s}) \in [0, 1]$ and $\theta_A^+(\check{s}) + \varphi_A^+(\check{s}) \le 1$.

3.2.11 Interval-Valued Degree of Hesitancy [18]

For an element s in the IVIFS, the intuitionistic fuzzy index or interval valued hesitancy degree, denoted as $\pi_A(\check{s})$ and is characterized as

$$\pi_A(\check{s}) = [\, 1 - \theta_A^+(\check{s}) - \varphi_A^+(\check{s}) \, , \, 1 - \theta_A^-(\check{s}) - \varphi_A^-(\check{s})].$$

3.2.12 Soft Set [4]

Soft set over a non-empty set S is characterized by a pair (j, Z), where j: Z→ $S^{\check{P}}$ is a mapping. Here, Z⊆ $\check{P}$ where $\check{P}$ is a set of parameters and $S^{\check{P}}$ represents power set of S. A parameterized family of subsets of the set S is referred to as a Soft Set. Each element $\check{p}$ in Z corresponds to a set of $\check{p}$-approximate elements within the soft set (j, Z), denoted by j($\check{p}$).

3.2.13 Relative Null Soft Set and Relative Whole Soft Set [33]

Consider S to be a universe set, the set of parameters $\check{P}$ and Z ⊆ $\check{P}$.

i) A soft set (j, Z) is said to be a relative null soft set (w.r.t. the parameter set Z), symbolized by $\emptyset_Z$, if the sets j($\check{p}$) are all empty for every $\check{p}$∈Z.
ii) A soft set (j, Z) is said to be a relative whole soft set (w.r.t the parameter set Z), symbolized by W_Z, if the sets j($\check{p}$) are all equal to the entire universe S for every $\check{p}$∈Z.

3.2.14 Soft Expert Set [19]

A soft expert set over a universe set S is denoted by the pair (j, A), where j: A → $S^{\check{P}}$ is a mapping. In this context, a SES can be seen as a SS, but with the set of parameter being substituted by Z = $\check{P} \times \tilde{E} \times$ O, where $\check{P}$ is set of parameters, $\tilde{E}$ is the set of experts, O is the set of opinions and A ⊆ Z.

3.3 IVIFSESs

In this section, certain fundamental definitions related to IVIFSESs are discussed. All definitions in this section are from [7].

Definition 3.3.1

Let S be the universe of discourse, $\check{P}$ and $\tilde{E}$ be the set of parameters and experts, respectively. A triplet (h, $\check{P}$, $\tilde{E}$) is IVIFSES which is characterized by a mapping h: $\check{P} \times \tilde{E} \rightarrow U_I(S)$, where $U_I(S)$ denotes the set of IVIFSs on the universe set S. For any attribute $\check{p}\in\check{P}$ and expert $\tilde{e}\in\tilde{E}$, h($\check{p}, \tilde{e}$) is defined as follows

$$h(\check{p}, \tilde{e}) = \{<\check{s}, [\theta^-_{(\check{p},\tilde{e})}(\check{s}),\theta^+_{(\check{p},\tilde{e})}(\check{s})], [\varphi^-_{(\check{p},\tilde{e})}(\check{s}), \varphi^+_{(\check{p},\tilde{e})}(\check{s})]>: \check{s}\in S\}.$$

Definition 3.3.2

The absolute IVIFSES over S is denoted and defined as $\Omega\ (\check{p}, \tilde{e}) = \{<\check{s},[\theta^-_{(\check{p},\tilde{e})}(\check{s}),\theta^+_{(\check{p},\tilde{e})}(\check{s})]= [0, 0], [\varphi^-_{(\check{p},\tilde{e})}(\check{s}), \varphi^+_{(\check{p},\tilde{e})}(\check{s})]= [1, 1]>:\check{s}\in S\},\forall\ \check{p}\in\check{P},\ \tilde{e}\in\tilde{E}$, and $\check{s}\in S$.

Definition 3.3.3

For an IVIFSES over S and for any $\check{p}$, $\check{p}' \in \check{P}$ and $\tilde{e}$, $\tilde{e}' \in \tilde{E}$, an element ($\check{p}$, $\tilde{e}$) is contained in ($\check{p}', \tilde{e}'$), denoted as $(\check{p}, \tilde{e}) \subseteq (\check{p}', \tilde{e}')$ if

i) $\theta^-_{(\check{p},\tilde{e})}(\check{s}) \leq \theta^-_{(\check{p}',\tilde{e}')}(\check{s}),\theta^+_{(\check{p},\tilde{e})}(\check{s}) \leq \theta^+_{(\check{p}',\tilde{e}')}(\check{s})$,

ii) $\varphi^-_{(\check{p},\tilde{e})}(\check{s}) \geq \varphi^-_{(\check{p}',\tilde{e}')}(\check{s}), \varphi^+_{(\check{p},\tilde{e})}(s) \geq \varphi^+_{(\check{p}',\tilde{e}')}(\check{s})$,

where $h(\check{p}, \tilde{e}) = \{<\check{s}, [\theta^-_{(\check{p},\tilde{e})}(\check{s}),\theta^+_{(\check{p},\tilde{e})}(\check{s})], [\varphi^-_{(\check{p},\tilde{e})}(\check{s}), \varphi^+_{(\check{p},\tilde{e})}(\check{s})]>: \check{s}\in S\}$

and $h(\check{p}', \tilde{e}') = \{<\check{s}, [\theta^-_{(\check{p}',\tilde{e}')}(\check{s}),\theta^+_{(\check{p}',\tilde{e}')}(\check{s})], [\varphi^-_{(\check{p}',\tilde{e}')}(\check{s}), \varphi^+_{(\check{p}',\tilde{e}')}(\check{s})]>: \check{s}\in S\}$.

Definition 3.3.4

For an IVIFSES over S and for any $\check{p}$, $\check{p}' \in \check{P}$and $\tilde{e}$, $\tilde{e}' \in \tilde{E}$, an element ($\check{p}$, $\tilde{e}$) is equal to ($\check{p}', \tilde{e}'$), denoted as ($\check{p}, \tilde{e}$) = ($\check{p}', \tilde{e}'$) if

i) $\theta^{-}_{(\check{p},\tilde{e})}(\check{s}) = \theta^{-}_{(\check{p}',\tilde{e}')}(\check{s}), \theta^{+}_{(\check{p},\tilde{e})}(\check{s}) = \theta^{+}_{(\check{p}',\tilde{e}')}(\check{s})$,

ii) $\varphi^{-}_{(\check{p},\tilde{e})}(\check{s}) = \varphi^{-}_{(\check{p}',\tilde{e}')}(\check{s}), \varphi^{+}_{(\check{p},\tilde{e})}(s) = \varphi^{+}_{(\check{p}',\tilde{e}')}(\check{s})$,

where $h(\check{p}, \tilde{e}) = \{<\check{s}, [\theta^{-}_{(\check{p},\tilde{e})}(\check{s}), \theta^{+}_{(\check{p},\tilde{e})}(\check{s})], [\varphi^{-}_{(\check{p},\tilde{e})}(\check{s}), \varphi^{+}_{(\check{p},\tilde{e})}(\check{s})]>: \check{s} \in S\}$

and $h(\check{p}', \tilde{e}') = \{<\check{s}, [\theta^{-}_{(\check{p}',\tilde{e}')}(\check{s}), \theta^{+}_{(\check{p}',\tilde{e}')}(\check{s})], [\varphi^{-}_{(\check{p}',\tilde{e}')}(\check{s}), \varphi^{+}_{(\check{p}',\tilde{e}')}(\check{s})]>: \check{s} \in S\}$.

Definition 3.3.5

For any two IVIFSESs (h_1, $\check{P}_1$, $\tilde{E}_1$) and (h_2, $\check{P}_2$, $\tilde{E}_2$) over S, (h_1, $\check{P}_1$, $\tilde{E}_1$) ⊆ (h_2, $\check{P}_2$, $\tilde{E}_2$) if the following axioms hold:

i) $\check{P}_1 \subseteq \check{P}_2$,

ii) $\tilde{E}_1 \subseteq \tilde{E}_2$,

iii) $h_1(\check{p}, \tilde{e}) \subseteq h_2(\check{p}, \tilde{e})$ $\forall\, \check{p} \in \check{P}_1, \tilde{e} \in \tilde{E}_1$,

where $h_1(\check{p}, \tilde{e}) = \{<\check{s}, [\theta^{-}_{1(\check{p},\tilde{e})}(\check{s}), \theta^{+}_{1(\check{p},\tilde{e})}(\check{s})], [\varphi^{-}_{1(\check{p},\tilde{e})}(\check{s}), \varphi^{+}_{1(\check{p},\tilde{e})}(\check{s})]>: \check{s} \in S\}$

and $h_2(\check{p}, \tilde{e}) = \{<\check{s}, [\theta^{-}_{2(\check{p},\tilde{e})}(\check{s}), \theta^{+}_{2(\check{p},\tilde{e})}(\check{s})], [\varphi^{-}_{2(\check{p},\tilde{e})}(\check{s}), \varphi^{+}_{2(\check{p},\tilde{e})}(\check{s})]>: \check{s} \in S\}$.

Definition 3.3.6

For any two IVIFSESs (h_1, $\check{P}_1$, $\tilde{E}_1$) and (h_2, $\check{P}_2$, $\tilde{E}_2$) over S, (h_1, $\check{P}_1$, $\tilde{E}_1$) is equal to (h_2, $\check{P}_2$, $\tilde{E}_2$) if the following axioms hold:

i) $\check{P}_1 = \check{P}_2$,

ii) $\tilde{E}_1 = \tilde{E}_2$,

iii) $h_1(\check{p}, \tilde{e}) = h_2(\check{p}, \tilde{e})$ $\forall\, \check{p} \in \check{P}_1, \tilde{e} \in \tilde{E}_1$,

where $h_1(\check{p}, \tilde{e}) = \{<\check{s}, [\theta^{-}_{1(\check{p},\tilde{e})}(\check{s}), \theta^{+}_{1(\check{p},\tilde{e})}(\check{s})], [\varphi^{-}_{1(\check{p},\tilde{e})}(\check{s}), \varphi^{+}_{1(\check{p},\tilde{e})}(\check{s})]>: \check{s} \in S\}$

and $h_2(\check{p}, \tilde{e}) = \{<\check{s}, [\theta^{-}_{2(\check{p},\tilde{e})}(\check{s}), \theta^{+}_{2(\check{p},\tilde{e})}(\check{s})], [\varphi^{-}_{2(\check{p},\tilde{e})}(\check{s}), \varphi^{+}_{2(\check{p},\tilde{e})}(\check{s})]>: \check{s} \in S\}$.

Definition 3.3.7

IVIFSES's complement is denoted by (h, $\check{P}$, $\tilde{E}$)^C and for all $\check{p}\in\check{P}$ and $\tilde{e}\in\tilde{E}$ is defined as follows:

$$(h, \check{P}, \tilde{E})^C = \{<\check{s}, [\varphi^-_{(\check{p},\tilde{e})}(\check{s}), \varphi^+_{(\check{p},\tilde{e})}(\check{s})], [\theta^-_{(\check{p},\tilde{e})}(\check{s}), \theta^+_{(\check{p},\tilde{e})}(\check{s})], >: \check{s}\in S\},$$

where h($\check{p}, \tilde{e}$) = $\{<\check{s}, [\theta^-_{(\check{p},\tilde{e})}(\check{s}), \theta^+_{(\check{p},\tilde{e})}(\check{s})], [\varphi^-_{(\check{p},\tilde{e})}(\check{s}), \varphi^+_{(\check{p},\tilde{e})}(\check{s})]>: \check{s}\in S\}$.

Definition 3.3.8

The union of any two IVIFSESs (h′, $\check{P}'$, $\tilde{E}'$) and (h″, $\check{P}''$, $\tilde{E}''$) over S is denoted as (h, $\check{P}$, $\tilde{E}$) = (h′, $\check{P}'$, $\tilde{E}'$) ∪ (h″, $\check{P}''$, $\tilde{E}''$) where $\check{P} = \check{P}' \cup \check{P}''$ and $\tilde{E} = \tilde{E}' \cup \tilde{E}''$ and for all $\check{p}\in\check{P}$ and $\tilde{e}\in\tilde{E}$, union is defined as

$$h(\check{p}, \tilde{e}) = \begin{cases} h'(\check{p}, \tilde{e}) & if\ (\check{p}, \tilde{e}) \in \left(\check{P}' \times \tilde{E}'\right)\backslash(\check{P}'' \times \tilde{E}'') \\ h''(\check{p}, \tilde{e}) & if\ (\check{p}, \tilde{e}) \in \left(\check{P}'' \times E''\right)\backslash(\check{P}' \times \tilde{E}') \\ < \check{s}, [\theta'^-_{(\check{p},\tilde{e})}(\check{s}) \vee \theta''^-_{(\check{p},\tilde{e})}(\check{s}), \theta'^+_{(\check{p},\tilde{e})}(\check{s}) \vee \theta''^+_{(\check{p},\tilde{e})}(\check{s})], & \\ [\varphi'^-_{(\check{p},\tilde{e})}(\check{s}) \wedge \varphi''^-_{(\check{p},\tilde{e})}(\check{s}), \varphi'^+_{(\check{p},\tilde{e})}(\check{s}) \wedge \varphi''^+_{(\check{p},\tilde{e})}(\check{s})] >, & \\ \text{if } (\check{p}, \tilde{e}) \in (\check{P}' \cap \check{P}'' \times \tilde{E}' \cap \tilde{E}'') & \end{cases}$$

where $h'(\check{p}, \tilde{e}) = \{<\check{s}, [\theta'^-_{(\check{p},\tilde{e})}(\check{s}), \theta'^+_{(\check{p},\tilde{e})}(\check{s})], [\varphi'^-_{(\check{p},\tilde{e})}(\check{s}), \varphi'^+_{(\check{p},\tilde{e})}(\check{s})]>: \check{s}\in S\}$

and $h''(\check{p}, \tilde{e}) = \{<\check{s}, [\theta''^-_{(\check{p},\tilde{e})}(\check{s}), \theta''^+_{(\check{p},\tilde{e})}(\check{s})], [\varphi''^-_{(\check{p},\tilde{e})}(\check{s}), \varphi''^+_{(\check{p},\tilde{e})}(\check{s})]>: \check{s}\in S\}$.

Definition 3.3.9

The intersection of any two IVIFSESs (h′, $\check{P}'$, $\tilde{E}'$) and (h″, $\check{P}''$, $\tilde{E}''$) over S is denoted as (h, $\check{P}$, $\tilde{E}$) = (h′, $\check{P}'$, $\tilde{E}'$) ∩ (h″, $\check{P}''$, $\tilde{E}''$) where $\check{P} = \check{P}' \cup \check{P}''$ and $\tilde{E} = \tilde{E}' \cup \tilde{E}''$ and for all $\check{p}\in\check{P}$ and $\tilde{e}\in\tilde{E}$, intersection is defined as

$$h(\check{p},\tilde{e}) = \begin{cases} h'(\check{p},\tilde{e}) & if\ \ (\check{p},\tilde{e}) \in (\check{P}' \times \tilde{E}')\setminus(\check{P}'' \times \tilde{E}'') \\ h''(\check{p},\tilde{e}) & if\ \ (\check{p},\tilde{e}) \in (\check{P}'' \times E'')\setminus(\check{P}' \times \tilde{E}') \\ <\check{s}, [\theta^{-}_{'(\check{p},\tilde{e})}(\check{s}) \wedge \theta^{-}_{''(\check{p},\tilde{e})}(\check{s}), \theta^{+}_{'(\check{p},\tilde{e})}(\check{s}) \wedge \theta^{+}_{''(\check{p},\tilde{e})}(\check{s})], \\ [\varphi^{-}_{'(\check{p},\tilde{e})}(\check{s}) \vee \varphi^{-}_{''(\check{p},\tilde{e})}(\check{s}), \varphi^{+}_{'(\check{p},\tilde{e})}(\check{s}) \vee \varphi^{+}_{''(\check{p},\tilde{e})}(\check{s})]>, \\ \text{if } (\check{p},\tilde{e}) \in (\check{P}' \cap \check{P}'' \times \tilde{E}' \cap \tilde{E}'') \end{cases}$$

where $h'(\check{p},\tilde{e}) = \{<\check{s}, [\theta^{-}_{'(\check{p},\tilde{e})}(\check{s}), \theta^{+}_{'(\check{p},\tilde{e})}(\check{s})], [\varphi^{-}_{'(\check{p},\tilde{e})}(\check{s}), \varphi^{+}_{'(\check{p},\tilde{e})}(\check{s})]>: \check{s} \in S\}$

and $h''(\check{p},\tilde{e}) = \{<\check{s}, [\theta^{-}_{''(\check{p},\tilde{e})}(\check{s}), \theta^{+}_{''(\check{p},\tilde{e})}(\check{s})], [\varphi^{-}_{''(\check{p},\tilde{e})}(\check{s}), \varphi^{+}_{''(\check{p},\tilde{e})}(\check{s})]>: \check{s} \in S\}$.

Definition 3.3.10

The sum of any two IVIFSESs (h′, $\check{P}'$, $\tilde{E}'$) and (h″, $\check{P}''$, $\tilde{E}''$) over S is denoted as (h′, $\check{P}'$, $\tilde{E}'$)+(h″, $\check{P}''$, $\tilde{E}''$) and is defined as

$$h'(\check{p},\tilde{e}) + h''(\check{p},\tilde{e}) = \{<\check{s}, [\theta^{-}_{'(\check{p},\tilde{e})}(\check{s}) + \theta^{-}_{''(\check{p},\tilde{e})}(\check{s}) - \theta^{-}_{'(\check{p},\tilde{e})}(\check{s})\theta^{-}_{''(\check{p},\tilde{e})}(\check{s}), \theta^{+}_{'(\check{p},\tilde{e})}(\check{s}) + \theta^{+}_{''(\check{p},\tilde{e})}(\check{s}) - \theta^{+}_{'(\check{p},\tilde{e})}(\check{s})\theta^{+}_{''(\check{p},\tilde{e})}(\check{s})], [\varphi^{-}_{'(\check{p},\tilde{e})}(\check{s})\varphi^{-}_{''(\check{p},\tilde{e})}(\check{s}), \varphi^{+}_{'(\check{p},\tilde{e})}(\check{s})\varphi^{+}_{''(\check{p},\tilde{e})}(\check{s})]>\},$$

where $h'(\check{p},\tilde{e}) = \{<\check{s}, [\theta^{-}_{'(\check{p},\tilde{e})}(\check{s}), \theta^{+}_{'(\check{p},\tilde{e})}(\check{s})], [\varphi^{-}_{'(\check{p},\tilde{e})}(\check{s}), \varphi^{+}_{'(\check{p},\tilde{e})}(\check{s})]>: \check{s} \in S\}$

and $h''(\check{p},\tilde{e}) = \{<\check{s}, [\theta^{-}_{''(\check{p},\tilde{e})}(\check{s}), \theta^{+}_{''(\check{p},\tilde{e})}(\check{s})], [\varphi^{-}_{''(\check{p},\tilde{e})}(\check{s}), \varphi^{+}_{''(\check{p},\tilde{e})}(\check{s})]>: \check{s} \in S\}$.

Definition 3.3.11

The product of IVIFSES (h,$\check{P}$, $\tilde{E}$) with any positive real number **r**> 0 is symbolized and defined as follows

$$\mathbf{r}h(\check{p},\tilde{e}) = \{<\check{s}, [1-(1-\theta^{-}_{(\check{p},\tilde{e})}(\check{s}))^{\mathbf{r}}, 1-(1-\theta^{+}_{(\check{p},\tilde{e})}(\check{s}))^{\mathbf{r}}], [(\varphi^{-}_{(\check{p},\tilde{e})}(\check{s}))^{\mathbf{r}}, (\varphi^{+}_{(\check{p},\tilde{e})}(\check{s}))^{\mathbf{r}}]>: \check{s} \in S\}.$$

Definition 3.3.12

The power of IVIFSES (h, $\breve{P}$, $\tilde{E}$) with any positive real number r > 0 is symbolized and defined as follows

$$(h(\breve{p},\tilde{e}))^{r} = \{<\breve{s}, [(\theta^{-}_{(\breve{p},\tilde{e})}(\breve{s}))^{r}, (\theta^{+}_{(\breve{p},\tilde{e})}(\breve{s}))^{r}, [1-(1-\varphi^{-}_{(\breve{p},\tilde{e})}(\breve{s}))^{r}, 1-(1-\varphi^{+}_{(\breve{p},\tilde{e})}(\breve{s}))^{r}] >: \breve{s} \in S\}.$$

Definition 3.3.13

The score § and accuracy Â function for an IVIFSES is defined as follows

$$§(h(\breve{p},\tilde{e})) = \frac{\theta^{-}_{(\breve{p},\tilde{e})}(\breve{s})+\theta^{+}_{(\breve{p},\tilde{e})}(\breve{s})-\varphi^{-}_{(\breve{p},\tilde{e})}(\breve{s})-\varphi^{+}_{(\breve{p},\tilde{e})}(\breve{s})}{2}$$

and

$$\hat{A}(h(\breve{p},\tilde{e})) = \frac{\theta^{-}_{(\breve{p},\tilde{e})}(\breve{s})+\theta^{+}_{(\breve{p},\tilde{e})}(\breve{s})+\varphi^{-}_{(\breve{p},\tilde{e})}(\breve{s})+\varphi^{+}_{(\breve{p},\tilde{e})}(\breve{s})}{2}$$

where $h(\breve{p},\tilde{e}) = \{<\breve{s}, [\theta^{-}_{(\breve{p},\tilde{e})}(\breve{s}),\theta^{+}_{(\breve{p},\tilde{e})}(\breve{s})], [\varphi^{-}_{(\breve{p},\tilde{e})}(\breve{s}),\varphi^{+}_{(\breve{p},\tilde{e})}(\breve{s})]>: \breve{s} \in S\}$, $§(h(\breve{p},\tilde{e})) \in [-1, 1]$ and $\hat{A}(h(\breve{p},\tilde{e})) \in [0, 1]$.

IVIFSESs can be ranked using score and accuracy function. The greater IVIFSES's element will have a larger accuracy and a larger score.

3.4 Q-ROFSs

Some basic definitions and concepts related to Q-ROFSs are discussed in the following section.

Definition 3.4.1

Let S be a finite universe of discussion, a Q-ROFS stated by Yager is $A = \{<\check{s}, \theta_A(\check{s}), \varphi_A(\check{s})>: \check{s} \in S\}$ where $\theta_A: S \rightarrow [0, 1]$ and $\varphi_A: S \rightarrow [0, 1]$ denotes the membership and non-membership degree of the element $\check{s} \in S$ to the set A, respectively, under the condition that $0 \leq (\theta_A(\check{s}))^Q + (\varphi_A(\check{s}))^Q \leq 1$, $(Q \geq 1)$. The hesitancy degree is given by $\pi_A(\check{s}) = ((\theta_A(\check{s}))^Q + (\varphi_A(\check{s}))^Q - (\theta_A(\check{s}))^Q(\varphi_A(\check{s}))^Q)^{1/Q}$. A Q-rung Orthopair Fuzzy number (Q-ROFN) $<\theta_A(\check{s}), \varphi_A(\check{s})>$ for convenience may be denoted as $A = <\theta_A, \varphi_A>$. [6]

3.4.2 Some Basic Operations of Q-ROFNs

Let $\alpha = <\theta, \varphi>$, $\alpha_1 = <\theta_1, \varphi_1>$, $\alpha_2 = <\theta_2, \varphi_2>$ be three q-ROFNs and $r \in \mathbb{R}$, then defined below are some basic operations of Q- ROFNs:

i) $\bar{\alpha} = <\varphi, \theta>$,

ii) $\alpha_1 \vee \alpha_2 = <max\{\theta_1, \theta_2\}, \min\{\varphi_1, \varphi_2\}>$,

iii) $\alpha_1 \wedge \alpha_2 = <min\{\theta_1, \theta_2\}, \max\{\varphi_1, \varphi_2\}>$,

iv) $\alpha_1 + \alpha_2 = <(\theta_1^Q + \theta_2^Q - \theta_1^Q\theta_2^Q)^{1/Q}, \varphi_1\varphi_2>$,

v) $\alpha_1 \times \alpha_2 = <\theta_1\theta_2, (\varphi_1^Q + \varphi_2^Q - \varphi_1^Q\varphi_2^Q)^{1/Q}>$,

vi) $r\alpha = <(1-(1-\theta^Q)^r, \varphi^r>$,

vii) $\alpha^r = <\theta^r, (1-(1-\varphi^Q)^r>$. [6]

Definition 3.4.3

The score § and accuracy Â function for a Q-ROFN $\alpha=<\theta, \varphi>$ is given by

$$\S(\alpha) = \theta^Q - \varphi^Q \text{ and } \hat{A}(\alpha) = \theta^Q + \varphi^Q, \text{ respectively. [6]}$$

Theorem 3.4.4

Let $\alpha =< \theta, \varphi>$, $\alpha_1 =< \theta_1, \varphi_1>$, $\alpha_2 =< \theta_2, \varphi_2>$ be three q-ROFNs and $r \in \mathbb{R}$,

i) If $\S(\alpha_1) > \S(\alpha_2)$, then $\alpha_1 > \alpha_2$,

ii) If $\S(\alpha_1) = \S(\alpha_2)$, then

1. If $\hat{A}(\alpha_1) > \hat{A}(\alpha_2)$, then $\alpha_1 > \alpha_2$,
2. If $\hat{A}(\alpha_1) = \hat{A}(\alpha_2)$, then $\alpha_1 = \alpha_2$. [6]

Definition 3.4.5

Consider a collection $\alpha_j = <\theta_j, \varphi_j>$, $(1 \le j \le n)$ of Q-ROFNs and $\delta: F^n \to F$, where

$$\delta(\alpha_1, \alpha_2, \dots, \alpha_n) = w_1\alpha_1 + w_2\alpha_2 + \dots + w_n\alpha_n = \langle \left(1 - \prod_{j=1}^{n}(1-\theta_j^Q)^{w_j}\right)^{1/Q}, \prod_{j=1}^{n}\varphi_j^{w_j} \rangle,$$

where F is set of all Q-ROFNs and a weight vector $w = (w_1, w_2, \dots, w_n)^T$ of $(\alpha_1, \alpha_2, \dots, \alpha_n)$ such that $0 \le w_k \le 1$ and $\sum_{j=1}^{n} w_k = 1$. Then, the δ is termed as Q-Rung orthopair fuzzy weighted averaging operator (Q-ROFWAO). [32]

In case where Q=1, Q-ROFWAO simplifies to Intuitionistic fuzzy weighted averaging operator (IFWAO).

Definition 3.4.6

Consider a collection $\alpha_j = <\theta_j, \varphi_j>$, $(1 \le j \le n)$ of Q-ROFNs and $\gamma: F^n \to F$, where

$$\gamma(\alpha_1, \alpha_2, \dots, \alpha_n) = \alpha_1^{w_1} \times \alpha_2^{w_2} \times \dots \times \alpha_n^{w_n} = \langle \prod_{j=1}^{n}\theta_j^{w_j}, \left(1 - \prod_{j=1}^{n}(1-\varphi_j^Q)^{w_j}\right)^{1/Q} \rangle,$$

where F is set of all Q-ROFNs and a weight vector w = $(w_1, w_2, \ldots, w_n)^T$ of $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ such that $0 \leq w_k \leq 1$ *and* $\sum_{j=1}^{n} w_k = 1$.Then, the γ is called Q-Rung orthopair fuzzy weighted geometric operator (Q-ROFWGO). [32]

In case where Q=1, Q-ROFWGO simplifies to Intuitionistic fuzzy weighted geometric operator (IFWGO).

CHAPTER 4

Q-RUNG ORTHOPAIR INTERVAL VALUED FUZZY SOFT EXPERT SETS

4.1 Introduction

In this engaging chapter, the exploration unfolds as the intricate domain of Q-Rung Orthopair Interval Valued Intuitionistic Fuzzy Soft Expert Sets (Q-ROIVFSESs) takes center stage. This endeavor involves the fusion of two distinct mathematical concepts—namely, Interval Valued Intuitionistic Fuzzy Soft Expert Sets (IVIFSESs) introduced by Qayyum [7] and the innovative Q-ROFSs by Yager [6].

Within these mathematical landscapes, fundamental definitions are meticulously formulated and novel operations are introduced. The chapter unfolds into two primary sections. The initial segment illuminates the structural aspects, containment principles and various operations such as addition, multiplication, power and scalar multiplication. The subsequent section delves into the properties surrounding the addition and multiplication of Q-ROIVFSESs, exploring concepts such as commutativity, inverses, identity and other intriguing facets that enrich the understanding of Q-ROIVFSESs.

4.2 Basic Definitions

Definition 4.2.1

Let S be a finite universe set of discussion and $\check{P}$, $\tilde{E}$ be the parameters and experts set respectively. Q-Rung Orthopair Interval Valued Fuzzy Soft expert set (Q-ROIVFSES) is quadruplet (h, $\check{P}$, $\tilde{E}$,Q) which is characterized by a mapping h: $\check{P} \times \tilde{E} \rightarrow Q_I(S)$ where $Q_I(S)$ is the set of all interval-valued Q- rung orthopair fuzzy set and $1 \leq Q \in \mathbb{N}$.

For some $\check{p} \in \check{P}$, $\tilde{e} \in \tilde{E}$, h($\check{p}$, $\tilde{e}$) is articulated as

$$h(\check{p}, \tilde{e}) = \{<\check{s}, [\theta^-_{(\check{p},\tilde{e})}(\check{s}), \theta^+_{(\check{p},\tilde{e})}(\check{s})], [\varphi^-_{(\check{p},\tilde{e})}(\check{s}), \varphi^+_{(\check{p},\tilde{e})}(\check{s})]>: \check{s} \in S\},$$

such that $(\theta^+_{(\check{p},\tilde{e})}(\check{s}))^Q + (\varphi^+_{(\check{p},\tilde{e})}(\check{s}))^Q \leq 1$, for some Q≥ 1and Q ∈ ℕ. The collection of all Q-ROIVFSES over a set of universe S be denoted by $\tilde{Q}_I(S)$. The hesitancy degree or non-determinacy index for each $\check{s}$∈S in the Q-ROIVFSES is given by

$$\pi_{h(\check{p},\tilde{e})}(\check{s}) = [\pi^-_{h(\check{p},\tilde{e})}(\check{s}), \quad \pi^+_{h(\check{p},\tilde{e})}(\check{s})]$$

$$= [(1 - (\theta^+_{(\check{p},\tilde{e})}(\check{s}))^Q - (\varphi^+_{(\check{p},\tilde{e})}(\check{s}))^Q)^{1/Q}, (1 - (\theta^-_{(\check{p},\tilde{e})}(\check{s}))^Q - (\varphi^-_{(\check{p},\tilde{e})}(\check{s}))^Q)^{1/Q}].$$

Theorem 4.2.2

If h($\check{p}$,$\tilde{e}$) shows Q-ROIVFSES over S and further Q′>Q, then h($\check{p}$,$\tilde{e}$) is also Q′-ROIVFSES over S.

Proof

Since, $h(\check{p}, \tilde{e}) = \{<\check{s}, [\theta^-_{(\check{p},\tilde{e})}(\check{s}), \theta^+_{(\check{p},\tilde{e})}(\check{s})], [\varphi^-_{(\check{p},\tilde{e})}(\check{s}), \varphi^+_{(\check{p},\tilde{e})}(\check{s})]>: \check{s} \in S\}$ is Q-ROIVFSES, then $(\theta^+_{(\check{p},\tilde{e})}(\check{s}))^Q + (\varphi^+_{(\check{p},\tilde{e})}(\check{s}))^Q \leq 1$, for some Q≥ 1.

Therefore, for $Q' >$Q, we have

$$(\theta^+_{(\check{p},\tilde{e})}(\check{s}))^{Q'} + (\varphi^+_{(\check{p},\tilde{e})}(\check{s}))^{Q'} \leq (\theta^+_{(\check{p},\tilde{e})}(\check{s}))^Q + (\varphi^+_{(\check{p},\tilde{e})}(\check{s}))^Q \leq 1.$$

Thus following statements can be concluded from above theorem:

i) Every IVIFSES shows Q-ROIVFSES for all Q≥ 1.

ii) If h($\breve{p}$, $\tilde{e}$) shows Q-ROIVFSES over S and further Q′<Q, then h($\breve{p}$,$\tilde{e}$) is not necessarily Q′-ROIVFSES over S.

Example 4.2.3

Consider three countries in the universe set S = {$\breve{s}_1$, $\breve{s}_2$, $\breve{s}_3$} and the set of parameter $\breve{P}$ = {$\breve{p}_1$=economic stability, $\breve{p}_2$=GDP, $\breve{p}_3$=literacy rate} and the set of experts $\tilde{E}$ = {$\tilde{e}_1$, $\tilde{e}_2$} and Q ≥ 2. Then Q-ROIVFSES (h, $\breve{P}$, $\tilde{E}$, Q) consist of expert opinions about these countries subjected to the given parameters as follows

h($\breve{p}_1$, $\tilde{e}_1$) = { <$\breve{s}_1$,[.2,.4],[.3,.7]>, <$\breve{s}_2$,[.4,.5],[.4,.7]>, <$\breve{s}_3$,[.3,.5],[.3,.4]>},

h($\breve{p}_2$, $\tilde{e}_1$) = { <$\breve{s}_1$,[.5,.8],[.3,.4]>, <$\breve{s}_2$,[.6,.8],[.3,.5]>, <$\breve{s}_3$,[.4,.7],[.3,.5]>},

h($\breve{p}_3$, $\tilde{e}_1$) = { <$\breve{s}_1$,[.7,.8],[.2,.4]>, <$\breve{s}_2$,[.8,.9],[.2,.3]>, <$\breve{s}_3$,[.4,.7],[.3,.4]>},

h($\breve{p}_1$, $\tilde{e}_2$) = { <$\breve{s}_1$,[.3,.4],[.4,.7]>, <$\breve{s}_2$,[.5,.7],[.4,.5]>, <$\breve{s}_3$,[.5,.6],[.4,.5]>},

h($\breve{p}_2$, $\tilde{e}_2$) = { <$\breve{s}_1$,[.4,.6],[.5,.7]>, <$\breve{s}_2$,[.5,.6],[.4,.7]>, <$\breve{s}_3$,[.4,.7],[.5,.6]>},

h($\breve{p}_3$, $\tilde{e}_2$) = { <$\breve{s}_1$,[.7,.8],[.3,.5]>, <$\breve{s}_2$,[.85,.95],[.1,.3]>, <$\breve{s}_3$,[.4,.6],[.4,.5]>}.

Definition 4.2.4

The absolute Q-ROIVFSES over S is symbolized as $\tilde{A}$($\breve{p}$, $\tilde{e}$) and is characterized as $\tilde{A}$($\breve{p}$, $\tilde{e}$) = {<$\breve{s}$, [1, 1], [0, 0]>: $\breve{s}$∈S} for all $\breve{p}$ ∈ $\breve{P}$ and $\tilde{e}$ ∈ $\tilde{E}$. That is $[\theta^-_{(\breve{p},\tilde{e})}(\breve{s}),\theta^+_{(\breve{p},\tilde{e})}(\breve{s})]$ = [1,1] and $[\varphi^-_{(\breve{p},\tilde{e})}(\breve{s}),\varphi^+_{(\breve{p},\tilde{e})}(\breve{s})]$> = [0, 0], ∀ $\breve{p}$ ∈ $\breve{P}$, $\tilde{e}$ ∈ $\tilde{E}$, $\breve{s}$∈S and Q≥ 1.

Example 4.2.5

Let S = {set of humans on earth}, $\breve{P}$= {$\breve{p}_1$=vertebrates, $\breve{p}_2$=mammals} and $\tilde{E}$ = { $\tilde{e}_1$, $\tilde{e}_2$}. Then we have absolute Q-ROIVFSES over S, denoted by

$\tilde{A}(\check{p}, \tilde{e})$ = {<$\check{s}$, [$\theta^{-}_{(\check{p},\tilde{e})}(\check{s})$,$\theta^{+}_{(\check{p},\tilde{e})}(\check{s})$] = [1, 1], [$\varphi^{-}_{(\check{p},\tilde{e})}(\check{s}), \varphi^{+}_{(\check{p},\tilde{e})}(\check{s})$]> = [0, 0]>: $\check{s}$∈S} for all $\check{p} \in \check{P}$, $\tilde{e} \in \tilde{E}$ and Q≥ 1.

Definition 4.2.6

The null Q-ROIVFSES over S is denoted and defined as $\Phi(\check{p}, \tilde{e})$ = {<$\check{s}$, [0, 0], [1, 1]>: $\check{s}$∈S} for all $\check{p} \in \check{P}$ and $\tilde{e} \in \tilde{E}$. That is [$\theta^{-}_{(\check{p},\tilde{e})}(\check{s})$,$\theta^{+}_{(\check{p},\tilde{e})}(\check{s})$] = [0, 0] and [$\varphi^{-}_{(\check{p},\tilde{e})}(\check{s}), \varphi^{+}_{(\check{p},\tilde{e})}(\check{s})$] = [1, 1], ∀ $\check{p} \in \check{P}$,$\tilde{e} \in \tilde{E}$, $\check{s}$∈S and Q≥ 1.

Definition 4.2.7

Consider two Q-ROIVFSESs (h, $\check{P}$, $\tilde{E}$, Q) and (h′, $\check{P}$, $\tilde{E}$, Q) over S, then for some $\check{p} \in \check{P}$, $\tilde{e} \in \tilde{E}$ and Q≥ 1, the join (⋁) and meet (⋀) for h($\check{p}$, $\tilde{e}$) and h′($\check{p}$, $\tilde{e}$) are defined as follows

h($\check{p}$, $\tilde{e}$) ⋁ h′($\check{p}$, $\tilde{e}$) = {<$\check{s}$, [$max(\theta^{-}_{(\check{p},\tilde{e})}(\check{s}),\theta^{-}_{\prime(\check{p},\tilde{e})}(\check{s})),max(\theta^{+}_{(\check{p},\tilde{e})}(\check{s}),\theta^{+}_{\prime(\check{p},\tilde{e})}(\check{s}))$], [$min(\varphi^{-}_{(\check{p},\tilde{e})}(\check{s}), \varphi^{-}_{\prime(\check{p},\tilde{e})}(\check{s})), min(\varphi^{+}_{(\check{p},\tilde{e})}(\check{s}), \varphi^{+}_{\prime(\check{p},\tilde{e})}(\check{s}))$]>: $\check{s}$∈S},

h($\check{p}$, $\tilde{e}$) ⋀ h′($\check{p}$, $\tilde{e}$) = {<$\check{s}$, [$min(\theta^{-}_{(\check{p},\tilde{e})}(\check{s}),\theta^{-}_{\prime(\check{p},\tilde{e})}(\check{s})),min(\theta^{+}_{(\check{p},\tilde{e})}(\check{s}),\theta^{+}_{\prime(\check{p},\tilde{e})}(\check{s}))$], [$max(\varphi^{-}_{(\check{p},\tilde{e})}(\check{s}), \varphi^{-}_{\prime(\check{p},\tilde{e})}(\check{s})), max(\varphi^{+}_{(\check{p},\tilde{e})}(\check{s}), \varphi^{+}_{\prime(\check{p},\tilde{e})}(\check{s}))$]>: $\check{s}$∈S} , respectively.

where h($\check{p}$, $\tilde{e}$) = {<$\check{s}$, [$\theta^{-}_{(\check{p},\tilde{e})}(\check{s})$,$\theta^{+}_{(\check{p},\tilde{e})}(\check{s})$], [$\varphi^{-}_{(\check{p},\tilde{e})}(\check{s}), \varphi^{+}_{(\check{p},\tilde{e})}(\check{s})$]>: $\check{s}$∈S} and h′($\check{p}$, $\tilde{e}$) = {<$\check{s}$, [$\theta^{-}_{\prime(\check{p},\tilde{e})}(\check{s})$,$\theta^{+}_{\prime(\check{p},\tilde{e})}(\check{s})$], [$\varphi^{-}_{\prime(\check{p},\tilde{e})}(\check{s}), \varphi^{+}_{\prime(\check{p},\tilde{e})}(\check{s})$]>: $\check{s}$∈S} such that $(\theta^{+}_{(\check{p},\tilde{e})}(\check{s}))^{Q} + (\varphi^{+}_{(\check{p},\tilde{e})}(\check{s}))^{Q} \leq 1$ and $(\theta^{+}_{\prime(\check{p},\tilde{e})}(\check{s}))^{Q} + (\varphi^{+}_{\prime(\check{p},\tilde{e})}(\check{s}))^{Q} \leq 1$, for some Q≥ 1and Q ∈ ℕ.

Definition 4.2.8

For a Q-ROIVFSES (h, $\check{P}$, $\tilde{E}$, Q) over a universe S, then for any $\check{p}_1$, $\check{p}_2$ ∈ $\check{P}$, $\tilde{e}_1$, $\tilde{e}_2$ ∈ $\tilde{E}$ and Q≥ 1, for h(p_1, e_1) and h(p_2, e_2), we define following relations:

i) h($\check{p}_1$, $\tilde{e}_1$) $\subset_{\theta^-}$ h($\check{p}_2$, $\tilde{e}_2$) iff $\theta^-_{(\check{p}_1,\tilde{e}_1)}(\check{s}) \leq \theta^-_{(\check{p}_2,\tilde{e}_2)}(\check{s})$, ∀$\check{s}$∈S,

ii) h($\check{p}_1$, $\tilde{e}_1$) $\subset_{\theta^+}$ h($\check{p}_2$, $\tilde{e}_2$) iff $\theta^+_{(\check{p}_1,\tilde{e}_1)}(\check{s}) \leq \theta^+_{(\check{p}_2,\tilde{e}_2)}(\check{s})$, ∀$\check{s}$∈S,

iii) h($\check{p}_1$, $\tilde{e}_1$) $\subset_{\varphi^-}$ h($\check{p}_2$, $\tilde{e}_2$) iff $\varphi^-_{(\check{p}_1,\tilde{e}_1)}(\check{s}) \geq \varphi^-_{(\check{p}_2,\tilde{e}_2)}(\check{s})$, ∀$\check{s}$∈S,

iv) h($\check{p}_1$, $\tilde{e}_1$) $\subset_{\varphi^+}$ h($\check{p}_2$, $\tilde{e}_2$) iff $\varphi^+_{(\check{p}_1,\tilde{e}_1)}(\check{s}) \geq \varphi^+_{(\check{p}_2,\tilde{e}_2)}(\check{s})$, , ∀$\check{s}$∈S.

v) h($\check{p}_1$, $\tilde{e}_1$) $\subset_{\theta}$ h($\check{p}_2$, $\tilde{e}_2$) iff h($\check{p}_1$, $\tilde{e}_1$) $\subset_{\theta^-}$ h($\check{p}_2$, $\tilde{e}_2$) and h($\check{p}_1$, $\tilde{e}_1$) $\subset_{\theta^+}$ h($\check{p}_2$, $\tilde{e}_2$),

vi) h($\check{p}_1$, $\tilde{e}_1$) $\subset_{\varphi}$ h($\check{p}_2$, $\tilde{e}_2$) iff h($\check{p}_1$, $\tilde{e}_1$) $\subset_{\varphi^-}$ h($\check{p}_2$, $\tilde{e}_2$) and h($\check{p}_1$, $\tilde{e}_1$) $\subset_{\varphi^+}$ h($\check{p}_2$, $\tilde{e}_2$).

vii) h($\check{p}_1$, $\tilde{e}_1$) ⊂ h($\check{p}_2$, $\tilde{e}_2$) iff h($\check{p}_1$, $\tilde{e}_1$) $\subset_{\theta}$ h($\check{p}_2$, $\tilde{e}_2$) and h($\check{p}_1$, $\tilde{e}_1$) $\subset_{\varphi}$ h($\check{p}_2$, $\tilde{e}_2$).

where h($\check{p}_1$, $\tilde{e}_1$) = {<$\check{s}$, [$\theta^-_{(\check{p}_1,\tilde{e}_1)}(\check{s}), \theta^+_{(\check{p}_1,\tilde{e}_1)}(\check{s})$], [$\varphi^-_{(\check{p}_1,\tilde{e}_1)}(\check{s}), \varphi^+_{(\check{p}_1,\tilde{e}_1)}(\check{s})$]>:$\check{s}$∈S }

and h($\check{p}_2$, $\tilde{e}_2$) = {<$\check{s}$, [$\theta^-_{(\check{p}_2,\tilde{e}_2)}(\check{s}), \theta^+_{(\check{p}_2,\tilde{e}_2)}(\check{s})$], [$\varphi^-_{(\check{p}_2,\tilde{e}_2)}(\check{s}), \varphi^+_{(\check{p}_2,\tilde{e}_2)}(\check{s})$]>:$\check{s}$∈S }.

Definition 4.2.9

For a Q-ROIVFSES (h, $\check{P}$, $\tilde{E}$, Q) over a universe S, then for any $\check{p}_1$, $\check{p}_2$ ∈ $\check{P}$, $\tilde{e}_1$, $\tilde{e}_2$ ∈ $\tilde{E}$ and Q≥ 1, an element h($\check{p}_1$, $\tilde{e}_1$) is said to be contained in h($\check{p}_2$, $\tilde{e}_2$), symbolized by h($\check{p}_1$, $\tilde{e}_1$) ⊆ h($\check{p}_2$, $\tilde{e}_2$) provided that the following two conditions hold true for all s ∈ S and for some Q≥ 1,

i) $\theta^-_{(\check{p}_1,\tilde{e}_1)}(\check{s}) \leq \theta^-_{(\check{p}_2,\tilde{e}_2)}(\check{s})$ and $\theta^+_{(\check{p}_1,\tilde{e}_1)}(\check{s}) \leq \theta^+_{(\check{p}_2,\tilde{e}_2)}(\check{s})$,

ii) $\varphi^-_{(\check{p}_1,\tilde{e}_1)}(\check{s}) \geq \varphi^-_{(\check{p}_2,\tilde{e}_2)}(\check{s})$ and $\varphi^+_{(\check{p}_1,\tilde{e}_1)}(\check{s}) \geq \varphi^+_{(\check{p}_2,\tilde{e}_2)}(\check{s})$.

Example 4.2.10

Consider h($\check{p}_1$, $\tilde{e}_1$) = {<$\check{s}_1$,[0.5,0.6],[0.2,0.4]>, <$\check{s}_2$,[0.7,0.8],[0.3,0.5]>} and h($\check{p}_2$, $\tilde{e}_2$) = {<$\check{s}_1$,[0.6,0.8],[0.1,0.3]>, <$\check{s}_2$,[0.8,0.9],[0.2,0.4]>}, for Q=2. Clearly h($\check{p}_1$, $\tilde{e}_1$) ⊆ h($\check{p}_2$, $\tilde{e}_2$).

Definition 4.2.11

For a Q-ROIVFSES (h, $\check{P}$, $\tilde{E}$, Q) over a universe S, then for any $\check{p}_1$, $\check{p}_2 \in \check{P}$, $\tilde{e}_1$, $\tilde{e}_2$ ∈ E and Q≥ 1, an element h($\check{p}_1$, $\tilde{e}_1$) is said to be equal to h($\check{p}_2$, $\tilde{e}_2$), symbolized by h($\check{p}_1$, $\tilde{e}_1$)= h($\check{p}_2$, $\tilde{e}_2$) if h($\check{p}_1$, $\tilde{e}_1$) ⊆ h($\check{p}_2$, $\tilde{e}_2$) and h($\check{p}_2$, $\tilde{e}_2$) ⊆ h($\check{p}_1$, $\tilde{e}_1$)or if for all $\check{s}$∈S,

i) $\theta^{-}_{(\check{p}_1,\tilde{e}_1)}(\check{s}) = \theta^{-}_{(\check{p}_2,\tilde{e}_2)}(\check{s})$ and $\theta^{+}_{(\check{p}_1,\tilde{e}_1)}(\check{s}) = \theta^{+}_{(\check{p}_2,\tilde{e}_2)}(\check{s})$,
ii) $\varphi^{-}_{(\check{p}_1,\tilde{e}_1)}(\check{s}) = \varphi^{-}_{(\check{p}_2,\tilde{e}_2)}(\check{s})$ and $\varphi^{+}_{(\check{p}_1,\tilde{e}_1)}(\check{s}) = \varphi^{+}_{(\check{p}_2,\tilde{e}_2)}(\check{s})$.

Example 4.2.12

Consider h($\check{p}_1$, $\tilde{e}_1$) = {<$\check{s}_1$,[0.5,0.8],[0.3,0.5]>, <$\check{s}_2$,[0.4,0.5],[0.2,0.3]>} and h($\check{p}_2$, $\tilde{e}_2$) = {<$\check{s}_1$,[0.5,0.8],[0.3,0.5]>, <$\check{s}_2$,[0.4,0.5],[0.2,0.3]>}. Here h($\check{p}_1$, $\tilde{e}_1$)= h($\check{p}_2$, $\tilde{e}_2$).

Definition 4.2.13

For two Q-ROIVFSESs (h, $\check{P}$, $\tilde{E}$, Q) and (h′, $\check{P}'$, $\tilde{E}'$, Q′) over S, (h, $\check{P}$, $\tilde{E}$, Q) is a subset of (h′, $\check{P}'$, $\tilde{E}'$, Q′), symbolized as (h, $\check{P}$, $\tilde{E}$, Q) ⊆ (h′, $\check{P}'$, $\tilde{E}'$, Q′) provided that the following two conditions hold true for all $\check{s}$∈S:

i) $\check{P} \subseteq \check{P}'$,
ii) $\tilde{E} \subseteq \tilde{E}'$,
iii) h($\check{p}$, $\tilde{e}$) ⊆ h′($\check{p}$, $\tilde{e}$) ∀$\check{s}$∈S, $\check{p} \in \check{P}$ and $\tilde{e} \in \tilde{E}$,
iv) min Q ≥ min Q',

where h($\check{p}$, $\tilde{e}$) = {<$\check{s}$, [$\theta^{-}_{(\check{p},\tilde{e})}(\check{s})$,$\theta^{+}_{(\check{p},\tilde{e})}(\check{s})$], [$\varphi^{-}_{(\check{p},\tilde{e})}(\check{s}), \varphi^{+}_{(\check{p},\tilde{e})}(\check{s})$]>: $\check{s}$∈S} and h′($\check{p}$, $\tilde{e}$) = {<$\check{s}$, [$\theta'^{-}_{(\check{p},\tilde{e})}(\check{s})$,$\theta'^{+}_{(\check{p},\tilde{e})}(\check{s})$], [$\varphi'^{-}_{(\check{p},\tilde{e})}(\check{s}), \varphi'^{+}_{(\check{p},\tilde{e})}(\check{s})$]>: $\check{s}$∈S} such that $(\theta^{+}_{(\check{p},\tilde{e})}(\check{s}))^{Q} + (\varphi^{+}_{(\check{p},\tilde{e})}(\check{s}))^{Q} \leq 1$ and $(\theta'^{+}_{(\check{p},\tilde{e})}(\check{s}))^{Q'} + (\varphi'^{+}_{(\check{p},\tilde{e})}(\check{s}))^{Q'} \leq 1$, for some Q, $Q' \geq$ 1and Q ∈ ℕ.

Example 4.2.14

Let S = {$\check{s}_1$, $\check{s}_2$}, $\check{P}$= {$\check{p}_1$, $\check{p}_2$} and $\tilde{E}$ = {$\tilde{e}_1$} be universe set, set of parameters and set of expert and Q-ROIVFSES (h, $\check{P}$, $\tilde{E}$, Q) over S, with Q ≥ 2 is given as

h($\check{p}_1$, $\tilde{e}_1$) = {<$\check{s}_1$,[0.6,0.7],[0.3,0.4]>, <$\check{s}_2$,[0.7,0.8],[0.4,0.5]>},
h($\check{p}_2$, $\tilde{e}_1$) = {<$\check{s}_1$,[0.5,0.7],[0.4,0.5]>, <$\check{s}_2$,[0.6,0.7],[0.5,0.6]>}.

Also for P′ = {$\check{p}_1$, $\check{p}_2$, $\check{p}_3$} and E′ = {$\tilde{e}_1$, $\tilde{e}_2$}, the Q-ROIVFSES (h′, $\check{P}'$, $\tilde{E}'$, Q′) over S, with Q′ ≥ 2 is

h′($\check{p}_1$, $\tilde{e}_1$) = {<$\check{s}_1$,[0.7,0.8],[0.2,0.3]>, <$\check{s}_2$,[0.8,0.9],[0.3,0.4]>},
h′($\check{p}_2$, $\tilde{e}_1$) = {<$\check{s}_1$,[0.6,0.8],[0.3,0.4]>, <$\check{s}_2$,[0.7,0.9],[0.3,0.4]>},
h′($\check{p}_3$, $\tilde{e}_1$) = {<$\check{s}_1$,[0.5,0.6],[0.2,0.4]>, <$\check{s}_2$,[0.7,0.8],[0.3,0.5]>},
h′($\check{p}_1$, $\tilde{e}_2$) = {<$\check{s}_1$,[0.6,0.8],[0.1,0.3]>, <$\check{s}_2$,[0.8,0.9],[0.2,0.4]>},
h′($\check{p}_2$, $\tilde{e}_2$) = {<$\check{s}_1$,[0.2,0.4],[0.5,0.7]>, <$\check{s}_2$,[0.6,0.8],[0.3,0.4]>},
h′($\check{p}_3$, $\tilde{e}_2$) = {<$\check{s}_1$,[0.5,0.6],[0.4,0.5]>, <$\check{s}_2$,[0.5,0.7],[0.3,0.4]>}.

Here it is clear that $\check{P} \subseteq \check{P}'$, $\tilde{E} \subseteq \tilde{E}'$, min Q ≥ min Q' and h($\check{p}$, $\tilde{e}$) ⊆ h′($\check{p}$, $\tilde{e}$) ∀ $\check{s}$ ∈ S, $\check{p} \in \check{P}$and $\tilde{e} \in \tilde{E}$. Hence (h, $\check{P}$, $\tilde{E}$, Q) ⊆ (h′, $\check{P}'$, $\tilde{E}'$, Q′).

Definition 4.2.15

For two Q-ROIVFSESs (h, $\check{P}$, $\tilde{E}$, Q) and (h′, $\check{P}'$, $\tilde{E}'$, Q′) over S, (h, $\check{P}$, $\tilde{E}$, Q) is equal to (h′, $\check{P}'$, $\tilde{E}'$, Q′), symbolized as (h, $\check{P}$, $\tilde{E}$, Q) = (h′, $\check{P}'$, $\tilde{E}'$, Q′) provided that the following two conditions hold true for all $\check{s}$∈S:

i) $\check{P} = \check{P}'$,
ii) $\tilde{E} = \tilde{E}'$,
iii) h($\check{p}$, $\tilde{e}$) = h′($\check{p}$, $\tilde{e}$) ∀ $\check{s}$∈S, $\check{p} \in \check{P}$ and $\tilde{e} \in \tilde{E}$,
iv) min Q = min Q'.

Definition 4.2.16

The complement of Q-ROIVFSES (h, $\check{P}$, $\tilde{E}$, Q) is symbolized by (h, $\check{P}$, $\tilde{E}$, Q)^C and for all $\check{p} \in \check{P}$ and $\tilde{e} \in \tilde{E}$ is articulated as follows

h$(\check{p}^C, \tilde{e})$ = {<$\check{s}$, $[\varphi^-_{(\check{p},\tilde{e})}(\check{s}), \varphi^+_{(\check{p},\tilde{e})}(\check{s})]$, $[\theta^-_{(\check{p},\tilde{e})}(\check{s}), \theta^+_{(\check{p},\tilde{e})}(\check{s})]$, >: $\check{s}$∈S},

where h$(\check{p}, \tilde{e})$ ={<$\check{s}$, $[\theta^-_{(\check{p},\tilde{e})}(\check{s}), \theta^+_{(\check{p},\tilde{e})}(\check{s})]$, $[\varphi^-_{(\check{p},\tilde{e})}(\check{s}), \varphi^+_{(\check{p},\tilde{e})}(\check{s})]$>: $\check{s}$∈S}.

Example 4.2.17

Let S = {$\check{s}_1$, $\check{s}_2$}, P = {$\check{p}_1$, $\check{p}_2$} and E = {$\tilde{e}_1$} be universe set, set of parameters and set of expert and Q-ROIVFSES (h, $\check{P}$, $\tilde{E}$, Q) over S, with Q ≥ 2 is given by

h($\check{p}_1$, $\tilde{e}_1$) = {<$\check{s}_1$,[0.5,0.6],[0.2,0.4]>, <$\check{s}_2$,[0.7,0.8],[0.3,0.5]>},
h($\check{p}_2$, $\tilde{e}_1$) = {<$\check{s}_1$,[0.6,0.8],[0.1,0.3]>, <$\check{s}_2$,[0.8,0.9],[0.2,0.4]>}.

Its complement which is symbolized by (h, P, E, Q)^C is

h($\check{p}_1^C$, $\tilde{e}_1$) = {<$\check{s}_1$,[0.2,0.4],[0.5,0.6]>, <$\check{s}_2$,[0.3,0.5],[0.7,0.8]>},
h($\check{p}_2^C$, $\tilde{e}_1$) = {<$\check{s}_1$,[0.1,0.3],[0.6,0.8]>, <$\check{s}_2$,[0.2,0.4],[0.8,0.9]>}.

Definition 4.2.18

For two Q-ROIVFSESs (h′, $\check{P}'$, $\tilde{E}'$, Q′) with Q′ ≥ 1 and (h″, $\check{P}''$, $\tilde{E}''$, Q″) with Q″≥ 1 over S. The union is symbolized as (h, $\check{P}$, $\tilde{E}$, Q) = (h′, $\check{P}'$, $\tilde{E}'$, Q′) ∪ (h″, $\check{P}''$, $\tilde{E}''$, Q″) with Q ≥ sup{min Q′, min Q″}, where $\check{P} = \check{P}' \cup \check{P}''$ *and* $\tilde{E} = \tilde{E}' \cup \tilde{E}''$ and for all $\check{p} \in \check{P}$ and $\tilde{e} \in \tilde{E}$, union is defined as follows:

$$h(\check{p},\tilde{e}) = \begin{cases} h'(\check{p},\tilde{e}) & if\ (\check{p},\tilde{e}) \in (\check{P}' \times \tilde{E}')\backslash(\check{P}" \times \tilde{E}") \\ h"(\check{p},\tilde{e}) & if\ (\check{p},\tilde{e}) \in (\check{P}" \times \tilde{E}")\backslash(\check{P}' \times \tilde{E}') \\ \{< \check{s}, [sup(\theta^-_{'(\check{p},\tilde{e})}(\check{s}), \theta^-_{"(\check{p},\tilde{e})}(\check{s})), \sup(\theta^+_{'(\check{p},\tilde{e})}(\check{s}), \theta^+_{"(\check{p},\tilde{e})}(\check{s}))], \\ [\inf(\varphi^-_{'(\check{p},\tilde{e})}(\check{s}), \varphi^-_{"(\check{p},\tilde{e})}(\check{s})), \inf(\varphi^+_{'(\check{p},\tilde{e})}(\check{s}), \varphi^+_{"(\check{p},\tilde{e})}(\check{s}))] >\} \\ \text{if } (\check{p},\tilde{e}) \in (\check{P}' \cap \check{P}" \times \tilde{E}' \cap \tilde{E}") \end{cases}$$

where $h'(\check{p}, \tilde{e}) = \{<\check{s}, [\theta^-_{'(\check{p},\tilde{e})}(\check{s}), \theta^+_{'(\check{p},\tilde{e})}(\check{s})], [\varphi^-_{'(\check{p},\tilde{e})}(\check{s}), \varphi^+_{'(\check{p},\tilde{e})}(\check{s})]>: \check{s}\in S\}$

and $h"(\check{p}, \tilde{e}) = \{<\check{s}, [\theta^-_{"(\check{p},\tilde{e})}(\check{s}), \theta^+_{"(\check{p},\tilde{e})}(\check{s})], [\varphi^-_{"(\check{p},\tilde{e})}(\check{s}), \varphi^+_{"(\check{p},\tilde{e})}(\check{s})]>: \check{s}\in S\}$.

Example 4.2.19

Let S = $\{\check{s}_1, \check{s}_2\}$, $\check{P}' = \{\check{p}_1, \check{p}_2\}$ and $\tilde{E}' = \{\tilde{e}_1\}$ be universe set, set of parameters and set of expert and Q-ROIVFSES (h′, $\check{P}'$, $\tilde{E}'$, Q′) over S, with Q′ ≥ 2 is given as

$h'(\check{p}_1, \tilde{e}_1) = \{<\check{s}_1,[0.6,0.7],[0.3,0.4]>, <\check{s}_2,[0.7,0.8],[0.4,0.5]>\}$,
$h'(\check{p}_2, \tilde{e}_1) = \{<\check{s}_1,[0.5,0.7],[0.4,0.5]>, <\check{s}_2,[0.6,0.7],[0.5,0.6]>\}$.

Also for $\check{P}''$ = {p1, p2, p3} and $\tilde{E}''$ = {e1, e2}, the Q-ROIVFSES (h″, $\check{P}''$, $\tilde{E}''$, Q″) over S, with Q″≥ 2 is

$h''(\check{p}_1, \tilde{e}_1) = \{<\check{s}_1,[0.7,0.8],[0.2,0.3]>, <\check{s}_2,[0.8,0.9],[0.3,0.4]>\}$,
$h''(\check{p}_2, \tilde{e}_1) = \{<\check{s}_1,[0.6,0.8],[0.3,0.4]>, <\check{s}_2,[0.7,0.9],[0.3,0.4]>\}$,
$h''(\check{p}_3, \tilde{e}_1) = \{<\check{s}_1,[0.5,0.6],[0.2,0.4]>, <\check{s}_2,[0.7,0.8],[0.3,0.5]>\}$,
$h''(\check{p}_1, \tilde{e}_2) = \{<\check{s}_1,[0.6,0.8],[0.1,0.3]>, <\check{s}_2,[0.8,0.9],[0.2,0.4]>\}$,
$h''(\check{p}_2, \tilde{e}_2) = \{<\check{s}_1,[0.2,0.4],[0.5,0.7]>, <\check{s}_2,[0.6,0.8],[0.3,0.4]>\}$,
$h''(\check{p}_3, \tilde{e}_2) = \{<\check{s}_1,[0.5,0.6],[0.4,0.5]>, <\check{s}_2,[0.5,0.7],[0.3,0.4]>\}$.

Then their union (h, $\check{P}$, $\tilde{E}$, Q) = (h′, P′, E′, Q′) ∪ (h″, P″, E″, Q″) with Q ≥ 2 is given as

$h(\check{p}_1, \tilde{e}_1) = \{<\check{s}_1,[0.7,0.8],[0.2,0.3]>, <\check{s}_2,[0.8,0.9],[0.3,0.4]>\}$,
$h(\check{p}_2, \tilde{e}_1) = \{<\check{s}_1,[0.6,0.8],[0.3,0.4]>, <\check{s}_2,[0.7,0.9],[0.3,0.4]>\}$,
$h(\check{p}_3, \tilde{e}_1) = \{<\check{s}_1,[0.5,0.6],[0.2,0.4]>, <\check{s}_2,[0.7,0.8],[0.3,0.5]>\}$,
$h(\check{p}_1, \tilde{e}_2) = \{<\check{s}_1,[0.6,0.8],[0.1,0.3]>, <\check{s}_2,[0.8,0.9],[0.2,0.4]>\}$,

h($\check{p}_2$, $\tilde{e}_2$) = {<$\check{s}_1$,[0.2,0.4],[0.5,0.7]>, <$\check{s}_2$,[0.6,0.8],[0.3,0.4]>},

h($\check{p}_3$, $\tilde{e}_2$) = {<$\check{s}_1$,[0.5,0.6],[0.4,0.5]>, <$\check{s}_2$,[0.5,0.7],[0.3,0.4]>}.

Clearly it obeys that if (h′, $\check{P}'$, $\tilde{E}'$, Q′) ⊆ (h″, $\check{P}''$, $\tilde{E}''$, Q″) then (h, $\check{P}$, $\tilde{E}$, Q) = (h′,$\check{P}'$,$\tilde{E}'$,Q′) ∪ (h″, $\check{P}''$, $\tilde{E}''$, Q″) = (h″, $\check{P}''$, $\tilde{E}''$, Q″).

Example 4.2.20

Let S = {$\check{s}_1$, $\check{s}_2$}, P′ = {$\check{p}_1$, $\check{p}_2$} and E′ = {$\tilde{e}_1$, $\tilde{e}_2$} be universe set, set of parameters and set of expert and Q-ROIVFSES (h′, $\check{P}'$, $\tilde{E}'$, Q′) over S, with Q′ ≥ 2 is given by

h′($\check{p}_1$, $\tilde{e}_1$) = {<$\check{s}_1$,[0.2,0.5],[0.4,0.6]>, <$\check{s}_2$,[0.3,0.5],[0.6,0.7]>},

h′($\check{p}_2$, $\tilde{e}_1$) = {<$\check{s}_1$,[0.6,0.9],[0.5,0.6]>, <$\check{s}_2$,[0.3,0.6],[0.4,0.5]>},

h′($\check{p}_1$, $\tilde{e}_2$) = {<$\check{s}_1$,[0.4,0.8],[0.3,0.4]>, <$\check{s}_2$,[0.5,0.7],[0.2,0.5]>},

h′($\check{p}_2$, $\tilde{e}_2$) = {<$\check{s}_1$,[0.7,0.9],[0.1,0.3]>, <$\check{s}_2$,[0.6,0.8],[0.0,0.3]>}.

Also if P″ = {$\check{p}_1$, $\check{p}_2$, $\check{p}_3$} and E″ = {$\tilde{e}_1$, $\tilde{e}_2$}, then Q-ROIVFSES (h″, P″, E″, Q″) over S, with Q″≥ 2 is

h″($\check{p}_1$, $\tilde{e}_1$) = {<$\check{s}_1$,[0.8,0.9],[0.2,0.3]>, <$\check{s}_2$,[0.5,0.7],[0.3,0.4]>},

h″($\check{p}_2$, $\tilde{e}_1$) = {<$\check{s}_1$,[0.6,0.8],[0.3,0.4]>, <$\check{s}_2$,[0.2,0.5],[0.4,0.6]>},

h″($\check{p}_3$, $\tilde{e}_1$) = {<$\check{s}_1$,[0.3,0.7],[0.3,0.5]>, <$\check{s}_2$,[0.3,0.4],[0.7,0.9]>},

h″($\check{p}_1$, $\tilde{e}_2$) = {<$\check{s}_1$,[0.7,0.9],[0.2,0.5]>, <$\check{s}_2$,[0.6,0.7],[0.1,0.3]>},

h″($\check{p}_2$, $\tilde{e}_2$) = {<$\check{s}_1$,[0.7,0.9],[0.2,0.4]>, <$\check{s}_2$,[0.3,0.7],[0.4,0.6]>},

h″($\check{p}_3$, $\tilde{e}_{21}$) = {<$\check{s}_1$,[0.5,0.8],[0.2,0.4]>, <$\check{s}_2$,[0.4,0.6],[0.6,0.7]>}.

Now, (h, $\check{P}$, $\tilde{E}$, Q) = (h′, P′, E′, Q′) ∪ (h″, P″, E″, Q″) with Q ≥ 2 is given by:

h($\check{p}_1$, $\tilde{e}_1$) = {<$\check{s}_1$,[0.8,0.9],[0.2,0.3]>, <$\check{s}_2$,[0.5,0.7],[0.3,0.4]>},

h($\check{p}_2$, $\tilde{e}_1$) = {<$\check{s}_1$,[0.6,0.9],[0.3,0.4]>, <$\check{s}_2$,[0.3,0.6],[0.4,0.5]>},

h($\check{p}_3$, $\tilde{e}_1$) = {<$\check{s}_1$,[0.3,0.7],[0.3,0.5]>, <$\check{s}_2$,[0.3,0.4],[0.7,0.9]>},

h($\check{p}_1$, $\tilde{e}_2$) = {<$\check{s}_1$,[0.7,0.9],[0.2,0.4]>, <$\check{s}_2$,[0.6,0.7],[0.1,0.3]>}, h($\check{p}_2, \tilde{e}_2$) = {<$\check{s}_1$,[0.7,0.9],[0.1,0.3]>, <$\check{s}_2$,[0.6,0.8],[0.0,0.3]>},

h($\check{p}_3, \tilde{e}_2$) = {<$\check{s}_1$,[0.5,0.8],[0.2,0.4]>, <$\check{s}_2$,[0.4,0.6],[0.6,0.7]>}.

Remark 4.2.21

The condition Q ≥sup{min Q′, min Q″} can be examined by an example.

Consider h′($\check{p}_1, \tilde{e}_1$) = {<$\check{s}_1$,[0.5,0.9],[0.5,0.6]>} with Q′≥ 3 and h″($\check{p}_1, \tilde{e}_1$) = {<$\check{s}_1$,[0.6,0.7],[0.4,0.6]>} with Q″≥ 2.

Then their union h($\check{p}_1, \tilde{e}_1$) = {<$\check{s}_1$,[0.6,0.9],[0.4,0.6]>} has Q≥ 3 as $0.9^3 + 0.6^3 \le 1$, whereas if Q≥ 2 then $0.9^2 + 0.6^2 \ge 1$.
And Q ≥ 3= sup{min Q′, min Q″} = sup {3, 2}.

Definition 4.2.22

For two Q-ROIVFSESs (h′, $\check{P}'$, $\tilde{E}'$, Q′) with Q′ ≥ 1 and (h″, $\check{P}''$, $\tilde{E}''$, Q″) with Q″≥ 1 over S, their intersection is symbolized as (h, $\check{P}$, $\tilde{E}$, Q) = (h′, $\check{P}'$, $\tilde{E}'$, Q′) ∩ (h″, $\check{P}''$, $\tilde{E}''$, Q″) with Q ≥ sup{min Q′, min Q″}, where $\check{P} = \check{P}' \cup \check{P}''$ and $\tilde{E} = \tilde{E}' \cup \tilde{E}''$ and for all $\check{p} \in \check{P}$ and $\tilde{e} \in \tilde{E}$, intersection is defined as below

$$h(\check{p}, \tilde{e}) = \begin{cases} h'(\check{p}, \tilde{e}) & if \;\; (\check{p}, \tilde{e}) \in (\check{P}' \times \tilde{E}')\backslash(\check{P}'' \times \tilde{E}'') \\ h''(\check{p}, \tilde{e}) & if \;\; (\check{p}, \tilde{e}) \in (\check{P}'' \times \tilde{E}'')\backslash(\check{P}' \times \tilde{E}') \\ \{< \check{s}, [\inf(\theta^-_{'(\check{p},\tilde{e})}(\check{s}), \theta^-_{''(\check{p},\tilde{e})}(\check{s})), \inf(\theta^+_{'(\check{p},\tilde{e})}(\check{s}), \theta^+_{''(\check{p},\tilde{e})}(\check{s}))], & \\ [\sup(\varphi^-_{'(\check{p},\tilde{e})}(\check{s}), \varphi^-_{''(\check{p},\tilde{e})}(\check{s})), \sup(\varphi^+_{'(\check{p},\tilde{e})}(\check{s}), \varphi^+_{''(\check{p},\tilde{e})}(\check{s}))] >\} & \\ \quad \text{if } (\check{p}, \tilde{e}) \in (\check{P}' \cap \check{P}'' \times \tilde{E}' \cap \tilde{E}'') & \end{cases}$$

where $h'(\check{p}, \tilde{e})$ = {<$\check{s}$, [$\theta^-_{'(\check{p},\tilde{e})}(\check{s})$,$\theta^+_{'(\check{p},\tilde{e})}(\check{s})$], [$\varphi^-_{'(\check{p},\tilde{e})}(\check{s})$, $\varphi^+_{'(\check{p},\tilde{e})}(\check{s})$]>: $\check{s}$∈S}

and $h''(\check{p}, \tilde{e})$ = {<$\check{s}$, [$\theta^-_{''(\check{p},\tilde{e})}(\check{s})$,$\theta^+_{''(\check{p},\tilde{e})}(\check{s})$], [$\varphi^-_{''(\check{p},\tilde{e})}(\check{s})$, $\varphi^+_{''(\check{p},\tilde{e})}(\check{s})$]>: $\check{s}$∈S}

Example 4.2.23

Consider two Q-ROIVFSESs (h′, $\check{P}'$, $\tilde{E}'$, Q′) and (h″, $\check{P}''$, $\tilde{E}''$, Q″) as in example 4.2.20. Then (h, $\check{P}$, $\tilde{E}$, Q) = (h′, $\check{P}'$, $\tilde{E}'$, Q′) ∩ (h″, $\check{P}''$, $\tilde{E}''$, Q″) with Q ≥ 2 is given as

h($\check{p}_1$, $\tilde{e}_1$) = {<$\check{s}_1$,[0.2,0.5],[0.4,0.6]>, <$\check{s}_2$,[0.3,0.5],[0.6,0.7]>},
h($\check{p}_2$, $\tilde{e}_1$) = {<$\check{s}_1$,[0.6,0.8],[0.5,0.6]>, <$\check{s}_2$,[0.2,0.5],[0.4,0.6]>},
h($\check{p}_3$, $\tilde{e}_1$) = {<$\check{s}_1$,[0.3,0.7],[0.3,0.5]>, <$\check{s}_2$,[0.3,0.4],[0.7,0.9]>},
h($\check{p}_1$, $\tilde{e}_2$) = {<$\check{s}_1$,[0.4,0.8],[0.3,0.5]>, <$\check{s}_2$,[0.5,0.7],[0.2,0.5]>},
h($\check{p}_2$, $\tilde{e}_2$) = {<$\check{s}_1$,[0.7,0.9],[0.2,0.4]>, <$\check{s}_2$,[0.3,0.7],[0.4,0.6]>},
h($\check{p}_3$, $\tilde{e}_2$) = {<$\check{s}_1$,[0.5,0.8],[0.2,0.4]>, <$\check{s}_2$,[0.4,0.6],[0.6,0.7]>}.

Definition 4.2.24

Let C_Q be any collection of Q-ROIVFSESs over a universe S, i.e., C_Q = {$(h_j, \check{P}_j, \tilde{E}_j, Q_j)$: $(h_j, \check{P}_j, \tilde{E}_j, Q_j) \in \tilde{Q}_I(S) \wedge$ j ∈ J} where J is an index set, $Q_j \geq 1$ and $\tilde{Q}_I(S)$ is the set of all Q-ROIVFSES over universe S. Then union of any arbitrary number of Q-ROIVFSESs is symbolized by $\bigcup_{j\in J} C_Q$ or $\bigcup_{j\in J}(h_j, \check{P}_j, \tilde{E}_j, Q_j)$ or $(h_{\cup j}, \check{P}_{\cup j}, \tilde{E}_{\cup j}, Q_{\cup j})$ where $\check{P}_{\cup j} = \bigcup_{j\in J} \check{P}_j$, $\tilde{E}_{\cup j} = \bigcup_{j\in J} \tilde{E}_j$, $Q_{\cup j} = \sup\{minQ_j, \forall \text{ j}\}$ and for all $\check{p} \in \check{P}_{\cup j}$ and $\tilde{e} \in \tilde{E}_{\cup j}$, union is characterized as

$$h_{\cup j}(\check{p}, \tilde{e}) = \begin{cases} h_j(\check{p}, \tilde{e}) \quad\quad if \ \ (\check{p}, \tilde{e}) \in (\check{P}_j \times \tilde{E}_j)\backslash(\check{P}_k \times \tilde{E}_k) \ such\ that\ j \neq k, \forall \text{ k} \in \text{J} \\ \\ \{< \check{s}, [\sup(\theta^-_{j(\check{p},\tilde{e})}(\check{s}), \theta^-_{k(\check{p},\tilde{e})}(\check{s})), \sup(\theta^+_{j(\check{p},\tilde{e})}(\check{s}), \theta^+_{k(\check{p},\tilde{e})}(\check{s}))], \\ [\inf(\varphi^-_{j(\check{p},\tilde{e})}(\check{s}), \varphi^-_{k(\check{p},\tilde{e})}(\check{s})), \inf(\varphi^+_{j(\check{p},\tilde{e})}(\check{s}), \varphi^+_{k(\check{p},\tilde{e})}(\check{s}))] >\} \\ if \ \ (\check{p}, \tilde{e}) \in (\check{P}_j \times \tilde{E}_j) \cap (\check{P}_k \times \tilde{E}_k) \ \ for\ some\ j, k \ \in \text{ J} \\ \\ \{< \check{s}, [\sup(\theta^-_{k(\check{p},\tilde{e})}(\check{s})), \sup(\theta^+_{k(\check{p},\tilde{e})}(\check{s}))], \\ [\inf(\varphi^-_{k(\check{p},\tilde{e})}(\check{s})), \inf(\varphi^+_{k(\check{p},\tilde{e})}(\check{s}))] >\} \\ \text{if}(\check{p}, \tilde{e}) \in \ \bigcap_{k\in J}(\check{P}_k \times \tilde{E}_k)\ for\ any\ arbitrary\ k \end{cases}$$

Where $h_k(\check{p}, \tilde{e}) = \{< \check{s}, [\theta^-_{k(\check{p},\tilde{e})}(\check{s}), \theta^+_{k(\check{p},\tilde{e})}(\check{s})], [\varphi^-_{k(\check{p},\tilde{e})}(\check{s}), \varphi^+_{k(\check{p},\tilde{e})}(\check{s})] >: \check{s} \in \text{S}\}$

Definition 4.2.25

Let C_Q be any collection of Q-ROIVFSESs over a universe S, i.e., $C_Q = \{(h_j, \check{P}_j, \tilde{E}_j, Q_j)$: $(h_j, \check{P}_j, \tilde{E}_j, Q_j) \in \tilde{Q}_I(S) \wedge j \in J\}$ where J is an index set, $Q_j \geq 1$ and $\tilde{Q}_I(S)$ is the set of all Q-ROIVFSES over universe S. Then intersection of any arbitrary number of Q-ROIVFSESs is symbolized by $\bigcap_{j\in J} C_Q$ or $\bigcap_{j\in J}(h_j, \check{P}_j, \tilde{E}_j, Q_j)$ or $(h_{\cap j}, \check{P}_{\cap j}, \tilde{E}_{\cap j}, Q_{\cap j})$, where $\check{P}_{\cap j} = \bigcup_{j\in J} \check{P}_j$, $\tilde{E}_{\cap j} = \bigcup_{j\in J} \tilde{E}_j$, $Q_{\cap j} = \sup\{minQ_j, \forall j\}$ and for all $\check{p} \in \check{P}_{\cap j}$ and $\tilde{e} \in \tilde{E}_{\cap j}$, intersection is characterized as

$$h_{\cap j}(\check{p}, \tilde{e}) = \begin{cases} h_j(\check{p}, \tilde{e}) & if\ (\check{p}, \tilde{e}) \in (\check{P}_j \times \tilde{E}_j)\backslash(\check{P}_k \times \tilde{E}_k)\ such\ that\ j \neq k, \forall\ k \in J \\ \{<\check{s}, [\inf(\theta^-_{j(\check{p},\tilde{e})}(\check{s}), \theta^-_{k(\check{p},\tilde{e})}(\check{s})), \inf(\theta^+_{j(\check{p},\tilde{e})}(\check{s}), \theta^+_{k(\check{p},\tilde{e})}(\check{s}))], \\ [\sup(\varphi^-_{j(\check{p},\tilde{e})}(\check{s}), \varphi^-_{k(\check{p},\tilde{e})}(\check{s})), \sup(\varphi^+_{j(\check{p},\tilde{e})}(\check{s}), \varphi^+_{k(\check{p},\tilde{e})}(\check{s}))] >\} \\ if\ (\check{p}, \tilde{e}) \in (\check{P}_j \times \tilde{E}_j) \cap (\check{P}_k \times \tilde{E}_k)\ for\ some\ j, k \in J \\ \{<\check{s}, [\inf(\theta^-_{k(\check{p},\tilde{e})}(\check{s})), \inf(\theta^+_{k(\check{p},\tilde{e})}(\check{s}))], \\ [\sup(\varphi^-_{k(\check{p},\tilde{e})}(\check{s})), \sup(\varphi^+_{k(\check{p},\tilde{e})}(\check{s}))] >\} \\ \text{if } (\check{p}, \tilde{e}) \in \bigcap_{k\in J}(\check{P}_k \times \tilde{E}_k)\ for\ any\ arbitrary\ k \end{cases}$$

where $h_k(\check{p}, \tilde{e}) = \{<\check{s}, [\theta^-_{k(\check{p},\tilde{e})}(\check{s}), \theta^+_{k(\check{p},\tilde{e})}(\check{s})], [\varphi^-_{k(\check{p},\tilde{e})}(\check{s}), \varphi^+_{k(\check{p},\tilde{e})}(\check{s})] >: \check{s} \in S\}$.

Definition 4.2.26

The extended sum of two Q-ROIVFSESs (h′, $\check{P}'$, $\tilde{E}'$, Q′) with Q′ $\geq$ 1 and (h″, $\check{P}''$, $\tilde{E}''$, Q″) with Q″$\geq$ 1 over S, is denoted as $(h_+, \check{P}_+, \tilde{E}_+, Q_+)$ = (h′, $\check{P}'$, $\tilde{E}'$, Q′) + (h″, $\check{P}''$, $\tilde{E}''$, Q″), where $h_+ = h' + h''$ and $Q_+ = \sup\{\min Q', \min Q''\}$ and is characterized as

$$h'(\check{p}, \tilde{e}) + h''(\check{p}, \tilde{e}) = \{<\check{s}, [\left(\theta^-_{\prime(\check{p},\tilde{e})}(\check{s})^Q + \theta^-_{\prime\prime(\check{p},\tilde{e})}(\check{s})^Q - \theta^-_{\prime(\check{p},\tilde{e})}(\check{s})^Q \theta^-_{\prime\prime(\check{p},\tilde{e})}(\check{s})^Q\right)^{1/Q},$$

$$\left(\theta^+_{\prime(\check{p},\tilde{e})}(\check{s})^Q + \theta^+_{\prime\prime(\check{p},\tilde{e})}(\check{s})^Q - \theta^+_{\prime(\check{p},\tilde{e})}(\check{s})^Q \theta^+_{\prime\prime(\check{p},\tilde{e})}(\check{s})^Q\right)^{1/Q}],$$

$$[\varphi^-_{\prime(\check{p},\tilde{e})}(\check{s})\varphi^-_{\prime\prime(\check{p},\tilde{e})}(\check{s}), \varphi^+_{\prime(\check{p},\tilde{e})}(\check{s})\varphi^+_{\prime\prime(\check{p},\tilde{e})}(\check{s})]>: \check{s} \in S\},$$

where $h'(\check{p}, \tilde{e}) = \{<\check{s}, [\theta^{-}_{'(\check{p},\tilde{e})}(\check{s}), \theta^{+}_{'(\check{p},\tilde{e})}(\check{s})], [\varphi^{-}_{'(\check{p},\tilde{e})}(\check{s}), \varphi^{+}_{'(\check{p},\tilde{e})}(\check{s})]>: \check{s}\in S\}$, $h''(\check{p}, \tilde{e}) = \{<\check{s}, [\theta^{-}_{''(\check{p},\tilde{e})}(\check{s}), \theta^{+}_{''(\check{p},\tilde{e})}(\check{s})], [\varphi^{-}_{''(\check{p},\tilde{e})}(\check{s}), \varphi^{+}_{''(\check{p},\tilde{e})}(\check{s})]>: \check{s}\in S\}$ and Q = inf{min Q′, min Q″}.

Remark 4.2.27

If Q′ = Q″, then Q = Q_+ and hence the extended sum reduces to simple sum or simply addition of two Q-ROIVFSESs. Extended sum is used when there are two different values of Q, that is Q′ ≠ Q″, otherwise we use simple sum, or (just) sum.

Example 4.2.28

Let S = $\{\check{s}_1, \check{s}_2, \check{s}_3\}$, $\check{P}'$ = $\{\check{p}_1\}$ and $\tilde{E}'$ = $\{\tilde{e}_1\}$ be universe set, set of parameters and set of expert and Q-ROIVFSES (h′, $\check{P}'$, $\tilde{E}'$, Q′) over S, with Q′ ≥ 2 is given by

h′($\check{p}_1$, $\tilde{e}_1$) = {<$\check{s}_1$,[.6,.7],[.2,.4]>, <$\check{s}_2$,[.3,.4],[.8,.9]>, <$\check{s}_3$,[.8,.9],[.2,.3]>}.

And for $\check{P}''$ = $\{\check{p}_1, \check{p}_2\}$ and $\tilde{E}''$ = $\{\tilde{e}_1\}$, then Q-ROIVFSES (h″, $\check{P}''$, $\tilde{E}''$, Q″) over S, with Q″≥ 3 is

h″($\check{p}_1$, $\tilde{e}_1$) = {<$\check{s}_1$,[.8,.9],[.5,.6]>, <$\check{s}_2$,[.6,.7],[.3,.4]>,<$\check{s}_3$,[.7,.8],[.6,.7]>},

h″($\check{p}_2$, $\tilde{e}_1$) = {<$\check{s}_1$,[.7,.8],[.6,.7]>, <$\check{s}_2$,[.7,.8],[.2,.3]>,<$\check{s}_3$,[.6,.7],[.6,.8]>}.

Then extended sum of (h′, $\check{P}'$, $\tilde{E}'$, Q′) and (h″, $\check{P}''$, $\tilde{E}''$, Q″) is given by

$h_{e+}(\check{p}_1, \tilde{e}_1)$ = h′($\check{p}_1$, $\tilde{e}_1$) + h″($\check{p}_1$, $\tilde{e}_1$) = {<$\check{s}_1$,[.88,.95],[.1,.24]>, <$\check{s}_2$,[.65,.76],[.24,.36]>, <$\check{s}_3$,[.90,.96],[.12,.21]>},

$h_{e+}(\check{p}_{12}, \tilde{e}_1)$ = h′($\check{p}_1$, $\tilde{e}_1$) + h″($\check{p}_2$, $\tilde{e}_1$) = {<$\check{s}_1$,[.82,.90],[.12,.28]>, <$\check{s}_2$,[.73, .84],[.16,.27]> , <$\check{s}_3$,[.88, .92],[.12,.24]>} and Q_+= 3, where h_{e+} indicates extended sum.

And simple sum of h″($\check{p}_1$, $\tilde{e}_1$) and h″($\check{p}_2$, $\tilde{e}_1$) of (h″, P″, E″, Q″) over S, with Q″≥ 3 is

$h_{+}(\check{p}_{12}, \tilde{e}_1)$ = h″($\check{p}_1$, $\tilde{e}_1$) + h″($\check{p}_2$, $\tilde{e}_1$) = {<$\check{s}_1$,[.88,.95],[.3,.42]>, <$\check{s}_2$,[.78,.88],[.06,.12]>, <$\check{s}_3$,[.78,.83],[.36,.56]>} and Q_+= Q″ = 3 = Q.

Definition 4.2.29

The extended product of two Q-ROIVFSESs (h′, $\check{P}'$, $\tilde{E}'$, Q′) with Q′ ≥ 1 and (h″, $\check{P}''$, $\tilde{E}''$, Q″) with Q″≥ 1 over S, is denoted as $(h_{\times}, \check{P}_{\times}, \tilde{E}_{\times}, Q_{\times})$ = (h′, $\check{P}'$, $\tilde{E}'$, Q′) × (h″, $\check{P}''$, $\tilde{E}''$, Q″), where $h_{\times} = h' \times h''$ and $Q_{\times}$= sup{min Q′, min Q″} and is characterized as:

$$h'(\check{p}, \tilde{e}) \times h''(\check{p}, \tilde{e}) = \{<\check{s}, [\theta^{-}_{'(\check{p},\tilde{e})}(\check{s})\theta^{-}_{''(\check{p},\tilde{e})}(\check{s}), \theta^{+}_{'(\check{p},\tilde{e})}(\check{s})\theta^{+}_{''(\check{p},\tilde{e})}(\check{s})],$$

$$[\left(\varphi^{-}_{'(\check{p},\tilde{e})}(\check{s})^{Q} + \varphi^{-}_{''(\check{p},\tilde{e})}(\check{s})^{Q} - \varphi^{-}_{'(\check{p},\tilde{e})}(\check{s})^{Q}\varphi^{-}_{''(\check{p},\tilde{e})}(\check{s})^{Q}\right)^{1/Q},$$

$$\left(\varphi^{+}_{'(\check{p},\tilde{e})}(\check{s})^{Q} + \varphi^{+}_{''(\check{p},\tilde{e})}(\check{s})^{Q} - \varphi^{+}_{'(\check{p},\tilde{e})}(\check{s})^{Q}\varphi^{+}_{''(\check{p},\tilde{e})}(\check{s})^{Q}\right)^{1/Q}]>: \check{s}\in S\},$$

where $h'(\check{p}, \tilde{e})$ = {<$\check{s}$, [$\theta^{-}_{'(\check{p},\tilde{e})}(\check{s})$,$\theta^{+}_{'(\check{p},\tilde{e})}(\check{s})$], [$\varphi^{-}_{'(\check{p},\tilde{e})}(\check{s}), \varphi^{+}_{'(\check{p},\tilde{e}v)}(\check{s})$]>: $\check{s}$∈S}, $h''(\check{p}, \tilde{e})$ = {<$\check{s}$, [$\theta^{-}_{''(\check{p},\tilde{e})}(\check{s})$,$\theta^{+}_{''(\check{p},\tilde{e})}(\check{s})$], [$\varphi^{-}_{''(\check{p},\tilde{e})}(\check{s}), \varphi^{+}_{''(\check{p},\tilde{e})}(\check{s})$]>: $\check{s}$∈S} and Q = inf{min Q′, min Q″}.

Remark 4.2.30

If Q′ = Q″, then Q = $Q_{\times}$ and hence the extended product reduces to simple product or simply product of two Q-ROIVFSESs. Extended product is used when there are two different values of Q, that is Q′ ≠ Q″, otherwise we use simple product, or (just) product.

Example 4.2.31

Let S = {$\check{s}_1$, $\check{s}_2$, $\check{s}_3$}, $\check{P}'$ = {$\check{p}_1$} and $\tilde{E}'$ = {$\tilde{e}_1$} be universe set, set of parameters and set of expert and Q-ROIVFSES (h′, $\check{P}'$, $\tilde{E}'$, Q′) over S, with Q′ ≥ 2 is given by

h′($\check{p}_1$, $\tilde{e}_1$) = {<$\check{s}_1$,[.6,.7],[.2,.4]>, <$\check{s}_2$,[.3,.4],[.8,.9]>, <$\check{s}_3$,[.8,.9],[.2,.3]>}.

And for P″ = {$\check{p}_1$, $\check{p}_2$} and E″ = {$\tilde{e}_1$}, then Q-ROIVFSES (h″, $\check{P}''$, $\tilde{E}''$, Q″) over S, with Q″≥ 3 is

h″($\check{p}_1$, $\tilde{e}_1$) = {<$\check{s}_1$,[.8,.9],[.5,.6]>, <$\check{s}_2$,[.6,.7],[.3, .4]>, <$\check{s}_3$,[.7,.8],[.6,.7]>},

h″($\check{p}_2$, $\tilde{e}_1$) = {<$\check{s}_1$,[.7,.8],[.6,.7]>, <$\check{s}_2$,[.7,.8],[.2,.3]>,<$\check{s}_3$,[.6,.7],[.6,.8]>}.

Then extended product of (h′, $\check{P}'$, $\tilde{E}'$, Q′) and (h″, $\check{P}''$, $\tilde{E}''$, Q″) is given by

$h_{e\times}(\check{p}_1, \tilde{e}_1)$ = h′($\check{p}_1$, $\tilde{e}_1$) × h″($\check{p}_1$, $\tilde{e}_1$) = {<$\check{s}_1$,[.48,.63],[.53,.68]>, <$\check{s}_2$,[.18,.28],[.82,.92]>, <$\check{s}_3$,[.56,.72],[.62,.73]>},

$h_{e\times}(\check{p}_{12}, \tilde{e}_1)$= h′($\check{p}_1$, $\tilde{e}_1$) × h″($\check{p}_2$, $\tilde{e}_1$)= {<$\check{s}_1$,[.42,.56],[.62,.76]>, <$\check{s}_2$,[.21, .32],[.81,.91]> , <$\check{s}_3$,[.48,.63],[.62,.82]>} and $Q_\times$= 3, where $h_{e\times}$ indicates extended product.

And simple product of h″($\check{p}_1$, $\tilde{e}_1$) and h″($\check{p}_2$, $\tilde{e}_1$) of (h″, $\check{P}''$, $\tilde{E}''$, Q″) over S, with Q″≥ 3 is $h_\times(\check{p}_{12}, \tilde{e}_1)$ = h″($\check{p}_1$, $\tilde{e}_1$) × h″($\check{p}_2$, $\tilde{e}_1$) = {<$\check{s}_1$,[.56,.72],[.68,.78]>, <$\check{s}_2$,[.42,.56], [.33,.45]>, <$\check{s}_3$,[.42,.56],[.73,.88]>} and $Q_\times$= Q″ = 3 = Q.

Definition 4.2.32

The product of a Q-ROIVFSES (h, $\check{P}$, $\tilde{E}$, Q) with Q′ ≥ 1 over a universe S, with any arbitrary positive real number $r > 0,$ is symbolized by r(h, $\check{P}, \tilde{E}$, Q) and defined for all h($\check{p}$, $\tilde{e}$) ∈ (h, $\check{P}$, $\tilde{E}$, Q) as follows

$$r\text{h}(\check{p}, \tilde{e}) = \{<\check{s}, [(1-(1-\theta^-_{(\check{p},\tilde{e})}(\check{s})^Q)^r)^{1/Q}, (1-(1-\theta^+_{(\check{p},\tilde{e})}(\check{s})^Q)^r)^{1/Q}], [\varphi^-_{(\check{p},\tilde{e})}(\check{s})^r, \varphi^+_{(\check{p},\tilde{e})}(\check{s})^r]>: \check{s} \in S\},$$

where $h(\check{p}, \tilde{e})$ = {<$\check{s}$, [$\theta^-_{(\check{p},\tilde{e})}(\check{s})$,$\theta^+_{(\check{p},\tilde{e})}(\check{s})$], [$\varphi^-_{(\check{p},\tilde{e})}(\check{s}), \varphi^+_{(\check{p},\tilde{e})}(\check{s})$]>: $\check{s}$∈S}.

Example 4.2.33

Consider Q-ROIVFSES (h, $\check{P}$, $\tilde{E}$, Q) over a universe S, where $\check{P}$ = {$\check{p}_1$, $\check{p}_2$}, $\tilde{E}$ = { $\tilde{e}_1$}, Q= 2 and

h($\check{p}_1$, $\tilde{e}_1$) = {<$\check{s}_1$,[0.6,0.7],[0.3,0.4]>, <$\check{s}_2$,[0.5,0.6],[0.4,0.5]>} and

h($\check{p}_2$, $\tilde{e}_1$) = {<$\check{s}_1$,[0.4,0.5],[0.6,0.8]>, <$\check{s}_2$,[0.3,0.4],[0.6,0.7]>}.

Then the product of Q-ROIVFSES with r = 5 is denoted by 5(h, $\check{P}, \tilde{E}$, Q) and given as

5h($\check{p}_1$, $\tilde{e}_1$) = {<$\check{s}_1$,[0.94,0.98],[0.002,0.010]>, <$\check{s}_2$,[0.87,0.94],[0.010,0.031]>},

5h($\check{p}_2$, $\tilde{e}_1$) = {<$\check{s}_1$,[0.76,0.87],[0.078,0.328]>, <$\check{s}_2$,[0.61,0.76],[0.078,0.168]>}.

Definition 4.2.34

The power of a Q-ROIVFSES (h, $\check{P}$, $\tilde{E}$, Q) with Q′ ≥ 1 over a universe S, with any arbitrary positive real number $r > 0$, is symbolized by $(h, \check{P}, \tilde{E}, Q)^r$ and characterized as follows:

$$(h(\check{p}, \tilde{e}))^r = \{<\check{s}, [\theta^-_{(\check{p},\tilde{e})}(\check{s})^r, \theta^+_{(\check{p},\tilde{e})}(\check{s})^r], [(1-(1-\varphi^-_{(\check{p},\tilde{e})}(\check{s})^Q)^r)^{1/Q}, (1-(1-\varphi^+_{(\check{p},\tilde{e})}(\check{s})^Q)^r)^{1/Q}]>: \check{s} \in S\},$$

$\forall\, h(\check{p}, \tilde{e}) = \{<\check{s}, [\theta^-_{(\check{p},\tilde{e})}(\check{s}), \theta^+_{(\check{p},\tilde{e})}(\check{s})], [\varphi^-_{(\check{p},\tilde{e})}(\check{s}), \varphi^+_{(\check{p},\tilde{e})}(\check{s})]>: \check{s} \in S\} \in$ (h, $\check{P}$, $\tilde{E}$, Q).

Example 4.2.35

Consider Q-ROIVFSES (h, $\check{P}$, $\tilde{E}$, Q) over a universe S, where $\check{P}$ = { $\check{p}_1$, $\check{p}_2$ }, E = {$\tilde{e}_1$} and Q= 2. And

$$h(\check{p}_1, \tilde{e}_1) = \{<\check{s}_1,[0.3,0.4],[0.6,0.8]>, <\check{s}_2,[0.4,0.5],[0.5,0.6]>\},$$
$$h(\check{p}_2, \tilde{e}_1) = \{<\check{s}_1,[0.6,0.8],[0.4,0.5]>, <\check{s}_2,[0.6,0.7],[0.3,0.4]>\}.$$

Then the power of Q-ROIVFSES with r = 5 is denoted by $(h, P, E, Q)^5$ and given as

$$(h(\check{p}_1, \tilde{e}_1))^5 = \{<\check{s}_1,[0.002,0.010],[0.94,0.98]>, <\check{s}_2,[0.010,0.031],[0.87,0.94]>\},$$
$$(h(\check{p}_2, \tilde{e}_1))^5 = \{<\check{s}_1,[0.078,0.328],[0.76,0.87]>, <\check{s}_2,[0.078,0.168],[0.61,0.76]>\}.$$

Definition 4.2.36

The score (§) and accuracy (Â) function for some $\check{s}$∈S of h($\check{p}$, $\tilde{e}$) ={<s, $[\theta^-_{(\check{p},\tilde{e})}(\check{s}), \theta^+_{(\check{p},\tilde{e})}(\check{s})]$, $[\varphi^-_{(\check{p},\tilde{e})}(\check{s}), \varphi^+_{(\check{p},\tilde{e})}(\check{s})]$>: $\check{s}$∈S} in Q-ROIVFSES (h, $\check{P}$, $\tilde{E}$, Q) with Q′ ≥ 1 over a universe S are respectively, defined as follows:

$$\S(h(\check{p}, \tilde{e}))_{\check{s}} = \frac{\theta^-_{(\check{p},\tilde{e})}(\check{s})^Q + \theta^+_{(\check{p},\tilde{e})}(\check{s})^Q - \varphi^-_{(\check{p},\tilde{e})}(\check{s})^Q - \varphi^+_{(\check{p},\tilde{e})}(\check{s})^Q}{2}$$

and

$$\hat{A}(h(\check{p},\tilde{e}))_{\check{s}} = \frac{\theta^{-}_{(\check{p},\tilde{e})}(\check{s})^{Q} + \theta^{+}_{(\check{p},\tilde{e})}(\check{s})^{Q} + \varphi^{-}_{(\check{p},\tilde{e})}(\check{s})^{Q} + \varphi^{+}_{(\check{p},\tilde{e})}(\check{s})^{Q}}{2}.$$

And $§(h(\check{p},\tilde{e}))_{\check{s}} \in [-1, 1]$, $\hat{A}(h(\check{p},\tilde{e}))_{\check{s}} \in [0, 1]\ \forall\ \check{s} \in S$.

Theorem 4.2.37

The larger accuracy (Â) and larger score (§) indicates the greater element of Q-ROIVFSES (h, $\check{P}$, $\tilde{E}$, Q). Q-ROIVFSESs can be ranked using score and accuracy function.

Let h($\check{p}_1$, $\tilde{e}_1$), h($\check{p}_2$, $\tilde{e}_2$) be two elements of Q-ROIVFSES (h, $\check{P}$, $\tilde{E}$, Q) and $§(h(\check{p}_1,\tilde{e}_1))_{\check{s}}$, $§(h(\check{p}_2,\tilde{e}_2))_{\check{s}}$ and $\hat{A}(h(\check{p}_1,\tilde{e}_1))_{\check{s}}$, $\hat{A}(h(\check{p}_2,\tilde{e}_2))_{\check{s}}$ be their score and accuracy function, respectively for some $\check{s} \in S$.

i) If $§(h(\check{p}_1,\tilde{e}_1))_{\check{s}} > §(h(\check{p}_2,\tilde{e}_2))_{\check{s}}$, then h($\check{p}_1$, $\tilde{e}_1$) > h($\check{p}_2$, $\tilde{e}_2$) for some $\check{s} \in S$.

ii) If $§(h(\check{p}_1,\tilde{e}_1))_{\check{s}} = §(h(\check{p}_2,\tilde{e}_2))_{\check{s}}$, then

   a. if $\hat{A}(h(\check{p}_1,\tilde{e}_1))_{\check{s}} > \hat{A}(h(\check{p}_2,\tilde{e}_2))_{\check{s}}$, then h($\check{p}_1$, $\tilde{e}_1$) > h($\check{p}_2$, $\tilde{e}_2$) for some $\check{s} \in S$,

   b. if $\hat{A}(h(\check{p}_1,\tilde{e}_1))_{\check{s}} = \hat{A}(h(\check{p}_2,\tilde{e}_2))_{\check{s}}$, then h($\check{p}_1$, $\tilde{e}_1$) = h($\check{p}_2$, $\tilde{e}_2$) for some $\check{s} \in S$

and vice versa.

Example 4.2.38

Consider Q-ROIVFSES (h, $\check{P}$, $\tilde{E}$, Q) with the set of parameter $\check{P} = \{\check{p}_1, \check{p}_2, \check{p}_3\}$, the set of experts E = $\{\tilde{e}_1, \tilde{e}_2\}$, Q = 2 and universe set S = $\{\check{s}_1, \check{s}_2\}$, as below:

h($\check{p}_1$, $\tilde{e}_1$) = {<$\check{s}_1$,[0.7,0.8],[0.1,0.3]>, <$\check{s}_2$,[0.3,0.5],[0.7,0.8]>},

h($\check{p}_2$, $\tilde{e}_1$) = {<$\check{s}_1$,[0.6,0.7],[0.3,0.5]>, <$\check{s}_2$,[0.5,0.6],[0.6,0.7]>},

h($\check{p}_3$, $\tilde{e}_1$) = {<$\check{s}_1$,[0.8,0.9],[0.2,0.3]>, <$\check{s}_2$,[0.5,0.6],[0.4,0.6]>},

h($\check{p}_1$, $\tilde{e}_2$) = {<$\check{s}_1$,[0.6,0.7],[0.4,0.6]>, <$\check{s}_2$,[0.2,0.4],[0.7,0.8]>},

h($\check{p}_2$, $\tilde{e}_2$) = {<$\check{s}_1$,[0.7,0.8],[0.3,0.4]>, <$\check{s}_2$,[0.4,0.5],[0.6,0.9]>},

h($\check{p}_3$, $\tilde{e}_2$) = {<$\check{s}_1$,[0.6,0.9],[0.2,0.5]>, <$\check{s}_2$,[0.1,0.2],[0.8,0.9]>}.

Next, calculating the score of above Q-ROIVFSES's elements for $\check{s}_1$, $\check{s}_2$∈S,

Score for $\check{s}_1$∈S	Score for $\check{s}_2$∈S
$\S(\mathrm{h}(\check{p}_1, \tilde{e}_1))_{\check{s}_1}$= 0.52	$\S(\mathrm{h}(\check{p}_1, \tilde{e}_1))_{\check{s}_2}$= -0.40
$\S(\mathrm{h}(\check{p}_2, \tilde{e}_1))_{\check{s}_1}$= 0.26	$\S(\mathrm{h}(\check{p}_2, \tilde{e}_1))_{\check{s}_2}$= -0.12
$\S(\mathrm{h}(\check{p}_3, \tilde{e}_1))_{\check{s}_1}$= 0.66	$\S(\mathrm{h}(\check{p}_3, \tilde{e}_1))_{\check{s}_2}$= 0.04
$\S(\mathrm{h}(\check{p}_1, \tilde{e}_2))_{\check{s}_1}$= 0.16	$\S(\mathrm{h}(\check{p}_1, \tilde{e}_2))_{\check{s}_2}$= -0.46
$\S(\mathrm{h}(\check{p}_2, \tilde{e}_2))_{\check{s}_1}$= 0.44	$\S(\mathrm{h}(\check{p}_2, \tilde{e}_2))_{\check{s}_2}$= -0.38
$\S(\mathrm{h}(\check{p}_3, \tilde{e}_2))_{\check{s}_1}$= 0.44	$\S(\mathrm{h}(\check{p}_3, \tilde{e}_2))_{\check{s}_2}$= -0.76

Table 4.1: Score for $\check{s}_1$, $\check{s}_2$∈ S

Upon organizing the score in ascending order, we obtain

For $\check{s}_1$∈S, 0.16< 0.26< 0.44= 0.44< 0.52< 0.66

$\Rightarrow \mathrm{h}(\check{p}_1, \tilde{e}_2) < \mathrm{h}(\check{p}_2, \tilde{e}_1) < \mathrm{h}(\check{p}_2, \tilde{e}_2) = \mathrm{h}(\check{p}_3, \tilde{e}_2) < \mathrm{h}(\check{p}_1, \tilde{e}_1) < \mathrm{h}(\check{p}_3, \tilde{e}_1)$.

For $\check{s}_2$∈S, -0.76< -0.46< -0.40< -0.38< -0.12< 0.04

$\Rightarrow \mathrm{h}(p_3, e_2) < \mathrm{h}(p_1, e_2) < \mathrm{h}(p_1, e_1) < \mathrm{h}(p_2, e_2) < \mathrm{h}(p_2, e_1) < \mathrm{h}(p_3, e_1)$.

This ranking is on the basis of score.

If for any $\check{s}$∈S, If $\S(\mathrm{h}(\check{p}_i, \tilde{e}_j))_{\check{s}}$= $\S(\mathrm{h}(\check{p}_k, \tilde{e}_l))_{\check{s}}$ for some (i, j) $\neq (k, l)$, then we can calculate their accuracies. Here for $\check{s}_1$∈S, $\mathrm{h}(\check{p}_2, \tilde{e}_2) = \mathrm{h}(\check{p}_3, \tilde{e}_2) = 0.44$ so calculating their accuracies

$\hat{A}(\mathrm{h}(\check{p}_2, \tilde{e}_2))_{\check{s}_1}$= 0.69, $\hat{A}(\mathrm{h}(\check{p}_3, \tilde{e}_2))_{\check{s}_1}$= 0.73.

$\Rightarrow \hat{A}(\mathrm{h}(\check{p}_3, \tilde{e}_2))_{\check{s}_1} > \hat{A}(\mathrm{h}(\check{p}_2, \tilde{e}_2))_{\check{s}_1}$.

4.3 Properties of addition (sum) and multiplication (product) for Q-ROIVFSESs

Let S, $\check{P}$, $\tilde{E}$ denote the universe of discourse, set of parameters, experts set respectively. And $\tilde{Q}_I(S)$ denotes the set of all Q-ROIVFSESs (h, $\check{P}$, $\tilde{E}$, Q) for some Q′ ≥ 1, where h: $\check{P} \times \tilde{E} \rightarrow Q_I(S)$ is a mapping and $Q_I(S)$ is the set of all interval-valued Q-ROFS over a universe S.

Properties of addition (+) for Q-ROIVFSESs are denoted by A1, A2, A3, A4 and A5 and that for multiplication (×) are symbolized by M1, M2, M3, M4 and M5.

Let $h_1(\check{p}, \tilde{e})$ = {<$\check{s}$, [$\theta^-_{1(\check{p},\tilde{e})}(\check{s})$,$\theta^+_{1(\check{p},\tilde{e})}(\check{s})$], [$\varphi^-_{1(\check{p},\tilde{e})}(\check{s}), \varphi^+_{1(\check{p},\tilde{e})}(\check{s})$]>: $\check{s}$∈ S},
$h_2(\check{p}, \tilde{e})$ = {<$\check{s}$, [$\theta^-_{2(\check{p},\tilde{e})}(\check{s})$,$\theta^+_{2(\check{p},\tilde{e})}(\check{s})$], [$\varphi^-_{2(\check{p},\tilde{e})}(\check{s}), \varphi^+_{2(\check{p},\tilde{e})}(\check{s})$]>: $\check{s}$∈ S} and
$h_3(\check{p}, \tilde{e})$ = {<$\check{s}$, [$\theta^-_{3(\check{p},\tilde{e})}(\check{s})$,$\theta^+_{3(\check{p},\tilde{e})}(\check{s})$], [$\varphi^-_{3(\check{p},\tilde{e})}(\check{s}), \varphi^+_{3(\check{p},\tilde{e})}(\check{s})$]>: $\check{s}$∈ S} ∈$\tilde{Q}_I(S)$ and hence

i) $\theta^+_{j(\check{p},\tilde{e})}(\check{s})^Q + \varphi^+_{j(\check{p},\tilde{e})}(\check{s})^Q \leq 1$,

ii) $\theta^-_{j(\check{p},\tilde{e})}(\check{s})$,$\theta^+_{j(\check{p},\tilde{e})}(\check{s})$,$\varphi^-_{j(\check{p},\tilde{e})}(\check{s})$,$\varphi^+_{j(\check{p},\tilde{e})}(\check{s})$∈ [0, 1], for j = 1, 2, 3.

Then ∀ $h_1(\check{p}, \tilde{e}), h_2(\check{p}, \tilde{e}), h_3(\check{p}, \tilde{e})$, we have,

4.3.1 (A1) Closure property

$h_1(\check{p}, \tilde{e})$ +$h_2(\check{p}, \tilde{e})$ = {<$\check{s}$,[$\left(\theta^-_{1(\check{p},\tilde{e})}(\check{s})^Q + \theta^-_{2(\check{p},\tilde{e})}(\check{s})^Q - \theta^-_{1(\check{p},\tilde{e})}(\check{s})^Q\theta^-_{2(\check{p},\tilde{e})}(\check{s})^Q\right)^{1/Q}$,
$(\theta^+_{1(\check{p},\tilde{e})}(\check{s})^Q + \theta^+_{2(\check{p},\tilde{e})}(\check{s})^Q - \theta^+_{1(\check{p},\tilde{e})}(\check{s})^Q\theta^+_{2(\check{p},\tilde{e})}(\check{s})^Q)^{1/Q}$],
[$\varphi^-_{1(\check{p},\tilde{e})}(\check{s})\varphi^-_{2(\check{p},\tilde{e})}(\check{s})$, $\varphi^+_{1(\check{p},\tilde{e})}(\check{s})\varphi^+_{2(\check{p},\tilde{e})}(\check{s})$]>: $\check{s}$∈ S} ∈$\tilde{Q}_I(S)$.

As $\left((\theta^+_{1(\check{p},\tilde{e})}(\check{s})^Q + \theta^+_{2(\check{p},\tilde{e})}(\check{s})^Q - \theta^+_{1(\check{p},\tilde{e})}(\check{s})^Q\theta^+_{2(\check{p},\tilde{e})}(\check{s})^Q)^{1/Q}\right)^Q + (\varphi^+_{1(\check{p},\tilde{e})}(\check{s})\varphi^+_{2(\check{p},\tilde{e})}(\check{s}))^Q \leq 1$.

Because

$$(\theta^{+}_{1(\check{p},\tilde{e})}(\check{s})^{Q} + \theta^{+}_{2(\check{p},\tilde{e})}(\check{s})^{Q} - \theta^{+}_{1(\check{p},\tilde{e})}(\check{s})^{Q}\theta^{+}_{2(\check{p},\tilde{e})}(\check{s})^{Q})^{1/Q^{Q}} + (\varphi^{+}_{1(\check{p},\tilde{e})}(\check{s})\varphi^{+}_{2(\check{p},\tilde{e})}(\check{s}))^{Q}$$

$= \theta^{+}_{1(\check{p},\tilde{e})}(\check{s})^{Q} + \theta^{+}_{2(\check{p},\tilde{e})}(\check{s})^{Q} - \theta^{+}_{1(\check{p},\tilde{e})}(\check{s})^{Q}\theta^{+}_{2(\check{p},\tilde{e})}(\check{s})^{Q} + (\varphi^{+}_{1(\check{p},\tilde{e})}(\check{s})^{Q} \times \varphi^{+}_{2(\check{p},\tilde{e})}(\check{s}))^{Q}$

$\leq \theta^{+}_{1(\check{p},\tilde{e})}(\check{s})^{Q} + \theta^{+}_{2(\check{p},\tilde{e})}(\check{s})^{Q} - \theta^{+}_{1(\check{p},\tilde{e})}(\check{s})^{Q}\theta^{+}_{2(\check{p},\tilde{e})}(\check{s})^{Q} + (1 - \theta^{+}_{1(\check{p},\tilde{e})}(\check{s})^{Q})(1 - \theta^{+}_{2(\check{p},\tilde{e})}(\check{s})^{Q})$

$= \theta^{+}_{1(\check{p},\tilde{e})}(\check{s})^{Q} + \theta^{+}_{2(\check{p},\tilde{e})}(\check{s})^{Q} - \theta^{+}_{1(\check{p},\tilde{e})}(\check{s})^{Q}\theta^{+}_{2(\check{p},\tilde{e})}(\check{s})^{Q} + 1 - \theta^{+}_{1(\check{p},\tilde{e})}(\check{s})^{Q} - \theta^{+}_{2(\check{p},\tilde{e})}(\check{s})^{Q} + \theta^{+}_{1(\check{p},\tilde{e})}(\check{s})\theta^{+}_{2(\check{p},\tilde{e})}(\check{s})^{Q}$

$= 1$

Proved using $\varphi^{+}_{1(\check{p},\tilde{e})}(\check{s})^{Q} \leq 1 - \theta^{+}_{1(\check{p},\tilde{e})}(\check{s})^{Q}$ and $\varphi^{+}_{2(\check{p},\tilde{e})}(\check{s}))^{Q} \leq 1 - \theta^{+}_{2(\check{p},\tilde{e})}(\check{s})^{Q}$.

4.3.2 (A2) Associativity

As

$\{h_1(\check{p},\tilde{e}) + h_2(\check{p},\tilde{e})\} + h_3(\check{p},\tilde{e}) =$ $\{<\check{s},[\left(\theta^{-}_{1(\check{p},\tilde{e})}(\check{s})^{Q} + \theta^{-}_{2(\check{p},\tilde{e})}(\check{s})^{Q} - \theta^{-}_{1(\check{p},\tilde{e})}(\check{s})^{Q}\theta^{-}_{2(\check{p},\tilde{e})}(\check{s})^{Q}\right)^{1/Q}$, $(\theta^{+}_{1(\check{p},\tilde{e})}(\check{s})^{Q} + \theta^{+}_{2(\check{p},\tilde{e})}(\check{s})^{Q} - \theta^{+}_{1(\check{p},\tilde{e})}(\check{s})^{Q}\theta^{+}_{2(\check{p},\tilde{e})}(\check{s})^{Q})^{1/Q}]$, $[\varphi^{-}_{1(\check{p},\tilde{e})}(\check{s})\varphi^{-}_{2(\check{p},\tilde{e})}(\check{s}), \varphi^{+}_{1(\check{p},\tilde{e})}(\check{s})\varphi^{+}_{2(\check{p},\tilde{e})}(\check{s})] >: \check{s} \in S\} + h_3(\check{p},\tilde{e})$

$=\{<\check{s},[\Bigg(\left(\theta^{-}_{1(\check{p},\tilde{e})}(\check{s})^{Q} + \theta^{-}_{2(\check{p},\tilde{e})}(\check{s})^{Q} - \theta^{-}_{1(\check{p},\tilde{e})}(\check{s})^{Q}\theta^{-}_{2(\check{p},\tilde{e})}(\check{s})^{Q}\right)^{1/Q^{Q}} + \theta^{-}_{3(\check{p},\tilde{e})}(\check{s})^{Q} - (\theta^{-}_{1(\check{p},\tilde{e})}(\check{s})^{Q} + \theta^{-}_{2(\check{p},\tilde{e})}(\check{s})^{Q} - \theta^{-}_{1(\check{p},\tilde{e})}(\check{s})^{Q}\theta^{-}_{2(\check{p},\tilde{e})}(\check{s})^{Q})\theta^{-}_{3(\check{p},\tilde{e})}(\check{s})^{Q}\Bigg)^{1/Q}, \Bigg(\left(\theta^{+}_{1(\check{p},\tilde{e})}(\check{s})^{Q} + \theta^{+}_{2(\check{p},\tilde{e})}(\check{s})^{Q} - \theta^{+}_{1(\check{p},\tilde{e})}(\check{s})^{Q}\theta^{+}_{2(\check{p},\tilde{e})}(\check{s})^{Q}\right)^{1/Q^{Q}} + \theta^{+}_{3(\check{p},\tilde{e})}(\check{s})^{Q} - (\theta^{+}_{1(\check{p},\tilde{e})}(\check{s})^{Q} + \theta^{+}_{2(\check{p},\tilde{e})}(\check{s})^{Q} - \theta^{+}_{1(\check{p},\tilde{e})}(\check{s})^{Q}\theta^{+}_{2(\check{p},\tilde{e})}(\check{s})^{Q})\theta^{+}_{3(\check{p},\tilde{e})}(\check{s})^{Q}\Bigg)^{1/Q}], [(\varphi^{-}_{1(\check{p},\tilde{e})}(\check{s})\varphi^{-}_{2(\check{p},\tilde{e})}(\check{s}))\varphi^{-}_{3(\check{p},\tilde{e})}(\check{s}), (\varphi^{+}_{1(\check{p},\tilde{e})}(\check{s})\varphi^{+}_{2(\check{p},\tilde{e})}(\check{s}))\varphi^{+}_{3(\check{p},\tilde{e})}(\check{s})] >: \check{s} \in S\}$

$$=\{<\breve{s},[\left(\theta^{-}_{1(\breve{p},\tilde{e})}(\breve{s})^{Q}+\theta^{-}_{2(\breve{p},\tilde{e})}(\breve{s})^{Q}-\theta^{-}_{1(\breve{p},\tilde{e})}(\breve{s})^{Q}\theta^{-}_{2(\breve{p},\tilde{e})}(\breve{s})^{Q}+\theta^{-}_{3(\breve{p},\tilde{e})}(\breve{s})^{Q}-\theta^{-}_{1(\breve{p},\tilde{e})}(\breve{s})^{Q}\theta^{-}_{3(\breve{p},\tilde{e})}(\breve{s})^{Q}-\theta^{-}_{2(\breve{p},\tilde{e})}(\breve{s})^{Q}\theta^{-}_{3(\breve{p},\tilde{e})}(\breve{s})^{Q}+\theta^{-}_{1(\breve{p},\tilde{e})}(\breve{s})^{Q}\theta^{-}_{2(\breve{p},\tilde{e})}(\breve{s})^{Q}\theta^{-}_{3(\breve{p},\tilde{e})}(\breve{s})^{Q}\right)^{1/Q},$$

$$\left(\theta^{+}_{1(\breve{p},\tilde{e})}(\breve{s})^{Q}+\theta^{+}_{2(\breve{p},\tilde{e})}(\breve{s})^{Q}-\mathrm{s}\theta^{+}_{1(\breve{p},\tilde{e})}(\breve{s})^{Q}\theta^{+}_{2(\breve{p},\tilde{e})}(\breve{s})^{Q}+\theta^{+}_{3(\breve{p},\tilde{e})}(\breve{s})^{Q}-\theta^{+}_{1(p,e)}(\breve{s})^{Q}\theta^{+}_{3(\breve{p},\tilde{e})}(\breve{s})^{Q}-\theta^{+}_{2(\breve{p},\tilde{e})}(\breve{s})^{Q}\theta^{+}_{3(\breve{p},\tilde{e})}(\breve{s})^{Q}+\theta^{+}_{1(\breve{p},\tilde{e})}(\breve{s})^{Q}\theta^{+}_{2(\breve{p},\tilde{e})}(\breve{s})^{Q}\theta^{+}_{3(\breve{p},\tilde{e})}(\breve{s})^{Q}\right)^{1/Q}],$$

$$[(\varphi^{-}_{1(\breve{p},\tilde{e})}(\breve{s})\varphi^{-}_{2(\breve{p},\tilde{e})}(\breve{s}))\varphi^{-}_{3(\breve{p},\tilde{e})}(\breve{s}),(\varphi^{+}_{1(\breve{p},\tilde{e})}(\breve{s})\varphi^{+}_{2(\breve{p},\tilde{e})}(\breve{s}))\varphi^{+}_{3(\breve{p},\tilde{e})}(\breve{s})]>:\breve{s}\in \mathrm{S}\}$$

$$=\{<\breve{s},[\left(\theta^{-}_{1(\breve{p},\tilde{e})}(\breve{s})^{Q}+\left(\theta^{-}_{2(\breve{p},\tilde{e})}(\breve{s})^{Q}+\theta^{-}_{3(\breve{p},\tilde{e})}(\breve{s})^{Q}-\theta^{-}_{2(\breve{p},\tilde{e})}(\breve{s})^{Q}\theta^{-}_{3(\breve{p},\tilde{e})}(\breve{s})^{Q}\right)^{1/Q^{Q}}-\theta^{-}_{1(\breve{p},\tilde{e})}(\breve{s})^{Q}\left(\theta^{-}_{2(\breve{p},\tilde{e})}(\breve{s})^{Q}+\theta^{-}_{3(\breve{p},\tilde{e})}(\breve{s})^{Q}-\theta^{-}_{2(\breve{p},\tilde{e})}(\breve{s})^{Q}\theta^{-}_{3(\breve{p},\tilde{e})}(\breve{s})^{Q}\right)^{1/Q^{Q}}\right)^{1/Q},$$

$$\left(\theta^{+}_{1(\breve{p},\tilde{e})}(\breve{s})^{Q}+\left(\theta^{+}_{2(\breve{p},\tilde{e})}(\breve{s})^{Q}+\theta^{+}_{3(\breve{p},\tilde{e})}(\breve{s})^{Q}-\theta^{+}_{2(\breve{p},\tilde{e})}(\breve{s})^{Q}\theta^{+}_{3(\breve{p},\tilde{e})}(\breve{s})^{Q}\right)^{1/Q^{Q}}-\theta^{+}_{1(\breve{p},\tilde{e})}(\breve{s})^{Q}\left(\theta^{+}_{2(\breve{p},\tilde{e})}(\breve{s})^{Q}+\theta^{+}_{3(\breve{p},\tilde{e})}(\breve{s})^{Q}-\theta^{+}_{2(\breve{p},\tilde{e})}(\breve{s})^{Q}\theta^{+}_{3(\breve{p},\tilde{e})}(\breve{s})^{Q}\right)^{1/Q^{Q}}\right)^{1/Q}],$$

$$[\varphi^{-}_{1(\breve{p},\tilde{e})}(\breve{s})(\varphi^{-}_{2(\breve{p},\tilde{e})}(\breve{s})\varphi^{-}_{3(\breve{p},\tilde{e})}(\breve{s})),\varphi^{+}_{1(\breve{p},\tilde{e})}(\breve{s})\ (\varphi^{+}_{2(\breve{p},\tilde{e})}(\breve{s})\varphi^{+}_{3(\breve{p},\tilde{e})}(\breve{s}))]>:\breve{s}\in \mathrm{S}\}$$

$$=h_1(p,e)+\{<\breve{s},[\left(\theta^{-}_{2(\breve{p},\tilde{e})}(\breve{s})^{Q}+\theta^{-}_{3(\breve{p},\tilde{e})}(\breve{s})^{Q}-\theta^{-}_{2(\breve{p},\tilde{e})}(\breve{s})^{Q}\theta^{-}_{3(\breve{p},\tilde{e})}(\breve{s})^{Q}\right)^{1/Q},\left(\theta^{+}_{2(\breve{p},\tilde{e})}(\breve{s})^{Q}+\theta^{+}_{3(\breve{p},\tilde{e})}(\breve{s})^{Q}-\theta^{+}_{2(\breve{p},\tilde{e})}(\breve{s})^{Q}\theta^{+}_{3(\breve{p},\tilde{e})}(\breve{s})^{Q}\right)^{1/Q}],$$

$$[\varphi^{-}_{2(\breve{p},\tilde{e})}(\breve{s})\varphi^{-}_{3(\breve{p},\tilde{e})}(\breve{s}),\varphi^{+}_{2(\breve{p},\tilde{e})}(\breve{s})\varphi^{+}_{3(\breve{p},\tilde{e})}(\breve{s})]>:\breve{s}\in \mathrm{S}\}$$

$$=h_1(\breve{p},\tilde{e})+\{h_2(\breve{p},\tilde{e})+h_3(\breve{p},\tilde{e})\}.$$

Hence associative law holds as

$$\{h_1(\breve{p},\tilde{e})+h_2(\breve{p},\tilde{e})\}+h_3(\breve{p},\tilde{e})=h_1(\breve{p},\tilde{e})+\{h_2(\breve{p},\tilde{e})+h_3(\breve{p},\tilde{e})\}.$$

4.3.3 (A3) Additive Identity

$$\Phi(\breve{p},\tilde{e})+\mathrm{h}(\breve{p},\tilde{e})=\mathrm{h}(\breve{p},\tilde{e})=\mathrm{h}(\breve{p},\tilde{e})+\Phi(\breve{p},\tilde{e}),$$

Where $\Phi(\check{p}, \tilde{e})$ = {<$\check{s}$, [0, 0], [1, 1]>: s ∈S} is null Q-ROIVFSES for all $\check{p} \in \check{P}$, $\tilde{e} \in \tilde{E}$ and Q ≥ 1 and h($\check{p}$, $\tilde{e}$) = {<$\check{s}$, $[\theta^{-}_{(\check{p},\tilde{e})}(\check{s}), \theta^{+}_{(\check{p},\tilde{e})}(\check{s})]$, $[\varphi^{-}_{(\check{p},\tilde{e})}(\check{s}), \varphi^{+}_{(\check{p},\tilde{e})}(\check{s})]$>: $\check{s}$∈S} ∈ Q-ROIVFSES (h, $\check{P}$, $\tilde{E}$, Q), then we have

As $\Phi(\check{p}, \tilde{e})$ + h($\check{p}$, $\tilde{e}$) = {<$\check{s}$, $[(0^{Q} + \theta^{-}_{(\check{p},\tilde{e})}(\check{s})^{Q} - (0^{Q} \times \theta^{-}_{(\check{p},\tilde{e})}(\check{s})^{Q}))^{1/Q}$,

$(0^{Q} + \theta^{+}_{(\check{p},\tilde{e})}(\check{s})^{Q} - (0^{Q} \times \theta^{+}_{(\check{p},\tilde{e})}(\check{s})^{Q}))^{1/Q}]$, $[1\times \varphi^{-}_{(\check{p},\tilde{e})}(\check{s}), 1\times \varphi^{+}_{(\check{p},\tilde{e})}(\check{s})]$>: $\check{s}$∈S}

= {<$\check{s}$,$[(\theta^{-}_{(\check{p},\tilde{e})}(\check{s})^{Q})^{1/Q}, (\theta^{+}_{(\check{p},\tilde{e})}(\check{s})^{Q})^{1/Q}]$, $[\varphi^{-}_{(\check{p},\tilde{e})}(\check{s}), \varphi^{+}_{(\check{p},\tilde{e})}(\check{s})]$>: $\check{s}$∈S}

= {<$\check{s}$, $[\theta^{-}_{(\check{p},\tilde{e})}(\check{s}), \theta^{+}_{(\check{p},\tilde{e})}(\check{s})]$, $[\varphi^{-}_{(\check{p},\tilde{e})}(\check{s}), \varphi^{+}_{(\check{p},\tilde{e})}(\check{s})]$>: $\check{s}$∈S}

= h($\check{p}$, $\tilde{e}$).

Similarly, h($\check{p}$, $\tilde{e}$) + $\Phi(\check{p}, \tilde{e})$ = h($\check{p}$, $\tilde{e}$). This implies that $\Phi(\check{p}, \tilde{e})$ is additive identity of Q-ROIVFSES over S.

4.3.4 (A4) Additive Inverse

Let us suppose that for each Q-ROIVFSES (h, $\check{P}$, $\tilde{E}$, Q) there exist a Q-ROIVFSES (h′, P′, E′, Q′) over S such that ∀h($\check{p}$, $\tilde{e}$) ∈ (h, P, E, Q) ∃ h′($\check{p}$, $\tilde{e}$) ∈ (h′, P′, E′, Q′) such that

h($\check{p}$, $\tilde{e}$) + h′($\check{p}$, $\tilde{e}$) = $\Phi(\check{p}, \tilde{e})$.

⇒ {<$\check{s}$, $[\theta^{-}_{(\check{p},\tilde{e})}(\check{s}), \theta^{+}_{(\check{p},\tilde{e})}(\check{s})]$, $[\varphi^{-}_{(\check{p},\tilde{e})}(\check{s}), \varphi^{+}_{(\check{p},\tilde{e})}(\check{s})]$>: $\check{s}$∈S} + {<$\check{s}$, $[\theta^{-}_{\prime(\check{p},\tilde{e})}(\check{s}), \theta^{+}_{\prime(\check{p},\tilde{e})}(\check{s})]$, $[\varphi^{-}_{\prime(\check{p},\tilde{e})}(\check{s}), \varphi^{+}_{\prime(\check{p},\tilde{e})}(\check{s})]$>: $\check{s}$∈S} = {<$\check{s}$, [0, 0], [1, 1]>: $\check{s}$∈S}.

⇒{<$\check{s}$,$[(\theta^{-}_{(\check{p},\tilde{e})}(\check{s})^{Q} + \theta^{-}_{\prime(\check{p},\tilde{e})}(\check{s})^{Q} - \theta^{-}_{(\check{p},\tilde{e})}(\check{s})^{Q} \times \theta^{-}_{\prime(\check{p},\tilde{e})}(\check{s})^{Q})^{1/Q}, (\theta^{+}_{(\check{p},\tilde{e})}(\check{s})^{Q}\theta^{+}_{\prime(\check{p},\tilde{e})}(\check{s})^{Q} -$
$\theta^{+}_{(\check{p},\tilde{e})}(\check{s})^{Q} \times \theta^{+}_{\prime(\check{p},\tilde{e})}(\check{s})^{Q})^{1/Q}]$, $[\varphi^{-}_{(\check{p},\tilde{e})}(\check{s})\varphi^{-}_{\prime(\check{p},\tilde{e})}(\check{s}), \varphi^{+}_{(\check{p},\tilde{e})}(\check{s})\varphi^{+}_{\prime(\check{p},\tilde{e})}(\check{s})]$>: $\check{s}$∈S} .

= {<$\check{s}$, [0, 0], [1, 1]>: $\check{s}$∈S}.

⇒ $(\theta^{-}_{(\check{p},\tilde{e})}(\check{s})^{Q} + \theta^{-}_{\prime(\check{p},\tilde{e})}(\check{s})^{Q} - \theta^{-}_{(\check{p},\tilde{e})}(\check{s})^{Q} \times \theta^{-}_{\prime(\check{p},\tilde{e})}(\check{s})^{Q})^{1/Q} = 0$,

$(\theta^{+}_{(\check{p},\tilde{e})}(\check{s})^{Q}\theta^{+}_{\prime(\check{p},\tilde{e})}(\check{s})^{Q} - \theta^{+}_{(\check{p},\tilde{e})}(\check{s})^{Q} \times \theta^{+}_{\prime(\check{p},\tilde{e})}(\check{s})^{Q})^{1/Q} = 0,$

$\varphi^{-}_{(\check{p},\tilde{e})}(\check{s})\varphi^{-}_{\prime(\check{p},\tilde{e})}(\check{s}) = 1.$

$\Rightarrow \varphi^{-}_{\prime(\check{p},\tilde{e})}(\check{s}) = \frac{1}{\varphi^{-}_{(\check{p},\tilde{e})}(\check{s})} \notin [0,1]$ because $\varphi^{-}_{(\check{p},\tilde{e})}(\check{s}) \in [0, 1],$

$\varphi^{+}_{(\check{p},\tilde{e})}(\check{s})\varphi^{+}_{\prime(\check{p},\tilde{e})}(\check{s}) = 1.$

$\Rightarrow \varphi^{+}_{\prime(\check{p},\tilde{e})}(\check{s}) = \frac{1}{\varphi^{+}_{(\check{p},\tilde{e})}(\check{s})} \notin [0,1]$ because $\varphi^{+}_{(\check{p},\tilde{e})}(\check{s}) \in [0, 1].$

This contradicts the fact that h′($\check{p}$, $\tilde{e}$) ∈ Q-ROIVFSES (h′, $\check{P}'$, $\tilde{E}'$, Q′).

Hence for all h($\check{p}$, $\tilde{e}$) ∈ (h, $\check{P}$, $\tilde{E}$, Q) there does not exist h′($\check{p}$, $\tilde{e}$) ∈ (h′, $\check{P}'$, $\tilde{E}'$, Q′) such that h($\check{p}$, $\tilde{e}$) + h′($\check{p}$, $\tilde{e}$) = Φ($\check{p}$, $\tilde{e}$), i.e., additive inverse does not exist.

4.3.5 (A5) Commutativity

As $h_1(\check{p}, \tilde{e}) + h_2(\check{p}, \tilde{e})$ = {<$\check{s}$,[$\left(\theta^{-}_{1(\check{p},\tilde{e})}(\check{s})^{Q} + \theta^{-}_{2(\check{p},\tilde{e})}(\check{s})^{Q} - \theta^{-}_{1(\check{p},\tilde{e})}(\check{s})^{Q}\theta^{-}_{2(\check{p},\tilde{e})}(\check{s})^{Q}\right)^{1/Q}$,

$(\theta^{+}_{1(\check{p},\tilde{e})}(\check{s})^{Q} + \theta^{+}_{2(\check{p},\tilde{e})}(\check{s})^{Q} - \theta^{+}_{1(\check{p},\tilde{e})}(\check{s})^{Q}\theta^{+}_{2(\check{p},\tilde{e})}(\check{s})^{Q})^{1/Q}$],

[$\varphi^{-}_{1(\check{p},\tilde{e})}(\check{s})\varphi^{-}_{2(\check{p},\tilde{e})}(\check{s}), \varphi^{+}_{1(\check{p},\tilde{e})}(\check{s})\varphi^{+}_{2(\check{p},\tilde{e})}(\check{s})$]>: $\check{s}$∈S}

= {<$\check{s}$,[$\left(\theta^{-}_{2(\check{p},\tilde{e})}(\check{s})^{Q} + \theta^{-}_{1(\check{p},\tilde{e})}(\check{s})^{Q} - \theta^{-}_{2(\check{p},\tilde{e})}(\check{s})^{Q}\theta^{-}_{1(\check{p},\tilde{e})}(\check{s})^{Q}\right)^{1/Q}$,

$(\theta^{+}_{2(\check{p},\tilde{e})}(\check{s})^{Q} + \theta^{+}_{1(\check{p},\tilde{e})}(\check{s})^{Q} - \theta^{+}_{2(\check{p},\tilde{e})}(\check{s})^{Q}\theta^{+}_{1(\check{p},\tilde{e})}(\check{s})^{Q})^{1/Q}$],

[$\varphi^{-}_{2(\check{p},\tilde{e})}(\check{s})\varphi^{-}_{1(\check{p},\tilde{e})}(\check{s}), \varphi^{+}_{2(\check{p},\tilde{e})}(\check{s})\varphi^{+}_{1(\check{p},\tilde{e})}(\check{s})$]>: $\check{s}$∈S}

= $h_2(\check{p}, \tilde{e}) + h_1(\check{p}, \tilde{e})$

Hence commutative property i.e.,$h_1(\check{p}, \tilde{e}) + h_2(\check{p}, \tilde{e}) = h_2(\check{p}, \tilde{e}) + h_1(\check{p}, \tilde{e})$ holdsfor all $h_1(\check{p}, \tilde{e})$,$h_2(\check{p}, \tilde{e})$∈ $\tilde{Q}_I(S)$.

4.3.6 (M1) Closure Property w.r.t Multiplication

$h_1(\check{p}, \tilde{e}) \times h_2(\check{p}, \tilde{e}) = \{<\check{s}, [\theta^{-}_{1(\check{p},\tilde{e})}(\check{s})\theta^{-}_{2(\check{p},\tilde{e})}(\check{s}), \theta^{+}_{1(\check{p},\tilde{e})}(\check{s})\theta^{+}_{2(\check{p},\tilde{e})}(\check{s})]$,

$[\left(\varphi^{-}_{1(\check{p},\tilde{e})}(\check{s})^Q + \varphi^{-}_{2(\check{p},\tilde{e})}(\check{s})^Q - \varphi^{-}_{1(\check{p},\tilde{e})}(\check{s})^Q\varphi^{-}_{2(\check{p},\tilde{e})}(\check{s})^Q\right)^{1/Q}$,

$(\varphi^{+}_{1(\check{p},\tilde{e})}(\check{s})^Q + \varphi^{+}_{2(\check{p},\tilde{e})}(\check{s})^Q - \varphi^{+}_{1(\check{p},\tilde{e})}(\check{s})^Q\varphi^{+}_{2(\check{p},\tilde{e})}(\check{s})^Q)^{1/Q}]>: \check{s}\in S\} \in \tilde{Q}_I(S)$.

As $(\theta^{+}_{1(\check{p},\tilde{e})}(\check{s})\theta^{+}_{2(\check{p},\tilde{e})}(\check{s}))^Q + {(\varphi^{+}_{1(\check{p},\tilde{e})}(\check{s})^Q + \varphi^{+}_{2(\check{p},\tilde{e})}(\check{s})^Q - \varphi^{+}_{1(\check{p},\tilde{e})}(\check{s})^Q\varphi^{+}_{2(\check{p},\tilde{e})}(\check{s})^Q)^{1/Q}}^Q \leq 1$

Because

$$(\theta^{+}_{1(\check{p},\tilde{e})}(\check{s})\theta^{+}_{2(\check{p},\tilde{e})}(\check{s}))^Q + {(\varphi^{+}_{1(\check{p},\tilde{e})}(\check{s})^Q + \varphi^{+}_{2(\check{p},\tilde{e})}(\check{s})^Q - \varphi^{+}_{1(\check{p},\tilde{e})}(\check{s})^Q\varphi^{+}_{2(\check{p},\tilde{e})}(\check{s})^Q)^{1/Q}}^Q$$

$$= (\theta^{+}_{1(\check{p},\tilde{e})}(\check{s})^Q \times \theta^{+}_{2(\check{p},\tilde{e})}(\check{s}))^Q + \varphi^{+}_{1(\check{p},\tilde{e})}(\check{s})^Q + \varphi^{+}_{2(\check{p},\tilde{e})}(\check{s})^Q - \varphi^{+}_{1(\check{p},\tilde{e})}(\check{s})^Q\varphi^{+}_{2(\check{p},\tilde{e})}(\check{s})^Q$$

$$\leq (1-\varphi^{+}_{1(\check{p},\tilde{e})}(\check{s})^Q)(1 - \varphi^{+}_{2(\check{p},\tilde{e})}(\check{s})^Q) + \varphi^{+}_{1(\check{p},\tilde{e})}(\check{s})^Q + \varphi^{+}_{2(\check{p},\tilde{e})}(\check{s})^Q - \varphi^{+}_{1(\check{p},\tilde{e})}(\check{s})^Q\varphi^{+}_{2(\check{p},\tilde{e})}(\check{s})^Q$$

$$= 1-\varphi^{+}_{1(\check{p},\tilde{e})}(\check{s})^Q - \varphi^{+}_{2(\check{p},\tilde{e})}(\check{s})^Q + \varphi^{+}_{1(\check{p},\tilde{e})}(\check{s})\varphi^{+}_{2(\check{p},\tilde{e})}(\check{s})^Q + \varphi^{+}_{1(\check{p},\tilde{e})}(\check{s})^Q + \varphi^{+}_{2(\check{p},\tilde{e})}(\check{s})^Q - \varphi^{+}_{1(\check{p},\tilde{e})}(\check{s})^Q\varphi^{+}_{2(\check{p},\tilde{e})}(\check{s})^Q$$

$$= 1.$$

Proved using $\theta^{+}_{1(\check{p},\tilde{e})}(\check{s})^Q \leq 1-\varphi^{+}_{1(\check{p},\tilde{e})}(\check{s})^Q$ and $\theta^{+}_{2(\check{p},\tilde{e})}(\check{s}))^Q \leq 1 - \varphi^{+}_{2(\check{p},\tilde{e})}(\check{s})^Q$.

4.3.7 (M2) Associativity

As

$\{h_1(\check{p},\tilde{e}) \times h_2(\check{p},\tilde{e})\} \times h_3(\check{p}, \tilde{e}) = \{<\check{s}, [\theta^{-}_{1(\check{p},\tilde{e})}(\check{s})\theta^{-}_{2(\check{p},\tilde{e})}(\check{s}), \theta^{+}_{1(\check{p},\tilde{e})}(\check{s})\theta^{+}_{2(\check{p},\tilde{e})}(\check{s})]$,

$[\left(\varphi^{-}_{1(\check{p},\tilde{e})}(\check{s})^Q + \varphi^{-}_{2(\check{p},\tilde{e})}(\check{s})^Q - \varphi^{-}_{1(\check{p},\tilde{e})}(\check{s})^Q\varphi^{-}_{2(\check{p},\tilde{e})}(\check{s})^Q\right)^{1/Q}$,

$(\varphi^{+}_{1(\check{p},\tilde{e})}(\check{s})^Q + \varphi^{+}_{2(\check{p},\tilde{e})}(\check{s})^Q - \varphi^{+}_{1(\check{p},\tilde{e})}(\check{s})^Q\varphi^{+}_{2(\check{p},\tilde{e})}(\check{s})^Q)^{1/Q}]>: \check{s}\in S\} \times h_3(\check{p}, \tilde{e})$.

$$=\{<\breve{s},[(\theta^{-}_{1(\breve{p},\tilde{e})}(\breve{s})\theta^{-}_{2(\breve{p},\tilde{e})}(\breve{s}))\theta^{-}_{3(\breve{p},\tilde{e})}(\breve{s}),(\theta^{+}_{1(\breve{p},\tilde{e})}(\breve{s})\theta^{+}_{2(\breve{p},\tilde{e})}(\breve{s}))\theta^{+}_{3(\breve{p},\tilde{e})}(\breve{s})],$$

$$[\Bigg(\left(\left(\varphi^{-}_{1(\breve{p},\tilde{e})}(\breve{s})^{Q}+\varphi^{-}_{2(\breve{p},\tilde{e})}(\breve{s})^{Q}-\varphi^{-}_{1(\breve{p},\tilde{e})}(\breve{s})^{Q}\varphi^{-}_{2(\breve{p},\tilde{e})}(\breve{s})^{Q}\right)^{1/Q}\right)^{Q}+\varphi^{-}_{3(\breve{p},\tilde{e})}(\breve{s})^{Q}-$$

$$\left(\varphi^{-}_{1(\breve{p},\tilde{e})}(\breve{s})^{Q}+\varphi^{-}_{2(\breve{p},\tilde{e})}(\breve{s})^{Q}-\varphi^{-}_{1(\breve{p},\tilde{e})}(\breve{s})^{Q}\varphi^{-}_{2(\breve{p},\tilde{e})}(\breve{s})^{Q}\right)\varphi^{-}_{3(\breve{p},\tilde{e})}(\breve{s})^{Q}\Bigg)^{1/Q},$$

$$\Bigg(\left(\left(\varphi^{+}_{1(\breve{p},\tilde{e})}(\breve{s})^{Q}+\varphi^{+}_{2(\breve{p},\tilde{e})}(\breve{s})^{Q}-\varphi^{+}_{1(\breve{p},\tilde{e})}(\breve{s})^{Q}\varphi^{+}_{2(\breve{p},\tilde{e})}(\breve{s})^{Q}\right)^{1/Q}\right)^{Q}+\varphi^{+}_{3(\breve{p},\tilde{e})}(\breve{s})^{Q}-$$

$$(\varphi^{+}_{1(\breve{p},\tilde{e})}(\breve{s})^{Q}+\varphi^{+}_{2(\breve{p},\tilde{e})}(\breve{s})^{Q}-\varphi^{+}_{1(\breve{p},\tilde{e})}(\breve{s})^{Q}\varphi^{+}_{2(\breve{p},\tilde{e})}(\breve{s})^{Q})\varphi^{+}_{3(\breve{p},\tilde{e})}(\breve{s})^{Q}\Bigg)^{1/Q}]>:\breve{s}\in S\}$$

$$=\{<\breve{s},(\theta^{-}_{1(\breve{p},\tilde{e})}(\breve{s})\theta^{-}_{2(\breve{p},\tilde{e})}(\breve{s}))\theta^{-}_{3(\breve{p},\tilde{e})}(\breve{s}),(\theta^{+}_{1(\breve{p},\tilde{e})}(\breve{s})\theta^{+}_{2(\breve{p},\tilde{e})}(\breve{s}))\theta^{+}_{3(\breve{p},\tilde{e})}(\breve{s})],$$

$$[\left(\varphi^{-}_{1(\breve{p},\tilde{e})}(\breve{s})^{Q}+\varphi^{-}_{2(\breve{p},\tilde{e})}(\breve{s})^{Q}-\varphi^{-}_{1(\breve{p},\tilde{e})}(\breve{s})^{Q}\varphi^{-}_{2(\breve{p},\tilde{e})}(\breve{s})^{Q}+\varphi^{-}_{3(\breve{p},\tilde{e})}(\breve{s})^{Q}-\right.$$

$$\varphi^{-}_{1(\breve{p},\tilde{e})}(\breve{s})^{Q}\varphi^{-}_{3(\breve{p},\tilde{e})}(\breve{s})^{Q}-\varphi^{-}_{2(\breve{p},\tilde{e})}(\breve{s})^{Q}\varphi^{-}_{3(\breve{p},\tilde{e})}(\breve{s})^{Q}+$$

$$\left.\varphi^{-}_{1(\breve{p},\tilde{e})}(\breve{s})^{Q}\varphi^{-}_{2(\breve{p},\tilde{e})}(\breve{s})^{Q}\varphi^{-}_{3(\breve{p},\tilde{e})}(\breve{s})^{Q}\right)^{1/Q},$$

$$\left(\varphi^{+}_{1(\breve{p},\tilde{e})}(\breve{s})^{Q}+\varphi^{+}_{2(\breve{p},\tilde{e})}(\breve{s})^{Q}-\varphi^{+}_{1(\breve{p},\tilde{e})}(\breve{s})^{Q}\varphi^{+}_{2(\breve{p},\tilde{e})}(\breve{s})^{Q}+\varphi^{+}_{3(\breve{p},\tilde{e})}(\breve{s})^{Q}-\right.$$

$$\varphi^{+}_{1(\breve{p},\tilde{e})}(\breve{s})^{Q}\varphi^{+}_{3(\breve{p},\tilde{e})}(\breve{s})^{Q}-\varphi^{+}_{2(\breve{p},\tilde{e})}(\breve{s})^{Q}\varphi^{+}_{3(\breve{p},\tilde{e})}(\breve{s})^{Q}+$$

$$\left.\varphi^{+}_{1(\breve{p},\tilde{e})}(\breve{s})^{Q}\varphi^{+}_{2(\breve{p},\tilde{e})}(\breve{s})^{Q}\varphi^{+}_{3(\breve{p},\tilde{e})}(\breve{s})^{Q}\right)^{1/Q}]>:\breve{s}\in S\}$$

$$=\{<\breve{s},[\theta^{-}_{1(\breve{p},\tilde{e})}(\breve{s})(\theta^{-}_{2(\breve{p},\tilde{e})}(\breve{s})\theta^{-}_{3(\breve{p},\tilde{e})}(\breve{s})),\theta^{+}_{1(\breve{p},\tilde{e})}(\breve{s})\left(\theta^{+}_{2(\breve{p},\tilde{e})}(\breve{s})\theta^{+}_{3(\breve{p},\tilde{e})}(\breve{s})\right)],$$

$$[\Bigg(\varphi^{-}_{1(\breve{p},\tilde{e})}(\breve{s})^{Q}+\left(\left(\varphi^{-}_{2(\breve{p},\tilde{e})}(\breve{s})^{Q}+\varphi^{-}_{3(\breve{p},\tilde{e})}(\breve{s})^{Q}-\varphi^{-}_{2(\breve{p},\tilde{e})}(\breve{s})^{Q}\varphi^{-}_{3(\breve{p},\tilde{e})}(\breve{s})^{Q}\right)^{1/Q}\right)^{Q}-$$

$$\varphi^{-}_{1(\breve{p},\tilde{e})}(\breve{s})^{Q}\left(\left(\varphi^{-}_{2(\breve{p},\tilde{e})}(\breve{s})^{Q}+\varphi^{-}_{3(\breve{p},\tilde{e})}(\breve{s})^{Q}-\varphi^{-}_{2(\breve{p},\tilde{e})}(\breve{s})^{Q}\varphi^{-}_{3(\breve{p},\tilde{e})}(\breve{s})^{Q}\right)^{1/Q}\right)^{Q}\Bigg)^{1/Q},$$

$$\Bigg(\varphi^{+}_{1(\breve{p},\tilde{e})}(\breve{s})^{Q}+\left(\left(\varphi^{+}_{2(\breve{p},\tilde{e})}(\breve{s})^{Q}+\varphi^{+}_{3(\breve{p},\tilde{e})}(\breve{s})^{Q}-\varphi^{+}_{2(\breve{p},\tilde{e})}(\breve{s})^{Q}\varphi^{+}_{3(\breve{p},\tilde{e})}(\breve{s})^{Q}\right)^{1/Q}\right)^{Q}-$$

$$\varphi^{+}_{1(\breve{p},\tilde{e})}(\breve{s})^{Q}\left(\left(\varphi^{+}_{2(\breve{p},\tilde{e})}(\breve{s})^{Q}+\varphi^{+}_{3(\breve{p},\tilde{e})}(\breve{s})^{Q}-\varphi^{+}_{2(\breve{p},\tilde{e})}(\breve{s})^{Q}\varphi^{+}_{3(\breve{p},\tilde{e})}(\breve{s})^{Q}\right)^{1/Q}\right)^{Q}\Bigg)^{1/Q}]>:$$

$$\breve{s}\in S\}$$

$= h_1(\check{p}, \tilde{e}) \times \{<\check{s},[\theta^-_{2(\check{p},\tilde{e})}(\check{s})\theta^-_{3(\check{p},\tilde{e})}(\check{s}),\theta^+_{2(\check{p},\tilde{e})}(\check{s})\theta^+_{3(\check{p},\tilde{e})}(\check{s})],$

$[\left(\varphi^-_{2(\check{p},\tilde{e})}(\check{s})^Q+\varphi^-_{3(\check{p},\tilde{e})}(\check{s})^Q - \varphi^-_{2(\check{p},\tilde{e})}(\check{s})^Q\varphi^-_{3(\check{p},\tilde{e})}(\check{s})^Q\right)^{1/Q},$

$(\varphi^+_{2(\check{p},\tilde{e})}(\check{s})^Q+\varphi^+_{3(\check{p},\tilde{e})}(\check{s})^Q - \varphi^+_{2(\check{p},\tilde{e})}(\check{s})^Q\varphi^+_{3(\check{p},\tilde{e})}(\check{s})^Q)^{1/Q}]>: \check{s}\in S\}$

$= h_1(\check{p}, \tilde{e}) \times \{h_2(\check{p}, \tilde{e}) \times h_3(\check{p}, \tilde{e})\}.$

Hence associative law holds as

$\{h_1(\check{p}, \tilde{e}) \times h_2(\check{p}, \tilde{e})\} \times h_3(\check{p}, \tilde{e}) = h_1(\check{p}, \tilde{e}) \times \{h_2(\check{p}, \tilde{e}) \times h_3(\check{p}, \tilde{e})\}.$

4.3.8 (M3) Multiplicative identity

$$\tilde{A}(\check{p}, \tilde{e}) \times h(\check{p}, \tilde{e}) = h(\check{p}, \tilde{e}) = h(\check{p}, \tilde{e}) \times \tilde{A}(\check{p}, \tilde{e}),$$

where $\tilde{A}(\check{p}, \tilde{e}) = \{<\check{s}, [1, 1], [0, 0]>: \check{s}\in S\}$ is the absolute Q-ROIVFSES is for all p∈ $\breve{P}$, e ∈ $\tilde{E}$ and Q≥ 1 and h(p, e) = $\{<\check{s}, [\theta^-_{(\check{p},\tilde{e})}(\check{s}),\theta^+_{(\check{p},\tilde{e})}(\check{s})], [\varphi^-_{(\check{p},\tilde{e})}(\check{s}), \varphi^+_{(\check{p},\tilde{e})}(\check{s})]>: \check{s}\in S\}\in$ Q-ROIVFSES (h, $\breve{P}$, $\tilde{E}$, Q), then we have

As $\tilde{A}(\check{p}, \tilde{e}) \times h(\check{p}, \tilde{e}) = \{<\check{s}, [1\times \theta^-_{(\check{p},\tilde{e})}(\check{s}), 1\times \theta^+_{(\check{p},\tilde{e})}(\check{s})],$

$[(0^Q + \varphi^-_{(\check{p},\tilde{e})}(\check{s})^Q - (0^Q \times \varphi^-_{(\check{p},\tilde{e})}(\check{s})^Q))^{1/Q},$

$(0^Q + \varphi^+_{(\check{p},\tilde{e})}(\check{s})^Q - (0^Q \times \varphi^+_{(\check{p},\tilde{e})}(\check{s})^Q))^{1/Q}]>: \check{s}\in S\}$

$= \{<\check{s},[\theta^-_{(\check{p},\tilde{e})}(\check{s}), \theta^+_{(\check{p},\tilde{e})}(\check{s})],[(\varphi^-_{(\check{p},\tilde{e})}(\check{s})^Q)^{1/Q},(\varphi^+_{(\check{p},\tilde{e})}(\check{s})^Q)^{1/Q}]>: \check{s}\in S\}$

$= \{<\check{s}, [\theta^-_{(\check{p},\tilde{e})}(\check{s}), \theta^+_{(\check{p},\tilde{e})}(\check{s})], [\varphi^-_{(\check{p},\tilde{e})}(\check{s}),\varphi^+_{(\check{p},\tilde{e})}(\check{s})]>: \check{s}\in S\}$

$= h(\check{p}, \tilde{e})$

Similarly, $h(\check{p}, \tilde{e}) \times \tilde{A}(\check{p}, \tilde{e}) = h(\check{p}, \tilde{e})$. This implies that $\tilde{A}(\check{p}, \tilde{e})$ is multiplicative identity of Q-ROIVFSES over S.

4.3.9 (M4) Multiplicative Inverse

Let us suppose that for each Q-ROIVFSES (h, P̆, Ẽ, Q) there exist a Q-ROIVFSES (h′, P̆′, Ẽ′, Q′) over S such that ∀ h$(\check{p}, \tilde{e})$ ∈ (h, P̆, Ẽ, Q) ∃ h′$(\check{p}, \tilde{e})$ ∈ (h′, P̆′, Ẽ′, Q′) such that

h$(\check{p}, \tilde{e})$ × h′$(\check{p}, \tilde{e})$ = $\tilde{A}(\check{p}, \tilde{e})$.

⇒ {<$\check{s}$, $[\theta^{-}_{(\check{p},\tilde{e})}(\check{s}),\theta^{+}_{(\check{p},\tilde{e})}(\check{s})]$, $[\varphi^{-}_{(\check{p},\tilde{e})}(\check{s}),\varphi^{+}_{(\check{p},\tilde{e})}(\check{s})]$>: $\check{s}$∈S} ×{<$\check{s}$, $[\theta'^{-}_{(\check{p},\tilde{e})}(\check{s}),\theta'^{+}_{(\check{p},\tilde{e})}(\check{s})]$, $[\varphi'^{-}_{(\check{p},\tilde{e})}(\check{s}),\varphi'^{+}_{(\check{p},\tilde{e})}(\check{s})]$>: $\check{s}$∈S} = {<$\check{s}$, [1, 1], [0, 0]>: $\check{s}$∈S}.

⇒{<$\check{s}$, $\left[\theta^{-}_{(\check{p},\tilde{e})}(\check{s})\theta'^{-}_{(\check{p},\tilde{e})}(\check{s}),\theta^{+}_{(\check{p},\tilde{e})}(\check{s})\theta'^{+}_{(\check{p},\tilde{e})}(\check{s})\right]$, $[(\varphi^{-}_{(\check{p},\tilde{e})}(\check{s})^{Q}+\varphi'^{-}_{(\check{p},\tilde{e})}(\check{s})^{Q}-\varphi^{-}_{(\check{p},\tilde{e})}(\check{s})^{Q}\times \varphi'^{-}_{(\check{p},\tilde{e})}(\check{s})^{Q})^{1/Q},(\varphi^{+}_{(\check{p},\tilde{e})}(\check{s})^{Q}\varphi'^{+}_{(\check{p},\tilde{e})}(\check{s})^{Q}-\varphi^{+}_{(\check{p},\tilde{e})}(\check{s})^{Q}\times\varphi'^{+}_{(\check{p},\tilde{e})}(\check{s})^{Q})^{1/Q}]$>: $\check{s}$∈S} = {<$\check{s}$, [1, 1], [0, 0]>: $\check{s}$∈S}.

$\Rightarrow \theta^{-}_{(\check{p},\tilde{e})}(\check{s})\theta'^{-}_{(\check{p},\tilde{e})}(\check{s}) = 1, \quad \theta^{+}_{(\check{p},\tilde{e})}(\check{s})\theta'^{+}_{(\check{p},\tilde{e})}(\check{s}) = 1,$

$(\varphi^{-}_{(\check{p},\tilde{e})}(\check{s})^{Q}+\varphi'^{-}_{(\check{p},\tilde{e})}(\check{s})^{Q}-\varphi^{-}_{(\check{p},\tilde{e})}(\check{s})^{Q}\times\varphi'^{-}_{(\check{p},\tilde{e})}(\check{s})^{Q})^{1/Q} = 0,$

$(\varphi^{+}_{(\check{p},\tilde{e})}(\check{s})^{Q}\varphi'^{+}_{(\check{p},\tilde{e})}(\check{s})^{Q}-\varphi^{+}_{(\check{p},\tilde{e})}(\check{s})^{Q}\times\varphi'^{+}_{(\check{p},\tilde{e})}(\check{s})^{Q})^{1/Q} = 0.$

$\Rightarrow \theta^{-}_{(\check{p},\tilde{e})}(\check{s})\theta'^{-}_{(\check{p},\tilde{e})}(\check{s}) = 1 \Rightarrow \theta'^{-}_{(\check{p},\tilde{e})}(\check{s}) = \frac{1}{\theta^{-}_{(\check{p},\tilde{e})}(\check{s})} \notin [0,1]$ because $\theta^{-}_{(p,e)}(\check{s}) \in [0,1]$.

And $\theta^{+}_{(\check{p},\tilde{e})}(\check{s})\theta'^{+}_{(\check{p},\tilde{e})}(\check{s}) = 1 \Rightarrow \theta'^{+}_{(\check{p},\tilde{e})}(\check{s}) = \frac{1}{\theta^{+}_{(\check{p},\tilde{e})}(\check{s})} \notin [0,1]$ because $\theta^{+}_{(p,e)}(\check{s}) \in [0,1]$.

This contradicts the fact that h′$(\check{p}, \tilde{e})$ ∈ Q-ROIVFSES (h′, P̆′, Ẽ′, Q′). Hence for all h$(\check{p}, \tilde{e})$ ∈ (h, P̆, Ẽ, Q) there does not exist h′$(\check{p}, \tilde{e})$ ∈ (h′, P̆′, Ẽ′, Q′) such that h$(\check{p}, \tilde{e})$ × h′$(\check{p}, \tilde{e})$ = $\tilde{A}(\check{p}, \tilde{e})$, i.e., multiplicative inverse does not exist.

4.3.10 (M5) Commutativity

As $h_1(\check{p},\tilde{e}) \times h_2(\check{p},\tilde{e})$ = {<$\check{s}$, $[\theta^{-}_{1(\check{p},\tilde{e})}(\check{s})\theta^{-}_{2(\check{p},\tilde{e})}(\check{s}),\theta^{+}_{1(\check{p},\tilde{e})}(\check{s})\theta^{+}_{2(\check{p},\tilde{e})}(\check{s})]$,

$$[\left(\varphi^{-}_{1(\check{p},\tilde{e})}(\check{s})^{Q}+\varphi^{-}_{2(\check{p},\tilde{e})}(\check{s})^{Q}-\varphi^{-}_{1(\check{p},\tilde{e})}(\check{s})^{Q}\varphi^{-}_{2(\check{p},\tilde{e})}(\check{s})^{Q}\right)^{1/Q},$$

$$(\varphi^{+}_{1(\check{p},\tilde{e})}(\check{s})^{Q}+\varphi^{+}_{2(\check{p},\tilde{e})}(\check{s})^{Q}-\varphi^{+}_{1(\check{p},\tilde{e})}(\check{s})^{Q}\varphi^{+}_{2(\check{p},\tilde{e})}(\check{s})^{Q})^{1/Q}]>: s∈S\}$$

$= \{<\breve{s}, [\theta^-_{2(\breve{p},\tilde{e})}(\breve{s})\theta^-_{1(\breve{p},\tilde{e})}(\mathrm{s}), \theta^+_{2(\breve{p},\tilde{e})}(\breve{s})\theta^+_{1(\breve{p},\tilde{e})}(\breve{s})],$

$[\left(\varphi^-_{2(\breve{p},\tilde{e})}(\breve{s})^Q + \varphi^-_{1(\breve{p},\tilde{e})}(\breve{s})^Q - \varphi^-_{2(\breve{p},\tilde{e})}(\breve{s})^Q\varphi^-_{1(\breve{p},\tilde{e})}(\breve{s})^Q\right)^{1/Q},$

$(\varphi^+_{2(\breve{p},\tilde{e})}(\breve{s})^Q + \varphi^+_{1(\breve{p},\tilde{e})}(\breve{s})^Q - \varphi^+_{2(\breve{p},\tilde{e})}(\breve{s})^Q\varphi^+_{1(\breve{p},\tilde{e})}(\breve{s})^Q)^{1/Q}]>: \breve{s}\in S\}$

$= h_2(\breve{p},\tilde{e}) \times h_1(\breve{p},\tilde{e}).$

Hence commutative property i.e., $h_1(\breve{p},\tilde{e}) \times h_2(\breve{p},\tilde{e}) = h_2(\breve{p},\tilde{e}) \times h_1(\breve{p},\tilde{e})$ holds for all $h_1(\breve{p},\tilde{e})$, $h_2(\breve{p},\tilde{e}) \in \tilde{Q}_I(S)$.

CHAPTER 5

AGGREGATION OPERATORS OF Q-ROIVFSESs AND DECISION ANALYSIS

5.1 Introduction

Building on the operational rules explained in the previous chapter regarding Q-ROIVFSESs, this chapter introduces and examines various aggregation operators in three distinct sections. The initial section focuses on the Q-Rung Orthopair Interval Valued Fuzzy Soft Expert Weighted Averaging Operator, followed by the subsequent section, which explores the Q-Rung Orthopair Interval Valued Fuzzy Soft Expert Weighted Geometric Operator. Each operator is explained with definitions and supported by practical examples.

Within these sections, the exploration encompasses different theorems dealing with essential attributes like idempotency, monotonicity and boundedness, considering different values of Q. The chapter also defines and discusses additional Aggregation Operators, such as the Q-Rung Orthopair Interval Valued Fuzzy Soft Expert Ordered Weighted Averaging Operator and the Q-Rung Orthopair Interval Valued Fuzzy Soft Expert Fusion Weighted Geometric Operator etc.

The zenith of the chapter is marked by a venture into decision analysis. An algorithm is formulated and applied to various real-world scenarios. In the subsequent section, three examples are presented, each scrutinizing the comparative effectiveness of diverse aggregation operators and exploring the consistency of results. The following example delves into the flexibility and sensitivity of parameter Q by experimenting with different values within a practical Multiple Criteria Decision Making (MCDM) context. The final example compares Q-ROIVFSEFWAO with IVIFSEFWAO, concluding the chapter with a synthesis of intellectual depth and analytical prowess.

5.2 Q-Rung Orthopair Interval Valued Fuzzy Soft Expert Weighted Averaging Operator (Q-ROIVFSEWAO)

Definition 5.2.1

Suppose $(h_k, \breve{P}_k, \tilde{E}_k, \mathrm{Q})_{(\theta_k,\, \varphi_k)} = \{< \check{s}, [\theta^-_{k(\check{p},\tilde{e})}(\check{s}),\ \theta^+_{k(\check{p},\tilde{e})}(\check{s})],\ [\varphi^-_{k(\check{p},\tilde{e})}(\check{s}), \varphi^+_{k(\check{p},\tilde{e})}(\check{s})] >: \check{s} \in \mathrm{S}\}$ with Q ≥ 1, be a Q-ROIVFSESs collection.

A mapping $\varpi : Q_I^{\breve{m}}(S) \rightarrow Q_I(S)$ is called Q-ROIVFSEWAO if it satisfies

$$\varpi\left(\left(h_k, \breve{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}\right) = \sum_{k=1}^{\breve{m}} \breve{W}_k \left(h_k, \breve{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)},$$

where $Q_I^{\breve{m}}(S)$ denote $\breve{m}$ copies of Q-ROIVFSES and weight vector $\breve{W} = (\breve{W}_1, \breve{W}_2, \ldots, \breve{W}_{\breve{m}})^{\mathrm{T}}$ of $\left(h_k, \breve{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}$, for $1 \leq k \leq \breve{m}$ satisfying the normalized condition i.e., $\sum_{k=1}^{\breve{m}} \breve{W}_k = 1$ and $\breve{W}_k \in [0, 1]$.

Note that $\left(\mathrm{h_k}, \breve{\mathrm{P}}_\mathrm{k}, \tilde{\mathrm{E}}_\mathrm{k}, \mathrm{Q}\right)_{(\theta_\mathrm{k},\, \varphi_\mathrm{k})}$ refers to a Q-ROIVFSES $\mathrm{h_k}(\check{\mathrm{p}}_\mathrm{i}, \tilde{\mathrm{e}}_\mathrm{i})$ for some $\check{\mathrm{p}}_\mathrm{i} \in \breve{\mathrm{P}}_\mathrm{k}, \tilde{\mathrm{e}}_\mathrm{i} \in \tilde{\mathrm{E}}_\mathrm{k}$ and it is a generalized way of writing and k indicate the element number.

Definition 5.2.2

If $\breve{W} = (\frac{1}{\breve{m}}, \frac{1}{\breve{m}}, \ldots, \frac{1}{\breve{m}})^{\mathrm{T}}$ in Definition 5.2.1 then Q-ROIVFSEWAO can be written as

$$\varpi\left(\left(h_k, \breve{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}\right) = \frac{1}{\breve{m}} \sum_{k=1}^{\breve{m}} \left(h_k, \breve{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}.$$

Under these circumstances ϖ simplifies to Q-ROIVFSE arithmetic mean operator ϖ_A.

Theorem 5.2.3

Consider $(h_k, \breve{P}_k, \tilde{E}_k, \mathrm{Q})_{(\theta_k,\, \varphi_k)} = \{< \check{s}, [\theta^-_{k(\check{p},\tilde{e})}(\check{s}),\ \theta^+_{k(\check{p},\tilde{e})}(\check{s})], [\varphi^-_{k(\check{p},\tilde{e})}(\check{s}), \varphi^+_{k(\check{p},\tilde{e})}(\check{s})] >: \check{s} \in \mathrm{S}\}$ with Q ≥ 1, be a Q-ROIVFSESs collection. Then, on the basis of operational rules characterized for Q-ROIVFSESs, for any k ∈ ℕ Q-ROIVFSEWAO, aggregation is also a Q-ROIVFSES and

$$\varpi\left((h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q})_{(\theta_k,\, \varphi_k)}\right) = <\left[\left(1 - \prod_{k=1}^{\breve{m}}\left(1 - (\theta^-_{k(\check{p},\tilde{e})}(\check{s}))^Q\right)^{\widetilde{W}_k}\right)^{1/Q},\ \left(1 - \prod_{k=1}^{\breve{m}}\left(1 - (\theta^+_{k(\check{p},\tilde{e})}(\check{s}))^Q\right)^{\widetilde{W}_k}\right)^{1/Q}\right], \left[\prod_{k=1}^{\breve{m}}(\varphi^-_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k},\ \prod_{k=1}^{\breve{m}}(\varphi^+_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}\right] >.$$

Proof

Proofing this result through mathematical induction, it is obvious that the result holds for k = 1, by using Definition 4.2.29 and Remark 4.2.30.

Now for k = 2

By using Definition 5.2.1, we have

$$\varpi\left((h_1, \check{P}_1, \tilde{E}_1, \mathrm{Q})_{(\theta_1,\, \varphi_1)}, (h_2, \check{P}_2, \tilde{E}_2, \mathrm{Q})_{(\theta_2,\, \varphi_2)}\right) = \sum_{k=1}^{2} \widetilde{W}_k (h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q})_{(\theta_k,\, \varphi_k)}$$

$$= \widetilde{W}_1 (h_1, \check{P}_1, \tilde{E}_1, \mathrm{Q})_{(\theta_1,\, \varphi_1)} + \widetilde{W}_2 (h_2, \check{P}_2, \tilde{E}_2, \mathrm{Q})_{(\theta_2,\, \varphi_2)}$$

$$= < [\left(1 - \left(1 - \theta^-_{1(\check{p},\tilde{e})}(\check{s})^Q\right)^{\widetilde{W}_1}\right)^{1/Q}, \left(1 - \left(1 - \theta^+_{1(\check{p},\tilde{e})}(\check{s})^Q\right)^{\widetilde{W}_1}\right)^{1/Q}], [\varphi^-_{1(\check{p},\tilde{e})}(\check{s})^{\widetilde{W}_1}, \varphi^+_{1(\check{p},\tilde{e})}(\check{s})^{\widetilde{W}_1}]> +$$

$$< [\left(1 - \left(1 - \theta^-_{2(\check{p},\tilde{e})}(\check{s})^Q\right)^{\widetilde{W}_2}\right)^{1/Q}, \left(1 - \left(1 - \theta^+_{2(\check{p},\tilde{e})}(\check{s})^Q\right)^{\widetilde{W}_2}\right)^{1/Q}], [\varphi^-_{2(\check{p},\tilde{e})}(s)^{\widetilde{W}_2}, \varphi^+_{2(\check{p},\tilde{e})}(\check{s})^{\widetilde{W}_2}]>$$

$$= < \left[\left(\left(\left(1 - \left(1 - \theta^-_{1(\check{p},\tilde{e})}(\check{s})^Q\right)^{\widetilde{W}_1}\right)^{1/Q}\right)^Q + \left(\left(1 - \left(1 - \theta^-_{2(\check{p},\tilde{e})}(\check{s})^Q\right)^{\widetilde{W}_2}\right)^{1/Q}\right)^Q - \left(\left(1 - \left(1 - \theta^-_{1(\check{p},\tilde{e})}(\check{s})^Q\right)^{\widetilde{W}_1}\right)^{1/Q}\right)^Q \left(\left(1 - \left(1 - \theta^-_{2(\check{p},\tilde{e})}(\check{s})^Q\right)^{\widetilde{W}_2}\right)^{1/Q}\right)^Q\right)^{1/Q}, \left(\left(\left(1 - \left(1 - \theta^+_{1(\check{p},\tilde{e})}(\check{s})^Q\right)^{\widetilde{W}}\right)^{1/Q}\right)^Q + \left(\left(1 - \left(1 - \theta^+_{2(\check{p},\tilde{e})}(\check{s})^Q\right)^{\widetilde{W}_2}\right)^{1/Q}\right)^Q - \left(\left(1 - \left(1 - \theta^+_{1(\check{p},\tilde{e})}(\check{s})^Q\right)^{\widetilde{W}_1}\right)^{1/Q}\right)^Q \left(\left(1 - \left(1 - \theta^+_{2(\check{p},\tilde{e})}(\check{s})^Q\right)^{\widetilde{W}_2}\right)^{1/Q}\right)^Q\right)^{1/Q}\right],$$

$$\left[\varphi^-_{1(\check{p},\tilde{e})}(\check{s})^{\widetilde{W}_1} \varphi^-_{2(\check{p},\tilde{e})}(\check{s})^{\widetilde{W}_2}, \varphi^+_{1(\check{p},\tilde{e})}(\check{s})^{\widetilde{W}_1} \varphi^+_{2(\check{p},\tilde{e})}(\check{s})^{\widetilde{W}_2}\right] >$$

$$= \quad < \Bigg[\Bigg(1-\left(1-\theta^{-}_{1(\check{p},\tilde{e})}(\check{s})^{Q}\right)^{\widetilde{W}_1}+\ 1-\left(1-\theta^{-}_{2(\check{p},\tilde{e})}(\check{s})^{Q}\right)^{\widetilde{W}_2}-\ 1+\left(1-\theta^{-}_{1(\check{p},\tilde{e})}(\check{s})^{Q}\right)^{\widetilde{W}_1}+ \left(1-\theta^{-}_{2(\check{p},\tilde{e})}(\check{s})^{Q}\right)^{\widetilde{W}_2}-\left(1-\theta^{-}_{1(\check{p},\tilde{e})}(\check{s})^{Q}\right)^{\widetilde{W}_1}\left(1-\theta^{-}_{2(\check{p},\tilde{e})}(\check{s})^{Q}\right)^{\widetilde{W}_2}\Bigg)^{1/Q},\Bigg(1-\left(1-\theta^{+}_{1(\check{p},\tilde{e})}(\check{s})^{Q}\right)^{\widetilde{W}_1}+\ 1-\left(1-\theta^{+}_{2(\check{p},\tilde{e})}(\check{s})^{Q}\right)^{\widetilde{W}_2}-\ 1+\left(1-\theta^{+}_{1(\check{p},\tilde{e})}(\check{s})^{Q}\right)^{\widetilde{W}_1}+\left(1-\theta^{+}_{2(\check{p},\tilde{e})}(\check{s})^{Q}\right)^{\widetilde{W}_2}-\left(1-\theta^{+}_{1(\check{p},\tilde{e})}(\check{s})^{Q}\right)^{\widetilde{W}_1}\left(1-\theta^{+}_{2(\check{p},\tilde{e})}(\check{s})^{Q}\right)^{W_2}\Bigg)^{1/Q}\Bigg], \left[\varphi^{-}_{1(\check{p},\tilde{e})}(\check{s})^{\widetilde{W}_1}\varphi^{-}_{2(\check{p},\tilde{e})}(\check{s})^{\widetilde{W}_2},\varphi^{+}_{1(\check{p},\tilde{e})}(\check{s})^{\widetilde{W}_1}\varphi^{+}_{2(\check{p},\tilde{e})}(\check{s})^{\widetilde{W}_2}\right]>$$

$$= \quad < \Bigg[\Bigg(1-\left(1-\theta^{-}_{1(\check{p},\tilde{e})}(\check{s})^{Q}\right)^{\widetilde{W}_1}\left(1-\theta^{-}_{2(\check{p},\tilde{e})}(\check{s})^{Q}\right)^{\widetilde{W}_2}\Bigg)^{1/Q},\Bigg(1-\left(1-\theta^{+}_{1(\check{p},\tilde{e})}(\check{s})^{Q}\right)^{\widetilde{W}_1}\left(1-\theta^{+}_{2(\check{p},\tilde{e})}(\check{s})^{Q}\right)^{\widetilde{W}_2}\Bigg)^{1/Q}\Bigg],\left[\varphi^{-}_{1(\check{p},\tilde{e})}(\check{s})^{\widetilde{W}_1}\varphi^{-}_{2(\check{p},\tilde{e})}(s)^{\widetilde{W}_2},\varphi^{+}_{1(\check{p},\tilde{e})}(\check{s})^{\widetilde{W}_1}\varphi^{+}_{2(\check{p},\tilde{e})}(\check{s})^{\widetilde{W}_2}\right]>$$

$$= \quad < \left[\left(1-\prod_{k=1}^{2}\left(1-(\theta^{-}_{k(\check{p},\tilde{e})}(\check{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q},\left(1-\prod_{k=1}^{2}\left(1-(\theta^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q}\right], \left[\prod_{k=1}^{2}(\varphi^{-}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k},\ \prod_{k=1}^{2}(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}\right]>.$$

Next, suppose that result holds for k = $\breve{m}$, that is

$$\varpi\left(\left(h_n,\check{P}_n,\tilde{E}_n,\mathrm{Q}\right)_{(\theta_n,\,\varphi_n)}\right)=\sum_{k=1}^{\breve{m}}\widetilde{W}_k\left(h_k,\check{P}_k,\tilde{E}_k,\mathrm{Q}\right)_{(\theta_k,\,\varphi_k)}$$

$$=<\left[\left(1-\prod_{k=1}^{\breve{m}}\left(1-\left(\theta^{-}_{k(\check{p},\tilde{e})}(\check{s})\right)^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q},\ \left(1-\prod_{k=1}^{\breve{m}}\left(1-\left(\theta^{+}_{k(\check{p},\tilde{e})}(\check{s})\right)^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q}\right], \left[\prod_{k=1}^{\breve{m}}(\varphi^{-}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k},\quad \prod_{k=1}^{\breve{m}}(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}\right]>$$

Further to show that result holds for k = $\breve{m}$+1,

$$\varpi\left(\left(h_1,\check{P}_1,\tilde{E}_1,\mathrm{Q}\right)_{(\theta_1,\,\varphi_1)},\left(h_2,\check{P}_2,\tilde{E}_2,\mathrm{Q}\right)_{(\theta_2,\,\varphi_2)},\ldots,\left(h_{\breve{m}+1},\check{P}_{\breve{m}+1},\tilde{E}_{\breve{m}+1},\mathrm{Q}\right)_{(\theta_{\overline{m}+1},\,\varphi_{\overline{m}+1})}\right)$$

$= \sum_{k=1}^{\breve{m}+1} \widetilde{W}_k \left(h_k, \breve{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k, \varphi_k)}$

$= \sum_{k=1}^{\breve{m}} \widetilde{W}_k \left(h_k, \breve{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k, \varphi_k)} + \widetilde{W}_{\breve{m}+1} \left(h_{\breve{m}+1}, \breve{P}_{\breve{m}+1}, \tilde{E}_{\breve{m}+1}, \mathrm{Q}\right)_{(\theta_{\breve{m}+1}, \varphi_{\breve{m}+1})}$

Again by using Definition (4.2.26) and (4.2.29), we have

$$\varpi\left(\left(h_{\breve{m}+1}, \breve{P}_{\breve{m}+1}, \tilde{E}_{\breve{m}+1}, \mathrm{Q}\right)_{(\theta_{\breve{m}+1}, \varphi_{\breve{m}+1})}\right) = < \left[\left(1 - \prod_{k=1}^{\breve{m}+1} \left(1 - \left(\theta^{-}_{k(\breve{p},\tilde{e})}(\breve{s})\right)^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q}, \left(1 - \prod_{k=1}^{\breve{m}+1} \left(1 - \left(\theta^{+}_{k(\breve{p},\tilde{e})}(\breve{s})\right)^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q}\right], \left[\prod_{k=1}^{\breve{m}+1} (\varphi^{-}_{k(\breve{p},\tilde{e})}(\breve{s}))^{\widetilde{W}_k}, \prod_{k=1}^{\breve{m}+1} (\varphi^{+}_{k(\breve{p},\tilde{e})}(\breve{s}))^{\widetilde{W}_k}\right] >.$$

Hence proved.

Next to prove that $\varpi\left(\left(h_k, \breve{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k, \varphi_k)}\right)$ is also Q-ROIVFSES. Let

$\theta^{-} = \left(1 - \prod_{k=1}^{\breve{m}} \left(1 - (\theta^{-}_{k(\breve{p},\tilde{e})}(\breve{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q}$, $\theta^{+} = \left(1 - \prod_{k=1}^{\breve{m}} \left(1 - (\theta^{+}_{k(\breve{p},\tilde{e})}(\breve{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q}$,

$\varphi^{-} = \prod_{k=1}^{\breve{m}} (\varphi^{-}_{k(\breve{p},\tilde{e})}(\breve{s}))^{\widetilde{W}_k}$ and $\varphi^{+} = \prod_{k=1}^{\breve{m}} (\varphi^{+}_{k(\breve{p},\tilde{e})}(\breve{s}))^{\widetilde{W}_k}$.

Since $0 \leq \theta^{-}_{k(\breve{p},\tilde{e})}(\breve{s}), \theta^{+}_{k(\breve{p},\tilde{e})}(\breve{s}), \varphi^{-}_{k(\breve{p},\tilde{e})}(\breve{s}), \varphi^{+}_{k(\breve{p},\tilde{e})}(\breve{s}) \leq 1$.

So, $0 \leq 1 - \theta^{-}_{k(\breve{p},\tilde{e})}(\breve{s})^{Q} \leq 1$

$\Rightarrow 0 \leq (1 - \theta^{-}_{k(\breve{p},\tilde{e})}(\breve{s})^{Q})^{\widetilde{W}_k} \leq 1$

$\Rightarrow 0 \leq \prod_{k=1}^{\breve{m}} \left(1 - (\theta^{-}_{k(\breve{p},\tilde{e})}(\breve{s}))^{Q}\right)^{\widetilde{W}_k} \leq 1$

$\Rightarrow 0 \leq 1 - \prod_{k=1}^{\breve{m}} \left(1 - (\theta^{-}_{k(\breve{p},\tilde{e})}(\breve{s}))^{Q}\right)^{\widetilde{W}_k} \leq 1$

$\Rightarrow 0 \leq \left(1 - \prod_{k=1}^{\breve{m}} \left(1 - (\theta^{-}_{k(\breve{p},\tilde{e})}(\breve{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q} \leq 1$

$\Rightarrow 0 \leq \theta^{-} \leq 1$

Similarly,

$\Rightarrow 0 \leq 1 - (\theta^{+}_{k(\breve{p},\tilde{e})}(\breve{s}))^{Q} \leq 1$

$\Rightarrow 0 \leq (1 - \theta^{+}_{k(\breve{p},\tilde{e})}(\breve{s})^{Q})^{\widetilde{W}_k} \leq 1$

$\Rightarrow 0 \leq \prod_{k=1}^{\breve{m}} \left(1 - (\theta^{+}_{k(\breve{p},\tilde{e})}(\breve{s}))^{Q}\right)^{\widetilde{W}_k} \leq 1$

$\Rightarrow 0 \leq 1 - \prod_{k=1}^{\breve{m}}\left(1 - (\theta^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{Q}\right)^{\widetilde{W}_k} \leq 1$

$\Rightarrow 0 \leq \left(1 - \prod_{k=1}^{\breve{m}}\left(1 - (\theta^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q} \leq 1$

$\Rightarrow 0 \leq \theta^{+} \leq 1$

Also, for φ^{-}, φ^{+} , we have

$0 \leq (\varphi^{-}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k} \leq 1$

$\Rightarrow 0 \leq \prod_{k=1}^{\breve{m}}(\varphi^{-}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k} \leq 1$

$\Rightarrow 0 \leq \varphi^{-} \leq 1$

Similarly,

$\Rightarrow 0 \leq (\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k} \leq 1$

$\Rightarrow 0 \leq \prod_{k=1}^{\breve{m}}(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k} \leq 1$

$\Rightarrow 0 \leq \varphi^{+} \leq 1$

Therefore $0 \leq \theta^{-}, \theta^{+}, \varphi^{-}, \varphi^{+} \leq 1$

Next, because $0 \leq \theta^{+}_{k(\check{p},\tilde{e})}(\check{s}), \varphi^{+}_{k(\check{p},\tilde{e})}(\check{s}) \leq 1$ and $\left(\theta^{+}_{k(\check{p},\tilde{e})}(\check{s})\right)^{Q} + \left(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s})\right)^{Q} \leq 1$

We have

$\left(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s})\right)^{Q} \leq 1 - \left(\theta^{+}_{k(\check{p},\tilde{e})}(\check{s})\right)^{Q}$

$$\Rightarrow \left(\left(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s})\right)^{Q}\right)^{\widetilde{W}_k} \leq \left(1 - \left(\theta^{+}_{k(\check{p},\tilde{e})}(\check{s})\right)^{Q}\right)^{\widetilde{W}_k}$$

$$\Rightarrow \prod_{k=1}^{\breve{m}}\left(\left(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s})\right)^{Q}\right)^{\widetilde{W}_k} \leq \prod_{k=1}^{\breve{m}}\left(1 - \left(\theta^{+}_{k(\check{p},\tilde{e})}(\check{s})\right)^{Q}\right)^{\widetilde{W}_k} \qquad \ldots\ (5.2.1)$$

Then consider

$$0 \leq (\theta^{+})^{Q} + (\varphi^{+})^{Q} = \left[\left(1 - \prod_{k=1}^{\breve{m}}\left(1 - (\theta^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q}\right]^{Q} + \left[\prod_{k=1}^{\breve{m}}(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}\right]^{Q}$$

$$= 1 - \prod_{k=1}^{\breve{m}}\left(1 - (\theta^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{Q}\right)^{\widetilde{W}_k} + \left[\prod_{k=1}^{\breve{m}}(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}\right]^{Q}$$

$\leq 1 - \prod_{k=1}^{\breve{m}}\left(1 - \left(\theta^{+}_{k(\check{p},\tilde{e})}(\check{s})\right)^{Q}\right)^{\widetilde{W}_k} + \prod_{k=1}^{\breve{m}}\left(1 - \left(\theta^{+}_{k(\check{p},\tilde{e})}(\check{s})\right)^{Q}\right)^{\widetilde{W}_k}$ using Inequality (5.2.1)

$= 1$

So, the aggregated results of Q-ROIVFSEWAO meets the following two condition

i) $0 \leq \theta^{-}, \theta^{+}, \varphi^{-}, \varphi^{+} \leq 1$,

ii) $0 \leq (\theta^{+})^{Q} + (\varphi^{+})^{Q} \leq 1$.

Thus, it is also a Q-ROIVFSES and theorem is proved.

Example 5.2.4

Let S = $\{\check{s}_1, \check{s}_2\}$, P = $\{\check{p}_1\}$, E = $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$be the universe, parameter and expert set respectively. Consider the Q-ROIVFSESs

$(h_1, \check{P}_1, \tilde{E}_1, Q) = h(\check{p}_1, \tilde{e}_1) = \{< \check{s}_1, [0.7, 0.8], [0.5, 0.6] >, < \check{s}_2, [0.6, 0.7], [0.6, 0.7] >\}$,

$(h_2, \check{P}_2, \tilde{E}_2, Q) = h(\check{p}_1, \tilde{e}_2) = \{< \check{s}_1, [0.3, 0.5], [0.5, 0.7] >, < \check{s}_2, [0.5, 0.6], [0.7, 0.8] >\}$,

$(h_3, \check{P}_3, \tilde{E}_3, Q) = h(\check{p}_1, \tilde{e}_3) = \{< \check{s}_1, [0.6, 0.7], [0.7, 0.8] >, < \check{s}_2, [0.5, 0.6], [0.3, 0.4] >\}$.

And $\widetilde{W} = (0.3, 0.3, 0.4)^{T}$ be the weight vector of $h(\check{p}_1, \tilde{e}_1)$, $h(\check{p}_1, \tilde{e}_2)$ and $h(\check{p}_1, \tilde{e}_3)$, without loss of generality take Q = 3 then Q-ROIVFSEWAO to aggregate the Q-ROIVFSENs as follows

$$\overline{\omega}\left(\left(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k, \varphi_k)}\right) = \left\{< \check{s}_i, \left[\left(1 - \prod_{k=1}^{3}\left(1 - \left(\theta^{-}_{k(\check{p},\tilde{e})}(\check{s}_i)\right)^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q},\right.\right.$$

$$\left.\left.\left(1 - \prod_{k=1}^{3}\left(1 - \left(\theta^{+}_{k(\check{p},\tilde{e})}(\check{s}_i)\right)^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q}\right], \left[\prod_{k=1}^{3}(\varphi^{-}_{k(\check{p},\tilde{e})}(\check{s}_i))^{\widetilde{W}_k}, \ \prod_{k=1}^{3}(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s}_i))^{\widetilde{W}_k}\right] >\right\},$$

where $i = 1, 2$

$=\{< \check{s}_1, [0.591, 0.701], [0.572, 0.705] >, < \check{s}_2, [0.535, 0.635], [0.479, 0.582] >\}$.

It is easy to prove that the Q-ROIVFSEWAO possesses the following properties:

Theorem 5.2.5 (Idempotency)

Consider $(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q})_{(\theta_k,\, \varphi_k)} = \{< \check{s}, [\theta^-_{k(\check{p},\tilde{e})}(\check{s}),\ \theta^+_{k(\check{p},\tilde{e})}(\check{s})], [\varphi^-_{k(\check{p},\tilde{e})}(\check{s}),\ \varphi^+_{k(\check{p},\tilde{e})}(\check{s})] >: \check{s} \in \mathrm{S}\}$ for k=1, 2, …, $\breve{m}$ with Q ≥ 1, be a Q-ROIVFSESs collection and if $(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q})_{(\theta_k,\, \varphi_k)} = (h, \check{P}, \tilde{E}, \mathrm{Q})_{(\theta,\varphi)} \forall$ k = 1, 2, …, $\breve{m}$ then

$$\varpi\left((h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q})_{(\theta_k,\, \varphi_k)}\right) = (h, \check{P}, \tilde{E}, \mathrm{Q})_{(\theta,\varphi)},$$

where $(h, \check{P}, \tilde{E}, \mathrm{Q})_{(\theta,\varphi)} = \{< \check{s}, [\theta^-_{(\check{p},\tilde{e})}(\check{s}),\ \theta^+_{(\check{p},\tilde{e})}(\check{s})], [\varphi^-_{(\check{p},\tilde{e})}(\check{s}),\ \varphi^+_{(\check{p},\tilde{e})}(\check{s})] >: \check{s} \in \mathrm{S}\}$.

Proof

For a Q-ROIVFSESs collection $(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q})_{(\theta_k,\, \varphi_k)} = \{< \check{s}, [\theta^-_{k(\check{p},\tilde{e})}(\check{s}),\ \theta^+_{k(\check{p},\tilde{e})}(\check{s})], [\varphi^-_{k(\check{p},\tilde{e})}(\check{s}),\ \varphi^+_{k(\check{p},\tilde{e})}(\check{s})] >: \check{s} \in \mathrm{S}\}$ for k=1, 2, …, $\breve{m}$ with Q ≥ 1, the Q-ROIVFSEWAO is given by

$$\varpi\left((h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q})_{(\theta_k,\, \varphi_k)}\right) = \sum_{k=1}^{\breve{m}} \widetilde{W}_k (h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q})_{(\theta_k,\, \varphi_k)}$$

$$= \left\{< \left[\left(1 - \prod_{k=1}^{\breve{m}}\left(1 - \left(\theta^-_{k(\check{p},\tilde{e})}(\check{s})\right)^Q\right)^{\widetilde{W}_k}\right)^{1/Q},\ \left(1 - \prod_{k=1}^{\breve{m}}\left(1 - \left(\theta^+_{k(\check{p},\tilde{e})}(\check{s})\right)^Q\right)^{\widetilde{W}_k}\right)^{1/Q}\right],\right.$$
$$\left.[\prod_{k=1}^{\breve{m}}(\varphi^-_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k},\ \prod_{k=1}^{\breve{m}}(\varphi^+_{k(\check{p},\tilde{e})})^{\widetilde{W}_k}] >\right\}$$

$$= \left\{< \left[\left(1 - \prod_{k=1}^{\breve{m}}\left(1 - \left(\theta^-_{(\check{p},\tilde{e})}(\check{s})\right)^Q\right)^{\widetilde{W}_k}\right)^{1/Q},\quad \left(-\prod_{k=1}^{\breve{m}}\left(1 - \left(\theta^+_{(\check{p},\tilde{e})}(\check{s})\right)^Q\right)^{\widetilde{W}_k}\right)^{1/Q}\right],\right.$$
$$\left.\left[\prod_{k=1}^{\breve{m}}(\varphi^-_{(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}, \prod_{k=1}^{\breve{m}}(\varphi^+_{(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}\right] >\right\}$$

$$= \left\{< \left[\left(1 - \left(1 - \left(\theta^-_{(\check{p},\tilde{e})}(\check{s})\right)^Q\right)^{\sum_{k=1}^{\breve{m}} \widetilde{W}_k}\right)^{1/Q}, \left(1 - \left(1 - \left(\theta^+_{(\check{p},\tilde{e})}(\check{s})\right)^Q\right)^{\sum_{k=1}^{\breve{m}} \widetilde{W}_k}\right)^{1/Q}\right],\right.$$
$$\left.\left[(\varphi^-_{(\check{p},\tilde{e})}(\check{s}))^{\sum_{k=1}^{\breve{m}} \widetilde{W}_k}, (\varphi^+_{(\check{p},\tilde{e})}(\check{s}))^{\sum_{k=1}^{\breve{m}} \widetilde{W}_k}\right] >\right\}$$

As $\sum_{k=1}^{\breve{m}} \widetilde{W}_k = 1$

$$= \left\{ < \left[\left(1 - \left(1 - \left(\theta^{-}_{(\check{p},\tilde{e})}(\breve{s})\right)^{Q}\right)^{1}\right)^{1/Q}, \left(1 - \left(1 - \left(\theta^{+}_{(\check{p},\tilde{e})}(\breve{s})\right)^{Q}\right)^{1}\right)^{1/Q} \right], \right.$$

$$\left. \left[(\varphi^{-}_{(\check{p},\tilde{e})}(\breve{s}))^{1}, (\varphi^{+}_{(\check{p},\tilde{e})}(\breve{s}))^{1} \right] > \right\}$$

$$= \left\{ < \left[\left(1 - 1 + \left(\theta^{-}_{(\check{p},\tilde{e})}(\breve{s})\right)^{Q}\right)^{1/Q}, \left(1 - 1 + \left(\theta^{+}_{(\check{p},\tilde{e})}(\breve{s})\right)^{Q}\right)^{1/Q} \right], \right.$$

$$\left. \left[(\varphi^{-}_{(\check{p},\tilde{e})}(\breve{s}))^{1}, (\varphi^{+}_{(\check{p},\tilde{e})}(\breve{s}))^{1} \right] > \right\}$$

$$= \left\{ < \left[\left(\left(\theta^{-}_{(\check{p},\tilde{e})}(\breve{s})\right)^{Q}\right)^{1/Q}, \left(\left(\theta^{+}_{(\check{p},\tilde{e})}(\breve{s})\right)^{Q}\right)^{1/Q} \right], \left[(\varphi^{-}_{(\check{p},\tilde{e})}(\breve{s}))^{1}, (\varphi^{+}_{(\check{p},\tilde{e})}(\breve{s}))^{1} \right] > \right\}$$

$$= \left\{ < \left[\theta^{-}_{(\check{p},\tilde{e})}(\breve{s}),\quad \theta^{+}_{(\check{p},\tilde{e})}(\breve{s})\right],\quad \left[\varphi^{-}_{(\check{p},\tilde{e})}(\breve{s}),\quad \varphi^{+}_{(\check{p},\tilde{e})}(\breve{s})\right] > \right\} = (h, \check{P}, \tilde{E}, \mathrm{Q})_{(\theta,\varphi)}$$

Theorem 5.2.6 (Monotonicity)

Suppose $(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q})_{(\theta_k, \varphi_k)}$ and $(\tilde{h}_k, \tilde{\check{P}}_k, \tilde{\tilde{E}}_k, \mathrm{Q})_{(\tilde{\theta}_k, \tilde{\varphi}_k)}$ for k = 1, 2, …, $\breve{m}$ be two sets of Q-ROIVFSESs. If $\theta^{-}_{k(\check{p},\tilde{e})}(\breve{s}) \geq \tilde{\theta}^{-}_{k(\check{p},\tilde{e})}(\breve{s})$, $\theta^{+}_{k(\check{p},\tilde{e})}(\breve{s}) \geq \tilde{\theta}^{+}_{k(\check{p},\tilde{e})}(\breve{s})$, $\varphi^{-}_{k(\check{p},\tilde{e})}(\breve{s}) \leq \tilde{\varphi}^{-}_{k(\check{p},\tilde{e})}(\breve{s})$ and $\varphi^{+}_{k(\check{p},\tilde{e})}(\breve{s}) \leq \varphi^{+}_{k(\check{p},\tilde{e})}(\breve{s}) \forall\ \mathrm{k} = 1, 2, \ldots, \breve{m}$ then $\overline{\omega}\left(\left(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k, \varphi_k)}\right) \geq \overline{\omega}\left(\left(\tilde{h}_k, \tilde{\check{P}}_k, \tilde{\tilde{E}}_k, \mathrm{Q}\right)_{(\tilde{\theta}_k, \tilde{\varphi}_k)}\right)$.

Proof

Since $\theta^{-}_{k(\check{p},\tilde{e})}(\breve{s}) \geq \tilde{\theta}^{-}_{k(\check{p},\tilde{e})}(\breve{s})$, $\theta^{+}_{k(\check{p},\tilde{e})}(\breve{s}) \geq \tilde{\theta}^{+}_{k(\check{p},\tilde{e})}(\breve{s})$, $\varphi^{-}_{k(\check{p},\tilde{e})}(\breve{s}) \leq \tilde{\varphi}^{-}_{k(\check{p},\tilde{e})}(\breve{s})$ and $\varphi^{+}_{k(\check{p},\tilde{e})}(\breve{s}) \leq \tilde{\varphi}^{+}_{k(\check{p},\tilde{e})}(\breve{s}) \forall\ \mathrm{k} = 1, 2, \ldots, \breve{m}$ then

$$-\,(\theta^{-}_{k(\check{p},\tilde{e})}(\breve{s}))^{Q} \leq -(\tilde{\theta}^{-}_{k(\check{p},\tilde{e})}(\breve{s}))^{Q}.$$

$$\Rightarrow 1 - (\theta^{-}_{k(\check{p},\tilde{e})}(\breve{s}))^{Q} \leq 1 - (\tilde{\theta}^{-}_{k(\check{p},\tilde{e})}(\breve{s}))^{Q}.$$

$$\Rightarrow \prod_{k=1}^{\breve{m}}\left(1-\left(\theta_{k(\check{p},\tilde{e})}^{-}(\breve{s})\right)^{Q}\right)^{\widetilde{W}_k} \leq \prod_{k=1}^{\breve{m}}\left(1-\left(\tilde{\theta}_{k(\check{p},\tilde{e})}^{-}(\breve{s})\right)^{Q}\right)^{\widetilde{W}_k}.$$

$$\Rightarrow \left(1-\prod_{k=1}^{\breve{m}}\left(1-\left(\theta_{k(\check{p},\tilde{e})}^{-}(\breve{s})\right)^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q} \geq \left(1-\prod_{k=1}^{\breve{m}}\left(1-\left(\tilde{\theta}_{k(\check{p},\tilde{e})}^{-}(\breve{s})\right)^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q}.$$

$$\Rightarrow \left(1-\prod_{k=1}^{\breve{m}}\left(1-\left(\theta_{k(\check{p},\tilde{e})}^{-}(\breve{s})\right)^{Q}\right)^{\widetilde{W}_k}\right)^{{1/Q}^{Q}} \geq \left(1-\prod_{k=1}^{\breve{m}}\left(1-\left(\tilde{\theta}_{k(\check{p},\tilde{e})}^{-}(\breve{s})\right)^{Q}\right)^{\widetilde{W}_k}\right)^{{1/Q}^{Q}} \quad \ldots (5.2.2)$$

Also

$$\prod_{k=1}^{\breve{m}}\left(\varphi_{k(\check{p},\tilde{e})}^{-}(\breve{s})\right)^{\widetilde{W}_k} \leq \prod_{k=1}^{\breve{m}}\left(\tilde{\varphi}_{k(\check{p},\tilde{e})}^{-}(\breve{s})\right)^{\widetilde{W}_k}.$$

$$\Rightarrow \left(\prod_{k=1}^{\breve{m}}\left(\varphi_{k(\check{p},\tilde{e})}^{-}(\breve{s})\right)^{\widetilde{W}_k}\right)^{Q} \leq \left(\prod_{k=1}^{\breve{m}}\left(\tilde{\varphi}_{k(\check{p},\tilde{e})}^{-}(\breve{s})\right)^{\widetilde{W}_k}\right)^{Q}.$$

$$\Rightarrow -\left(\prod_{k=1}^{\breve{m}}\left(\varphi_{k(\check{p},\tilde{e})}^{-}(\breve{s})\right)^{\widetilde{W}_k}\right)^{Q} \geq -\left(\prod_{k=1}^{\breve{m}}\left(\tilde{\varphi}_{k(\check{p},\tilde{e})}^{-}(\breve{s})\right)^{\widetilde{W}_k}\right)^{Q} \quad \ldots (5.2.3)$$

Combining (5.2.2) and (5.2.3),

$$\left(1-\prod_{k=1}^{\breve{m}}\left(1-\left(\theta_{k(\check{p},\tilde{e})}^{-}(\breve{s})\right)^{Q}\right)^{\widetilde{W}_k}\right)^{{1/Q}^{Q}} - \left(\prod_{k=1}^{\breve{m}}\left(\varphi_{k(\check{p},\tilde{e})}^{-}(\breve{s})\right)^{\widetilde{W}_k}\right)^{Q} \geq$$

$$\left(1-\prod_{k=1}^{\breve{m}}\left(1-\left(\tilde{\theta}_{k(\check{p},\tilde{e})(\breve{s})}^{-}\right)^{Q}\right)^{\widetilde{W}_k}\right)^{{1/Q}^{Q}} - \left(\prod_{k=1}^{\breve{m}}\left(\tilde{\varphi}_{k(\check{p},\tilde{e})}^{-}(\breve{s})\right)^{\widetilde{W}_k}\right)^{Q} \quad \ldots (5.2.4)$$

Similarly,

$$\left(1-\prod_{k=1}^{\breve{m}}\left(1-\left(\theta_{k(p,e)}^{+}(\breve{s})\right)^{Q}\right)^{\widetilde{W}_k}\right)^{{1/Q}^{Q}} - \left(\prod_{k=1}^{\breve{m}}\left(\varphi_{k(p,e)}^{+}(\breve{s})\right)^{\widetilde{W}_k}\right)^{Q} \geq$$

$$\left(1-\prod_{k=1}^{\breve{m}}\left(1-\left(\tilde{\theta}_{k(p,e)}^{+}(\breve{s})\right)^{Q}\right)^{\widetilde{W}_k}\right)^{{1/Q}^{Q}} - \left(\prod_{k=1}^{\breve{m}}\left(\tilde{\varphi}_{k(p,e)}^{+}(\breve{s})\right)^{\widetilde{W}_k}\right)^{Q} \quad \ldots (5.2.5)$$

Adding (5.2.4) and (5.2.5)

$$\Rightarrow \left(1-\prod_{k=1}^{\breve{m}}\left(1-\left(\theta_{k(\check{p},\tilde{e})}^{-}(\breve{s})\right)^{Q}\right)^{\widetilde{W}_k}\right)^{{1/Q}^{Q}} - \left(\prod_{k=1}^{\breve{m}}\left(\varphi_{k(\check{p},\tilde{e})}^{-}(\breve{s})\right)^{\widetilde{W}_k}\right)^{Q} + \left(1-\prod_{k=1}^{\breve{m}}\left(1-\left(\theta_{k(\check{p},\tilde{e})}^{+}(\breve{s})\right)^{Q}\right)^{\widetilde{W}_k}\right)^{{1/Q}^{Q}} - \left(\prod_{k=1}^{\breve{m}}\left(\varphi_{k(\check{p},\tilde{e})}^{+}(\breve{s})\right)^{\widetilde{W}_k}\right)^{Q} \geq \left(1-\prod_{k=1}^{\breve{m}}\left(1-\left(\tilde{\theta}_{k(\check{p},\tilde{e})}^{-}(\breve{s})\right)^{Q}\right)^{\widetilde{W}_k}\right)^{{1/Q}^{Q}} - \left(\prod_{k=1}^{\breve{m}}\left(\tilde{\varphi}_{k(\check{p},\tilde{e})}^{-}(\breve{s})\right)^{\widetilde{W}_k}\right)^{Q} + \left(1-\prod_{k=1}^{\breve{m}}\left(1-\left(\tilde{\theta}_{k(\check{p},\tilde{e})}^{+}(\breve{s})\right)^{Q}\right)^{\widetilde{W}_k}\right)^{{1/Q}^{Q}} - \left(\prod_{k=1}^{\breve{m}}\left(\tilde{\varphi}_{k(\check{p},\tilde{e})}^{+}(\breve{s})\right)^{\widetilde{W}_k}\right)^{Q} \quad \ldots (5.2.6)$$

$$\Rightarrow \S\left(\overline{\omega}\left(\left(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k, \varphi_k)}\right)\right) \geq \S\left(\overline{\omega}\left(\left(\tilde{h}_k, \tilde{\check{P}}_k, \tilde{\tilde{E}}_k, \mathrm{Q}\right)_{(\tilde{\theta}_k, \tilde{\varphi}_k)}\right)\right)$$ by using Definition 4.2.36

Now by using Theorem 4.2.37

1. If $\S\left(\overline{\omega}\left(\left(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k, \varphi_k)}\right)\right) > \S\left(\overline{\omega}\left(\left(\tilde{h}_k, \tilde{\check{P}}_k, \tilde{\tilde{E}}_k, \mathrm{Q}\right)_{(\tilde{\theta}_k, \tilde{\varphi}_k)}\right)\right)$ then

   $$\overline{\omega}\left(\left(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k, \varphi_k)}\right) > \overline{\omega}\left(\left(\tilde{h}_k, \tilde{\check{P}}_k, \tilde{\tilde{E}}_k, \mathrm{Q}\right)_{(\tilde{\theta}_k, \tilde{\varphi}_k)}\right). \qquad \ldots (5.2.7)$$

2. And if $\S\left(\overline{\omega}\left(\left(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k, \varphi_k)}\right)\right) = \S\left(\overline{\omega}\left(\left(\tilde{h}_k, \tilde{\check{P}}_k, \tilde{\tilde{E}}_k, \mathrm{Q}\right)_{(\tilde{\theta}_k, \tilde{\varphi}_k)}\right)\right)$

Then (5.2.6) reduces as

$$\left(1 - \prod_{k=1}^{\breve{m}}\left(1 - (\theta^{-}_{k(\check{p},\tilde{e})}(\check{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q^{Q}} - \left(\prod_{k=1}^{\breve{m}}(\varphi^{-}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}\right)^{Q}$$

$$+\left(1 - \prod_{k=1}^{\breve{m}}\left(1 - (\theta^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q^{Q}} - \left(\prod_{k=1}^{\breve{m}}(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}\right)^{Q} =$$

$$\left(1 - \prod_{k=1}^{\breve{m}}\left(1 - (\tilde{\theta}^{-}_{k(\check{p},\tilde{e})}(\check{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q^{Q}} - \left(\prod_{k=1}^{\breve{m}}(\tilde{\varphi}^{-}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}\right)^{Q} +$$

$$\left(1 - \prod_{k=1}^{\breve{m}}\left(1 - (\tilde{\theta}^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q^{Q}} - \left(\prod_{k=1}^{\breve{m}}(\tilde{\varphi}^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}\right)^{Q}$$

Then by condition,

$\theta^{-}_{k(\check{p},\tilde{e})}(\check{s}) \geq \tilde{\theta}^{-}_{k(\check{p},\tilde{e})}(\check{s})$, $\theta^{+}_{k(\check{p},\tilde{e})}(\check{s}) \geq \tilde{\theta}^{+}_{k(\check{p},\tilde{e})}(\check{s})$, $\varphi^{-}_{k(\check{p},\tilde{e})}(\check{s}) \leq \tilde{\varphi}^{-}_{k(\check{p},\tilde{e})}(\check{s})$ and $\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s}) \leq \varphi^{+}_{k(\check{p},\tilde{e})}(\check{s})$ ∀ k, we have

$$\left(1 - \prod_{k=1}^{\breve{m}}\left(1 - (\theta^{-}_{k(\check{p},\tilde{e})}(\check{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q} = \left(1 - \prod_{k=1}^{\breve{m}}\left(1 - (\tilde{\theta}^{-}_{k(\check{p},\tilde{e})}(\check{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q},$$

$$\left(1 - \prod_{k=1}^{\breve{m}}\left(1 - (\theta^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q} = \left(1 - \prod_{k=1}^{\breve{m}}\left(1 - (\tilde{\theta}^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q},$$

$$\prod_{k=1}^{\breve{m}}(\varphi^{-}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k} = \prod_{k=1}^{\breve{m}}(\tilde{\varphi}^{-}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}$$

and

$$\prod_{k=1}^{\breve{m}}(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k} = \prod_{k=1}^{\breve{m}}(\tilde{\varphi}^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}.$$

$$\Rightarrow\left(1-\prod_{k=1}^{\breve{m}}\left(1-\left(\theta^{-}_{k(\check{p},\tilde{e})}(\check{s})\right)^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q^{Q}}+\left(\prod_{k=1}^{\breve{m}}\left(\varphi^{-}_{k(\check{p},\tilde{e})}(\check{s})\right)^{\widetilde{W}_k}\right)^{Q}$$

$$+\left(1-\prod_{k=1}^{\breve{m}}\left(1-\left(\theta^{+}_{k(\check{p},\tilde{e})}(\check{s})\right)^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q^{Q}}+\left(\prod_{k=1}^{\breve{m}}\left(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s})\right)^{\widetilde{W}_k}\right)^{Q}=$$

$$\left(1-\prod_{k=1}^{\breve{m}}\left(1-\left(\tilde{\theta}^{-}_{k(\check{p},\tilde{e})}(\check{s})\right)^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q^{Q}}+\left(\prod_{k=1}^{\breve{m}}\left(\tilde{\varphi}^{-}_{k(\check{p},\tilde{e})}(\check{s})\right)^{\widetilde{W}_k}\right)^{Q}+$$

$$\left(1-\prod_{k=1}^{\breve{m}}\left(1-\left(\tilde{\theta}^{+}_{k(\check{p},\tilde{e})}(\check{s})\right)^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q^{Q}}+\left(\prod_{k=1}^{\breve{m}}\left(\tilde{\varphi}^{+}_{k(\check{p},\tilde{e})}(\check{s})\right)^{\widetilde{W}_k}\right)^{Q}.$$

$$\Rightarrow \hat{A}\left(\overline{\omega}\left(\left(h_k,\check{P}_k,\tilde{E}_k,\mathrm{Q}\right)_{(\theta_k,\varphi_k)}\right)\right)=\hat{A}\left(\overline{\omega}\left(\left(\tilde{h}_k,\tilde{\check{P}}_k,\tilde{\tilde{E}}_k,\mathrm{Q}\right)_{(\tilde{\theta}_k,\tilde{\varphi}_k)}\right)\right).$$

Then $\overline{\omega}\left(\left(h_k,\check{P}_k,\tilde{E}_k,\mathrm{Q}\right)_{(\theta_k,\varphi_k)}\right)=\overline{\omega}\left(\left(\tilde{h}_k,\tilde{\check{P}}_k,\tilde{\tilde{E}}_k,\mathrm{Q}\right)_{(\tilde{\theta}_k,\tilde{\varphi}_k)}\right)$ … (5.2.8)

Combining (5.2.7) and (5.2.8) implies

$$\overline{\omega}\left(\left(h_k,\check{P}_k,\tilde{E}_k,\mathrm{Q}\right)_{(\theta_k,\varphi_k)}\right)\geq\overline{\omega}\left(\left(\tilde{h}_k,\tilde{\check{P}}_k,\tilde{\tilde{E}}_k,\mathrm{Q}\right)_{(\tilde{\theta}_k,\tilde{\varphi}_k)}\right).$$

Theorem 5.2.7 (Boundedness)

Consider a collection of Q-ROIVFSESs $\left(h_k,\check{P}_k,\tilde{E}_k,\mathrm{Q}\right)_{(\theta_k,\varphi_k)}$ and $\left(h_i,\check{P}_i,\tilde{E}_i,\mathrm{Q}\right)_{(\theta_i,\varphi_i)}=\{<$ $\check{s},[\min_{1\leq k\leq\breve{m}}\theta^{-}_{k}(\check{s}),\min_{1\leq k\leq\breve{m}}\theta^{+}_{k}(\check{s})],[\max_{1\leq k\leq\breve{m}}\varphi^{-}_{k}(\check{s}),\max_{1\leq k\leq\breve{m}}\varphi^{+}_{k}(\check{s})]>\}$ and

$\left(h_s,\check{P}_s,\tilde{E}_s,\mathrm{Q}\right)_{(\theta_s,\varphi_s)}=\{<$ $\check{s},[\max_{1\leq k\leq\breve{m}}\theta^{-}_{k}(\check{s}),\max_{1\leq k\leq\breve{m}}\theta^{+}_{k}(\check{s})],[\min_{1\leq k\leq\breve{m}}\varphi^{-}_{k}(\check{s}),\min_{1\leq k\leq\breve{m}}\varphi^{+}_{k}(\check{s})]>\}$ then $\left(h_i,\check{P}_i,\tilde{E}_i,\mathrm{Q}\right)_{(\theta_i,\varphi_i)}\leq\overline{\omega}\left(\left(h_k,\check{P}_k,\tilde{E}_k,\mathrm{Q}\right)_{(\theta_k,\varphi_k)}\right)\leq\left(h_s,\check{P}_s,\tilde{E}_s,\mathrm{Q}\right)_{(\theta_s,\varphi_s)}$.

Proof

For membership degree of $\overline{\omega}\left(\left(h_k, \breve{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}\right)$ we get

$$\min_{1\le k\le \breve{m}}(\theta_k^+(\breve{s}))^{Q} \le (\theta_k^+(\breve{s}))^{Q} \le \max_{1\le k\le \breve{m}}(\theta_k^+(\breve{s}))^{Q}.$$

$$\Rightarrow \left(1-\prod_{k=1}^{\breve{m}}\left(1-\min_{1\le k\le \breve{m}}(\theta_{k(p,e)}^+(\breve{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q} \le \left(1-\prod_{k=1}^{\breve{m}}\left(1-(\theta_{k(p,e)}^+(\breve{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q} \le \left(1-\prod_{k=1}^{\breve{m}}\left(1-\max_{1\le k\le \breve{m}}(\theta_{k(p,e)}^+(\breve{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q}.$$

$$\Rightarrow \left(1-\left(1-\min_{1\le k\le \breve{m}}(\theta_{k(\breve{p},\tilde{e})}^+(\breve{s}))^{Q}\right)^{\sum_{k=1}^{\breve{m}}\widetilde{W}_k}\right)^{1/Q} \le \left(1-\prod_{k=1}^{\breve{m}}\left(1-(\theta_{k(\breve{p},\tilde{e})}^+(\breve{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q} \le \left(1-\left(1-\max_{1\le k\le \breve{m}}(\theta_{k(\breve{p},\tilde{e})}^+(\breve{s}))^{Q}\right)^{\sum_{k=1}^{\breve{m}}\widetilde{W}_k}\right)^{1/Q}.$$

As $\sum_{k=1}^{\breve{m}}\widetilde{W}_k = 1$

$$\Rightarrow \left(1-\left(1-\min_{1\le k\le \breve{m}}(\theta_{k(\breve{p},\tilde{e})}^+(\breve{s}))^{Q}\right)^{1}\right)^{1/Q} \le \left(1-\prod_{k=1}^{\breve{m}}\left(1-(\theta_{k(\breve{p},\tilde{e})}^+(\breve{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q} \le \left(1-\left(1-\max_{1\le k\le \breve{m}}(\theta_{k(\breve{p},\tilde{e})}^+(\breve{s}))^{Q}\right)^{1}\right)^{1/Q}.$$

$$\Rightarrow \left(1-1+\min_{1\le k\le \breve{m}}(\theta_{k(\breve{p},\tilde{e})}^+(\breve{s}))^{Q}\right)^{1/Q} \le \left(1-\prod_{k=1}^{\breve{m}}\left(1-(\theta_{k(\breve{p},\tilde{e})}^+(\breve{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q} \le \left(1-1+\max_{1\le k\le \breve{m}}(\theta_{k(\breve{p},\tilde{e})}^+(\breve{s}))^{Q}\right)^{1/Q}.$$

$$\Rightarrow \min_{1\le k\le \breve{m}}(\theta_{k(\breve{p},\tilde{e})}^+(\breve{s})) \le \left(1-\prod_{k=1}^{\breve{m}}\left(1-(\theta_{k(\breve{p},\tilde{e})}^+(\breve{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q} \le \max_{1\le k\le \breve{m}}(\theta_{k(\breve{p},\tilde{e})}^+(\breve{s}). \qquad \dots (5.2.9)$$

Similarly,

$$\min_{1\le k\le \breve{m}}(\theta_{k(\breve{p},\tilde{e})}^-(\breve{s})) \le \left(1-\prod_{k=1}^{\breve{m}}\left(1-(\theta_{k(\breve{p},\tilde{e})}^-(\breve{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q} \le \max_{1\le k\le \breve{m}}(\theta_{k(\breve{p},\tilde{e})}^-(\breve{s})) \qquad \dots (5.2.10)$$

For non-membership degree of $\overline{\omega}\left(\left(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}\right)$

$\prod_{k=1}^{\breve{m}} \min_{1\le k\le \breve{m}}(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k} \le \prod_{k=1}^{\breve{m}}(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k} \le \prod_{k=1}^{\breve{m}} \max_{1\le k\le \breve{m}}(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}.$

$$\Rightarrow \min_{1\le k\le \breve{m}}(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{\sum_{k=1}^{\breve{m}}\widetilde{W}_k} \le \prod_{k=1}^{\breve{m}}(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k} \le \max_{1\le k\le \breve{m}}(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{\sum_{k=1}^{\breve{m}}\widetilde{W}_k}$$

$$\Rightarrow \min_{1\le k\le \breve{m}}(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s})) \le \prod_{k=1}^{\breve{m}}(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k} \le \max_{1\le k\le \breve{m}}(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s})).$$

$$\Rightarrow (\min_{1\le k\le \breve{m}}(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s})))^{Q} \le \left(\prod_{k=1}^{\breve{m}}(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}\right)^{Q} \le (\max_{1\le k\le \breve{m}}(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s})))^{Q}.$$

$$\Rightarrow -(\min_{1\le k\le \breve{m}}(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s})))^{Q} \ge -\left(\prod_{k=1}^{\breve{m}}(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}\right)^{Q} \ge -(\max_{1\le k\le \breve{m}}(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s})))^{Q}.$$

$$\Rightarrow -(\max_{1\le k\le \breve{m}}(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s})))^{Q} \le -\left(\prod_{k=1}^{\breve{m}}(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}\right)^{Q} \le -(\min_{1\le k\le \breve{m}}(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s})))^{Q}. \quad \ldots\ (5.2.11)$$

Similarly,

$$\Rightarrow -(\max_{1\le k\le \breve{m}}(\varphi^{-}_{k(\check{p},\tilde{e})}(\check{s})))^{Q} \le -\left(\prod_{k=1}^{\breve{m}}(\varphi^{-}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}\right)^{Q} \le -(\min_{1\le k\le \breve{m}}(\varphi^{-}_{k(\check{p},\tilde{e})}(\check{s})))^{Q}. \quad \ldots\ (5.2.12)$$

Adding Inequalities (5.2.9) - (5.2.12) and dividing by 2

$$\frac{\left(\min_{1\le k\le \breve{m}}(\theta^{+}_{k(\check{p},\tilde{e})}(\check{s}))\right)^{Q} + \left(\min_{1\le k\le \breve{m}}(\theta^{-}_{k(\check{p},\tilde{e})}(\check{s}))\right)^{Q} - (\max_{1\le k\le \breve{m}}(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s})))^{Q} - (\max_{1\le k\le \breve{m}}(\varphi^{-}_{k(\check{p},\tilde{e})}(\check{s})))^{Q}}{2}$$

$$\le \frac{\left(\left(1-\prod_{k=1}^{\breve{m}}\left(1-(\theta^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q}\right)^{Q} + \left(\left(1-\prod_{k=1}^{\breve{m}}\left(1-(\theta^{-}_{k(\check{p},\tilde{e})}(\check{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q}\right)^{Q} - \left(\prod_{k=1}^{\breve{m}}(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}\right)^{Q} - \left(\prod_{k=1}^{\breve{m}}(\varphi^{-}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}\right)^{Q}}{2}$$

$$\le \frac{\left(\max_{1\le k\le \breve{m}}(\theta^{+}_{k(\check{p},\tilde{e})}(\check{s}))\right)^{Q} + \left(\max_{1\le k\le \breve{m}}(\theta^{-}_{k(\check{p},\tilde{e})}(\check{s}))\right)^{Q} - (\min_{1\le k\le \breve{m}}(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s})))^{Q} - (\min_{1\le k\le \breve{m}}(\varphi^{-}_{k(\check{p},\tilde{e})}(\check{s})))^{Q}}{2}$$

$$\Rightarrow \S\left(\left(h_i, \check{P}_i, \tilde{E}_i, \mathrm{Q}\right)_{(\theta_i,\, \varphi_i)}\right) \le \S\left(\overline{\omega}\left(\left(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}\right)\right) \le \S\left(\left(h_s, \check{P}_s, \tilde{E}_s, \mathrm{Q}\right)_{(\theta_s,\, \varphi_s)}\right)$$

Then by using theorem 4.2.37

$$\Rightarrow \left(h_i, \check{P}_i, \tilde{E}_i, \mathrm{Q}\right)_{(\theta_i,\, \varphi_i)} \le \overline{\omega}\left(\left(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}\right) \le \left(h_s, \check{P}_s, \tilde{E}_s, \mathrm{Q}\right)_{(\theta_s,\, \varphi_s)}.$$

Example 5.2.8

Let S = {$\check{s}_1$, $\check{s}_2$ }, P = {$\check{p}_1$}, E = {$\tilde{e}_1$, $\tilde{e}_2$, $\tilde{e}_3$}be the universe, parameter and expert set respectively. Consider the Q-ROIVFSESs

$(h_1, \check{P}_1, \tilde{E}_1, Q)$ = $h(\check{p}_1, \tilde{e}_1)$ = $\{< \check{s}_1, [0.7, 0.8], [0.5, 0.6] >, < \check{s}_2, [0.6, 0.7], [0.6, 0.7] >\}$,

$(h_2, \check{P}_2, \tilde{E}_2, Q)$ = $h(\check{p}_1, \tilde{e}_2)$ = $\{< \check{s}_1, [0.3, 0.5], [0.5, 0.7] >, < \check{s}_2, [0.5, 0.6], [0.7, 0.8] >\}$,

$(h_3, \check{P}_3, \tilde{E}_3, Q)$ = $h(\check{p}_1, \tilde{e}_3)$ = $\{< \check{s}_1, [0.6, 0.7], [0.7, 0.8] >, < \check{s}_2, [0.5, 0.6], [0.3, 0.4] >\}$.

Then $\left(h_i, \check{P}_i, \tilde{E}_i, \mathrm{Q}\right)_{(\theta_i,\, \varphi_i)}$=$h(\check{p}_i, \tilde{e}_i)$=$\{< \check{s}_1, [0.3, 0.5], [0.7, 0.8] >, < \check{s}_2, [0.5, 0.6], [0.7, 0.8] >\}$

And $\left(h_s, \check{P}_s, \tilde{E}_s, \mathrm{Q}\right)_{(\theta_s,\, \varphi_s)}$=$h(\check{p}_s, \tilde{e}_s)$=$\{< \check{s}_1, [0.7, 0.8], [0.5, 0.6] >, < \check{s}_2, [0.6, 0.7], [0.3, 0.4] >\}$

And $\widetilde{W} = (0.3, 0.3, 0.4)^T$ be the weight vector of $h(\check{p}_1, \tilde{e}_1)$, $h(\check{p}_1, \tilde{e}_2)$ and $h(\check{p}_1, \tilde{e}_3)$, without loss of generality take Q = 3 then Q-ROIVFSEWAO to aggregate the Q-ROIVFSENs as follows

$$\overline{\omega}\left(\left(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}\right) = \left\{< \check{s}_i, \left[\left(1 - \prod_{k=1}^{3}\left(1 - \left(\theta^-_{k(\check{p},\tilde{e})}(\check{s}_i)\right)^Q\right)^{\widetilde{W}_k}\right)^{1/Q},\right.\right.$$

$$\left.\left(1 - \prod_{k=1}^{3}\left(1 - \left(\theta^+_{k(\check{p},\tilde{e})}(\check{s}_i)\right)^Q\right)^{\widetilde{W}_k}\right)^{1/Q}\right],\ \left[\prod_{k=1}^{3}(\varphi^-_{k(\check{p},\tilde{e})}(\check{s}_i))^{\widetilde{W}_k},\right.$$

$$\left.\left.\prod_{k=1}^{3}(\varphi^+_{k(\check{p},\tilde{e})}(\check{s}_i))^{\widetilde{W}_k}\right] >\right\}, \qquad \text{where } i = 1, 2$$

$$=\{< \check{s}_1, [0.591, 0.701], [0.572, 0.705] >, < \check{s}_2, [0.535, 0.635], [0.479, 0.582] >\}.$$

Then

$$§(\mathrm{h}(\check{p}_i, \tilde{e}_i))_{\check{s}_1} = -0.3515, §\left(\overline{\omega}\left(\left(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}\right)\right)_{\check{s}_1} = 0.0066, §(\mathrm{h}(\check{p}_s, \tilde{e}_s))_{\check{s}_1} = 0.257.$$

And

$$§(\mathrm{h}(\check{p}_i, \tilde{e}_i))_{\check{s}_2} = -0.0635,\ §\left(\overline{\omega}\left(\left(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}\right)\right)_{\check{s}_2} = 0.052,\ §(\mathrm{h}(\check{p}_s, \tilde{e}_s))_{\check{s}_2} = 0.234.$$

This implies that

$$\S\left(\mathrm{h}(\check{p}_i,\tilde{e}_i)\right)_{\check{s}_1} \leq \S\left(\varpi\left(\left(h_k,\check{P}_k,\tilde{E}_k,\mathrm{Q}\right)_{(\theta_k,\,\varphi_k)}\right)\right)_{\check{s}_1} \leq \S\left(\mathrm{h}(\check{p}_s,\tilde{e}_s)\right)_{\check{s}_1}$$

and $\S(\mathrm{h}(\check{p}_i,\tilde{e}_i))_{\check{s}_2} \leq \S\left(\varpi\left(\left(h_k,\check{P}_k,\tilde{E}_k,\mathrm{Q}\right)_{(\theta_k,\,\varphi_k)}\right)\right)_{\check{s}_2} \leq \S(\mathrm{h}(\check{p}_s,\tilde{e}_s))_{\check{s}_2}$.

Therefore, $\left(h_i,\check{P}_i,\tilde{E}_i,\mathrm{Q}\right)_{(\theta_i,\,\varphi_i)} \leq \varpi\left(\left(h_k,\check{P}_k,\tilde{E}_k,\mathrm{Q}\right)_{(\theta_k,\,\varphi_k)}\right) \leq \left(h_s,\check{P}_s,\tilde{E}_s,\mathrm{Q}\right)_{(\theta_s,\,\varphi_s)}$.

Definition 5.2.9

Special Cases of Q-ROIVFSEWAO (Definition 5.2.1) by taking different values of Q

1. When Q=1, Q-ROIVFSEWAO will reduce to Interval Valued Intuitionistic Fuzzy Soft Expert Weighted Averaging Operator (IVIFSEWAO) [7]which is characterized as follows

$$\varpi\left(\left(h_k,\check{P}_k,\tilde{E}_k,1\right)_{(\theta_k,\,\varphi_k)}\right) = <\left[1-\prod_{k=1}^{\breve{m}}\left(1-\theta^-_{k(\check{p},\tilde{e})}(\check{s})\right)^{\widetilde{W}_k},\ 1-\prod_{k=1}^{\breve{m}}\left(1-\theta^+_{k(\check{p},\tilde{e})}(\check{s})\right)^{\widetilde{W}_k}\right],\ \left[\prod_{k=1}^{\breve{m}}(\varphi^-_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k},\ \prod_{k=1}^{\breve{m}}(\varphi^+_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}\right]>.$$

2. When Q=2, Q-ROIVFSES Weighted Averaging Operator will reduce to Interval Valued Pythagorean Fuzzy Soft Expert Weighted Averaging Operator (IVPFSEWAO)which is characterized as follows

$$\varpi\left(\left(h_k,\check{P}_k,\tilde{E}_k,2\right)_{(\theta_k,\,\varphi_k)}\right) = <\left[\left(1-\prod_{k=1}^{\breve{m}}\left(1-(\theta^-_{k(\check{p},\tilde{e})}(\check{s}))^2\right)^{\widetilde{W}_k}\right)^{1/2},\ \left(1-\prod_{k=1}^{\breve{m}}\left(1-(\theta^+_{k(\check{p},\tilde{e})}(\check{s}))^2\right)^{\widetilde{W}_k}\right)^{1/2}\right],\ \left[\prod_{k=1}^{\breve{m}}(\varphi^-_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k},\quad \prod_{k=1}^{\breve{m}}(\varphi^+_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}\right]>$$

$$= <\left[\sqrt[2]{1-\prod_{k=1}^{\breve{m}}\left(1-(\theta^-_{k(\check{p},\tilde{e})}(\check{s}))^2\right)^{\widetilde{W}_k}},\ \sqrt[2]{1-\prod_{k=1}^{\breve{m}}\left(1-(\theta^+_{k(\check{p},\tilde{e})}(\check{s}))^2\right)^{\widetilde{W}_k}}\right],\ \left[\prod_{k=1}^{\breve{m}}(\varphi^-_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k},\quad \prod_{k=1}^{\breve{m}}(\varphi^+_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}\right]>$$

5.3 Q-Rung Orthopair Interval Valued Fuzzy Soft Expert Weighted Geometric Operator (Q-ROIVFSEWGO)

Definition 5.3.1

Suppose $\left(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}$ = $\{< \check{s}, \left[\theta^{-}_{k(\check{p},\tilde{e})}(\check{s}),\ \theta^{+}_{k(\check{p},\tilde{e})}(\check{s})\right], \left[\varphi^{-}_{k(\check{p},\tilde{e})}(\check{s}),\right.$ $\left.\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s})\right] >: \check{s} \in \mathrm{S}\}$ with Q ≥ 1, be a Q-ROIVFSESs collection. A mapping $\underline{\omega} : Q_I^{\breve{m}}(S) \rightarrow Q_I(S)$ is called Q-ROIVFSEWGO if it satisfies

$$\underline{\omega}\left(\left(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}\right) = \prod_{k=1}^{\breve{m}} \left(\left(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}\right)^{\widetilde{W}_k}$$

$$= \left(\left(h_1, \check{P}_1, \tilde{E}_1, \mathrm{Q}\right)_{(\theta_1,\, \varphi_1)}\right)^{\widetilde{W}_1} \times \left(\left(h_2, \check{P}_2, \tilde{E}_2, \mathrm{Q}\right)_{(\theta_2,\, \varphi_2)}\right)^{\widetilde{W}_2} \times \ldots \times \left(\left(h_m, \check{P}_m, \tilde{E}_m, \mathrm{Q}\right)_{(\theta_m,\, \varphi_m)}\right)^{\widetilde{W}_m},$$

where $Q_I^{\breve{m}}(S)$ denote $\breve{m}$ copies of Q-ROIVFSES and $\widetilde{W} = (\widetilde{W}_1, \widetilde{W}_2, \ldots, \widetilde{W}_m)^{\mathrm{T}}$ is a weight vector of $\left(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}$, for $1 \leq k \leq \breve{m}$ satisfying the normalized condition i.e., $\sum_{k=1}^{\breve{m}} \widetilde{W}_k = 1$ and $\widetilde{W}_k \in [0, 1]$. Note that $\left(\mathrm{h_k}, \check{\mathrm{P}}_\mathrm{k}, \tilde{\mathrm{E}}_\mathrm{k}, \mathrm{Q}\right)_{(\theta_\mathrm{k},\, \varphi_\mathrm{k})}$ refers to a Q-ROIVFSES $\mathrm{h_k}(\check{\mathrm{p}}_\mathrm{i}, \tilde{\mathrm{e}}_\mathrm{i})$ for some $\mathrm{p_i} \in \mathrm{P_k}$, $\mathrm{e_i} \in \mathrm{E_k}$ and it is a generalized way of writing and k indicate the element number.

Definition 5.3.2

If $\widetilde{W} = (\frac{1}{\breve{m}}, \frac{1}{\breve{m}}, \ldots, \frac{1}{\breve{m}})^{\mathrm{T}}$ is a weight vector in Definition 5.2.1 then Q-ROIVFSE weighted geometric operator can be written as

$$\underline{\omega}\left(\left(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}\right) = \left(\prod_{k=1}^{\breve{m}} \left(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}\right)^{1/\breve{m}}.$$

In this case $\underline{\omega}$ degenerates to Q-ROIVFSE geometric mean operator $\underline{\omega}_G$.

Theorem 5.3.3

Consider $(h_k, \check{P}_k, \tilde{E}_k, \text{Q})_{(\theta_k,\, \varphi_k)}$ = $\{< \check{s}, [\theta^-_{k(\check{p},\tilde{e})}(\check{s}),\ \theta^+_{k(\check{p},\tilde{e})}(\check{s})], [\varphi^-_{k(\check{p},\tilde{e})}(\check{s}),$ $\varphi^+_{k(\check{p},\tilde{e})}(\check{s})] >: \check{s} \in \text{S}\}$ with Q ≥ 1, be a Q-ROIVFSESs collection. Then, on the basis of operational rules characterized for Q-ROIVFSESs, for any k ∈ ℕ Q-ROIVFSE weighted geometric operator, aggregation is also a Q-ROIVFSES and

$$\underline{\omega}\left((h_k, \check{P}_k, \tilde{E}_k, \text{Q})_{(\theta_k,\, \varphi_k)}\right) = < \left[\prod_{k=1}^{\breve{m}} (\theta^-_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k},\quad \prod_{k=1}^{\breve{m}} (\theta^+_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}\right],$$
$$\left[\left(1 - \prod_{k=1}^{\breve{m}} \left(1 - (\varphi^-_{k(\check{p},\tilde{e})}(\check{s}))^Q\right)^{\widetilde{W}_k}\right)^{1/Q},\quad \left(1 - \prod_{k=1}^{\breve{m}} \left(1 - (\varphi^+_{k(\check{p},\tilde{e})}(\check{s}))^Q\right)^{\widetilde{W}_k}\right)^{1/Q}\right] >$$

Proof

Proofing this result using mathematical induction, the result holds for k = 1, by using Definition 4.2.29 and Remark 4.2.30.

Now for k = 2

By using Definition 5.3.1, we have

$$\underline{\omega}\left((h_1, \check{P}_1, \tilde{E}_1, \text{Q})_{(\theta_1,\, \varphi_1)}, (h_2, \check{P}_2, \tilde{E}_2, \text{Q})_{(\theta_2,\, \varphi_2)}\right) = \prod_{k=1}^{2} \left((h_k, \check{P}_k, \tilde{E}_k, \text{Q})_{(\theta_k,\, \varphi_k)}\right)^{\widetilde{W}_k}$$

$$= \left((h_1, \check{P}_1, \tilde{E}_1, \text{Q})_{(\theta_1,\, \varphi_1)}\right)^{\widetilde{W}_1} \times \left((h_2, \check{P}_2, \tilde{E}_2, \text{Q})_{(\theta_2,\, \varphi_2)}\right)^{\widetilde{W}_2}$$

$$=< [\theta^-_{1(\check{p},\tilde{e})}(\check{s})^{\widetilde{W}_1}, \theta^+_{1(\check{p},\tilde{e})}(s)^{\widetilde{W}_1}], \left[\left(1 - \left(1 - \varphi^-_{1(\check{p},\tilde{e})}(\check{s})^Q\right)^{\widetilde{W}_1}\right)^{1/Q}, \left(1 - \left(1 - \varphi^+_{1(\check{p},\tilde{e})}(\check{s})^Q\right)^{\widetilde{W}_1}\right)^{1/Q}\right] > \times < [\theta^-_{2(\check{p},\tilde{e})}(\check{s})^{\widetilde{W}_2}, \theta^+_{2(\check{p},\tilde{e})}(\check{s})^{\widetilde{W}_2}], \left[\left(1 - \left(1 - \varphi^-_{2(\check{p},\tilde{e})}(\check{s})^Q\right)^{\widetilde{W}_2}\right)^{1/Q}, \left(1 - \left(1 - \varphi^+_{2(\check{p},\tilde{e})}(\check{s})^Q\right)^{\widetilde{W}_2}\right)^{1/Q}\right] >$$

$$=< \left[\theta^{-}_{1(\check{p},\tilde{e})}(\check{s})^{\widetilde{W}_1}\theta^{-}_{2(\check{p},\tilde{e})}(\check{s})^{\widetilde{W}_2}, \theta^{+}_{1(\check{p},\tilde{e})}(\check{s})^{\widetilde{W}_1}\theta^{+}_{2(\check{p},\tilde{e})}(\check{s})^{\widetilde{W}_2}\right], \left[\left(\left(\left(1-\left(1-\varphi^{-}_{1(\check{p},\tilde{e})}(\check{s})^{Q}\right)^{\widetilde{W}_1}\right)^{1/Q}\right)^{Q} + \left(\left(1-\left(1-\varphi^{-}_{2(\check{p},\tilde{e})}(\check{s})^{Q}\right)^{\widetilde{W}_2}\right)^{1/Q}\right)^{Q} - \left(\left(1-\left(1-\varphi^{-}_{1(\check{p},\tilde{e})}(\check{s})^{Q}\right)^{\widetilde{W}_1}\right)^{1/Q}\right)^{Q}\left(\left(1-\left(1-\varphi^{-}_{2(\check{p},\tilde{e})}(\check{s})^{Q}\right)^{\widetilde{W}_2}\right)^{1/Q}\right)^{Q}\right)^{1/Q}, \left(\left(\left(1-\left(1-\varphi^{+}_{1(\check{p},\tilde{e})}(\check{s})^{Q}\right)^{\widetilde{W}_1}\right)^{1/Q}\right)^{Q} + \left(\left(1-\left(1-\varphi^{+}_{2(\check{p},\tilde{e})}(\check{s})^{Q}\right)^{\widetilde{W}_2}\right)^{1/Q}\right)^{Q} - \left(\left(1-\left(1-\varphi^{+}_{1(\check{p},\tilde{e})}(\check{s})^{Q}\right)^{\widetilde{W}_1}\right)^{1/Q}\right)^{Q}\left(\left(1-\left(1-\varphi^{+}_{2(\check{p},\tilde{e})}(\check{s})^{Q}\right)^{\widetilde{W}_2}\right)^{1/Q}\right)^{Q}\right)^{1/Q}\right] >$$

$$=< \left[\theta^{-}_{1(\check{p},\tilde{e})}(\check{s})^{\widetilde{W}_1}\theta^{-}_{2(\check{p},\tilde{e})}(\check{s})^{\widetilde{W}_2}, \theta^{+}_{1(\check{p},\tilde{e})}(\check{s})^{\widetilde{W}_1}\theta^{+}_{2(\check{p},\tilde{e})}(\check{s})^{\widetilde{W}_2}\right], \left[\left(1-\left(1-\varphi^{-}_{1(\check{p},\tilde{e})}(\check{s})^{Q}\right)^{\widetilde{W}_1} + 1-\left(1-\varphi^{-}_{2(\check{p},\tilde{e})}(\check{s})^{Q}\right)^{\widetilde{W}_2} - 1 + \left(1-\varphi^{-}_{1(\check{p},\tilde{e})}(\check{s})^{Q}\right)^{\widetilde{W}_1} + \left(1-\varphi^{-}_{2(\check{p},\tilde{e})}(\check{s})^{Q}\right)^{\widetilde{W}_2} - \left(1-\varphi^{-}_{1(\check{p},\tilde{e})}(\check{s})^{Q}\right)^{\widetilde{W}_1}\left(1-\varphi^{-}_{2(\check{p},\tilde{e})}(\check{s})^{Q}\right)^{\widetilde{W}_2}\right)^{1/Q}, \left(1-\left(1-\varphi^{+}_{1(\check{p},\tilde{e})}(\check{s})^{Q}\right)^{\widetilde{W}_1} + 1-\left(1-\varphi^{+}_{2(\check{p},\tilde{e})}(\check{s})^{Q}\right)^{\widetilde{W}_2} - 1 + \left(1-\varphi^{+}_{1(\check{p},\tilde{e})}(\check{s})^{Q}\right)^{\widetilde{W}_1} + \left(1-\varphi^{+}_{2(\check{p},\tilde{e})}(\check{s})^{Q}\right)^{\widetilde{W}_2} - \left(1-\varphi^{+}_{1(\check{p},\tilde{e})}(\check{s})^{Q}\right)^{\widetilde{W}_1}\left(1-\varphi^{+}_{2(\check{p},\tilde{e})}(\check{s})^{Q}\right)^{\widetilde{W}_2}\right)^{1/Q}\right] >$$

$$=< \left[\theta^{-}_{1(\check{p},\tilde{e})}(\check{s})^{\widetilde{W}_1}\theta^{-}_{2(\check{p},\tilde{e})}(\check{s})^{\widetilde{W}_2}, \theta^{+}_{1(\check{p},\tilde{e})}(\check{s})^{\widetilde{W}_1}\theta^{+}_{2(\check{p},\tilde{e})}(\check{s})^{\widetilde{W}_2}\right], \left[\left(1-\left(1-\varphi^{-}_{1(\check{p},\tilde{e})}(\check{s})^{Q}\right)^{\widetilde{W}_1}\left(1-\varphi^{-}_{2(\check{p},\tilde{e})}(\check{s})^{Q}\right)^{\widetilde{W}_2}\right)^{1/Q}, \left(1-\left(1-\varphi^{+}_{1(\check{p},\tilde{e})}(\check{s})^{Q}\right)^{\widetilde{W}_1}\left(1-\varphi^{+}_{2(\check{p},\tilde{e})}(\check{s})^{Q}\right)^{\widetilde{W}_2}\right)^{1/Q}\right] >$$

$=< [\prod_{k=1}^{2}(\theta^{-}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k},\ \prod_{k=1}^{2}(\theta^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}],\ \left[\left(1-\prod_{k=1}^{2}\left(1-\right.\right.\right.$

$\left.\left.\left.(\varphi^{-}_{k(\check{p},\tilde{e})}(\check{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q},\left(1-\prod_{k=1}^{2}\left(1-(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q}\right]>.$

Next, suppose that result holds for k = $\breve{m}$, that is

$$\underline{\omega}\left(\left(h_{\breve{m}},\check{P}_{\breve{m}},\tilde{E}_{\breve{m}},\mathrm{Q}\right)_{(\theta_{\breve{m}},\,\varphi_{\breve{m}})}\right)=\prod_{k=1}^{\breve{m}}(\left(h_k,\check{P}_k,\tilde{E}_k,\mathrm{Q}\right)_{(\theta_k,\,\varphi_k)})^{\widetilde{W}_k}$$

$$=<\left[\prod_{k=1}^{\breve{m}}(\theta^{-}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k},\quad \prod_{k=1}^{\breve{m}}(\theta^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}\right],$$

$$\left[\left(1-\prod_{k=1}^{\breve{m}}\left(1-\left(\varphi^{-}_{k(\check{p},\tilde{e})}(\check{s})\right)^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q},\left(1-\prod_{k=1}^{\breve{m}}\left(1-\left(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s})\right)^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q}\right]>.$$

Further to show that result holds for k = $\breve{m}$+1,

$$\underline{\omega}\left(\left(h_1,\check{P}_1,\tilde{E}_1,\mathrm{Q}\right)_{(\theta_1,\,\varphi_1)},\left(h_2,\check{P}_2,\tilde{E}_2,\mathrm{Q}\right)_{(\theta_2,\,\varphi_2)},\ldots,\left(h_{\breve{m}+1},\check{P}_{\breve{m}+1},\tilde{E}_{\breve{m}+1},\mathrm{Q}\right)_{(\theta_{\breve{m}+1},\,\varphi_{\breve{m}+1})}\right)$$

$$=\prod_{k=1}^{\breve{m}+1}\left(\left(h_k,\check{P}_k,\tilde{E}_k,\mathrm{Q}\right)_{(\theta_k,\,\varphi_k)}\right)^{\widetilde{W}_k}$$

$$=\prod_{k=1}^{\breve{m}}\left(\left(h_k,\check{P}_k,\tilde{E}_k,\mathrm{Q}\right)_{(\theta_k,\,\varphi_k)}\right)^{\widetilde{W}_k}\times\ \left((h_{\breve{m}+1},P_{\breve{m}+1},E_{\breve{m}+1},\mathrm{Q})_{(\theta_{\breve{m}+1},\,\varphi_{\breve{m}+1})}\right)^{\widetilde{W}_{\breve{m}+1}}.$$

Again by using Definition (4.2.26) and product (4.2.29), we have

$$\underline{\omega}\left(\left(h_{\breve{m}+1},\check{P}_{\breve{m}+1},\tilde{E}_{\breve{m}+1},\mathrm{Q}\right)_{(\theta_{\breve{m}+1},\,\varphi_{\breve{m}+1})}\right).$$

$$=<[\prod_{k=1}^{\breve{m}+1}(\theta^{-}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k},\ \prod_{k=1}^{\breve{m}+1}(\theta^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}],$$

$$\left[\left(1-\prod_{k=1}^{\breve{m}+1}\left(1-\left(\varphi^{-}_{k(\check{p},\tilde{e})}(\check{s})\right)^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q},\left(1-\prod_{k=1}^{\breve{m}+1}\left(1-\left(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s})\right)^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q}\right]>$$

Hence proved.

Next to prove that $\underline{\omega}\left(\left(h_k, \breve{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}\right)$ is also Q-ROIVFSES. Let

$\theta^- = \prod_{k=1}^{\breve{m}}(\theta^-_{k(\breve{p},\tilde{e})}(\breve{s}))^{\widetilde{W}_k}$, $\theta^+ = \prod_{k=1}^{\breve{m}}(\theta^+_{k(\breve{p},\tilde{e})}(\breve{s}))^{\widetilde{W}_k}$

$\varphi^- = \left(1 - \prod_{k=1}^{\breve{m}}\left(1 - (\varphi^-_{k(\breve{p},\tilde{e})}(\breve{s}))^Q\right)^{\widetilde{W}_k}\right)^{1/Q}$, $\varphi^+ = \left(1 - \prod_{k=1}^{\breve{m}}\left(1 - (\varphi^+_{k(\breve{p},\tilde{e})}(\breve{s}))^Q\right)^{\widetilde{W}_k}\right)^{1/Q}$.

Since $0 \le \theta^-_{k(\breve{p},\tilde{e})}(\breve{s}), \theta^+_{k(\breve{p},\tilde{e})}(\breve{s}), \varphi^-_{k(\breve{p},\tilde{e})}(\breve{s}), \varphi^+_{k(\breve{p},\tilde{e})}(\breve{s}) \le 1$.

So, for θ^-, θ^+ we have

$0 \le (\theta^-_{k(\breve{p},\tilde{e})}(\breve{s}))^{\widetilde{W}_k} \le 1$

$\Rightarrow 0 \le \prod_{k=1}^{\breve{m}}(\theta^-_{k(\breve{p},\tilde{e})}(\breve{s}))^{\widetilde{W}_k} \le 1$

$\Rightarrow 0 \le \theta^- \le 1$

Similarly,

$0 \le (\theta^+_{k(\breve{p},\tilde{e})}(\breve{s}))^{\widetilde{W}_k} \le 1$

$\Rightarrow 0 \le \prod_{k=1}^{\breve{m}}(\theta^+_{k(\breve{p},\tilde{e})}(\breve{s}))^{\widetilde{W}_k} \le 1$

$\Rightarrow 0 \le \theta^+ \le 1$

Also, for φ^-, φ^+ , we have

$0 \le 1 - (\varphi^-_{k(\breve{p},\tilde{e})}(\breve{s}))^Q \le 1$

$\Rightarrow 0 \le (1 - (\varphi^-_{k(\breve{p},\tilde{e})}(\breve{s}))^Q)^{\widetilde{W}_k} \le 1$

$\Rightarrow 0 \le \prod_{k=1}^{\breve{m}}\left(1 - (\varphi^-_{k(\breve{p},\tilde{e})}(\breve{s}))^Q\right)^{\widetilde{W}_k} \le 1$

$\Rightarrow 0 \le 1 - \prod_{k=1}^{\breve{m}}\left(1 - (\varphi^-_{k(\breve{p},\tilde{e})}(\breve{s}))^Q\right)^{\widetilde{W}_k} \le 1$

$\Rightarrow 0 \le \left(1 - \prod_{k=1}^{\breve{m}}\left(1 - (\varphi^-_{k(\breve{p},\tilde{e})}(\breve{s}))^Q\right)^{\widetilde{W}_k}\right)^{1/Q} \le 1$

$\Rightarrow 0 \le \varphi^- \le 1$

Similarly,

$0 \le 1 - (\varphi^+_{k(\breve{p},\tilde{e})}(\breve{s}))^Q \le 1$

$\Rightarrow 0 \le (1 - (\varphi^+_{k(\breve{p},\tilde{e})}(\breve{s}))^Q)^{\widetilde{W}_k} \le 1$

$\Rightarrow 0 \le \prod_{k=1}^{\breve{m}}\left(1 - (\varphi^+_{k(\breve{p},\tilde{e})}(\breve{s}))^Q\right)^{\widetilde{W}_k} \le 1$

$\Rightarrow 0 \le 1 - \prod_{k=1}^{\breve{m}}\left(1 - (\varphi^+_{k(\breve{p},\tilde{e})}(\breve{s}))^Q\right)^{\widetilde{W}_k} \le 1$

$\Rightarrow 0 \le \left(1 - \prod_{k=1}^{m}\left(1 - (\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q} \le 1$

$\Rightarrow 0 \le \varphi^{+} \le 1.$

Therefore $0 \le \theta^{-}, \theta^{+}, \varphi^{-}, \varphi^{+} \le 1.$

Next, because $0 \le \theta^{+}_{k(\check{p},\tilde{e})}(\check{s}), \varphi^{+}_{k(\check{p},\tilde{e})}(\check{s}) \le 1$ and $\left(\theta^{+}_{k(\check{p},\tilde{e})}(\check{s})\right)^{Q} + \left(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s})\right)^{Q} \le 1,$

We have

$\left(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s})\right)^{Q} \left(\theta^{+}_{k(\check{p},\tilde{e})}(\check{s})\right)^{Q} \le 1 - \left(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s})\right)^{Q}$

$\Rightarrow \left(\left(\theta^{+}_{k(\check{p},\tilde{e})}(\check{s})\right)^{Q}\right)^{\widetilde{W}_k} \le \left(1 - \left(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s})\right)^{Q}\right)^{\widetilde{W}_k}$

$\Rightarrow \prod_{k=1}^{m}\left(\left(\theta^{+}_{k(\check{p},\tilde{e})}(\check{s})\right)^{Q}\right)^{\widetilde{W}_k} \le \prod_{k=1}^{m}\left(1 - \left(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s})\right)^{Q}\right)^{\widetilde{W}_k}.$ … (5.3.1)

Then consider

$0 \le (\theta^{+})^{Q} + (\varphi^{+})^{Q} = \left[\prod_{k=1}^{m}(\theta^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}\right]^{Q} + \left[\left(1 - \prod_{k=1}^{m}\left(1 - (\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q}\right]^{Q}$

$$= \left[\prod_{k=1}^{m}(\theta^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}\right]^{Q} + 1 - \prod_{k=1}^{m}\left(1 - (\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{Q}\right)^{\widetilde{W}_k}$$

$\le \prod_{k=1}^{m}\left(1 - \left(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s})\right)^{Q}\right)^{\widetilde{W}_k} + 1 - \prod_{k=1}^{m}\left(1 - \left(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s})\right)^{Q}\right)^{\widetilde{W}_k}$ using Inequality (5.3.1)

$= 1.$

So, the aggregated results of Q-ROIVFSEWAO meets the following two condition

iii) $0 \le \theta^{-}, \theta^{+}, \varphi^{-}, \varphi^{+} \le 1,$

iv) $0 \le (\theta^{+})^{Q} + (\varphi^{+})^{Q} \le 1.$

Thus, it is also a Q-ROIVFSES and theorem is proved.

Example 5.3.4

Let S = $\{\check{s}_1, \check{s}_2\}$, P = $\{\check{p}_1\}$, E = $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$be the universe, parameter and expert set respectively. Consider the Q-ROIVFSESs

$(h_1, \check{P}_1, \tilde{E}_1, Q) = h(\check{p}_1, \tilde{e}_1) = \{< \check{s}_1, [0.7, 0.8], [0.5, 0.6] >, < \check{s}_2, [0.6, 0.7], [0.6, 0.7] >\}$,

$(h_2, \check{P}_2, \tilde{E}_2, Q) = h(\check{p}_1, \tilde{e}_2) = \{< \check{s}_1, [0.3, 0.5], [0.5, 0.7] >, < \check{s}_2, [0.5, 0.6], [0.7, 0.8] >\}$,

$(h_3, \check{P}_3, \tilde{E}_3, Q) = h(\check{p}_1, \tilde{e}_3) = \{< \check{s}_1, [0.6, 0.7], [0.7, 0.8] >, < \check{s}_2, [0.5, 0.6], [0.3, 0.4] >\}$.

And $\widetilde{W} = (0.3, 0.3, 0.4)^T$ be the weight vector of $h(\check{p}_1, \tilde{e}_1)$, $h(\check{p}_1, \tilde{e}_2)$ and $h(\check{p}_1, \tilde{e}_3)$, without loss of generality take Q = 3 then Q-ROIVFSEWGO to aggregate the Q-ROIVFSENs as follows

$$\underline{\omega}\left(\left(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}\right) = \left\{< \check{s}_i, \left[\prod_{k=1}^{3}(\theta^{-}_{k(\check{p},\tilde{e})}(\check{s}_i))^{\widetilde{W}_k},\ \prod_{k=1}^{3}(\theta^{+}_{k(\check{p},\tilde{e})}(\check{s}_i))^{\widetilde{W}_k}\right],\right.$$

$$\left.\left[\left(1-\prod_{k=1}^{3}\left(1-\left(\varphi^{-}_{k(\check{p},\tilde{e})}(\check{s}_i)\right)^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q},\ \left(1-\prod_{k=1}^{3}\left(1-\left(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s}_i)\right)^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q}\right] >\right\}$$

where $i = 1, 2$

$=\{< \check{s}_1, [0.5104, 0.6586], [0.6034, 0.7274] >, < \check{s}_2, [0.5281, 0.6284], [0.5743, 0.675] >\}$

It is easy to prove that the Q-ROIVFSEWGO has the following properties.

Theorem 5.3.5 (Idempotency)

Consider $(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q})_{(\theta_k,\, \varphi_k)} = \{< \check{s}, [\theta^{-}_{k(\check{p},\tilde{e})}(\check{s}),\ \theta^{+}_{k(\check{p},\tilde{e})}(\check{s})], [\varphi^{-}_{k(\check{p},\tilde{e})}(\check{s}),\ \varphi^{+}_{k(\check{p},\tilde{e})}(\check{s})] >: \check{s} \in \mathrm{S}\}$ for k=1, 2, …, $\breve{m}$ with Q ≥ 1, be a Q-ROIVFSESs collection and if $(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q})_{(\theta_k,\, \varphi_k)} = (h, \check{P}, \tilde{E}, \mathrm{Q})_{(\theta,\varphi)} \forall$ k = 1, 2, …, $\breve{m}$ then

$$\underline{\omega}\left(\left(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}\right) = (h, \check{P}, \tilde{E}, \mathrm{Q})_{(\theta,\varphi)},$$

where $(h, \check{P}, \tilde{E}, \mathrm{Q})_{(\theta,\varphi)} = \{< \check{s}, [\theta^{-}_{(\check{p},\tilde{e})}(\check{s}),\ \theta^{+}_{(\check{p},\tilde{e})}(\check{s})], [\varphi^{-}_{(\check{p},\tilde{e})}(\check{s}),\ \varphi^{+}_{(\check{p},\tilde{e})}(\check{s})] >: \check{s} \in \mathrm{S}\}$.

Proof

For a Q-ROIVFSESs collection $(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q})_{(\theta_k,\, \varphi_k)} = \{< \check{s}, [\theta^{-}_{k(\check{p},\tilde{e})}(\check{s}),\ \theta^{+}_{k(\check{p},\tilde{e})}(\check{s})], [\varphi^{-}_{k(\check{p},\tilde{e})}(\check{s}),\ \varphi^{+}_{k(\check{p},\tilde{e})}(\check{s})] >: \check{s} \in \mathrm{S}\}$ for k=1, 2, …, $\breve{m}$ with Q ≥ 1, the Q-ROIVFSE weighted geometric operator is given by

$$\underline{\omega}\left(\left(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}\right) = \prod_{k=1}^{\breve{m}}(\left(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)})^{\widetilde{W}_k}$$

$$= \left\{< \left[\prod_{k=1}^{\breve{m}}(\theta^-_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k},\ \prod_{k=1}^{\breve{m}}(\theta^+_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}\right],\ \left[\left(1 - \prod_{k=1}^{\breve{m}}\left(1 - \left(\varphi^-_{k(\check{p},\tilde{e})}(\check{s})\right)^Q\right)^{\widetilde{W}_k}\right)^{1/Q},\ \left(1 - \prod_{k=1}^{\breve{m}}\left(1 - \left(\varphi^+_{k(\check{p},\tilde{e})}(\check{s})\right)^Q\right)^{\widetilde{W}_k}\right)^{1/Q}\right] >\right\}$$

$$= \left\{< \left[\prod_{k=1}^{\breve{m}}(\theta^-_{(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}, \prod_{k=1}^{\breve{m}}(\theta^+_{(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}\right],\ \left[\left(1 - \prod_{k=1}^{\breve{m}}\left(1 - \left(\varphi^-_{(\check{p},\tilde{e})}(\check{s})\right)^Q\right)^{\widetilde{W}_k}\right)^{1/Q},\ \left(1 - \prod_{k=1}^{\breve{m}}\left(1 - \left(\varphi^+_{(\check{p},\tilde{e})}(\check{s})\right)^Q\right)^{\widetilde{W}_k}\right)^{1/Q}\right] >\right\}$$

$$= \left\{< \left[(\theta^-_{(\check{p},\tilde{e})}(\check{s}))^{\sum_{k=1}^{\breve{m}}\widetilde{W}_k}, (\theta^+_{(\check{p},\tilde{e})}(\check{s}))^{\sum_{k=1}^{\breve{m}}\widetilde{W}_k}\right],\ \left[\left(1 - \left(1 - \left(\varphi^-_{(\check{p},\tilde{e})}(\check{s})\right)^Q\right)^{\sum_{k=1}^{\breve{m}}\widetilde{W}_k}\right)^{1/Q}, \left(1 - \left(1 - \left(\varphi^+_{(\check{p},\tilde{e})}(\check{s})\right)^Q\right)^{\sum_{k=1}^{\breve{m}}\widetilde{W}_k}\right)^{1/Q}\right] >\right\}$$

As $\sum_{k=1}^{\breve{m}}\widetilde{W}_k = 1$

$$= \left\{< \left[(\theta^-_{(\check{p},\tilde{e})}(\check{s}))^1, (\theta^+_{(\check{p},\tilde{e})}(\check{s}))^1\right],\ \left[\left(1 - \left(1 - \left(\varphi^-_{(\check{p},\tilde{e})}(\check{s})\right)^Q\right)^1\right)^{1/Q}, \left(1 - \left(1 - \left(\varphi^+_{(\check{p},\tilde{e})}(\check{s})\right)^Q\right)^1\right)^{1/Q}\right] >\right\}$$

$$= \left\{< \left[(\theta^-_{(\check{p},\tilde{e})}(\check{s}))^1, (\theta^+_{(\check{p},\tilde{e})}(\check{s}))^1\right],\ \left[\left(1 - 1 + \left(\varphi^-_{(\check{p},\tilde{e})}(\check{s})\right)^Q\right)^{1/Q}, \left(1 - 1 + \left(\varphi^+_{(\check{p},\tilde{e})}(\check{s})\right)^Q\right)^{1/Q}\right] >\right\}$$

$$= \left\{< \left[(\theta^-_{(\check{p},\tilde{e})}(\check{s}))^1, (\theta^+_{(\check{p},\tilde{e})}(\check{s}))^1\right], \left[\left(\left(\varphi^-_{(\check{p},\tilde{e})}(\check{s})\right)^Q\right)^{1/Q}, \left(\left(\varphi^+_{(\check{p},\tilde{e})}(\check{s})\right)^Q\right)^{1/Q}\right] >\right\}$$

$$= \left\{< \left[\theta^-_{(\check{p},\tilde{e})}(\check{s}),\ \theta^+_{(\check{p},\tilde{e})}(\check{s})\right], \left[\varphi^-_{(\check{p},\tilde{e})}(\check{s}),\ \varphi^+_{(\check{p},\tilde{e})}(\check{s})\right] >\right\} = (h, \check{P}, \tilde{E}, \mathrm{Q})_{(\theta,\varphi)}.$$

Theorem 5.3.6 (Monotonicity)

Suppose $(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q})_{(\theta_k, \varphi_k)}$ and $(\tilde{h}_k, \tilde{\check{P}}_k, \tilde{\tilde{E}}_k, \mathrm{Q})_{(\tilde{\theta}_k, \tilde{\varphi}_k)}$ for k = 1, 2, …, $\breve{m}$ be two sets of Q-ROIVFSESs. If $\theta^-_{k(\check{p},\tilde{e})} \geq \tilde{\theta}^-_{k(\check{p},\tilde{e})}$, $\theta^+_{k(\check{p},\tilde{e})} \geq \tilde{\theta}^+_{k(\check{p},\tilde{e})}$, $\varphi^-_{k(\check{p},\tilde{e})} \leq \tilde{\varphi}^-_{k(\check{p},\tilde{e})}$ and $\varphi^+_{k(\check{p},\tilde{e})} \leq \tilde{\varphi}^+_{k(\check{p},\tilde{e})} \forall\ \mathrm{k} = 1, 2, \dots, \breve{m}$ then $\underline{\omega}\left(\left(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k, \varphi_k)}\right) \geq \underline{\omega}\left(\left(\tilde{h}_k, \tilde{\check{P}}_k, \tilde{\tilde{E}}_k, \mathrm{Q}\right)_{(\tilde{\theta}_k, \tilde{\varphi}_k)}\right)$.

Proof

Since $\theta^-_{k(\check{p},\tilde{e})}(\check{s}) \geq \tilde{\theta}^-_{k(\check{p},\tilde{e})}(\check{s})$, $\theta^+_{k(\check{p},\tilde{e})}(\check{s}) \geq \tilde{\theta}^+_{k(\check{p},\tilde{e})}(\check{s})$, $\varphi^-_{k(\check{p},\tilde{e})}(\check{s}) \leq \tilde{\varphi}^-_{k(\check{p},\tilde{e})}(\check{s})$ and $\varphi^+_{k(\check{p},\tilde{e})}(\check{s}) \leq \tilde{\varphi}^+_{k(\check{p},\tilde{e})}(\check{s}) \forall\ \mathrm{k} = 1, 2, \dots, \breve{m}$ then

$$\prod_{k=1}^{\breve{m}} (\theta^-_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k} \geq \prod_{k=1}^{\breve{m}} (\tilde{\theta}^-_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}.$$

$$\Rightarrow \left(\prod_{k=1}^{\breve{m}} (\theta^-_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}\right)^Q \geq \left(\prod_{k=1}^{\breve{m}} (\tilde{\theta}^-_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}\right)^Q. \quad \dots (5.3.2)$$

Also

$$-(\varphi^-_{k(\check{p},\tilde{e})}(\check{s}))^Q \geq -(\tilde{\varphi}^-_{k(\check{p},\tilde{e})}(\check{s}))^Q.$$

$$\Rightarrow 1 - (\varphi^-_{k(\check{p},\tilde{e})}(\check{s}))^Q \geq 1 - (\tilde{\varphi}^-_{k(\check{p},\tilde{e})}(\check{s}))^Q.$$

$$\Rightarrow \prod_{k=1}^{\breve{m}} \left(1 - (\varphi^-_{k(\check{p},\tilde{e})}(\check{s}))^Q\right)^{\widetilde{W}_k} \geq \prod_{k=1}^{\breve{m}} \left(1 - (\tilde{\varphi}^-_{k(\check{p},\tilde{e})}(\check{s}))^Q\right)^{\widetilde{W}_k}.$$

$$\Rightarrow \left(1 - \prod_{k=1}^{\breve{m}} \left(1 - (\varphi^-_{k(\check{p},\tilde{e})}(\check{s}))^Q\right)^{\widetilde{W}_k}\right)^{1/Q} \leq \left(1 - \prod_{k=1}^{\breve{m}} \left(1 - (\tilde{\varphi}^-_{k(\check{p},\tilde{e})}(\check{s}))^Q\right)^{\widetilde{W}_k}\right)^{1/Q}.$$

$$\Rightarrow \left(1 - \prod_{k=1}^{\breve{m}} \left(1 - (\varphi^-_{k(\check{p},\tilde{e})}(\check{s}))^Q\right)^{\widetilde{W}_k}\right)^{1/Q^{\,Q}} \leq \left(1 - \prod_{k=1}^{\breve{m}} \left(1 - (\tilde{\varphi}^-_{k(\check{p},\tilde{e})}(\check{s}))^Q\right)^{\widetilde{W}_k}\right)^{1/Q^{\,Q}}.$$

$$\Rightarrow -\left(1 - \prod_{k=1}^{\breve{m}} \left(1 - (\varphi^-_{k(\check{p},\tilde{e})}(\check{s}))^Q\right)^{\widetilde{W}_k}\right)^{1/Q^{\,Q}} \geq -\left(1 - \prod_{k=1}^{\breve{m}} \left(1 - (\tilde{\varphi}^-_{k(\check{p},\tilde{e})}(\check{s}))^Q\right)^{\widetilde{W}_k}\right)^{1/Q^{\,Q}}. \quad \dots (5.3.3)$$

Combining (5.3.2) and (5.3.3),

$$\left(\prod_{k=1}^{\breve{m}} (\theta^-_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}\right)^Q - \left(1 - \prod_{k=1}^{\breve{m}} \left(1 - (\varphi^-_{k(\check{p},\tilde{e})}(\check{s}))^Q\right)^{\widetilde{W}_k}\right)^{1/Q^{\,Q}} \geq$$

$$\left(\prod_{k=1}^{\breve{m}} (\tilde{\theta}^-_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}\right)^Q - \left(1 - \prod_{k=1}^{\breve{m}} \left(1 - (\tilde{\varphi}^-_{k(\check{p},\tilde{e})}(\check{s}))^Q\right)^{\widetilde{W}_k}\right)^{1/Q^{\,Q}}. \quad \dots (5.3.4)$$

Similarly,

$$\left(\prod_{k=1}^{m}(\theta^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}\right)^{Q} - \left(1-\prod_{k=1}^{m}\left(1-(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q^{Q}} \geq$$

$$\left(\prod_{k=1}^{m}(\tilde{\theta}^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}\right)^{Q} - \left(1-\prod_{k=1}^{m}\left(1-(\tilde{\varphi}^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q^{Q}} \quad \ldots \text{ (5.3.5)}$$

Adding (5.3.4) and (5.3.5)

$$\Rightarrow \left(\prod_{k=1}^{m}(\theta^{-}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}\right)^{Q} + \left(\prod_{k=1}^{m}(\theta^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}\right)^{Q} - \left(1-\prod_{k=1}^{m}\left(1-(\varphi^{-}_{k(\check{p},\tilde{e})}(\check{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q^{Q}} - \left(1-\prod_{k=1}^{m}\left(1-(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q^{Q}} \geq$$

$$\left(\prod_{k=1}^{m}(\tilde{\theta}^{-}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}\right)^{Q} + \left(\prod_{k=1}^{m}(\tilde{\theta}^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}\right)^{Q} - \left(1-\prod_{k=1}^{m}\left(1-(\tilde{\varphi}^{-}_{k(\check{p},\tilde{e})}(\check{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q^{Q}} - \left(1-\prod_{k=1}^{m}\left(1-(\tilde{\varphi}^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q^{Q}}$$

… (5.2.6)

$\Rightarrow \S\left(\underline{\omega}\left(\left(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}\right)\right) \geq \S\left(\underline{\omega}\left(\left(\tilde{h}_k, \tilde{\check{P}}_k, \tilde{\tilde{E}}_k, \mathrm{Q}\right)_{(\tilde{\theta}_k, \tilde{\varphi}_k)}\right)\right)$ by using Definition 4.2.36.

Now by using Theorem 4.2.37,

1. If $\S\left(\underline{\omega}\left(\left(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}\right)\right) > \S\left(\underline{\omega}\left(\left(\tilde{h}_k, \tilde{\check{P}}_k, \tilde{\tilde{E}}_k, \mathrm{Q}\right)_{(\tilde{\theta}_k, \tilde{\varphi}_k)}\right)\right)$ then
   $\underline{\omega}\left(\left(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}\right) > \underline{\omega}\left(\left(\tilde{h}_k, \tilde{\check{P}}_k, \tilde{\tilde{E}}_k, \mathrm{Q}\right)_{(\tilde{\theta}_k, \tilde{\varphi}_k)}\right)$. … (5.3.7)
2. And if $\S\left(\underline{\omega}\left(\left(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}\right)\right) = \S\left(\underline{\omega}\left(\left(\tilde{h}_k, \tilde{\check{P}}_k, \tilde{\tilde{E}}_k, \mathrm{Q}\right)_{(\tilde{\theta}_k, \tilde{\varphi}_k)}\right)\right)$.

Then (5.3.6) reduces as

$$\left(\prod_{k=1}^{m}(\theta^{-}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}\right)^{Q} + \left(\prod_{k=1}^{m}(\theta^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}\right)^{Q} - \left(1-\prod_{k=1}^{m}\left(1-(\varphi^{-}_{k(\check{p},\tilde{e})}(\check{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q^{Q}} - \left(1-\prod_{k=1}^{m}\left(1-(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q^{Q}} =$$

$$\left(\prod_{k=1}^{m}(\tilde{\theta}^{-}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}\right)^{Q} + \left(\prod_{k=1}^{m}(\tilde{\theta}^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}\right)^{Q} - \left(1-\prod_{k=1}^{m}\left(1-(\tilde{\varphi}^{-}_{k(\check{p},\tilde{e})}(\check{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q^{Q}} - \left(1-\prod_{k=1}^{m}\left(1-(\tilde{\varphi}^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q^{Q}}.$$

Then by condition,

$\theta^{-}_{k(\check{p},\tilde{e})}(\check{s}) \geq \tilde{\theta}^{-}_{k(\check{p},\tilde{e})}(\check{s})$, $\theta^{+}_{k(\check{p},\tilde{e})}(\check{s}) \geq \tilde{\theta}^{+}_{k(\check{p},\tilde{e})}(\check{s})$, $\varphi^{-}_{k(\check{p},\tilde{e})}(\check{s}) \leq \tilde{\varphi}^{-}_{k(\check{p},\tilde{e})}(\check{s})$ and $\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s}) \leq \tilde{\varphi}^{+}_{k(\check{p},\tilde{e})}(\check{s}) \forall$ k, we have

$$\prod_{k=1}^{\breve{m}}(\theta^{-}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k} = \prod_{k=1}^{\breve{m}}(\tilde{\theta}^{-}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}, \prod_{k=1}^{\breve{m}}(\theta^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k} = \prod_{k=1}^{\breve{m}}(\tilde{\theta}^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k},$$

$$\left(1 - \prod_{k=1}^{\breve{m}}\left(1 - (\varphi^{-}_{k(\check{p},\tilde{e})}(\check{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q} = \left(1 - \prod_{k=1}^{\breve{m}}\left(1 - (\tilde{\varphi}^{-}_{k(\check{p},\tilde{e})}(\check{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q},$$

and $\left(1 - \prod_{k=1}^{\breve{m}}\left(1 - (\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q} = \left(1 - \prod_{k=1}^{\breve{m}}\left(1 - (\tilde{\varphi}^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q}$.

$$\Rightarrow \left(\prod_{k=1}^{\breve{m}}(\theta^{-}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}\right)^{Q} + \left(\prod_{k=1}^{\breve{m}}(\theta^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}\right)^{Q} + \left(1 - \prod_{k=1}^{\breve{m}}\left(1 - (\varphi^{-}_{k(\check{p},\tilde{e})}(\check{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q^{\,Q}} + \left(1 - \prod_{k=1}^{\breve{m}}\left(1 - (\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q^{\,Q}} = \left(\prod_{k=1}^{\breve{m}}(\tilde{\theta}^{-}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}\right)^{Q} + \left(\prod_{k=1}^{\breve{m}}(\tilde{\theta}^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}\right)^{Q} + \left(1 - \prod_{k=1}^{\breve{m}}\left(1 - (\tilde{\varphi}^{-}_{k(\check{p},\tilde{e})}(\check{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q^{\,Q}} + \left(1 - \prod_{k=1}^{\breve{m}}\left(1 - (\tilde{\varphi}^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q^{\,Q}}.$$

$$\Rightarrow \hat{A}\left(\underline{\omega}\left(\left(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k, \varphi_k)}\right)\right) = \hat{A}\left(\underline{\omega}\left(\left(\tilde{h}_k, \tilde{\check{P}}_k, \tilde{\tilde{E}}_k, \mathrm{Q}\right)_{(\tilde{\theta}_k, \tilde{\varphi}_k)}\right)\right).$$

Then

$$\underline{\omega}\left(\left(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k, \varphi_k)}\right) = \underline{\omega}\left(\left(\tilde{h}_k, \tilde{\check{P}}_k, \tilde{\tilde{E}}_k, \mathrm{Q}\right)_{(\tilde{\theta}_k, \tilde{\varphi}_k)}\right) \ldots \quad (5.3.8)$$

Combining (5.3.7) and (5.3.8) implies

$$\underline{\omega}\left(\left(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k, \varphi_k)}\right) \geq \underline{\omega}\left(\left(\tilde{h}_k, \tilde{\check{P}}_k, \tilde{\tilde{E}}_k, \mathrm{Q}\right)_{(\tilde{\theta}_k, \tilde{\varphi}_k)}\right).$$

Theorem 5.3.7 (Boundedness)

Consider a collection of Q-ROIVFSESs $\left(h_k, \breve{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}$ and $\left(h_i, \breve{P}_i, \tilde{E}_i, \mathrm{Q}\right)_{(\theta_i,\, \varphi_i)} = \{< \breve{s}, [\min_{1\le k\le \breve{m}} \theta_k^-(\breve{s}), \min_{1\le k\le \breve{m}} \theta_k^+(\breve{s})], [\max_{1\le k\le \breve{m}} \varphi_k^-(\breve{s}), \max_{1\le k\le \breve{m}} \varphi_k^+(\breve{s})] >\}$ and $\left(h_s, \breve{P}_s, \tilde{E}_s, \mathrm{Q}\right)_{(\theta_s,\, \varphi_s)} = \{< \breve{s}, [\max_{1\le k\le \breve{m}} \theta_k^-(\breve{s}), \max_{1\le k\le \breve{m}} \theta_k^+(\breve{s})], [\min_{1\le k\le \breve{m}} \varphi_k^-(\breve{s}), \min_{1\le k\le \breve{m}} \varphi_k^+(\breve{s})] >\}$ then $\left(h_i, \breve{P}_i, \tilde{E}_i, \mathrm{Q}\right)_{(\theta_i,\, \varphi_i)} \le \underline{\omega}\left(\left(h_k, \breve{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}\right) \le \left(h_s, \breve{P}_s, \tilde{E}_s, \mathrm{Q}\right)_{(\theta_s,\, \varphi_s)}$.

Proof

For membership degree of $\underline{\omega}\left(\left(h_k, \breve{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}\right)$ we get

$$\prod_{k=1}^{\breve{m}} \min_{1\le k\le \breve{m}} (\theta^+_{k(\breve{p},\tilde{e})}(\breve{s}))^{\widetilde{W}_k} \le \prod_{k=1}^{\breve{m}} (\theta^+_{k(\breve{p},\tilde{e})}(\breve{s}))^{\widetilde{W}_k} \le \prod_{k=1}^{\breve{m}} \max_{1\le k\le \breve{m}} (\theta^+_{k(\breve{p},\tilde{e})}(\breve{s}))^{\widetilde{W}_k}$$

$$\Rightarrow \min_{1\le k\le \breve{m}} (\theta^+_{k(\breve{p},\tilde{e})}(\breve{s}))^{\sum_{k=1}^{\breve{m}} \widetilde{W}_k} \le \prod_{k=1}^{\breve{m}} (\theta^+_{k(\breve{p},\tilde{e})}(\breve{s}))^{\widetilde{W}_k} \le \max_{1\le k\le \breve{m}} (\theta^+_{k(\breve{p},\tilde{e})}(\breve{s}))^{\sum_{k=1}^{\breve{m}} \widetilde{W}_k}$$

$$\Rightarrow \min_{1\le k\le \breve{m}} (\theta^+_{k(\breve{p},\tilde{e})}(\breve{s})) \le \prod_{k=1}^{\breve{m}} (\theta^+_{k(\breve{p},\tilde{e})}(\breve{s}))^{\widetilde{W}_k} \le \max_{1\le k\le \breve{m}} (\theta^+_{k(\breve{p},\tilde{e})}(\breve{s}))$$

$$\Rightarrow (\min_{1\le k\le \breve{m}} (\theta^+_{k(\breve{p},\tilde{e})}(\breve{s})))^Q \le \left(\prod_{k=1}^{\breve{m}} (\theta^+_{k(\breve{p},\tilde{e})}(\breve{s}))^{\widetilde{W}_k}\right)^Q \le (\max_{1\le k\le \breve{m}} (\theta^+_{k(\breve{p},\tilde{e})}(\breve{s})))^Q \quad \dots (5.3.9)$$

Similarly,

$$(\min_{1\le k\le \breve{m}} (\theta^-_{k(\breve{p},\tilde{e})}(\breve{s})))^Q \le \left(\prod_{k=1}^{\breve{m}} (\theta^-_{k(\breve{p},\tilde{e})}(\breve{s}))^{\widetilde{W}_k}\right)^Q \le (\max_{1\le k\le \breve{m}} (\theta^-_{k(\breve{p},\tilde{e})}(\breve{s})))^Q \quad \dots (5.3.10)$$

For non-membership degree of $\underline{\omega}\left((h_k, P_k, E_k, \mathrm{Q})_{(\theta_k,\, \varphi_k)}\right)$

$$\Rightarrow \min_{1\le k\le \breve{m}} (\varphi_k^+(\breve{s}))^{\mathrm{Q}} \le (\varphi_k^+(\breve{s}))^{\mathrm{Q}} \le \max_{1\le k\le \breve{m}} (\varphi_k^+(\breve{s}))^{\mathrm{Q}}.$$

$$\Rightarrow \left(1-\prod_{k=1}^{\breve{m}}\left(1-\min_{1\le k\le \breve{m}}(\varphi^{+}_{k(\breve{p},\tilde{e})}(\breve{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q} \le \left(1-\prod_{k=1}^{\breve{m}}\left(1-(\varphi^{+}_{k(\breve{p},\tilde{e})}(\breve{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q} \le \left(1-\prod_{k=1}^{\breve{m}}\left(1-\max_{1\le k\le \breve{m}}(\varphi^{+}_{k(\breve{p},\tilde{e})}(\breve{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q}.$$

$$\Rightarrow \left(1-\left(1-\min_{1\le k\le \breve{m}}(\varphi^{+}_{k(\breve{p},\tilde{e})}(\breve{s}))^{Q}\right)^{\sum_{k=1}^{\breve{m}}\widetilde{W}_k}\right)^{1/Q} \le \left(1-\prod_{k=1}^{\breve{m}}\left(1-(\varphi^{+}_{k(\breve{p},\tilde{e})}(\breve{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q} \le \left(1-\left(1-\max_{1\le k\le \breve{m}}(\varphi^{+}_{k(\breve{p},\tilde{e})}(\breve{s}))^{Q}\right)^{\sum_{k=1}^{\breve{m}}\widetilde{W}_k}\right)^{1/Q}.$$

As $\sum_{k=1}^{\breve{m}}\widetilde{W}_k = 1$

$$\Rightarrow \left(1-\left(1-\min_{1\le k\le \breve{m}}(\varphi^{+}_{k(\breve{p},\tilde{e})}(\breve{s}))^{Q}\right)^{1}\right)^{1/Q} \le \left(1-\prod_{k=1}^{\breve{m}}\left(1-(\varphi^{+}_{k(\breve{p},\tilde{e})}(\breve{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q} \le \left(1-\left(1-\max_{1\le k\le \breve{m}}(\varphi^{+}_{k(\breve{p},\tilde{e})}(\breve{s}))^{Q}\right)^{1}\right)^{1/Q}.$$

$$\Rightarrow \left(1-1+\min_{1\le k\le \breve{m}}(\varphi^{+}_{k(\breve{p},\tilde{e})}(\breve{s}))^{Q}\right)^{1/Q} \le \left(1-\prod_{k=1}^{\breve{m}}\left(1-(\varphi^{+}_{k(\breve{p},\tilde{e})}(\breve{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q} \le \left(1-1+\max_{1\le k\le \breve{m}}(\varphi^{+}_{k(\breve{p},\tilde{e})}(\breve{s}))^{Q}\right)^{1/Q}.$$

$$\Rightarrow \min_{1\le k\le \breve{m}}(\varphi^{+}_{k(\breve{p},\tilde{e})}(\breve{s})) \le \left(1-\prod_{k=1}^{\breve{m}}\left(1-(\varphi^{+}_{k(\breve{p},\tilde{e})}(\breve{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q} \le \max_{1\le k\le \breve{m}}(\varphi^{+}_{k(\breve{p},\tilde{e})}(\breve{s})).$$

$$\Rightarrow -\min_{1\le k\le \breve{m}}(\varphi^{+}_{k(\breve{p},\tilde{e})}(\breve{s})) \ge -\left(1-\prod_{k=1}^{\breve{m}}\left(1-(\varphi^{+}_{k(\breve{p},\tilde{e})}(\breve{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q} \ge -\max_{1\le k\le \breve{m}}(\varphi^{+}_{k(\breve{p},\tilde{e})}(\breve{s})).$$

$$\Rightarrow -\max_{1\le k\le \breve{m}}(\varphi^{+}_{k(\breve{p},\tilde{e})}(\breve{s})) \le -\left(1-\prod_{k=1}^{\breve{m}}\left(1-(\varphi^{+}_{k(\breve{p},\tilde{e})}(\breve{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q} \le -\min_{1\le k\le \breve{m}}(\varphi^{+}_{k(\breve{p},\tilde{e})}(\breve{s}). \qquad \ldots (5.3.11)$$

Similarly,

$$\Rightarrow -\max_{1\le k\le \breve{m}}(\varphi^{-}_{k(\breve{p},\tilde{e})}(\breve{s})) \le -\left(1-\prod_{k=1}^{\breve{m}}\left(1-(\varphi^{-}_{k(\breve{p},\tilde{e})}(\breve{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q} \le -\min_{1\le k\le \breve{m}}(\varphi^{-}_{k(\breve{p},\tilde{e})}(\breve{s})). \qquad \ldots (5.3.12)$$

Adding Inequalities (5.3.9) - (5.3.12) and dividing by 2,

$$\frac{\left(\min_{1\le k\le \breve{m}}(\theta^{+}_{k(\breve{p},\tilde{e})}(\breve{s}))\right)^{Q} + \left(\min_{1\le k\le \breve{m}}(\theta^{-}_{k(\breve{p},\tilde{e})}(\breve{s}))\right)^{Q} - (\max_{1\le k\le \breve{m}}(\varphi^{+}_{k(\breve{p},\tilde{e})}(\breve{s})))^{Q} - (\max_{1\le k\le \breve{m}}(\varphi^{-}_{k(\breve{p},\tilde{e})}(\breve{s})))^{Q}}{2}$$

$$\le \frac{\left(\prod_{k=1}^{\breve{m}}(\theta^{+}_{k(\breve{p},\tilde{e})}(\breve{s}))^{\widetilde{W}_k}\right)^{Q} + \left(\prod_{k=1}^{\breve{m}}(\theta^{-}_{k(\breve{p},\tilde{e})}(\breve{s}))^{\widetilde{W}_k}\right)^{Q} - \left(\left(1-\prod_{k=1}^{\breve{m}}\left(1-(\varphi^{+}_{k(\breve{p},\tilde{e})}(\breve{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q}\right)^{Q} - \left(\left(1-\prod_{k=1}^{\breve{m}}\left(1-(\varphi^{-}_{k(\breve{p},\tilde{e})}(\breve{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q}\right)^{Q}}{2}$$

$$\le \frac{\left(\max_{1\le k\le \breve{m}}(\theta^{+}_{k(\breve{p},\tilde{e})}(\breve{s}))\right)^{Q} + \left(\max_{1\le k\le \breve{m}}(\theta^{-}_{k(\breve{p},\tilde{e})}(\breve{s}))\right)^{Q} - (\min_{1\le k\le \breve{m}}(\varphi^{+}_{k(\breve{p},\tilde{e})}(\breve{s})))^{Q} - (\min_{1\le k\le \breve{m}}(\varphi^{-}_{k(\breve{p},\tilde{e})}(\breve{s})))^{Q}}{2}$$

$$\Rightarrow \S\left(\left(h_i, \breve{P}_i, \tilde{E}_i, \mathrm{Q}\right)_{(\theta_i,\, \varphi_i)}\right) \le \S\left(\underline{\omega}\left(\left(h_k, \breve{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}\right)\right) \le \S\left(\left(h_s, \breve{P}_s, \tilde{E}_s, \mathrm{Q}\right)_{(\theta_s,\, \varphi_s)}\right)$$

Then by using Theorem 4.2.37

$$\Rightarrow \left(h_i, \breve{P}_i, \tilde{E}_i, \mathrm{Q}\right)_{(\theta_i,\, \varphi_i)} \le \underline{\omega}\left(\left(h_k, \breve{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}\right) \le \left(h_s, \breve{P}_s, \tilde{E}_s, \mathrm{Q}\right)_{(\theta_s,\, \varphi_s)}.$$

Example 5.3.8

Let S = $\{\breve{s}_1, \breve{s}_2\}$, P = $\{\breve{p}_1\}$, E = $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$be the universe, parameter and expert set respectively. Consider the Q-ROIVFSESs

$(h_1, \breve{P}_1, \tilde{E}_1, Q) = h(\breve{p}_1, \tilde{e}_1) = \{< \breve{s}_1, [0.7, 0.8], [0.5, 0.6] >, < \breve{s}_2, [0.6, 0.7], [0.6, 0.7] >\}$,

$(h_2, \breve{P}_2, \tilde{E}_2, Q) = h(\breve{p}_1, \tilde{e}_2) = \{< \breve{s}_1, [0.3, 0.5], [0.5, 0.7] >, < \breve{s}_2, [0.5, 0.6], [0.7, 0.8] >\}$,

$(h_3, \breve{P}_3, \tilde{E}_3, Q) = h(\breve{p}_1, \tilde{e}_3) = \{< \breve{s}_1, [0.6, 0.7], [0.7, 0.8] >, < \breve{s}_2, [0.5, 0.6], [0.3, 0.4] >\}$.

Then $\left(h_i, \breve{P}_i, \tilde{E}_i, \mathrm{Q}\right)_{(\theta_i,\, \varphi_i)} = h(\breve{p}_i, \tilde{e}_i) = \{< \breve{s}_1, [0.3, 0.5], [0.7, 0.8] >, < \breve{s}_2, [0.5, 0.6], [0.7, 0.8] >\}$

and $\left(h_s, \breve{P}_s, \tilde{E}_s, \mathrm{Q}\right)_{(\theta_s,\, \varphi_s)} = h(\breve{p}_s, \tilde{e}_s) = \{< \breve{s}_1, [0.7, 0.8], [0.5, 0.6] >, < \breve{s}_2, [0.6, 0.7], [0.3, 0.4] >\}$.

And $\widetilde{W} = (0.3, 0.3, 0.4)^T$ be the weight vector of $h(\breve{p}_1, \tilde{e}_1)$, $h(\breve{p}_1, \tilde{e}_2)$ and $h(\breve{p}_1, \tilde{e}_3)$, without loss of generality take Q = 3 then Q-ROIVFSEWGO to aggregate the Q-ROIVFSENs as follows

$$\underline{\omega}\left(\left(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}\right) = \left\{ < \check{s}_i, \left[\prod_{k=1}^{3}(\theta^{-}_{k(\check{p},\tilde{e})}(\check{s}_i))^{\widetilde{W}_k},\ \prod_{k=1}^{3}(\theta^{+}_{k(\check{p},\tilde{e})}(\check{s}_i))^{\widetilde{W}_k}\right],\right.$$

$$\left.\left[\left(1-\prod_{k=1}^{3}\left(1-\left(\varphi^{-}_{k(\check{p},\tilde{e})}(\check{s}_i)\right)^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q},\ \left(1-\prod_{k=1}^{3}\left(1-\left(\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s}_i)\right)^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q}\right] > \right\}.$$

where $i = 1, 2$

$=\{< \check{s}_1, [0.5104, 0.6586], [0.6034, 0.7274] >, < \check{s}_2, [0.5281, 0.6284], [0.5743, 0.675] >\}.$

Then

$$\S(\mathrm{h}(\check{p}_i, \tilde{e}_i))_{\check{s}_1} = -0.3515, \qquad \S\left(\underline{\omega}\left(\left(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}\right)\right)_{\check{s}_1} = -0.09297,$$

$\S(\mathrm{h}(\check{p}_s, \tilde{e}_s))_{\check{s}_1} = 0.257$

And

$$\S(\mathrm{h}(\check{p}_i, \tilde{e}_i))_{\check{s}_2} = -0.0635, \qquad \S\left(\underline{\omega}\left(\left(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}\right)\right)_{\check{s}_2} = -0.05077,$$

$\S(\mathrm{h}(\check{p}_s, \tilde{e}_s))_{\check{s}_2} = 0.234$

This implies that

$$\S\left(\mathrm{h}(\check{p}_i, \tilde{e}_i)\right)_{\check{s}_1} \le \S\left(\underline{\omega}\left(\left(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}\right)\right)_{\check{s}_1} \le \S\left(\mathrm{h}(\check{p}_s, \tilde{e}_s)\right)_{\check{s}_1}$$

and $\S\left(\mathrm{h}(\check{p}_i, \tilde{e}_i)\right)_{\check{s}_2} \le \S\left(\underline{\omega}\left(\left(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}\right)\right)_{\check{s}_2} \le \S\left(\mathrm{h}(\check{p}_s, \tilde{e}_s)\right)_{\check{s}_2}.$

Therefore, $\left(h_i, \check{P}_i, \tilde{E}_i, \mathrm{Q}\right)_{(\theta_i,\, \varphi_i)} \le \underline{\omega}\left(\left(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}\right) \le \left(h_s, \check{P}_s, \tilde{E}_s, \mathrm{Q}\right)_{(\theta_s,\, \varphi_s)}.$

Definition 5.3.9

Special Cases of Q-ROIVFSEWGO (Definition 5.3.1) by taking different values of Q

1. When Q=1, Q-ROIVFSEWGO will reduce to Interval Valued Intuitionistic Fuzzy Soft Expert Weighted Geometric Operator (IVIFSEWGO) which is characterized as follows

$$\underline{\omega}\left(\left(h_k, \check{P}_k, \tilde{E}_k, 1\right)_{(\theta_k,\, \varphi_k)}\right) = < \left[\prod_{k=1}^{\check{m}}(\theta^{-}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k},\ \prod_{k=1}^{\check{m}}(\theta^{+}_{k(\check{p},\tilde{e})}(\check{s}))^{\widetilde{W}_k}\right],$$

$$\left[1-\prod_{k=1}^{\check{m}}\left(1-\varphi^{-}_{k(\check{p},\tilde{e})}(\check{s})\right)^{\widetilde{W}_k},\ 1-\prod_{k=1}^{\check{m}}\left(1-\varphi^{+}_{k(\check{p},\tilde{e})}(\check{s})\right)^{\widetilde{W}_k}\right] >.$$

2. When Q=2, Q-ROIVFSESWGO will reduce to Interval Valued Pythagorean Fuzzy Soft Expert Weighted Geometric Operator (IVPFSEWGO) which is defined as follows

$$\underline{\omega}\left(\left(h_k, \breve{P}_k, \tilde{E}_k, 2\right)_{(\theta_k,\, \varphi_k)}\right) = < [\prod_{k=1}^{\breve{m}}(\theta^-_{k(\breve{p},\tilde{e})}(\breve{s}))^{\widetilde{W}_k},\ \prod_{k=1}^{\breve{m}}(\theta^+_{k(\breve{p},\tilde{e})}(\breve{s}))^{\widetilde{W}_k}],$$

$$\left[\left(1-\prod_{k=1}^{\breve{m}}\left(1-\left(\varphi^-_{k(\breve{p},\tilde{e})}(\breve{s})\right)^2\right)^{\widetilde{W}_k}\right)^{1/2},\ \left(1-\prod_{k=1}^{\breve{m}}\left(1-\left(\varphi^+_{k(\breve{p},\tilde{e})}(\breve{s})\right)^2\right)^{\widetilde{W}_k}\right)^{1/2}\right]>.$$

$$= < [\prod_{k=1}^{\breve{m}}(\theta^-_{k(\breve{p},\tilde{e})}(\breve{s}))^{\widetilde{W}_k},\ \prod_{k=1}^{\breve{m}}(\theta^+_{k(\breve{p},\tilde{e})}(\breve{s}))^{\widetilde{W}_k}],$$

$$\left[\sqrt[2]{1-\prod_{k=1}^{\breve{m}}\left(1-(\varphi^-_{k(\breve{p},\tilde{e})}(\breve{s}))^2\right)^{\widetilde{W}_k}},\ \sqrt[2]{1-\prod_{k=1}^{\breve{m}}\left(1-(\varphi^+_{k(\breve{p},\tilde{e})}(\breve{s}))^2\right)^{\widetilde{W}_k}}\right]>.$$

5.4 Some Other Aggregation Operators

Definition 5.4.1

Suppose $\left(h_k, \breve{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)} = \{< \breve{s}, [\theta^-_{k(\breve{p},\tilde{e})},\ \theta^+_{k(\breve{p},\tilde{e})}], [\varphi^-_{k(\breve{p},\tilde{e})},\ \varphi^+_{k(\breve{p},\tilde{e})}] >: \breve{s} \in \mathrm{S}\}$ with Q $\geq$ 1, where k = 1, 2, …, $\breve{m}$ be a Q-ROIVFSESs collection. A mapping $\overline{\omega_o} : Q_I^{\breve{m}}(S) \rightarrow Q_I(S)$ is called Q-Rung Orthopair Interval Valued Fuzzy Soft Expert Ordered Weighted Averaging Operator (Q-ROIVFSEOWAO) if it satisfies

$$\overline{\omega_o}\left(\left(h_k, \breve{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}\right) = \sum_{i=1}^{\breve{m}} \boldsymbol{W}_i\left(\boldsymbol{h}_i, \breve{\boldsymbol{P}}_i, \widetilde{\boldsymbol{E}}_i, \mathrm{Q}\right)_{(\boldsymbol{\theta}_i,\, \boldsymbol{\varphi}_i)},$$

where $Q_I^{\breve{m}}(S)$ denote $\breve{m}$ copies of Q-ROIVFSES and $\boldsymbol{W}_i = (\boldsymbol{W}_1, \boldsymbol{W}_2, \ldots, \boldsymbol{W}_{\breve{m}})^{\mathrm{T}}$ is a position weight vector of $\left(\boldsymbol{h}_i, \breve{\boldsymbol{P}}_i, \widetilde{\boldsymbol{E}}_i, \mathrm{Q}\right)_{(\boldsymbol{\theta}_i,\, \boldsymbol{\varphi}_i)}$, for $1 \leq i \leq \breve{m}$ satisfying the normalized condition i.e., $\sum_{i=1}^{\breve{m}} \boldsymbol{W}_i = 1$ and $\boldsymbol{W}_i \in [0, 1]$.

$\left(\boldsymbol{h}_i, \breve{\boldsymbol{P}}_i, \widetilde{\boldsymbol{E}}_i, \mathrm{Q}\right)_{(\boldsymbol{\theta}_i,\, \boldsymbol{\varphi}_i)} = \{< \breve{s}, [\boldsymbol{\theta}^-_{i(\breve{p},\tilde{e})},\ \boldsymbol{\theta}^+_{i(\breve{p},\tilde{e})}], [\boldsymbol{\varphi}^-_{i(\breve{p},\tilde{e})},\ \boldsymbol{\varphi}^+_{i(\breve{p},\tilde{e})}] >: \breve{s} \in \mathrm{S}\}$ is the i-th largest of $\breve{m}$ Q-ROIVFSESs $\left(h_k, \breve{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}$, for $1 \leq k \leq \breve{m}$, which can be determined by using ranking method of Q-ROIVFSESs such as Score or accuracy function.

Remark 5.4.2

If $\boldsymbol{W}_i = (\frac{1}{\breve{m}}, \frac{1}{\breve{m}}, \ldots, \frac{1}{\breve{m}})^{\mathrm{T}}$ then Q-ROIVFSEOWAO $\overline{\omega_o}$ degenerates to the Q-Rung Orthopair Interval Valued Fuzzy Soft Expert Arithmetic Mean Operator (Q-ROIVFSEAMO).

Theorem 5.4.3

Consider $(h_k, \breve{P}_k, \tilde{E}_k, \mathrm{Q})_{(\theta_k, \varphi_k)} = \{< \breve{s}, [\theta^-_{k(\breve{p},\tilde{e})},\ \theta^+_{k(\breve{p},\tilde{e})}], [\varphi^-_{k(\breve{p},\tilde{e})},\ \varphi^+_{k(\breve{p},\tilde{e})}] >: \breve{s} \in \mathrm{S}\}$ with $\mathrm{Q} \geq 1$, be a Q-ROIVFSESs collection. Then , on the basis of operational rules characterized for Q-ROIVFSESs, for any $k \in \mathbb{N}$ Q-ROIVFSE ordered weighted averaging operator, aggregation is also a Q-ROIVFSES and

$$\overline{\omega_o}\left((h_k, \breve{P}_k, \tilde{E}_k, \mathrm{Q})_{(\theta_k, \varphi_k)}\right) = \ < \left[\left(1 - \prod_{i=1}^{\breve{m}}\left(1 - \left(\boldsymbol{\theta}^-_{i(\breve{p},\tilde{e})}\right)^Q\right)^{\boldsymbol{W}_i}\right)^{1/Q},\ \left(1 - \prod_{i=1}^{\breve{m}}\left(1 - (\boldsymbol{\theta}^+_{i(\breve{p},\tilde{e})})^Q\right)^{\boldsymbol{W}_i}\right)^{1/Q}\right],\ \left[\prod_{i=1}^{\breve{m}}(\boldsymbol{\varphi}^-_{i(\breve{p},\tilde{e})})^{\boldsymbol{W}_i},\ \prod_{i=1}^{\breve{m}}(\boldsymbol{\varphi}^+_{i(\breve{p},\tilde{e})})^{\boldsymbol{W}_i}\right] >,$$

where $\boldsymbol{W}_i$ is position weight vector and $\left(\boldsymbol{h}_i, \breve{\boldsymbol{P}}_i, \tilde{\boldsymbol{E}}_i, \mathrm{Q}\right)_{(\boldsymbol{\theta}_i, \boldsymbol{\varphi}_i)} = \{< \breve{s}, [\boldsymbol{\theta}^-_{i(\breve{p},\tilde{e})},\ \boldsymbol{\theta}^+_{i(\breve{p},\tilde{e})}]$, $[\boldsymbol{\varphi}^-_{i(\breve{p},\tilde{e})},\ \boldsymbol{\varphi}^+_{i(\breve{p},\tilde{e})}] >: \breve{s} \in \mathrm{S}\}$ is the i-th largest of $\breve{m}$ Q-ROIVFSESs $\left(h_k, \breve{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k, \varphi_k)}$, for $1 \leq k \leq \breve{m}$, which can be determined by using ranking method of Q-ROIVFSESs such as Score or accuracy function and $\widetilde{W}_k$ is weight vector.

Proof

It is straight forward by using mathematical induction, Definition 5.4.1, 4.2.26, 4.2.29, 4.2.32, 4.2.34 and Remark 4.2.27, 4.2.30.

Remark 5.4.4

The Q-ROIVFSEWAO ϖ considers importance of aggregated Q-ROIVFSESs themselves. The Q-ROIVFSEOWAO $\overline{\omega_o}$ concerns with the significance of positional ranking orders within the aggregated Q-ROIVFSESs.

Definition 5.4.5

Suppose $\left(h_k, \breve{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)} = \{< \breve{s}, \left[\theta^-_{k(\breve{p},\tilde{e})},\ \theta^+_{k(\breve{p},\tilde{e})}\right], \left[\varphi^-_{k(\breve{p},\tilde{e})},\ \varphi^+_{k(\breve{p},\tilde{e})}\right] >: \breve{s} \in \mathrm{S}\}$ with Q $\geq$ 1, where k = 1, 2, …, $\breve{m}$ be a Q-ROIVFSESs collection. A mapping $\overline{\omega_f} : Q_I^{\breve{m}}(S) \rightarrow Q_I(S)$ is called Q-Rung Orthopair Interval Valued Fuzzy Soft Expert fusion weighted averaging operator (Q-ROIVFSEFWAO) if it satisfies

$$\overline{\omega_f}\left(\left(h_k, \breve{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}\right) = \sum_{i=1}^{\breve{m}} \boldsymbol{W}_i \left(\underline{\boldsymbol{h}}_i, \underline{\breve{\boldsymbol{P}}}_i, \underline{\tilde{\boldsymbol{E}}}_i, \mathrm{Q}\right)_{\left(\underline{\boldsymbol{\theta}}_i, \underline{\boldsymbol{\varphi}}_i\right)},$$

where $Q_I^{\breve{m}}(S)$ denote $\breve{m}$ copies of Q-ROIVFSES and $\boldsymbol{W}_i = (\boldsymbol{W}_1, \boldsymbol{W}_2, \dots, \boldsymbol{W}_{\breve{m}})^{\mathrm{T}}$ is a position weight vector of $\left(\underline{\boldsymbol{h}}_i, \underline{\breve{\boldsymbol{P}}}_i, \underline{\tilde{\boldsymbol{E}}}_i, \mathrm{Q}\right)_{\left(\underline{\boldsymbol{\theta}}_i, \underline{\boldsymbol{\varphi}}_i\right)}$, for $1 \leq i \leq \breve{m}$ satisfying the normalized condition i.e., $\sum_{i=1}^{\breve{m}} \boldsymbol{W}_i = 1$ and $\boldsymbol{W}_i \in [0, 1]$.The Q-ROIVFSES of $\left(h_k, \breve{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}$ weighted with $\breve{m}\widetilde{W}_k$ and is symbolized by $\left(\underline{h}_k, \underline{\breve{P}}_k, \underline{\tilde{E}}_k, \mathrm{Q}\right)_{\left(\underline{\theta}_k, \underline{\varphi}_k\right)} = \breve{m}\widetilde{W}_k\left(h_k, \breve{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}$ where $\widetilde{W}_k = (\widetilde{W}_1, \widetilde{W}_2, \dots, \widetilde{W}_{\breve{m}})^{\mathrm{T}}$ is a weight vector of $\left(h_k, \breve{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}$ and $\left(\underline{\boldsymbol{h}}_i, \underline{\breve{\boldsymbol{P}}}_i, \underline{\tilde{\boldsymbol{E}}}_i, \mathrm{Q}\right)_{\left(\underline{\boldsymbol{\theta}}_i, \underline{\boldsymbol{\varphi}}_i\right)} = \{< \breve{s}, \left[\underline{\boldsymbol{\theta}}^-_{i(\breve{p},\tilde{e})},\ \underline{\boldsymbol{\theta}}^+_{i(\breve{p},\tilde{e})}\right], \left[\underline{\boldsymbol{\varphi}}^-_{i(\breve{p},\tilde{e})},\ \underline{\boldsymbol{\varphi}}^+_{i(\breve{p},\tilde{e})}\right] >: \breve{s} \in \mathrm{S}\}$ is the i-th largest of m Q-ROIVFSESs $\left(\underline{h}_k, \underline{\breve{P}}_k, \underline{\tilde{E}}_k, \mathrm{Q}\right)_{\left(\underline{\theta}_k, \underline{\varphi}_k\right)}$, for $1 \leq k \leq \breve{m}$, which can be determined by using ranking method of Q-ROIVFSESs such as Score or accuracy function.

Remarks 5.4.6

1. If $\boldsymbol{W}_i = (\frac{1}{\breve{m}}, \frac{1}{\breve{m}}, \dots, \frac{1}{\breve{m}})^{\mathrm{T}}$ is a position weight vector then Q-ROIVFSEFWAO$\overline{\omega_f}$degenerates to Q-ROIVFSEWAOϖ.

2. If $\widetilde{W}_k = (\frac{1}{\breve{m}}, \frac{1}{\breve{m}}, \ldots, \frac{1}{\breve{m}})^{T}$ is the weight vector then Q-ROIVFSEFWAO $\overline{\omega_f}$ degenerates to Q-ROIVFSEOWAO $\overline{\omega_o}$.
3. So, $\overline{\omega_f}$ is a generalization of ϖ and $\overline{\omega_o}$. $\overline{\omega_f}$ is concerned with both the characteristics of ϖ and $\overline{\omega_o}$.

Theorem 5.4.7

Consider $(h_k, \breve{P}_k, \tilde{E}_k, \mathrm{Q})_{(\theta_k, \varphi_k)} = \{< \breve{s}, [\theta^{-}_{k(\breve{p},\tilde{e})}, \theta^{+}_{k(\breve{p},\tilde{e})}], [\varphi^{-}_{k(\breve{p},\tilde{e})}, \varphi^{+}_{k(\breve{p},\tilde{e})}] >: \breve{s} \in \mathrm{S}\}$ with Q $\geq$ 1, be a Q-ROIVFSESs collection. Then , on the basis of operational rules defined for Q-ROIVFSESs, for any k $\in \mathbb{N}$ Q-ROIVFSE fusion weighted averaging operator $\overline{\omega_f}$ aggregation is also a Q-ROIVFSES and

$$\overline{\omega_f}\left((h_k, \breve{P}_k, \tilde{E}_k, \mathrm{Q})_{(\theta_k, \varphi_k)}\right) = < \left[\left(1 - \prod_{i=1}^{\breve{m}}\left(1 - (\underline{\boldsymbol{\theta}}^{-}_{i(\breve{p},\tilde{e})})^{Q}\right)^{\boldsymbol{W}_i}\right)^{1/Q}, \left(1 - \prod_{i=1}^{\breve{m}}\left(1 - (\underline{\boldsymbol{\theta}}^{+}_{i(\breve{p},\tilde{e})})^{Q}\right)^{\boldsymbol{W}_i}\right)^{1/Q}\right], \left[\prod_{i=1}^{\breve{m}}(\underline{\boldsymbol{\varphi}}^{-}_{i(\breve{p},\tilde{e})})^{\boldsymbol{W}_i}, \prod_{i=1}^{\breve{m}}(\underline{\boldsymbol{\varphi}}^{+}_{i(\breve{p},\tilde{e})})^{\boldsymbol{W}_i}\right] >,$$

where $\boldsymbol{W}_i$ is position weight vector and $(\underline{\boldsymbol{h}}_i, \underline{\breve{\boldsymbol{P}}}_i, \underline{\tilde{\boldsymbol{E}}}_i, \mathrm{Q})_{(\underline{\boldsymbol{\theta}}_i, \underline{\boldsymbol{\varphi}}_i)} = \{< \breve{s}, [\underline{\boldsymbol{\theta}}^{-}_{i(\breve{p},\tilde{e})}, \underline{\boldsymbol{\theta}}^{+}_{i(\breve{p},\tilde{e})}], [\underline{\boldsymbol{\varphi}}^{-}_{i(\breve{p},\tilde{e})}, \underline{\boldsymbol{\varphi}}^{+}_{i(\breve{p},\tilde{e})}] >: \breve{s} \in \mathrm{S}\}$ is the i-th largest of m Q-ROIVFSESs $(\underline{h}_k, \underline{\breve{P}}_k, \underline{\tilde{E}}_k, \mathrm{Q})_{(\underline{\theta}_k, \underline{\varphi}_k)} = \breve{m}\widetilde{W}_k(h_k, \breve{P}_k, \tilde{E}_k, \mathrm{Q})_{(\theta_k, \varphi_k)}$ which can be ascertained through the application of ranking method of Q-ROIVFSESs such as Score or accuracy function and $\widetilde{W}_k$ is weight vector.

Proof

It is straight forward through mathematical induction, Definition 5.3.5, 4.2.26, 4.2.29, 4.2.32, 4.2.34 and Remark 4.2.27, 4.2.30.

Definition 5.4.8

Suppose $\left(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)} = \{< \check{s}, \left[\theta^-_{k(\check{p},\tilde{e})},\ \theta^+_{k(\check{p},\tilde{e})}\right], \left[\varphi^-_{k(\check{p},\tilde{e})},\ \varphi^+_{k(\check{p},\tilde{e})}\right] >: \check{s} \in \mathrm{S}\}$ with Q $\geq$ 1, where k = 1, 2, …, $\breve{m}$ be a Q-ROIVFSESs collection. A mapping $\overline{\omega_o}$: $Q_I^{\breve{m}}(S) \rightarrow Q_I(S)$ is called Q-Rung Orthopair Interval Valued Fuzzy Soft Expert generalized ordered weighted averaging operator (Q-ROIVFSEGOWAO) if it satisfies:

$$G\overline{\omega_o}\left(\left(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}\right) = \sqrt[g]{\sum_{i=1}^{\breve{m}} \boldsymbol{W}_i (\left(\boldsymbol{h}_i, \check{\boldsymbol{P}}_i, \tilde{\boldsymbol{E}}_i, \mathrm{Q}\right)_{(\boldsymbol{\theta}_i,\, \boldsymbol{\varphi}_i)})^g},$$

where $Q_I^{\breve{m}}(S)$ denote $\breve{m}$ copies of Q-ROIVFSES andposition weight vector of $\left(\boldsymbol{h}_i, \check{\boldsymbol{P}}_i, \tilde{\boldsymbol{E}}_i, \mathrm{Q}\right)_{(\boldsymbol{\theta}_i,\, \boldsymbol{\varphi}_i)}be\boldsymbol{W}_i$ =$(\boldsymbol{W}_1, \boldsymbol{W}_2, \ldots, \boldsymbol{W}_{\breve{m}})^{\mathrm{T}}$, for 1 $\leq i \leq$ $\breve{m}$, satisfying the normalized condition i.e., $\sum_{i=1}^{\breve{m}} \boldsymbol{W}_i = 1$ and $\boldsymbol{W}_i \in$ [0, 1] and the parameter$g > 0$serves as a control parameter and its selection is based on specified condition. $\left(\boldsymbol{h}_i, \check{\boldsymbol{P}}_i, \tilde{\boldsymbol{E}}_i, \mathrm{Q}\right)_{(\boldsymbol{\theta}_i,\, \boldsymbol{\varphi}_i)} = \{< \check{s}, \left[\boldsymbol{\theta}^-_{i(\check{p},\tilde{e})},\ \boldsymbol{\theta}^+_{i(\check{p},\tilde{e})}\right], \left[\boldsymbol{\varphi}^-_{i(\check{p},\tilde{e})},\ \boldsymbol{\varphi}^+_{i(\check{p},\tilde{e})}\right] >: \check{s} \in \mathrm{S}\}$ is the i-th largest of $\breve{m}$ Q-ROIVFSESs $\left(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}$, for 1 $\leq k \leq$ $\breve{m}$, which can be ascertained through the application of ranking method of Q-ROIVFSESs such as Score or accuracy function.

Corollary 5.4.9

If $g = 1$ then Q-ROIVFSEGOWAO $G\overline{\omega_o}$ degenerates to Q-ROIVFSEOWAO $\overline{\omega_o}$.

Theorem 5.4.10

Consider $\left(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)} = \{< \check{s}, \left[\theta^-_{k(\check{p},\tilde{e})},\ \theta^+_{k(\check{p},\tilde{e})}\right], \left[\varphi^-_{k(\check{p},\tilde{e})},\ \varphi^+_{k(\check{p},\tilde{e})}\right] >: \check{s} \in \mathrm{S}\}$ with Q $\geq$ 1, be a Q-ROIVFSESs collection. Then , on the basis of operational rules characterized for Q-ROIVFSESs, for any k $\in \mathbb{N}$ Q-ROIVFSEGOWAO $G\overline{\omega_o}$ aggregation is also a Q-ROIVFSES and

$$G\overline{\omega_o}\left(\left(h_k, \breve{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}\right) =$$

$$< \left[\sqrt[g]{\left(1 - \prod_{i=1}^{\widetilde{m}} \left(1 - \left(\boldsymbol{\theta}^{-}_{i(\breve{p},\tilde{e})}\right)^{Qg}\right)^{\boldsymbol{W}_i}\right)^{1/Q}},\ \sqrt[g]{\left(1 - \prod_{i=1}^{\widetilde{m}} \left(1 - \left(\boldsymbol{\theta}^{+}_{i(\breve{p},\tilde{e})}\right)^{Qg}\right)^{\boldsymbol{W}_i}\right)^{1/Q}} \right],$$

$$\left[\left(1 - \sqrt[g]{1 - \prod_{i=1}^{\widetilde{m}}\left(1 - \left(1 - (\boldsymbol{\varphi}^{-}_{i(\breve{p},\tilde{e})})^{Q}\right)^{g}\right)^{\boldsymbol{W}_i}}\right)^{1/Q},\ \left(1 - \sqrt[g]{1 - \prod_{i=1}^{\widetilde{m}}\left(1 - \left(1 - (\boldsymbol{\varphi}^{+}_{i(\breve{p},\tilde{e})})^{Q}\right)^{g}\right)^{\boldsymbol{W}_i}}\right)^{1/Q}\right] >,$$

where $\boldsymbol{W}_i$ is position weight vector and $\left(\boldsymbol{h}_i, \breve{\boldsymbol{P}}_i, \tilde{\boldsymbol{E}}_i, \mathrm{Q}\right)_{(\boldsymbol{\theta}_i,\, \boldsymbol{\varphi}_i)} = \{< \check{s}, \left[\boldsymbol{\theta}^{-}_{i(\breve{p},\tilde{e})},\ \boldsymbol{\theta}^{+}_{i(\breve{p},\tilde{e})}\right], \left[\boldsymbol{\varphi}^{-}_{i(\breve{p},\tilde{e})},\ \boldsymbol{\varphi}^{+}_{i(\breve{p},\tilde{e})}\right] >: \check{s} \in \mathrm{S}\}$ is the i-th largest of $\widetilde{m}$ Q-ROIVFSESs $\left(h_k, \breve{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}$, for $1 \le k \le \widetilde{m}$, which can be ascertained through the application of ranking method of Q-ROIVFSESs such as Score or accuracy function and $\widetilde{W}_k$ is weight vector.

Proof

It is straight forward by using mathematical induction, Definition 5.4.5, 4.2.26, 4.2.29, 4.2.32, 4.2.34 and Remark 4.2.27, 4.2.30.

Definition 5.4.11

Suppose $\left(h_k, \breve{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)} = \{< \check{s}, \left[\theta^{-}_{k(\breve{p},\tilde{e})},\ \theta^{+}_{k(\breve{p},\tilde{e})}\right], \left[\varphi^{-}_{k(\breve{p},\tilde{e})},\ \varphi^{+}_{k(\breve{p},\tilde{e})}\right] >: \check{s} \in \mathrm{S}\}$ with Q $\geq$ 1, where k = 1, 2, …, $\widetilde{m}$ be a Q-ROIVFSESs collection. A mapping $\overline{\omega_f}$: $Q_I^{\widetilde{m}}(S) \rightarrow Q_I(S)$ is called Q-Rung Orthopair Interval Valued Fuzzy Soft Expert generalized fusion weighted averaging operator (Q-ROIVFSEGFWAO) if it satisfies

$$G\overline{\omega_f}\left(\left(h_k, \breve{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}\right) = \sqrt[g]{\sum_{i=1}^{\widetilde{m}} \boldsymbol{W}_i \left(\left(\underline{\boldsymbol{h}}_i, \underline{\breve{\boldsymbol{P}}}_i, \underline{\tilde{\boldsymbol{E}}}_i, \mathrm{Q}\right)_{\left(\underline{\boldsymbol{\theta}}_i, \underline{\boldsymbol{\varphi}}_i\right)}\right)^{g}},$$

where $Q_I^{\widetilde{m}}(S)$ denote m copies of Q-ROIVFSES andposition weight vector of $\left(\boldsymbol{h}_i, \breve{\boldsymbol{P}}_i, \tilde{\boldsymbol{E}}_i, \mathrm{Q}\right)_{(\boldsymbol{\theta}_i,\, \boldsymbol{\varphi}_i)}be\boldsymbol{W}_i$ =$(\boldsymbol{W}_1, \boldsymbol{W}_2, \dots, \boldsymbol{W}_{\widetilde{m}})^{\mathrm{T}}$, for $1 \le i \le \widetilde{m}$ satisfying the normalized condition i.e., $\sum_{i=1}^{\breve{m}} \boldsymbol{W}_i = 1$ and $\boldsymbol{W}_i \in [0, 1]$ and the parameter $g > 0$ serves as a control parameter and its selection is based on specified condition.

The Q-ROIVFSES of $(h_k, P_k, E_k, \mathrm{Q})_{(\theta_k, \varphi_k)}$ weighted with $\breve{m}\widetilde{W}_k$ and is symbolized by $\left(\underline{h}_k, \underline{\breve{P}}_k, \underline{\tilde{E}}_k, \mathrm{Q}\right)_{\left(\underline{\theta}_k, \underline{\varphi}_k\right)} = \breve{m}\widetilde{W}_k\left(h_k, \breve{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k, \varphi_k)}$ where $\widetilde{W}_k = \left(\widetilde{W}_1, \widetilde{W}_2, \ldots, \widetilde{W}_{\breve{m}}\right)^{\mathrm{T}}$ is a weight vector of $\left(h_k, \breve{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k, \varphi_k)}$ and $\left(\underline{\boldsymbol{h}}_i, \underline{\breve{\boldsymbol{P}}}_i, \underline{\tilde{\boldsymbol{E}}}_i, \mathrm{Q}\right)_{\left(\underline{\boldsymbol{\theta}}_i, \underline{\boldsymbol{\varphi}}_i\right)} = \{< \breve{s}, \left[\underline{\boldsymbol{\theta}}^{-}_{i(\breve{p},\tilde{e})}, \ \underline{\boldsymbol{\theta}}^{+}_{i(\breve{p},\tilde{e})}\right],$ $\left[\underline{\boldsymbol{\varphi}}^{-}_{i(\breve{p},\tilde{e})}, \ \underline{\boldsymbol{\varphi}}^{+}_{i(\breve{p},\tilde{e})}\right] >: \breve{s} \in \mathrm{S}\}$ is the i-th largest of $\breve{m}$ Q-ROIVFSESs $\left(\underline{h}_k, \underline{\breve{P}}_k, \underline{\tilde{E}}_k, \mathrm{Q}\right)_{\left(\underline{\theta}_k, \underline{\varphi}_k\right)}$ for $1 \leq k \leq \breve{m}$, which can be determined by using ranking method of Q-ROIVFSESs such as Score or accuracy function.

Remark 5.4.12

If $\widetilde{W}_k = (\frac{1}{\breve{m}}, \frac{1}{\breve{m}}, \ldots, \frac{1}{\breve{m}})^{\mathrm{T}}$ is the weight vector then Q-ROIVFSEGFWAO $G\overline{\omega_f}$ degenerates to Q-ROIVFSEGOWAO $G\overline{\omega_o}$.

Theorem 5.4.13

Consider $\left(h_k, \breve{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k, \varphi_k)} = \{< \breve{s}, \left[\theta^{-}_{k(\breve{p},\tilde{e})}, \ \theta^{+}_{k(\breve{p},\tilde{e})}\right], \left[\varphi^{-}_{k(\breve{p},\tilde{e})}, \ \varphi^{+}_{k(\breve{p},\tilde{e})}\right] >: \breve{s} \in \mathrm{S}\}$ with $\mathrm{Q} \geq 1$, be a Q-ROIVFSESs collection. Then , on the basis of operational rules defined for Q-ROIVFSESs, for any $k \in \mathbb{N}$ Q-ROIVFSE generalized fusion weighted averaging operator $G\overline{\omega_f}$ aggregation is also a Q-ROIVFSES and

$$G\overline{\omega_f}\left(\left(h_k,\breve{P}_k,\tilde{E}_k,\mathrm{Q}\right)_{(\theta_k,\varphi_k)}\right)=<\left[\sqrt[g]{\left(1-\prod_{i=1}^{\breve{m}}\left(1-\left(\underline{\boldsymbol{\theta}}^{-}_{i(\breve{p},\breve{e})}\right)^{Qg}\right)^{\boldsymbol{W}_i}\right)^{1/Q}},\right.$$

$$\left.\sqrt[g]{\left(1-\prod_{i=1}^{\breve{m}}\left(1-\left(\underline{\boldsymbol{\theta}}^{+}_{i(\breve{p},\breve{e})}\right)^{Qg}\right)^{\boldsymbol{W}_i}\right)^{1/Q}}\right],\ \left[\left(1-\sqrt[g]{1-\prod_{i=1}^{\breve{m}}\left(1-\left(1-(\underline{\boldsymbol{\varphi}}^{-}_{i(\breve{p},\breve{e})})^{Q}\right)^{g}\right)^{\boldsymbol{W}_i}}\right)^{1/Q},\right.$$

$$\left.\left(1-\sqrt[g]{1-\prod_{i=1}^{\breve{m}}\left(1-\left(1-(\underline{\boldsymbol{\varphi}}^{+}_{i(\breve{p},\breve{e})})^{Q}\right)^{g}\right)^{\boldsymbol{W}_i}}\right)^{1/Q}\right]>,$$

where $\boldsymbol{W}_i$ is position weight vector and $\left(\underline{\boldsymbol{h}}_i,\underline{\breve{\boldsymbol{P}}}_i,\underline{\tilde{\boldsymbol{E}}}_i,\mathrm{Q}\right)_{\left(\underline{\boldsymbol{\theta}}_i,\underline{\boldsymbol{\varphi}}_i\right)}=\{<\breve{s},[\underline{\boldsymbol{\theta}}^{-}_{i(\breve{p},\breve{e})},\ \underline{\boldsymbol{\theta}}^{+}_{i(\breve{p},\breve{e})}],$ $[\underline{\boldsymbol{\varphi}}^{-}_{i(\breve{p},\breve{e})},\ \underline{\boldsymbol{\varphi}}^{+}_{i(\breve{p},\breve{e})}]>:\ \breve{s}\ \in\ \mathrm{S}\}$ is the i-th largest of $\breve{m}$ Q-ROIVFSESs $\left(\underline{h}_k,\underline{\breve{P}}_k,\underline{\tilde{E}}_k,\mathrm{Q}\right)_{\left(\underline{\theta}_k,\underline{\varphi}_k\right)}=$ $\breve{m}\widetilde{W}_k\left(h_k,\breve{P}_k,\tilde{E}_k,\mathrm{Q}\right)_{(\theta_k,\varphi_k)}$, for $1\le k\le\ \breve{m}$, which can be ascertained through the application of ranking method of Q-ROIVFSESs such as Score or accuracy function and $\widetilde{W}_k$ is weight vector.

Proof

It is straight forward by employing mathematical induction, Definition 5.4.11, 4.2.26, 4.2.29, 4.2.32, 4.2.34 and Remark 4.2.27, 4.2.30.

Corollary 5.4.14

If $g=1$ then Q-ROIVFSEGFWAO $G\overline{\omega_f}$ degenerates to Q-ROIVFSEFWAO $\overline{\omega_f}$.

Definition 5.4.15

Suppose $\left(h_k,\breve{P}_k,\tilde{E}_k,\mathrm{Q}\right)_{(\theta_k,\varphi_k)}=\{<\breve{s},\left[\theta^{-}_{k(\breve{p},\breve{e})},\ \theta^{+}_{k(\breve{p},\breve{e})}\right],\left[\varphi^{-}_{k(\breve{p},\breve{e})},\ \varphi^{+}_{k(\breve{p},\breve{e})}\right]>:\ \breve{s}\ \in\ \mathrm{S}\}$ with Q $\geq$ 1, where k = 1, 2, …, $\breve{m}$ be a Q-ROIVFSESs collection. A mapping $\underline{\omega_o}: Q_I^{\breve{m}}(S)\rightarrow Q_I(S)$ is called Q-Rung Orthopair Interval Valued Fuzzy Soft Expert ordered weighted geometric operator (Q-ROIVFSEOWGO) if it satisfies

$$\underline{\omega_o}\left(\left(h_k,\breve{P}_k,\tilde{E}_k,\mathrm{Q}\right)_{(\theta_k,\,\varphi_k)}\right)=\prod_{i=1}^{\breve{m}}\left(\left(\boldsymbol{h}_i,\breve{\boldsymbol{P}}_i,\widetilde{\boldsymbol{E}}_i,\mathrm{Q}\right)_{(\boldsymbol{\theta}_i,\,\boldsymbol{\varphi}_i)}\right)^{\boldsymbol{W}_i}$$

where $\boldsymbol{W}_i$ =$(\boldsymbol{W}_1,\boldsymbol{W}_2,\ldots,\boldsymbol{W}_{\breve{m}})^{\mathrm{T}}$ is a position weight vector of $\left(\boldsymbol{h}_i,\breve{\boldsymbol{P}}_i,\widetilde{\boldsymbol{E}}_i,\mathrm{Q}\right)_{(\boldsymbol{\theta}_i,\,\boldsymbol{\varphi}_i)}$, for 1 $\leq i \leq$ $\breve{m}$ satisfying the normalized condition i.e., $\sum_{i=1}^{\breve{m}}\boldsymbol{W}_i=1$ and $\boldsymbol{W}_i\in$ [0, 1]. $\left(\boldsymbol{h}_i,\breve{\boldsymbol{P}}_i,\widetilde{\boldsymbol{E}}_i,\mathrm{Q}\right)_{(\boldsymbol{\theta}_i,\,\boldsymbol{\varphi}_i)}$ = $\{<\check{s},[\boldsymbol{\theta}^-_{i(\breve{p},\tilde{e})},\ \boldsymbol{\theta}^+_{i(\breve{p},\tilde{e})}],[\boldsymbol{\varphi}^-_{i(\breve{p},\tilde{e})},\ \boldsymbol{\varphi}^+_{i(\breve{p},\tilde{e})}]>:\check{s}\in \mathrm{S}\}$ is the i-th largest of $\breve{m}$ Q-ROIVFSESs $\left(h_k,\breve{P}_k,\tilde{E}_k,\mathrm{Q}\right)_{(\theta_k,\,\varphi_k)}$, for 1 $\leq k\leq$ $\breve{m}$, which can be ascertained through the application of ranking method of Q-ROIVFSESs such as Score or accuracy function.

Theorem 5.4.16

Consider $(h_k,\breve{P}_k,\tilde{E}_k,\mathrm{Q})_{(\theta_k,\,\varphi_k)}$ = $\{<\check{s},[\theta^-_{k(\breve{p},\tilde{e})},\ \theta^+_{k(\breve{p},\tilde{e})}],[\varphi^-_{k(\breve{p},\tilde{e})},\ \varphi^+_{k(\breve{p},\tilde{e})}]>:\check{s}\in \mathrm{S}\}$ with Q $\geq$ 1, be a Q-ROIVFSESs collection. Then , on the basis of operational rules characterized for Q-ROIVFSESs, for any k ∈ ℕ Q-ROIVFSEOWGO aggregation is also a Q-ROIVFSES and

$$\underline{\omega_o}\left(\left(h_k,\breve{P}_k,\tilde{E}_k,\mathrm{Q}\right)_{(\theta_k,\,\varphi_k)}\right)=<\left[\prod_{i=1}^{\breve{m}}(\boldsymbol{\theta}^-_{i(\breve{p},\tilde{e})})^{\boldsymbol{W}_i},\quad \prod_{i=1}^{\breve{m}}(\boldsymbol{\theta}^+_{i(\breve{p},\tilde{e})})^{\boldsymbol{W}_i}\right],$$
$$\left[\left(1-\prod_{i=1}^{\breve{m}}\left(1-(\boldsymbol{\varphi}^-_{i(\breve{p},\tilde{e})})^{Q}\right)^{\boldsymbol{W}_i}\right)^{1/Q},\ \left(1-\prod_{i=1}^{\breve{m}}\left(1-(\boldsymbol{\varphi}^+_{i(\breve{p},\tilde{e})})^{Q}\right)^{\boldsymbol{W}_i}\right)^{1/Q}\right]>,$$

where $\boldsymbol{W}_i$ is position weight vector and $\left(\boldsymbol{h}_i,\breve{\boldsymbol{P}}_i,\widetilde{\boldsymbol{E}}_i,\mathrm{Q}\right)_{(\boldsymbol{\theta}_i,\,\boldsymbol{\varphi}_i)}$ = $\{<\check{s},[\boldsymbol{\theta}^-_{i(\breve{p},\tilde{e})},\ \boldsymbol{\theta}^+_{i(\breve{p},\tilde{e})}],$ $[\boldsymbol{\varphi}^-_{i(\breve{p},\tilde{e})},\ \boldsymbol{\varphi}^+_{i(\breve{p},\tilde{e})}]>:\check{s}\in \mathrm{S}\}$ is the i-th largest of $\breve{m}$ Q-ROIVFSESs $\left(h_k,\breve{P}_k,\tilde{E}_k,\mathrm{Q}\right)_{(\theta_k,\,\varphi_k)}$, for 1 $\leq k\leq$ $\breve{m}$, which can be ascertained through the application of ranking method of Q-ROIVFSESs such as Score or accuracy function.

Proof

It is straight forward by employing mathematical induction, Definition 5.4.15, 4.2.26, 4.2.29, 4.2.32, 4.2.34 and Remark 4.2.27, 4.2.30.

Remarks 5.4.17

1. The Q-ROIVFSEWGO $\underline{\omega}$ considers importance of aggregated Q-ROIVFSESs themselves. The Q-ROIVFSEOWGO $\underline{\omega_o}$ concerns with the significance of positional ranking orders within the aggregated Q-ROIVFSESs.
2. If $\boldsymbol{W}_i=(\frac{1}{\breve{m}},\frac{1}{\breve{m}},\dots,\frac{1}{\breve{m}})^{\mathrm{T}}$ is a position weight vector then Q-ROIVFSEOWGO degenerates to Q-ROIVFSEGMO $\underline{\omega_G}$.

Definition 5.4.18

Suppose $\left(h_k,\breve{P}_k,\tilde{E}_k,\mathrm{Q}\right)_{(\theta_k,\,\varphi_k)}=\{<\breve{s},\left[\theta^-_{k(\breve{p},\tilde{e})},\ \theta^+_{k(\breve{p},\tilde{e})}\right],\left[\varphi^-_{k(\breve{p},\tilde{e})},\ \varphi^+_{k(\breve{p},\tilde{e})}\right]>:\breve{s}\in \mathrm{S}\}$ with Q $\geq$ 1, where k = 1, 2, …, $\breve{m}$ be a Q-ROIVFSESs collection. A mapping $\underline{\omega_f}$: $Q_I^{\breve{m}}(S)\rightarrow Q_I(S)$ is called Q-Rung Orthopair Interval Valued Fuzzy Soft Expert fusion weighted geometric operator (Q-ROIVFSEFWGO) if it satisfies

$$\underline{\omega_f}\left(\left(h_k,\breve{P}_k,\tilde{E}_k,\mathrm{Q}\right)_{(\theta_k,\,\varphi_k)}\right)=\prod_{i=1}^{\breve{m}}\left(\left(\underline{\boldsymbol{h}}_i,\underline{\breve{\boldsymbol{P}}}_i,\underline{\tilde{\boldsymbol{E}}}_i,\mathrm{Q}\right)_{\left(\underline{\boldsymbol{\theta}}_i,\underline{\boldsymbol{\varphi}}_i\right)}\right)^{\boldsymbol{W}_i},$$

where $\boldsymbol{W}_i=(\boldsymbol{W}_1,\boldsymbol{W}_2,\dots,\boldsymbol{W}_{\breve{m}})^{\mathrm{T}}$ is a position weight vector of $\left(\underline{\boldsymbol{h}}_i,\underline{\breve{\boldsymbol{P}}}_i,\underline{\tilde{\boldsymbol{E}}}_i,\mathrm{Q}\right)_{\left(\underline{\boldsymbol{\theta}}_i,\underline{\boldsymbol{\varphi}}_i\right)}$, for $1\leq i\leq \breve{m}$ satisfying the normalized condition i.e., $\sum_{i=1}^{\breve{m}}\boldsymbol{W}_i=1$ and $\boldsymbol{W}_i\in[0,1]$.

The Q-ROIVFSES of $\left(h_k,\breve{P}_k,\tilde{E}_k,\mathrm{Q}\right)_{(\theta_k,\,\varphi_k)}$ weighted with $\breve{m}\widetilde{W}_k$ and is symbolized by $\left(\underline{h}_k,\underline{\breve{P}}_k,\underline{\tilde{E}}_k,\mathrm{Q}\right)_{\left(\underline{\theta}_k,\underline{\varphi}_k\right)}=\left(\left(h_k,\breve{P}_k,\tilde{E}_k,\mathrm{Q}\right)_{(\theta_k,\,\varphi_k)}\right)^{\breve{m}\widetilde{W}_k}$ where $\widetilde{W}_k=(\widetilde{W}_1,\widetilde{W}_2,\dots,\widetilde{W}_m)^{\mathrm{T}}$ is a weight vector of $\left(h_k,\breve{P}_k,\tilde{E}_k,\mathrm{Q}\right)_{(\theta_k,\,\varphi_k)}$ and $\left(\underline{\boldsymbol{h}}_i,\underline{\breve{\boldsymbol{P}}}_i,\underline{\tilde{\boldsymbol{E}}}_i,\mathrm{Q}\right)_{\left(\underline{\boldsymbol{\theta}}_i,\underline{\boldsymbol{\varphi}}_i\right)}=\{<\breve{s},\left[\underline{\boldsymbol{\theta}}^-_{i(\breve{p},\tilde{e})},\ \underline{\boldsymbol{\theta}}^+_{i(\breve{p},\tilde{e})}\right],\left[\underline{\boldsymbol{\varphi}}^-_{i(\breve{p},\tilde{e})},\ \underline{\boldsymbol{\varphi}}^+_{i(\breve{p},\tilde{e})}\right]>:\breve{s}\in \mathrm{S}\}$ is the i-th largest of $\breve{m}$ Q-ROIVFSESs $\left(\underline{h}_k,\underline{\breve{P}}_k,\underline{\tilde{E}}_k,\mathrm{Q}\right)_{\left(\underline{\theta}_k,\underline{\varphi}_k\right)}$, for $1\leq k\leq \breve{m}$, which can be ascertained through the application of ranking method of Q-ROIVFSESs such as Score or accuracy function.

Remark 5.4.19

1. If $\boldsymbol{W}_i = (\frac{1}{\breve{m}}, \frac{1}{\breve{m}}, \ldots, \frac{1}{\breve{m}})^{\mathrm{T}}$ is a position weight vector then Q-ROIVFSEFWGO $\underline{\omega_f}$ degenerates to Q-ROIVFSEWGO.
2. If $\widetilde{W}_k = (\frac{1}{\breve{m}}, \frac{1}{\breve{m}}, \ldots, \frac{1}{\breve{m}})^{\mathrm{T}}$ is the weight vector then Q-ROIVFSEFWGO $\underline{\omega_f}$ degenerates to Q-ROIVFSEOWGO $\underline{\omega_o}$.
3. So, $\underline{\omega_f}$ is a generalization of $\underline{\omega}$ and $\underline{\omega_o}$. $\underline{\omega_f}$ is concerned with both the characteristics of $\underline{\omega}$ and $\underline{\omega_o}$.

Theorem 5.4.20

Consider $(h_k, \breve{P}_k, \tilde{E}_k, \mathrm{Q})_{(\theta_k, \varphi_k)} = \{< \breve{s}, [\theta^-_{k(\breve{p},\tilde{e})}, \ \theta^+_{k(\breve{p},\tilde{e})}], [\varphi^-_{k(\breve{p},\tilde{e})}, \ \varphi^+_{k(\breve{p},\tilde{e})}] >: \breve{s} \in \mathrm{S}\}$ with Q ≥ 1, be a Q-ROIVFSESs collection. Then , on the basis of operational rules defined for Q-ROIVFSESs, for any k $\in \mathbb{N}$ Q-ROIVFSEFWGO $\underline{\omega_f}$ aggregation is also a Q-ROIVFSES and

$$\underline{\omega_f}\left(\left(h_k, \breve{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k, \varphi_k)}\right) = < \left[\prod_{i=1}^{\breve{m}} (\underline{\boldsymbol{\theta}}^-_{i(\breve{p},\tilde{e})})^{\boldsymbol{W}_i}, \quad \prod_{i=1}^{\breve{m}} (\underline{\boldsymbol{\theta}}^+_{i(\breve{p},\tilde{e})})^{\boldsymbol{W}_i}\right],$$

$$\left[\left(1 - \prod_{i=1}^{\breve{m}}\left(1 - (\underline{\boldsymbol{\varphi}}^-_{i(\breve{p},\tilde{e})})^{Q}\right)^{\boldsymbol{W}_i}\right)^{1/Q}, \left(1 - \prod_{i=1}^{\breve{m}}\left(1 - (\underline{\boldsymbol{\varphi}}^+_{i(\breve{p},\tilde{e})})^{Q}\right)^{\boldsymbol{W}_i}\right)^{1/Q}\right] >,$$

where $\boldsymbol{W}_i$ is position weight vector and $\left(\underline{\boldsymbol{h}}_i, \underline{\boldsymbol{P}}_i, \underline{\boldsymbol{E}}_i, \mathrm{Q}\right)_{\left(\underline{\boldsymbol{\theta}}_i, \underline{\boldsymbol{\varphi}}_i\right)} = \{< \breve{s}, [\underline{\boldsymbol{\theta}}^-_{i(\breve{p},\tilde{e})}, \ \underline{\boldsymbol{\theta}}^+_{i(\breve{p},\tilde{e})}],$ $[\underline{\boldsymbol{\varphi}}^-_{i(\breve{p},\tilde{e})}, \ \underline{\boldsymbol{\varphi}}^+_{i(\breve{p},\tilde{e})}] >: \breve{s} \in \mathrm{S}\}$ is the i-th largest of $\breve{m}$ Q-ROIVFSESs $\left(\underline{h}_k, \underline{\breve{P}}_k, \underline{\tilde{E}}_k, \mathrm{Q}\right)_{\left(\underline{\theta}_k, \underline{\varphi}_k\right)} = \left(\left(h_k, \breve{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k, \varphi_k)}\right)^{\breve{m}\widetilde{W}_k}$ which can be ascertained through the application of ranking method of Q-ROIVFSESs such as Score or accuracy function.

Proof

It is straight forward by employing mathematical induction, Definition 5.4.18, 4.2.26, 4.2.29, 4.2.32, 4.2.34 and Remark 4.2.27, 4.2.30.

5.5 Multi-Criteria Decision Making of Q-ROIVFSESs with Q-Rung Orthopair Interval Valued Fuzzy Soft Expert Aggregation Operators

Let S $= \{\breve{s}_i : 1 \leq i \leq l\}$be a finite universe of discourse set and $\breve{P} = \{\breve{p}_j : 1 \leq j \leq \breve{m}\}$, $\tilde{E} = \{\tilde{e}_k : 1 \leq k \leq \tilde{n}\}$ be the set of parameters and experts respectively. Experts opinion corresponding to each of the parameter is expressed in the structured form of a Q-ROIVFSES (h, $\breve{P}$, $\tilde{E}$, Q) which is characterized by a mapping h: $\breve{P} \times \tilde{E} \rightarrow Q_I(S)$ where $Q_I(S)$ is the set of all interval-valued Q-ROFS and $1 \leq Q \in \mathbb{N}$.

5.5.1 Algorithm for Multi-Criteria Decision Making

The algorithm and process of Q-ROIVFSE aggregation operator method for multi-criteria DMPs with Q-ROIVFSESs can be succinctly summarized as outlined below:

i) **Step 1: Data Collection**

Employ the evaluations provided by the experts in the form of Q-ROIVFSESs to assess the opinion regarding the specified options and criteria.

ii) **Step 2: Arrangement of Data**

Separate and analyze the individual opinion of each expert, discerning their unique insights and evaluations.

$\tilde{e}_i$	$\check{s}_1$	.	.	.	$\check{s}_l$
$\check{p}_1$	$< \check{s}_1, \left[\theta^{-}_{(\check{p}_1,\tilde{e}_i)}, \theta^{+}_{(\check{p}_1,\tilde{e}_i)}\right],$ $[\varphi^{-}_{(\check{p}_1,\tilde{e}_i)}, \varphi^{+}_{(\check{p}_1,\tilde{e}_i)}] >$	.	.	.	$< \check{s}_l, \left[\theta^{-}_{(\check{p}_1,\tilde{e}_i)}, \theta^{+}_{(\check{p}_1,\tilde{e}_i)}\right],$ $[\varphi^{-}_{(\check{p}_1,\tilde{e}_i)}, \varphi^{+}_{(\check{p}_1,\tilde{e}_i)}] >$
$\check{p}_2$	$< \check{s}_1, \left[\theta^{-}_{(\check{p}_2,\tilde{e}_i)}, \theta^{+}_{(\check{p}_2,\tilde{e}_i)}\right],$ $[\varphi^{-}_{(\check{p}_2,\tilde{e}_i)}, \varphi^{+}_{(\check{p}_2,\tilde{e}_i)}] >$	.	.	.	$< \check{s}_l, \left[\theta^{-}_{(\check{p}_2,\tilde{e}_i)}, \theta^{+}_{(\check{p}_2,\tilde{e}_i)}\right],$ $[\varphi^{-}_{(\check{p}_2,\tilde{e}_i)}, \varphi^{+}_{(\check{p}_2,\tilde{e}_i)}] >$
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
$\check{p}_m$	$< \check{s}_1, \left[\theta^{-}_{(\check{p}_m,\tilde{e}_i)}, \theta^{+}_{(\check{p}_m,\tilde{e}_i)}\right],$ $[\varphi^{-}_{(\check{p}_m,\tilde{e}_i)}, \varphi^{+}_{(\check{p}_m,\tilde{e}_i)}] >$	.	.	.	$< \check{s}_l, \left[\theta^{-}_{(\check{p}_m,\tilde{e}_i)}, \theta^{+}_{(\check{p}_m,\tilde{e}_i)}\right],$ $[\varphi^{-}_{(\check{p}_m,\tilde{e}_i)}, \varphi^{+}_{(\check{p}_m,\tilde{e}_i)}] >$

Table 5.1: Expert Opinion and Parameters

iii) **Step 3: Weight Vector**

Assign weight to each criteria.

iv) **Step 4: Position Weight Vector**

Allocate a position weight vector. The role of weight vector is to eliminate the impact of individual perception on overall comprehensive assessment.

v) **Step 5: Aggregation of Parameters**

Aggregate parameters by using Q-ROIVFSE aggregation operator.

vi) **Step 6: Finding Accuracy**

Determine the accuracy of each member of the set S in accordance with the evaluations provided by each expert.

vii) **Step 7: Average Accuracy**

Compute the average accuracy for each element of the set S.

viii) **Step 8: Generate Chain**

Create a non-decreasing chain of these average values.

ix) **Step 9: Conclusion**

Make Conclusive remarks from the chain about the data.

Note: Step 4 is only included in case of Fusion Weighted averaging operator and Fusion Weighted Geometric Operator.

Example 5.5.2

Let us consider the case of COVID-19 and the universe of discourse be S = $\{\breve{s}_1 = \text{Itlay},\ \breve{s}_2 = \text{Russia}, \breve{s}_3 = \text{United States}, \breve{s}_4 = \text{Turkey}, \breve{s}_5 = \text{United Kingdom}\}$ which is the set of countries which are most affected by COVID-19 pandemic, E = $\{\tilde{e}_1, \tilde{e}_2\}$ be the set of two experts from World Health Organization (WHO) and Center for Disease Control and Prevention (CDCP). Among many parameters e.g., total confirmed cases, total deaths, case fatality rate (CFR) = $\frac{\text{total deaths}}{\text{confirmed cases}} \times 100$, total test conducted, test positivity rate (TPR), active cases and recovered cases, vaccination rates etc. we choose three parameters $\{\breve{p}_1 = \text{total death cases}, \breve{p}_2 = \text{CFR}, \breve{p}_3 = \text{TPR}\}$. The two experts evaluate some data and express their evaluation in form of Q-ROIVFSESs. Calculate the comprehensive assessment of the experts regarding the most affected country by employing different Q-ROIVFSE aggregation operators like Q-ROIVFSEWAO, Q-ROIVFSEOWAO, Q-ROIVFSEFWAO, Q-ROIVFSEWGO, Q-ROIVFSEOWGO and Q-ROIVFSEFWGO.

Solution

Step 1: Utilize the expert assessments, presented as Q-ROIVFSESs, to derive expert opinions concerning the provided alternatives and criteria.

$(\check{p}_1,\tilde{e}_1)$={$<\check{s}_1,[.4,.5],[.5,.6]>,<\check{s}_2,[.6,.8],[.3,.5]>,<\check{s}_3,[.8,.9],[.1,.3]>,$
$<\check{s}_4,[.7,.8],[.4,.5]>,<\check{s}_5,[.7,.8],[.2,.4]>$},

$(\check{p}_2,\tilde{e}_1)$={$<\check{s}_1,[.5,.6],[.6,.7]>,<\check{s}_2,[.7,.8],[.3,.4]>,<\check{s}_3,[.7,.9],[.1,.2]>,$
$<\check{s}_4,[.6,.8],[.2,.5]>,<\check{s}_5,[.6,.7],[.4,.5]>$},

$(\check{p}_3,\tilde{e}_1)$={$<\check{s}_1,[.5,.7],[.4,.5]>,<\check{s}_2,[.8,.9],[.2,.3]>,<\check{s}_3,[.7,.8],[.3,.4]>,$
$<\check{s}_4,[.6,.7],[.3,.4]>,<\check{s}_5,[.6,.8],[.3,.5]>$},

$(\check{p}_1,\tilde{e}_2)$={$<\check{s}_1,[.5,.6],[.6,.7]>,<\check{s}_2,[.6,.8],[.4,.5]>,<\check{s}_3,[.7,.8],[.3,.4]>,$
$<\check{s}_4,[.6,.7],[.5,.6]>,<\check{s}_5,[.7,.8],[.4,.6]>$},

$(\check{p}_2,\tilde{e}_2)$={$<\check{s}_1,[.4,.5],[.5,.6]>,<\check{s}_2,[.8,.9],[.2,.3]>,<\check{s}_3,[.8,.9],[.2,.4]>,$
$<\check{s}_4,[.6,.7],[.4,.5]>,<\check{s}_5,[.6,.8],[.3,.5]>$},

$(\check{p}_3,\tilde{e}_2)$={$<\check{s}_1,[.6,.7],[.6,.7]>,<\check{s}_2,[.7,.8],[.3,.4]>,<\check{s}_3,[.7,.9],[.2,.3]>,$
$<\check{s}_4,[.6,.7],[.5,.6]>,<\check{s}_5,[.7,.8],[.4,.5]>$},

Taking Q = 3, without loss of generality.

Step 2:

$\tilde{e}_1$	$\breve{p}_1$	$\breve{p}_2$	$\breve{p}_3$
$\breve{s}_1$	[.4,.5],[.5,.6]	[.5,.6],[.6,.7]	[.5,.7],[.4,.5]
$\breve{s}_2$	[.6,.8],[.3,.5]	[.7,.8],[.3,.4]	[.8,.9],[.2,.3]
$\breve{s}_3$	[.8,.9],[.1,.3]	[.7,.9],[.1,.2]	[.7,.8],[.3,.4]
$\breve{s}_4$	[.7,.8],[.4,.5]	[.6,.8],[.2,.5]	[.6,.7],[.3,.4]
$\breve{s}_5$	[.7,.8],[.2,.4]	[.6,.7],[.4,.5]	[.6,.8],[.3,.5]

$\tilde{e}_2$	$\breve{p}_1$	$\breve{p}_2$	$\breve{p}_3$
$\breve{s}_1$	[.5,.6],[.6,.7]	[.4,.5],[.5,.6]	[.6,.7],[.6,.7]
$\breve{s}_2$	[.6,.8],[.4,.5]	[.8,.9],[.2,.3]	[.7,.8],[.3,.4]
$\breve{s}_3$	[.7,.8],[.3,.4]	[.8,.9],[.2,.4]	[.7,.9],[.2,.3]
$\breve{s}_4$	[.6,.7],[.5,.6]	[.6,.7],[.4,.5]	[.6,.7],[.5,.6]
$\breve{s}_5$	[.7,.8],[.4,.6]	[.6,.8],[.3,.5]	[.7,.8],[.4,.5]

Table 5.2: Expert Assessments of COVID-19 Pandemic

Step 3: Let $\widetilde{W}$ $=(0.3, 0.4, 0.3)^{\mathrm{T}}$ be a weight vector of criteria satisfying the normalized condition.

Step 4: Only in case of Q-ROIVFSEFWAO and Q-ROIVFSEFWGO. Let $\boldsymbol{W}$ $=(0.4, 0.3, 0.3)^{\mathrm{T}}$ be a position weight vector. The role of weight vector is to eliminate the impact of individual perception on overall comprehensive assessment.

Q-ROIVFSE Weighted averaging Operator

$$\overline{\omega}\left(\left(h_k, \breve{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}\right) = <\left[\left(1-\prod_{k=1}^{\breve{m}}\left(1-\left(\theta^{-}_{k(\breve{p},\tilde{e})}(\breve{s})\right)^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q}, \left(1-\prod_{k=1}^{\breve{m}}\left(1-(\theta^{+}_{k(\breve{p},\tilde{e})}(\breve{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q}\right],\ \left[\prod_{k=1}^{\breve{m}}(\varphi^{-}_{k(\breve{p},\tilde{e})}(\breve{s}))^{\widetilde{W}_k},\ \prod_{k=1}^{\breve{m}}(\varphi^{+}_{k(\breve{p},\tilde{e})}(\breve{s}))^{\widetilde{W}_k}\right] >.$$

Step 5: Aggregate criteria by using Q-ROIVFSEWAO.

Here, for $\breve{s}_1$ corresponding to $\tilde{e}_1$

$= \quad <\left[(1-((1-0.4^3)^{0.3}(1-0.5^3)^{0.4}(1-0.5^3)^{0.3}))^{1/3},\ (1-((1-0.5^3)^{0.3}(1-0.6^3)^{0.4}(1-0.7^3)^{0.3}))^{1/3}\right], [(0.5)^{0.3}(0.6)^{0.4}(0.4)^{0.3}, (0.6)^{0.3}(0.7)^{0.4}(0.5)^{0.3}] >$

$= < [0.4749, 0.6141], [0.5030, 0.6042] >.$

	$\tilde{e}_1$	$\tilde{e}_2$
$\breve{s}_1$	[0.4749, 0.6141], [0.5030, 0.6042]	[0.5069, 0.6065], [0.5578, 0.6581]
$\breve{s}_2$	[0.7155, 0.8392], [0.2656, 0.3923]	[0.7274, 0.8501], [0.2780, 0.3812]
$\breve{s}_3$	[0.7917, 0.8779], [0.1390, 0.2781]	[0.7469, 0.8647], [0.2259, 0.3669]
$\breve{s}_4$	[0.6354, 0.7755], [0.2781, 0.4676]	[0.6, 0.7], [0.4573, 0.5578]
$\breve{s}_5$	[0.6354, 0.7665], [0.2980, 0.4676]	[0.6656, 0.8], [0.3565, 0.5281]

Table 5.3: Aggregate criteria by Q-ROIVFSEWAO

Step 6: Determine the accuracy of each member of the set S in accordance with the evaluations provided by each expert.

	$\tilde{e}_1$	$\tilde{e}_2$
$\breve{s}_1$	0.3432	0.4060
$\breve{s}_2$	0.5182	0.6102
$\breve{s}_3$	0.5985	0.5621
$\breve{s}_4$	0.4233	0.4141
$\breve{s}_5$	0.4177	0.4997

Table 5.4: Each Member's Accuracy per Each Expert

Step 7: Calculate average accuracy for each element of S,

	$\breve{s}_1$	$\breve{s}_2$	$\breve{s}_3$	$\breve{s}_4$	$\breve{s}_5$
Average	0.3746	0.5642	0.5803	0.4187	0.4587

Table 5.5: Average Accuracy for all Expert

Step 8: Non-increasing chain of these averages is

$0.5803 > 0.5642 > 0.4587 > 0.4187 > 0.3746$ which implies that

$$\breve{s}_3 > \breve{s}_2 > \breve{s}_5 > \breve{s}_4 > \breve{s}_1.$$

Step 9: Hence $\breve{s}_3$ is most affected one. That is United States is most affected country by COVID-19 as compared to Itlay, Russia, Turkey or United Kingdom. In accordance to expert opinion about the set of attributes, like total death cases, CFR and TPR.

Q-ROIVFSE Ordered Weighted averaging Operator

$$\overline{\omega_o}\left(\left(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\,\varphi_k)}\right) = <\left[\left(1 - \prod_{i=1}^{\widehat{m}}\left(1 - (\boldsymbol{\theta}^{-}_{i(\check{p},\tilde{e})}(\breve{s}))^{Q}\right)^{\boldsymbol{W}_i}\right)^{1/Q},\ \left(1 - \prod_{i=1}^{\widehat{m}}\left(1 - (\boldsymbol{\theta}^{+}_{i(\check{p},\tilde{e})}(\breve{s}))^{Q}\right)^{\boldsymbol{W}_i}\right)^{1/Q}\right],\ \left[\prod_{i=1}^{\widehat{m}}(\boldsymbol{\varphi}^{-}_{i(\check{p},\tilde{e})}(\breve{s}))^{\boldsymbol{W}_i},\ \prod_{i=1}^{\widehat{m}}(\boldsymbol{\varphi}^{+}_{i(\check{p},\tilde{e})}(\breve{s}))^{\boldsymbol{W}_i}\right] >.$$

Step 5: Aggregate criteria by using Q-ROIVFSE ordered weighted averaging Operator. But before that we find score § of each of above elements in table 5.2 by using Definition (4.2.36) as below,

$\tilde{e}_1$	$\breve{p}_1$	$\breve{p}_2$	$\breve{p}_3$
$\breve{s}_1$	-0.076	-0.109	0.140
$\breve{s}_2$	0.288	0.382	0.603
$\breve{s}_3$	0.606	0.536	0.382
$\breve{s}_4$	0.333	0.298	0.234
$\breve{s}_5$	0.392	0.185	0.288

$\tilde{e}_2$	$\breve{p}_1$	$\breve{p}_2$	$\breve{p}_3$
$\breve{s}_1$	-0.109	-0.076	0
$\breve{s}_2$	0.270	0.603	0.382
$\breve{s}_3$	0.382	0.584	0.518
$\breve{s}_4$	0.181	0.185	0.109
$\breve{s}_5$	0.288	0.288	0.333

Table 5.6: *S*core of Each Element

By using ranking method of Q-ROIVFSESs as Score function, $\left(\boldsymbol{h}_i, \breve{\boldsymbol{P}}_i, \widetilde{\boldsymbol{E}}_i, \mathrm{Q}\right)_{(\boldsymbol{\theta}_i, \boldsymbol{\varphi}_i)}$ = $\{< \breve{s}, \left[\boldsymbol{\theta}^-_{i(\breve{p},\tilde{e})}, \ \boldsymbol{\theta}^+_{i(\breve{p},\tilde{e})}\right], \left[\boldsymbol{\varphi}^-_{i(\breve{p},\tilde{e})}, \ \boldsymbol{\varphi}^+_{i(\breve{p},\tilde{e})}\right] >: \breve{s} \in \mathrm{S}\}$ is the i-th largest of $\breve{m}$ Q-ROIVFSESs $\left(h_k, \breve{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k, \varphi_k)}$, for 1 $\leq k \leq$ $\breve{m}$, is determined as follows,

$\tilde{e}_1$	$\breve{p}_1$	$\breve{p}_2$	$\breve{p}_3$
$\breve{s}_1$	$\breve{p}_3$ [.5,.7],[.4,.5]	$\breve{p}_1$ [.4,.5],[.5,.6]	$\breve{p}_2$ [.5,.6],[.6,.7]
$\breve{s}_2$	$\breve{p}_3$ [.8,.9],[.2,.3]	$\breve{p}_2$ [.7,.8],[.3,.4]	$\breve{p}_1$ [.6,.8],[.3,.5]
$\breve{s}_3$	$\breve{p}_1$ [.8,.9],[.1,.3]	$\breve{p}_2$ [.7,.9],[.1,.2]	$\breve{p}_3$ [.7,.8],[.3,.4]
$\breve{s}_4$	$\breve{p}_1$ [.7,.8],[.4,.5]	$\breve{p}_2$ [.6,.8],[.2,.5]	$\breve{p}_3$ [.6,.7],[.3,.4]
$\breve{s}_5$	$\breve{p}_1$ [.7,.8],[.2,.4]	$\breve{p}_3$ [.6,.8],[.3,.5]	$\breve{p}_2$ [.6,.7],[.4,.5]

$\tilde{e}_2$	$\breve{p}_1$	$\breve{p}_2$	$\breve{p}_3$
$\breve{s}_1$	$\breve{p}_3$ [.6,.7],[.6,.7]	$\breve{p}_2$ [.4,.5],[.5,.6]	$\breve{p}_1$ [.5,.6],[.6,.7]
$\breve{s}_2$	$\breve{p}_2$ [.8,.9],[.2,.3]	$\breve{p}_3$ [.7,.8],[.3,.4]	$\breve{p}_1$ [.6,.8],[.4,.5]
$\breve{s}_3$	$\breve{p}_2$ [.8,.9],[.2,.4]	$\breve{p}_3$ [.7,.9],[.2,.3]	$\breve{p}_1$ [.7,.8],[.3,.4]
$\breve{s}_4$	$\breve{p}_2$ [.6,.7],[.4,.5]	$\breve{p}_1$ [.6,.7],[.5,.6]	$\breve{p}_3$ [.6,.7],[.5,.6]
$\breve{s}_5$	$\breve{p}_3$ [.7,.8],[.4,.5]	$\breve{p}_2$ [.6,.8],[.3,.5]	$\breve{p}_1$ [.7,.8],[.4,.6]

Table 5.7: ith largest of 3 Q-ROIVFSESs

Further aggregate criteria by using Q-ROIVFSEOWAO. Here, for $\breve{s}_1$ corresponding to expert $\tilde{e}_1$,

$$= \quad < \left[(1-((1-0.5^3)^{0.3}(1-0.4^3)^{0.4}(1-0.5^3)^{0.3}))^{1/3},\ (1-((1-0.7^3)^{0.3}(1-0.5^3)^{0.4}(1-0.6^3)^{0.3}))^{1/3}\right], [(0.4)^{0.3}(0.5)^{0.4}(0.6)^{0.3}, (0.5)^{0.3}(0.6)^{0.4}(0.7)^{0.3}] >$$

$$= < [0.4658, 0.6065], [0.4939, 0.5950] >.$$

	$\tilde{e}_1$	$\tilde{e}_2$
$\breve{s}_1$	$[0.4658, 0.6065], [0.4939, 0.5950]$	$[0.5069, 0.6065], [0.5578, 0.6581]$
$\breve{s}_2$	$[0.7155, 0.8392], [0.2656, 0.3923]$	$[0.7155, 0.8392], [0.2896, 0.3923]$
$\breve{s}_3$	$[0.7917, 0.8779], [0.1390, 0.2781]$	$[0.7362, 0.8779], [0.2259, 0.3565]$
$\breve{s}_4$	$[0.6354, 0.7755], [0.2781, 0.4676]$	$[0.6, 0.7], [0.4676, 0.5681]$
$\breve{s}_5$	$[0.6354, 0.7755], [0.2896, 0.4676]$	$[0.6656, 0.8], [0.3565, 0.5281]$

Table 5.8: Aggregate criteria by Q-ROIVFSEOWAO

Step 6: Determine the accuracy of each member of the set S in accordance with the evaluations provided by each expert.

	$\tilde{e}_1$	$\tilde{e}_2$
$\breve{s}_1$	0.3276	0.4060
$\breve{s}_2$	0.5182	0.4363
$\breve{s}_3$	0.5985	0.5094
$\breve{s}_4$	0.4233	0.4223
$\breve{s}_5$	0.4247	0.4997

Table 5.9: Each Member's Accuracy per Each Expert

Step 7: Calculate average accuracy for each element of S.

	$\breve{s}_1$	$\breve{s}_2$	$\breve{s}_3$	$\breve{s}_4$	$\breve{s}_5$
Average	0.3668	0.4772	0.5540	0.4228	0.4622

Table 5.10: Average Accuracy for all Expert

Step 8: Non-increasing chain of these averages is

$0.5540 > 0.4772 > 0.4622 > 0.4228 > 0.3668$ which implies that

$$\breve{s}_3 > \breve{s}_2 > \breve{s}_5 > \breve{s}_4 > \breve{s}_1.$$

Step 9: Hence $\check{s}_3$ is most affected one. That is United States is most affected country by COVID-19 as compared to Itlay, Russia, Turkey or United Kingdom. In accordance to expert opinion about the set of attributes, like total death cases, CFR and TPR.

Q-ROIVFSE Fusion Weighted averaging Operator

$$\overline{\omega_f}\left(\left(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}\right) = <\left[\left(1 - \prod_{i=1}^{\breve{m}}\left(1 - (\underline{\boldsymbol{\theta}}^{-}_{i(\check{p},\tilde{e})}(\check{s}))^{Q}\right)^{\boldsymbol{W}_i}\right)^{1/Q},\ \left(1 - \prod_{i=1}^{\breve{m}}\left(1 - (\underline{\boldsymbol{\theta}}^{+}_{i(\check{p},\tilde{e})}(\check{s}))^{Q}\right)^{\boldsymbol{W}_i}\right)^{1/Q}\right],\ \left[\prod_{i=1}^{\breve{m}}(\underline{\boldsymbol{\varphi}}^{-}_{i(\check{p},\tilde{e})}(\check{s}))^{\boldsymbol{W}_i},\ \prod_{i=1}^{\breve{m}}(\underline{\boldsymbol{\varphi}}^{+}_{i(\check{p},\tilde{e})}(\check{s}))^{\boldsymbol{W}_i}\right] >.$$

Step 5: Aggregate criteria by using Q-ROIVFSEFWAO,

In this example, $\breve{m}$ = 3 and $\boldsymbol{W}$ =$(0.4, 0.3, 0.3)^{\mathrm{T}}$ is position weight vector, then for $\check{s}_1$, $\check{p}_1$ and $\tilde{e}_1$

$$\left(\underline{h_1}, \underline{\check{P}_1}, \underline{\tilde{E}_1}, 3\right)_{\left(\underline{\theta_1}, \underline{\varphi_1}\right)} = \underline{\mathrm{h}}\left(\underline{\underline{\check{p}_1}}, \underline{\underline{\tilde{e}_1}}\right) = 3(0.3) < \check{s}_1, [0.4,0.5], [0.5,0.6] >$$

$$= \ 0.9 < \check{s}_1, [0.4,0.5], [0.5,0.6] >$$

$$= < \check{s}_1, \left[(1-(1-0.4^3)^{0.9})^{1/3}, (1-(1-0.5^3)^{0.9})^{1/3}\right] [0.5^{0.9}, 0.6^{0.9}] >$$

$$= < \check{s}_1, [0.3866,0.4838], [0.5359,0.6314] >.$$

Similarly,

$\underline{\underline{\tilde{e}_1}}$	$\underline{\underline{\check{p}_1}}$	$\underline{\underline{\check{p}_2}}$	$\underline{\underline{\check{p}_3}}$
$\check{s}_1$	[.3866,.4838], [.5359,.6314]	[.5290,.6327], [.5417,.6518]	[.4838,.6803], [.4384,.5359]
$\check{s}_2$	[.5816,.7806], [.3384,.5359]	[.7343,.8326], [.2358,.333]	[.7806,.8842], [.2349,.3384]
$\check{s}_3$	[.7806,.8842], [.1259,.3384]	[.7343,.9249], [.0631,.1450]	[.6803,.7806], [.3384,.4384]
$\check{s}_4$	[.6803,.7806], [.4384,.5359]	[.6327,.8326], [.1450,.4353]	[.5816,.6803], [.3384,.4384]
$\check{s}_5$	[.6803,.7806], [.2349,.4384]	[.6327,.7343], [.333,.4353]	[.5816,.7806], [.3384,.5359]

$\underline{\underline{\tilde{e}_2}}$	$\underline{\underline{\check{p}_1}}$	$\underline{\underline{\check{p}_2}}$	$\underline{\underline{\check{p}_3}}$
$\check{s}_1$	[.4838,.5816], [.6314,.7254]	[.4241,.5290], [.4353,.5417]	[.5816,.6803], [.6314,.7254]
$\check{s}_2$	[.5816,.7806], [.4384,.5359]	[.8326,.9294], [.1450,.2358]	[.6803,.7806], [.3384,.4384]
$\check{s}_3$	[.6803,.7806], [.3384,.4384]	[.8326,.9294], [.1450,.333]	[.6803,.8842], [.2349,.3384]
$\check{s}_4$	[.5816,.6803], [.5359,.6314]	[.6327,.7343], [.333,.4353]	[.5186,.6803], [.5359,.6314]
$\check{s}_5$	[.6803,.7806], [.4384,.6314]	[.6327,.8326], [.2358,.4353]	[.6803,.7806], [.4384,.5359]

Table 5.11: Aggregate criteria by Q-ROIVFSEFWAO

Next, finding the score of each of above element by Definition (4.2.36) as follows:

$\underline{\underline{\tilde{e}_1}}$	$\underline{\underline{\check{p}_1}}$	$\underline{\underline{\check{p}_2}}$	$\underline{\underline{\check{p}_3}}$
$\check{s}_1$	-0.1173	-0.0173	0.0950
$\check{s}_2$	0.2399	0.4615	0.5576
$\check{s}_3$	0.5631	0.5919	0.3335
$\check{s}_4$	0.2762	0.3725	0.1943
$\check{s}_5$	0.3466	0.2649	0.2399

$\underline{\underline{\tilde{e}_2}}$	$\underline{\underline{\check{p}_1}}$	$\underline{\underline{\check{p}_2}}$	$\underline{\underline{\check{p}_3}}$
$\check{s}_1$	-0.1617	-0.0086	-0.0609
$\check{s}_2$	0.2171	0.6761	0.3337
$\check{s}_3$	0.1567	0.6642	0.4772
$\check{s}_4$	0.0530	0.2649	0.0530
$\check{s}_5$	0.2273	0.3674	0.2762

Table 5.12: Score of Each Element

Finding the ith largest of 3 Q-ROIVFSESs

$\underline{\tilde{e}_1}$	$\underline{\check{p}_1}$	$\underline{\check{p}_2}$	$\underline{\check{p}_3}$
$\check{s}_1$	$\underline{\underline{\check{p}_3}}$ [0.4838,0.6803], [0.4384,0.5359]	$\underline{\underline{\check{p}_2}}$ [0.5290,0.6327], [0.5417,0.6518]	$\underline{\underline{\check{p}_1}}$ [0.3866,0.4838], [0.5359,0.6314]
$\check{s}_2$	$\underline{\underline{\check{p}_3}}$ [0.7806,0.8842], [0.2349,0.3384]	$\underline{\underline{\check{p}_2}}$ [0.7343,0.8326], [0.2358,0.333]	$\underline{\underline{\check{p}_1}}$ [0.5816,0.7806], [0.3384,0.5359]
$\check{s}_3$	$\underline{\underline{\check{p}_2}}$ [0.7343,0.9249], [0.0631,0.1450]	$\underline{\underline{\check{p}_1}}$ [0.7806,0.8842], [0.1259,0.3384]	$\underline{\underline{\check{p}_3}}$ [0.6803,0.7806], [0.3384,0.4384]

$\check{s}_4$	$\underline{\underline{\check{p}_2}}$ [0.6327,0.8326], [0.1450,0.4353]	$\underline{\underline{\check{p}_1}}$ [0.6803,0.7806], [0.4384,0.5359]	$\underline{\underline{\check{p}_3}}$ [0.5816,0.6803], [0.3384,0.4384]
$\check{s}_5$	$\underline{\underline{\check{p}_1}}$ [0.6803,0.7806], [0.2349,0.4384]	$\underline{\underline{\check{p}_2}}$ [0.6327,0.7343], [0.333,0.4353]	$\underline{\underline{\check{p}_3}}$ [0.5816,0.7806], [0.3384,0.5359]

$\underline{\tilde{e}_2}$	$\underline{\check{p}_1}$	$\underline{\check{p}_2}$	$\underline{\check{p}_3}$
$\check{s}_1$	$\underline{\underline{\check{p}_2}}$ [0.4241,0.5290], [0.4353,0.5417]	$\underline{\underline{\check{p}_3}}$ [0.5816,0.6803], [0.6314,0.7254]	$\underline{\underline{\check{p}_1}}$ [0.4838,0.5816], [0.6314,0.7254]
$\check{s}_2$	$\underline{\underline{\check{p}_2}}$ [0.8326,0.9294], [0.1450,0.2358]	$\underline{\underline{\check{p}_3}}$ [0.6803,0.7806], [0.3384,0.4384]	$\underline{\underline{\check{p}_1}}$ [0.5816,0.7806], [0.4384,0.5359]
$\check{s}_3$	$\underline{\underline{\check{p}_2}}$ [0.8326,0.9294], [0.1450,0.333]	$\underline{\underline{\check{p}_3}}$ [0.6803,0.8842], [0.2349,0.3384]	$\underline{\underline{\check{p}_1}}$ [0.6803,0.7806], [0.3384,0.4384]
$\check{s}_4$	$\underline{\underline{\check{p}_2}}$ [0.6327,0.7343], [0.333,0.4353]	$\underline{\underline{\check{p}_3}}$ [0.5186,0.6803], [0.5359,0.6314]	$\underline{\underline{\check{p}_1}}$ [0.5816,0.6803], [0.5359,0.6314]
$\check{s}_5$	$\underline{\underline{\check{p}_2}}$ [0.6327,0.8326], [0.2358,0.4353]	$\underline{\underline{\check{p}_3}}$ [0.6803,0.7806], [0.4384,0.5359]	$\underline{\underline{\check{p}_1}}$ [0.6803,0.7806], [0.4384,0.6314]

Table 5.13: ith largest of 3 Q-ROIVFSESs

Further aggregate Criteria ($\underline{\check{p}_1}, \underline{\check{p}_2}, \underline{\check{p}_3}$) by using Q-ROIVFSEFWAO.

For example, for $\breve{s}_1$ corresponding to expert $\underline{\tilde{e}_1}$, we have

$$=<\left[(1-((1-0.4838^3)^{0.4}(1-0.5290^3)^{0.3}(1-0.3886^3)^{0.3})\,)^{1/3},\ (1-((1-0.6803^3)^{0.4}(1-0.6327^3)^{0.3}(1-0.4838^3)^{0.3})\,)^{1/3}\right],$$
$$[(0.4384)^{0.4}(0.5417)^{0.3}(0.5359)^{0.3},(0.5359)^{0.4}(0.6518)^{0.3}(0.6314)^{0.3}]>$$

$$=<[0.4762,0.6217],[0.4961,0.5970]>.$$

Similarly,

	$\underline{\tilde{e}_1}=\tilde{e}_1$	$\underline{\tilde{e}_2}=\tilde{e}_2$
$\breve{s}_1$	$[0.4762,0.6217],[0.4961,0.5970]$	$[0.5000,0.5999],[0.5441,0.6454]$
$\breve{s}_2$	$[0.7232,0.8445],[0.2624,0.3866]$	$[0.7414,0.8605],[0.2606,0.3633]$
$\breve{s}_3$	$[0.7360,0.8839],[0.1285,0.2606]$	$[0.7578,0.8839],[0.2161,0.3634]$
$\breve{s}_4$	$[0.6351,0.7823],[0.2606,0.4643]$	$[0.6035,0.7037],[0.4430,0.5441]$
$\breve{s}_5$	$[0.6403,0.7680],[0.2910,0.4646]$	$[0.6626,0.8036],[0.3421,0.5180]$

Table 5.14: Aggregate Criteria ($\underline{\breve{p}_1},\underline{\breve{p}_2},\underline{\breve{p}_3}$) by Q-ROIVFSEFWAO

Step 6: Determine the accuracy of each member of the set S in accordance with the evaluations provided by each expert.

	$\breve{s}_1$	$\breve{s}_2$	$\breve{s}_3$	$\breve{s}_4$	$\breve{s}_5$
$\tilde{e}_1$	0.3416	0.5282	0.5545	0.4264	0.4202
$\tilde{e}_2$	0.3854	0.5552	0.5919	0.4081	0.4944

Table 5.15: Each Member's Accuracy per Each Expert

Step 7: Finding average accuracy for all expert of each element of S,

	$\breve{s}_1$	$\breve{s}_2$	$\breve{s}_3$	$\breve{s}_4$	$\breve{s}_5$
Average	0.3635	0.5417	0.5732	0.4172	0.4573

Table 5.16: Average Accuracy for all Expert

Step 8: Non-increasing chain of these averages is

$0.5732 > 0.5417 > 0.4573 > \ 0.4172 > 0.3635$ which implies that

$$\breve{s}_3 > \breve{s}_2 > \breve{s}_5 > \breve{s}_4 > \breve{s}_1.$$

Step 9: Hence $\breve{s}_3$ is most affected one. That is United States is most affected country by COVID-19 as compared to Itlay, Russia, Turkey or United Kingdom. In accordance to expert opinion about the set of attributes, like total death cases, CFR and TPR.

Q-ROIVFSE Weighted Geometric Operator

$$\underline{\omega}\left(\left(h_k, \breve{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}\right) = < [\prod_{k=1}^{\tilde{m}}(\theta^{-}_{k(\breve{p},\tilde{e})}(\breve{s}))^{\widetilde{W}_k},\ \prod_{k=1}^{\tilde{m}}(\theta^{+}_{k(\breve{p},\tilde{e})}(\breve{s}))^{\widetilde{W}_k}],$$
$$\left[\left(1 - \prod_{k=1}^{\tilde{m}}\left(1 - (\varphi^{-}_{k(\breve{p},\tilde{e})}(\breve{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q},\ \left(1 - \prod_{k=1}^{\tilde{m}}\left(1 - (\varphi^{+}_{k(\breve{p},\tilde{e})}(\breve{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q}\right] >.$$

Step 5: Aggregate criteria by using Q-ROIVFSEWGO.

Here, for $\breve{s}_1$ corresponding to $\tilde{e}_1$

$$= \quad < [(0.4)^{0.3}(0.5)^{0.4}(0.5)^{0.3}, (0.5)^{0.3}(0.6)^{0.4}(0.7)^{0.3}], \Big[(1 - ((1-0.5^3)^{0.3}(1 - 0.6^3)^{0.4}(1-0.4^3)^{0.3}))^{1/3},\ (1 - ((1-0.6^3)^{0.3}(1-0.7^3)^{0.4}(1-0.5^3)^{0.3}))^{1/3}\Big] >$$

$= < [0.4676, 0.5950], [0.5260, 0.6258] >.$

	$\tilde{e}_1$	$\tilde{e}_2$
$\breve{s}_1$	$[0.4676, 0.5950], [0.5260, 0.6258]$	$[0.4830, 0.5842], [0.5654, 0.6656]$
$\breve{s}_2$	$[0.6957, 0.8288], [0.2274, 0.4160]$	$[0.7050, 0.8386], [0.3134, 0.4090]$
$\breve{s}_3$	$[0.7286, 0.8688], [0.2070, 0.3134]$	$[0.7384, 0.8688], [0.2395, 0.3757]$
$\breve{s}_4$	$[0.6284, 0.7686], [0.3134, 0.4749]$	$[0.6, 0.7], [0.4658, 0.5654]$
$\breve{s}_5$	$[0.6284, 0.7584], [0.3314, 0.4749]$	$[0.6581, 0.8], [0.3668, 0.5353]$

Table 5.17: Aggregate criteria by Q-ROIVFSEWGO

Step 6: Determine the accuracy of each member of the set S in accordance with the evaluations provided by each expert.

	$\tilde{e}_1$	$\tilde{e}_2$
$\check{s}_1$	0.3517	0.3938
$\check{s}_2$	0.4949	0.5197
$\check{s}_3$	0.5411	0.5626
$\check{s}_4$	0.4200	0.4204
$\check{s}_5$	0.4139	0.4999

Table 5.18: Each Member's Accuracy per Each Expert pert

Step 7: Calculate average accuracy for each element of S,

	$\check{s}_1$	$\check{s}_2$	$\check{s}_3$	$\check{s}_4$	$\check{s}_5$
Average	0.3728	0.5073	0.5518	0.4202	0.4569

Table 5.19: Average Accuracy for all Expert

Step 8: Non-increasing chain of these averages is

$0.5518 > 0.5073 > 0.4569 > 0.4202 > 0.3728$ which implies that

$$\check{s}_3 > \check{s}_2 > \check{s}_5 > \check{s}_4 > \check{s}_1.$$

Step 9: Hence $\check{s}_3$ is most affected one. That is United States is most affected country by COVID-19 as compared to Itlay, Russia, Turkey or United Kingdom. In accordance to expert opinion about the set of attributes, like total death cases, CFR and TPR.

Q-ROIVFSE Ordered Weighted Geometric Operator

$$\underline{\omega_o}\left(\left(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}\right) = < [\prod_{i=1}^{\check{m}} (\boldsymbol{\theta}^{-}_{i(\check{p},\check{e})}(\check{s}))^{\boldsymbol{w}_i},\ \prod_{i=1}^{\check{m}} (\boldsymbol{\theta}^{+}_{i(\check{p},\check{e})}(\check{s}))^{\boldsymbol{w}_i}],$$

$$\left[\left(1 - \prod_{i=1}^{\check{m}}\left(1 - (\boldsymbol{\varphi}^{-}_{i(\check{p},\check{e})}(\check{s}))^{Q}\right)^{\boldsymbol{w}_i}\right)^{1/Q}, \left(1 - \prod_{i=1}^{\check{m}}\left(1 - (\boldsymbol{\varphi}^{+}_{i(\check{p},\check{e})}(\check{s}))^{Q}\right)^{\boldsymbol{w}_i}\right)^{1/Q}\right] >.$$

Step 5: Aggregate criteria by using Q-ROIVFSE ordered weighted geometric Operator. But before that we find score § of each of above elements in table 5.2 by using Definition (4.2.36) as below,

$\tilde{e}_1$	$\breve{p}_1$	$\breve{p}_2$	$\breve{p}_3$
$\breve{s}_1$	-0.076	-0.109	0.140
$\breve{s}_2$	0.288	0.382	0.603
$\breve{s}_3$	0.606	0.536	0.382
$\breve{s}_4$	0.333	0.298	0.234
$\breve{s}_5$	0.392	0.185	0.288

$\tilde{e}_2$	$\breve{p}_1$	$\breve{p}_2$	$\breve{p}_3$
$\breve{s}_1$	-0.109	-0.076	0
$\breve{s}_2$	0.270	0.603	0.382
$\breve{s}_3$	0.382	0.584	0.518
$\breve{s}_4$	0.181	0.185	0.109
$\breve{s}_5$	0.288	0.288	0.333

Table 5.20: *S*core of Each Element

By using ranking method of Q-ROIVFSESs as Score function, $\left(\boldsymbol{h}_i, \breve{\boldsymbol{P}}_i, \widetilde{\boldsymbol{E}}_i, \mathrm{Q}\right)_{(\boldsymbol{\theta}_i, \boldsymbol{\varphi}_i)}$ = $\{< \breve{s}, \left[\boldsymbol{\theta}^-_{i(\breve{p},\tilde{e})}, \ \boldsymbol{\theta}^+_{i(\breve{p},\tilde{e})}\right], \left[\boldsymbol{\varphi}^-_{i(\breve{p},\tilde{e})}, \ \boldsymbol{\varphi}^+_{i(\breve{p},\tilde{e})}\right] >: \breve{s} \in \mathrm{S}\}$ is the i-th largest of $\breve{m}$ Q-ROIVFSESs $\left(h_k, \breve{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k, \varphi_k)}$, for $1 \leq k \leq \breve{m}$, is determined as follows,

$\tilde{e}_1$	$\breve{p}_1$	$\breve{p}_2$	$\breve{p}_3$
$\breve{s}_1$	$\breve{p}_3$ [.5,.7],[.4,.5]	$\breve{p}_1$ [.4,.5],[.5,.6]	$\breve{p}_2$ [.5,.6],[.6,.7]
$\breve{s}_2$	$\breve{p}_3$ [.8,.9],[.2,.3]	$\breve{p}_2$ [.7,.8],[.3,.4]	$\breve{p}_1$ [.6,.8],[.3,.5]
$\breve{s}_3$	$\breve{p}_1$ [.8,.9],[.1,.3]	$\breve{p}_2$ [.7,.9],[.1,.2]	$\breve{p}_3$ [.7,.8],[.3,.4]
$\breve{s}_4$	$\breve{p}_1$ [.7,.8],[.4,.5]	$\breve{p}_2$ [.6,.8],[.2,.5]	$\breve{p}_3$ [.6,.7],[.3,.4]
$\breve{s}_5$	$\breve{p}_1$ [.7,.8],[.2,.4]	$\breve{p}_3$ [.6,.8],[.3,.5]	$\breve{p}_2$ [.6,.7],[.4,.5]

$\tilde{e}_2$	$\breve{p}_1$	$\breve{p}_2$	$\breve{p}_3$
$\breve{s}_1$	$\breve{p}_3$ [.6,.7],[.6,.7]	$\breve{p}_2$ [.4,.5],[.5,.6]	$\breve{p}_1$ [.5,.6],[.6,.7]
$\breve{s}_2$	$\breve{p}_2$ [.8,.9],[.2,.3]	$\breve{p}_3$ [.7,.8],[.3,.4]	$\breve{p}_1$ [.6,.8],[.4,.5]
$\breve{s}_3$	$\breve{p}_2$ [.8,.9],[.2,.4]	$\breve{p}_3$ [.7,.9],[.2,.3]	$\breve{p}_1$ [.7,.8],[.3,.4]
$\breve{s}_4$	$\breve{p}_2$ [.6,.7],[.4,.5]	$\breve{p}_1$ [.6,.7],[.5,.6]	$\breve{p}_3$ [.6,.7],[.5,.6]
$\breve{s}_5$	$\breve{p}_3$ [.7,.8],[.4,.5]	$\breve{p}_2$ [.6,.8],[.3,.5]	$\breve{p}_1$ [.7,.8],[.4,.6]

Table 5.21: ith largest of 3 Q-ROIVFSESs

Further aggregate criteria by using Q-ROIVFSEOWGO. Here, for example for $\breve{s}_1$ corresponding to expert $\tilde{e}_1$,

$$= \quad < [(0.5)^{0.3}(0.4)^{0.4}(0.5)^{0.3}, (0.7)^{0.3}(0.5)^{0.4}(0.6)^{0.3}], \Big[(1-((1-0.4^3)^{0.3}(1-0.5^3)^{0.4}(1-0.6^3)^{0.3}))^{1/3},\ (1-((1-0.5^3)^{0.3}(1-0.6^3)^{0.4}(1-0.7^3)^{0.3}))^{1/3}\Big] >$$

$= < [0.4573, 0.5842], [0.5143, 0.6141] >.$

	$\tilde{e}_1$	$\tilde{e}_2$
$\breve{s}_1$	$[0.4573, 0.5842], [0.5143, 0.6141]$	$[0.4830, 0.5842], [0.5654, 0.6656]$
$\breve{s}_2$	$[0.6957, 0.8288], [0.2274, 0.4160]$	$[0.6957, 0.8288], [0.3196, 0.4160]$
$\breve{s}_3$	$[0.7286, 0.8688], [0.2070, 0.3134]$	$[0.7286, 0.8688], [0.2395, 0.3668]$
$\breve{s}_4$	$[0.6284, 0.7686], [0.3134, 0.4749]$	$[0.6, 0.7], [0.4749, 0.5746]$
$\breve{s}_5$	$[0.6284, 0.7686], [0.3757, 0.5353]$	$[0.6581, 0.8], [0.3668, 0.5353]$

Table 5.22: Aggregate criteria by Q-ROIVFSEOWGO

Step 6: Determine the accuracy of each member of the set S in accordance with the evaluations provided by each expert

	$\tilde{e}_1$	$\tilde{e}_2$
$\breve{s}_1$	0.3313	0.3938
$\breve{s}_2$	0.4949	0.5053
$\breve{s}_3$	0.5411	0.5528
$\breve{s}_4$	0.4200	0.4279
$\breve{s}_5$	0.4542	0.4999

Table 5.23: Aggregate Criteria ($\underline{\breve{p}_1}, \underline{\breve{p}_2}, \underline{\breve{p}_3}$) by Q-ROIVFSEFWGO

Step 7: Calculate average accuracy for each element of S

	$\breve{s}_1$	$\breve{s}_2$	$\breve{s}_3$	$\breve{s}_4$	$\breve{s}_5$
Average	0.3626	0.5001	0.5470	0.4240	0.4770

Table 5.24: Average Accuracy for all Expert

Step 8: Non-increasing chain of these averages is

$0.5470 > 0.5001 > 0.4770 > 0.4240 > 0.3626$ which implies that

$$\breve{s}_3 > \breve{s}_2 > \breve{s}_5 > \breve{s}_4 > \breve{s}_1.$$

Step 9: Hence $\check{s}_3$ is most affected one. That is United States is most affected country by COVID-19 as compared to Itlay, Russia, Turkey or United Kingdom. In accordance to expert opinion about the set of attributes, like total death cases, CFR and TPR.

Q-ROIVFSE Fusion Weighted Geometric Operator

$$\underline{\omega_f}\left(\left(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\,\varphi_k)}\right) = <\left[\prod_{i=1}^{\breve{m}}(\underline{\boldsymbol{\theta}}^{-}_{i(\check{p},\tilde{e})}(\check{s}))^{\boldsymbol{W}_i},\ \prod_{i=1}^{\breve{m}}(\underline{\boldsymbol{\theta}}^{+}_{i(\check{p},\tilde{e})}(\check{s}))^{\boldsymbol{W}_i}\right],$$

$$\left[\left(1-\prod_{i=1}^{\breve{m}}\left(1-(\underline{\boldsymbol{\varphi}}^{-}_{i(\check{p},\tilde{e})}(\check{s}))^{Q}\right)^{\boldsymbol{W}_i}\right)^{1/Q},\ \left(1-\prod_{i=1}^{\breve{m}}\left(1-(\underline{\boldsymbol{\varphi}}^{+}_{i(\check{p},\tilde{e})}(\check{s}))^{Q}\right)^{\boldsymbol{W}_i}\right)^{1/Q}\right]>.$$

Step 5: Aggregate criteria by using Q-ROIVFSE fusion weighted geometric operator,

In this example, $\breve{m}$ = 3 and $\boldsymbol{W}$ =$(0.4, 0.3, 0.3)^{\mathrm{T}}$ is position weight vector, then for $\check{s}_1$, $\check{p}_1$ and $\tilde{e}_1$

$$\left(\underline{\underline{h_1}}, \underline{\underline{\check{P}_1}}, \underline{\underline{\tilde{E}_1}}, 3\right)_{\left(\underline{\underline{\theta_1}},\underline{\underline{\varphi_1}}\right)} = \underline{\mathrm{h}}\left(\underline{\underline{\check{p}_1}}, \underline{\underline{\tilde{e}_1}}\right) = 3(0.3) < \check{s}_1, [0.4,0.5], [0.5,0.6] >$$

$$= \ 0.9 < \check{s}_1, [0.4,0.5], [0.5,0.6] >$$

$$= < \check{s}_1, \left[(1-(1-0.4^3)^{0.9})^{1/3}, (1-(1-0.5^3)^{0.9})^{1/3}\right] [0.5^{0.9}, 0.6^{0.9}] >$$

$$= < \check{s}_1, [0.3866,0.4838], [0.5359,0.6314] >.$$

Similarly,

$\underline{\underline{\tilde{e}_1}}$	$\underline{\underline{\check{p}_1}}$	$\underline{\underline{\check{p}_2}}$	$\underline{\underline{\check{p}_3}}$
$\check{s}_1$	[.3866,.4838], [.5359,.6314]	[.5290,.6327], [.5417,.6518]	[.4838,.6803], [.4384,.5359]
$\check{s}_2$	[.5816,.7806], [.3384,.5359]	[.7343,.8326], [.2358,.333]	[.7806,.8842], [.2349,.3384]
$\check{s}_3$	[.7806,.8842], [.1259,.3384]	[.7343,.9249], [.0631,.1450]	[.6803,.7806], [.3384,.4384]
$\check{s}_4$	[.6803,.7806], [.4384,.5359]	[.6327,.8326], [.1450,.4353]	[.5816,.6803], [.3384,.4384]
$\check{s}_5$	[.6803,.7806], [.2349,.4384]	[.6327,.7343], [.333,.4353]	[.5816,.7806], [.3384,.5359]

$\underline{\underline{\tilde{e}_2}}$	$\underline{\underline{\check{p}_1}}$	$\underline{\underline{\check{p}_2}}$	$\underline{\underline{\check{p}_3}}$
$\check{s}_1$	[.4838,.5816], [.6314,.7254]	[.4241,.5290], [.4353,.5417]	[.5816,.6803], [.6314,.7254]
$\check{s}_2$	[.5816,.7806], [.4384,.5359]	[.8326,.9294], [.1450,.2358]	[.6803,.7806], [.3384,.4384]
$\check{s}_3$	[.6803,.7806], [.3384,.4384]	[.8326,.9294], [.1450,.333]	[.6803,.8842], [.2349,.3384]
$\check{s}_4$	[.5816,.6803], [.5359,.6314]	[.6327,.7343], [.333,.4353]	[.5186,.6803], [.5359,.6314]
$\check{s}_5$	[.6803,.7806], [.4384,.6314]	[.6327,.8326], [.2358,.4353]	[.6803,.7806], [.4384,.5359]

Table 5.25: Aggregate criteria by Q-ROIVFSEFWGO

Next, finding the score of each of above element by Definition (4.2.36) as follows:

$\underline{\underline{\tilde{e}_1}}$	$\underline{\underline{\check{p}_1}}$	$\underline{\underline{\check{p}_2}}$	$\underline{\underline{\check{p}_3}}$
$\check{s}_1$	-0.1173	-0.0173	0.0950
$\check{s}_2$	0.2399	0.4615	0.5576
$\check{s}_3$	0.5631	0.5919	0.3335
$\check{s}_4$	0.2762	0.3725	0.1943
$\check{s}_5$	0.3466	0.2649	0.2399

$\underline{\underline{\tilde{e}_2}}$	$\underline{\underline{\check{p}_1}}$	$\underline{\underline{\check{p}_2}}$	$\underline{\underline{\check{p}_3}}$
$\check{s}_1$	-0.1617	-0.0086	-0.0609
$\check{s}_2$	0.2171	0.6761	0.3337
$\check{s}_3$	0.1567	0.6642	0.4772
$\check{s}_4$	0.0530	0.2649	0.0530
$\check{s}_5$	0.2273	0.3674	0.2762

Table 5.26: *S*core of Each Element

Finding the ith largest of 3 Q-ROIVFSESs

$\underline{\tilde{e}_1}$	$\underline{\check{p}_1}$	$\underline{\check{p}_2}$	$\underline{\check{p}_3}$
$\check{s}_1$	$\underline{\underline{\check{p}_3}}$ [0.4838,0.6803], [0.4384,0.5359]	$\underline{\underline{\check{p}_2}}$ [0.5290,0.6327], [0.5417,0.6518]	$\underline{\underline{\check{p}_1}}$ [0.3866,0.4838], [0.5359,0.6314]
$\check{s}_2$	$\underline{\underline{\check{p}_3}}$ [0.7806,0.8842], [0.2349,0.3384]	$\underline{\underline{\check{p}_2}}$ [0.7343,0.8326], [0.2358,0.333]	$\underline{\underline{\check{p}_1}}$ [0.5816,0.7806], [0.3384,0.5359]
$\check{s}_3$	$\underline{\underline{\check{p}_2}}$ [0.7343,0.9249], [0.0631,0.1450]	$\underline{\underline{\check{p}_1}}$ [0.7806,0.8842], [0.1259,0.3384]	$\underline{\underline{\check{p}_3}}$ [0.6803,0.7806], [0.3384,0.4384]

$\check{s}_4$	$\underline{\underline{\check{p}_2}}$ [0.6327,0.8326], [0.1450,0.4353]	$\underline{\underline{\check{p}_1}}$ [0.6803,0.7806], [0.4384,0.5359]	$\underline{\underline{\check{p}_3}}$ [0.5816,0.6803], [0.3384,0.4384]
$\check{s}_5$	$\underline{\underline{\check{p}_1}}$ [0.6803,0.7806], [0.2349,0.4384]	$\underline{\underline{\check{p}_2}}$ [0.6327,0.7343], [0.333,0.4353]	$\underline{\underline{\check{p}_3}}$ [0.5816,0.7806], [0.3384,0.5359]

$\underline{\tilde{e}_2}$	$\underline{\check{p}_1}$	$\underline{\check{p}_2}$	$\underline{\check{p}_3}$
$\check{s}_1$	$\underline{\underline{\check{p}_2}}$ [0.4241,0.5290], [0.4353,0.5417]	$\underline{\underline{\check{p}_3}}$ [0.5816,0.6803], [0.6314,0.7254]	$\underline{\underline{\check{p}_1}}$ [0.4838,0.5816], [0.6314,0.7254]
$\check{s}_2$	$\underline{\underline{\check{p}_2}}$ [0.8326,0.9294], [0.1450,0.2358]	$\underline{\underline{\check{p}_3}}$ [0.6803,0.7806], [0.3384,0.4384]	$\underline{\underline{\check{p}_1}}$ [0.5816,0.7806], [0.4384,0.5359]
$\check{s}_3$	$\underline{\underline{\check{p}_2}}$ [0.8326,0.9294], [0.1450,0.333]	$\underline{\underline{\check{p}_3}}$ [0.6803,0.8842], [0.2349,0.3384]	$\underline{\underline{\check{p}_1}}$ [0.6803,0.7806], [0.3384,0.4384]
$\check{s}_4$	$\underline{\underline{\check{p}_2}}$ [0.6327,0.7343], [0.333,0.4353]	$\underline{\underline{\check{p}_3}}$ [0.5186,0.6803], [0.5359,0.6314]	$\underline{\underline{\check{p}_1}}$ [0.5816,0.6803], [0.5359,0.6314]
$\check{s}_5$	$\underline{\underline{\check{p}_2}}$ [0.6327,0.8326], [0.2358,0.4353]	$\underline{\underline{\check{p}_3}}$ [0.6803,0.7806], [0.4384,0.5359]	$\underline{\underline{\check{p}_1}}$ [0.6803,0.7806], [0.4384,0.6314]

Table 5.27: ith largest of 3 Q-ROIVFSESs

Further aggregate Criteria $(\underline{\check{p}_1}, \underline{\check{p}_2}, \underline{\check{p}_3})$ by using Q-ROIVFSE fusion weighted geometric operator.

For example, for $\check{s}_1$ corresponding to expert $\underline{\tilde{e}_1}$, we have

$$= \ < [(0.4838)^{0.4}(0.5290)^{0.3}(0.3866)^{0.3}, (0.6803)^{0.4}(0.6327)^{0.3}(0.4838)^{0.3}], \Big[(1 - ((1 - 0.4384^3)^{0.4}(1 - 0.5417^3)^{0.3}(1 - 0.5359^3)^{0.3})\,)^{1/3},\ (1 - ((1 - 0.5359^3)^{0.4}(1 - 0.6518^3)^{0.3}(1 - 0.6314^3)^{0.3})\,)^{1/3}\Big] >$$

$$= < [0.4646, 0.6010], [0.5043, 0.6056] >.$$

Similarly,

	$\underline{\tilde{e}_1} = \tilde{e}_1$	$\underline{\tilde{e}_2} = \tilde{e}_2$
$\check{s}_1$	$[0.4646, 0.6010], [0.5043, 0.6056]$	$[0.4850, 0.5869], [0.5730, 0.6700]$
$\check{s}_2$	$[0.7016, 0.8365], [0.2751, 0.4204]$	$[0.7037, 0.8354], [0.3383, 0.4283]$
$\check{s}_3$	$[0.7310, 0.8672], [0.2319, 0.3383]$	$[0.7376, 0.8672], [0.2564, 0.3730]$
$\check{s}_4$	$[0.6305, 0.7686], [0.3383, 0.4718]$	$[0.6015, 0.7014], [0.4776, 0.5730]$
$\check{s}_5$	$[0.6351, 0.7664], [0.3035, 0.4721]$	$[0.6608, 0.8010], [0.3836, 0.5402]$

Table 5.28: Aggregate Criteria $(\underline{\check{p}_1}, \underline{\check{p}_2}, \underline{\check{p}_3})$ by Q-ROIVFSEFWGO

Step 6: Determine the accuracy of each member of the set S in accordance with the evaluations provided by each expert.

	$\check{s}_1$	$\check{s}_2$	$\check{s}_3$	$\check{s}_4$	$\check{s}_5$
$\tilde{e}_1$	0.3339	0.5129	0.5470	0.4242	0.4198
$\tilde{e}_2$	0.4026	0.5244	0.5611	0.4299	0.5083

Table 5.29: Each Member’s Accuracy per Each Expert

Step 7: Finding average accuracy for all expert of each element of S,

	$\check{s}_1$	$\check{s}_2$	$\check{s}_3$	$\check{s}_4$	$\check{s}_5$
Average	0.3682	0.5186	0.5540	0.4270	0.4640

Table 5.30: Average Accuracy for all Expert

Step 8: Non-increasing chain of these averages is

$$0.5540 > 0.5186 > 0.4640 > \ 0.4270 > 0.3682 \text{ which implies that}$$

$$\breve{s}_3 > \breve{s}_2 > \breve{s}_5 > \breve{s}_4 > \breve{s}_1.$$

Step 9: Hence $\breve{s}_3$ is most affected one. That is United States is most affected country by COVID-19 as compared to Itlay, Russia, Turkey or United Kingdom. In accordance to expert opinion about the set of attributes, like total death cases, CFR and TPR.

Conclusion:

In studying different Q-ROIVFSE aggregation operators for the impact of COVID-19 on a list of selected countries like Italy, Russia, the United States, Turkey and the United Kingdom, experts from the World Health Organization (WHO) and the Center for Disease Control and Prevention (CDCP) focused on key factors or parameters like total death cases, case fatality rate (CFR) and test positivity rate (TPR).

After analyzing the data using various aggregation operators, it became evident that, according to all the aggregation operators considered (Q-ROIVFSEWAO, Q-ROIVFSEOWAO, Q-ROIVFSEFWAO, Q-ROIVFSEWGO, Q-ROIVFSEOWGO and Q-ROIVFSEFWGO), the $\breve{s}_3$ =United States stands out as the country most severely affected by COVID-19. This result shows consistency for different evaluation methods and aggregation operators.

While some aggregation operators, like Q-ROIVFSEWAO and Q-ROIVFSEWGO, are simpler to calculate, others like Q-ROIVFSEOWAO, Q-ROIVFSEFWAO, Q-ROIVFSEOWGO and Q-ROIVFSEFWGO are important for removing individual biases and providing a more objective assessment.

In summary, the study not only compares different aggregation operators but also highlights the various ways experts can combine their evaluations to achieve a fair and comprehensive understanding.

Example 5.5.3

In order to further analyze the flexibility and sensitivity of parameter Q, we set the different values of Q to sort the new practical MCDM, for this we use data of example 5.4.2, which is as follows:

Let us consider the case of COVID-19 and the universe of discourse be S = $\{\check{s}_1 = \text{Itlay},\ \check{s}_2 = \text{Russia}, \check{s}_3 = \text{United States}, \check{s}_4 = \text{Turkey}, \check{s}_5 = \text{United Kingdom}\}$ which is the set of countries which are most affected by COVID-19 pandemic, E = $\{\tilde{e}_1,\ \tilde{e}_2\}$be the set of two experts from World Health Organization (WHO) and Center for Disease Control and Prevention (CDCP). Among many parameters e.g., total confirmed cases, total deaths, case fatality rate (CFR) = $\frac{\text{total deaths}}{\text{confirmed cases}} \times 100$, total test conducted, test positivity rate (TPR), active cases and recovered cases, vaccination rates etc. we choose three parameters $\{\check{p}_1 = \text{total death cases}, \check{p}_2 = \text{CFR}, \check{p}_3 = \text{TPR}\}$. The two experts evaluate some data and express their evaluation in form of Q-ROIVFSESs.

Calculate the comprehensive assessment of the experts regarding the most affected country by employing two different Q-ROIVFSE aggregation operators which are Q-ROIVFSE Weighted averaging Operator and Q-ROIVFSE Weighted Geometric Operator, setting Q=2, 3, 5,10, 20.

Solution

Step 1: Utilize the expert assessments, presented as Q-ROIVFSESs, to derive expert opinions concerning the provided alternatives and criteria.

$(\check{p}_1,\tilde{e}_1)$=$\{< \check{s}_1, [.4, .5], [.5, .6] >, < \check{s}_2, [.6, .8], [.3, .5] >, < \check{s}_3, [.8, .9], [.1, .3] >,$
$< \check{s}_4, [.7, .8], [.4, .5] >, < \check{s}_5, [.7, .8], [.2, .4] >\},$

$(\check{p}_2,\tilde{e}_1)$=$\{< \check{s}_1, [.5, .6], [.6, .7] >, < \check{s}_2, [.7, .8], [.3, .4] >, < \check{s}_3, [.7, .9], [.1, .2] >,$
$< \check{s}_4, [.6, .8], [.2, .5] >, < \check{s}_5, [.6, .7], [.4, .5] >\},$

$(\check{p}_3,\tilde{e}_1)$={$<\check{s}_1,[.5,.7],[.4,.5]>,<\check{s}_2,[.8,.9],[.2,.3]>,<\check{s}_3,[.7,.8],[.3,.4]>,$
$<\check{s}_4,[.6,.7],[.3,.4]>,<\check{s}_5,[.6,.8],[.3,.5]>$},

$(\check{p}_1,\tilde{e}_2)$={$<\check{s}_1,[.5,.6],[.6,.7]>,<\check{s}_2,[.6,.8],[.4,.5]>,<\check{s}_3,[.7,.8],[.3,.4]>,$
$<\check{s}_4,[.6,.7],[.5,.6]>,<\check{s}_5,[.7,.8],[.4,.6]>$},

$(\check{p}_2,\tilde{e}_2)$={$<\check{s}_1,[.4,.5],[.5,.6]>,<\check{s}_2,[.8,.9],[.2,.3]>,<\check{s}_3,[.8,.9],[.2,.4]>,$
$<\check{s}_4,[.6,.7],[.4,.5]>,<\check{s}_5,[.6,.8],[.3,.5]>$},

$(\check{p}_3,\tilde{e}_2)$={$<\check{s}_1,[.6,.7],[.6,.7]>,<\check{s}_2,[.7,.8],[.3,.4]>,<\check{s}_3,[.7,.9],[.2,.3]>,$
$<\check{s}_4,[.6,.7],[.5,.6]>,<\check{s}_5,[.7,.8],[.4,.5]>$}.

Taking Q = 2,3,5,10,20 to observe flexibility and sensitivity of parameter Q.

Step 2:

$\tilde{e}_1$	$\check{p}_1$	$\check{p}_2$	$\check{p}_3$
$\check{s}_1$	[.4,.5],[.5,.6]	[.5,.6],[.6,.7]	[.5,.7],[.4,.5]
$\check{s}_2$	[.6,.8],[.3,.5]	[.7,.8],[.3,.4]	[.8,.9],[.2,.3]
$\check{s}_3$	[.8,.9],[.1,.3]	[.7,.9],[.1,.2]	[.7,.8],[.3,.4]
$\check{s}_4$	[.7,.8],[.4,.5]	[.6,.8],[.2,.5]	[.6,.7],[.3,.4]
$\check{s}_5$	[.7,.8],[.2,.4]	[.6,.7],[.4,.5]	[.6,.8],[.3,.5]

$\tilde{e}_2$	$\check{p}_1$	$\check{p}_2$	$\check{p}_3$
$\check{s}_1$	[.5,.6],[.6,.7]	[.4,.5],[.5,.6]	[.6,.7],[.6,.7]
$\check{s}_2$	[.6,.8],[.4,.5]	[.8,.9],[.2,.3]	[.7,.8],[.3,.4]
$\check{s}_3$	[.7,.8],[.3,.4]	[.8,.9],[.2,.4]	[.7,.9],[.2,.3]
$\check{s}_4$	[.6,.7],[.5,.6]	[.6,.7],[.4,.5]	[.6,.7],[.5,.6]
$\check{s}_5$	[.7,.8],[.4,.6]	[.6,.8],[.3,.5]	[.7,.8],[.4,.5]

Table 5.31: Expert Assessments of COVID-19 Pandemic

Step 3: Let $W=(0.3,0.4,0.3)^{T}$ be a weight vector of criteria satisfying the normalized condition.

Step 4: Here we will take Q-ROIVFSE Weighted averaging Operator and Q-ROIVFSE Weighted geometric Operator for different values of Q, so we will skip step 4, that is there is no need to take $\boldsymbol{W}$ =$(0.4, 0.3, 0.3)^{\mathrm{T}}$ as a position weight vector. The role of weight vector is to eliminate the impact of individual perception on overall comprehensive assessment.

Q-ROIVFSE Weighted averaging Operator

$$\varpi\left(\left(h_k, \breve{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}\right) = <\left[\left(1-\prod_{k=1}^{\breve{m}}\left(1-(\theta^-_{k(\breve{p},\tilde{e})}(\breve{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q},\ \left(1-\prod_{k=1}^{\breve{m}}\left(1-(\theta^+_{k(\breve{p},\tilde{e})}(\breve{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q}\right],\ \left[\prod_{k=1}^{\breve{m}}(\varphi^-_{k(\breve{p},\tilde{e})}(\breve{s}))^{\widetilde{W}_k},\ \prod_{k=1}^{\breve{m}}(\varphi^+_{k(\breve{p},\tilde{e})}(\breve{s}))^{\widetilde{W}_k}\right] >.$$

Q = 2

Step 5: Aggregate criteria by using Q-ROIVFSEWAO.Here, for $\breve{s}_1$ corresponding to $\tilde{e}_1$,

$$= \quad <\left[(1-((1-0.4^2)^{0.3}(1-0.5^2)^{0.4}(1-0.5^2)^{0.3}))^{1/2},\ (1-((1-0.5^2)^{0.3}(1-0.6^2)^{0.4}(1-0.7^2)^{0.3}))^{1/2}\right], [(0.5)^{0.3}(0.6)^{0.4}(0.4)^{0.3}, (0.6)^{0.3}(0.7)^{0.4}(0.5)^{0.3}] >$$

$$= < [0.4733, 0.61074], [0.5030, 0.6042] >.$$

	$\tilde{e}_1$	$\tilde{e}_2$
$\breve{s}_1$	$[0.4733,0.61074], [0.5030, 0.6042]$	$[0.5017,0.6025], [0.5578, 0.6581]$
$\breve{s}_2$	$[0.7129,0.8383], [0.2656, 0.3923]$	$[0.7246,0.8492], [0.2780, 0.3812]$
$\breve{s}_3$	$[0.7352,0.8774], [0.1390, 0.2781]$	$[0.7459,0.8774], [0.2259, 0.3669]$
$\breve{s}_4$	$[0.6341,0.7748], [0.2781, 0.4676]$	$[0.6, 0.7], [0.4573, 0.5578]$
$\breve{s}_5$	$[0.6341, 0.7656], [0.2980, 0.4676]$	$[0.6645,0.8], [0.3565, 0.5281]$

Table 5.32: Aggregate criteria by Q-ROIVFSEWAO for Q=2

Step 6: Determine the accuracy of each member of the set S in accordance with the evaluations provided by each expert.

	$\tilde{e}_1$	$\tilde{e}_2$
$\breve{s}_1$	0.6075	0.6795
$\breve{s}_2$	0.7177	0.7417
$\breve{s}_3$	0.7035	0.7559
$\breve{s}_4$	0.6492	0.6851
$\breve{s}_5$	0.6478	0.7438

Table 5.33: Each Member's Accuracy per Each Expert

Step 7: Calculate average accuracy for each element of S,

	$\breve{s}_1$	$\breve{s}_2$	$\breve{s}_3$	$\breve{s}_4$	$\breve{s}_5$
Average	0.6435	0.7297	0.7297	0.6672	0.6958

Table 5.34: Average Accuracy for all Expert

Step 8: Non-increasing chain of these averages is

$0.7279 = 0.7297 > 0.6958 > 0.6672 > 0.6435$ which implies that

$$\breve{s}_3 > \breve{s}_2 > \breve{s}_5 > \breve{s}_4 > \breve{s}_1.$$

Step 9: Hence $\breve{s}_3$ is most affected one. That is United States is most affected country by COVID-19 as compared to Itlay, Russia, Turkey or United Kingdom. In accordance to expert opinion about the set of attributes, like total death cases, CFR and TPR.

<u>Q = 3</u>

Step 5: Aggregate criteria by using Q-ROIVFSE weighted averaging Operator.

Here, for $\breve{s}_1$ corresponding to $\tilde{e}_1$,

$$= \; < \left[(1-((1-0.4^3)^{0.3}(1-0.5^3)^{0.4}(1-0.5^3)^{0.3}))^{1/3},\ (1-((1-0.5^3)^{0.3}(1-0.6^3)^{0.4}(1-0.7^3)^{0.3}))^{1/3}\right], [(0.5)^{0.3}(0.6)^{0.4}(0.4)^{0.3}, (0.6)^{0.3}(0.7)^{0.4}(0.5)^{0.3}] >$$

$= < [0.4749, 0.6141], [0.5030, 0.6042] >$.

	$\tilde{e}_1$	$\tilde{e}_2$
$\check{s}_1$	$[0.4749, 0.6141], [0.5030, 0.6042]$	$[0.5069, 0.6065], [0.5578, 0.6581]$
$\check{s}_2$	$[0.7155, 0.8392], [0.2656, 0.3923]$	$[0.7274, 0.8501], [0.2780, 0.3812]$
$\check{s}_3$	$[0.7917, 0.8779], [0.1390, 0.2781]$	$[0.7469, 0.8647], [0.2259, 0.3669]$
$\check{s}_4$	$[0.6354, 0.7755], [0.2781, 0.4676]$	$[0.6, 0.7], [0.4573, 0.5578]$
$\check{s}_5$	$[0.6354, 0.7665], [0.2980, 0.4676]$	$[0.6656, 0.8], [0.3565, 0.5281]$

Table 5.35: Aggregate criteria by Q-ROIVFSEWAO for Q=3

Step 6: Determine the accuracy of each member of the set S in accordance with the evaluations provided by each expert.

	$\tilde{e}_1$	$\tilde{e}_2$
$\check{s}_1$	0.3432	0.4060
$\check{s}_2$	0.5182	0.6102
$\check{s}_3$	0.5985	0.5621
$\check{s}_4$	0.4233	0.4141
$\check{s}_5$	0.4177	0.4997

Table 5.36: Each Member's Accuracy per Each Expert

Step 7: Calculate average accuracy for each element of S,

	$\check{s}_1$	$\check{s}_2$	$\check{s}_3$	$\check{s}_4$	$\check{s}_5$
Average	0.3746	0.5642	0.5803	0.4187	0.4587

Table 5.37: Aggregate criteria by Q-ROIVFSEWAO for Q = 2

Step 8: Non-increasing chain of these averages is

$0.5803 > 0.5642 > 0.4587 > 0.4187 > 0.3746$ which implies that

$$\check{s}_3 > \check{s}_2 > \check{s}_5 > \check{s}_4 > \check{s}_1.$$

Step 9: Hence $\check{s}_3$ is most affected one. That is United States is most affected country by COVID-19 as compared to Itlay, Russia, Turkey or United Kingdom. In accordance to expert opinion about the set of attributes, like total death cases, CFR and TPR.

<u>Q = 5</u>

Step 5: Aggregate criteria by using Q-ROIVFSE weighted averaging Operator.

Here, for $\check{s}_1$ corresponding to $\tilde{e}_1$,

$$= < \left[(1-((1-0.4^5)^{0.3}(1-0.5^5)^{0.4}(1-0.5^5)^{0.3}))^{1/5},\ (1-((1-0.5^5)^{0.3}(1-0.6^5)^{0.4}(1-0.7^5)^{0.3}))^{1/5}\right], [(0.5)^{0.3}(0.6)^{0.4}(0.4)^{0.3}, (0.6)^{0.3}(0.7)^{0.4}(0.5)^{0.3}] >$$

$= < [0.4782, 0.6212], [0.5030, 0.6042] >.$

	$\tilde{e}_1$	$\tilde{e}_2$
$\check{s}_1$	$[0.4782,0.6212], [0.5030, 0.6042]$	$[0.5178,0.6151], [0.5578, 0.6581]$
$\check{s}_2$	$[0.7212,0.8410], [0.2656, 0.3923]$	$[0.7333, 0.8519], [0.2780, 0.3812]$
$\check{s}_3$	$[0.7384,0.8790], [0.1390, 0.2781]$	$[0.7492,0.8790], [0.2259, 0.3669]$
$\check{s}_4$	$[0.6382,0.7770], [0.2781, 0.4676]$	$[0.6, 0.7], [0.4573, 0.5578]$
$\check{s}_5$	$[0.6382,0.7684], [0.2980, 0.4676]$	$[0.6680,0.8], [0.3565, 0.5281]$

Table 5.38: Aggregate criteria by Q-ROIVFSEWAO for Q = 5

Step 6: Determine the accuracy of each member of the set S in accordance with the evaluations provided by each expert.

	$\tilde{e}_1$	$\tilde{e}_2$
$\check{s}_1$	0.1151	0.1514
$\check{s}_2$	0.3132	0.3352
$\check{s}_3$	0.3730	0.3840
$\check{s}_4$	0.2065	0.1599
$\check{s}_5$	0.1992	0.2538

Table 5.39: Each Member’s Accuracy per Each Expert

Step 7: Calculate average accuracy for each element of S,

	$\breve{s}_1$	$\breve{s}_2$	$\breve{s}_3$	$\breve{s}_4$	$\breve{s}_5$
Average	0.1332	0.3242	0.3758	0.1830	0.2265

Table 5.40: Average Accuracy for all Expert

Step 8: Non-increasing chain of these averages is

$0.3758 > 0.3242 > 0.2265 > 0.1830 > 0.1332$ which implies that

$$\breve{s}_3 > \breve{s}_2 > \breve{s}_5 > \breve{s}_4 > \breve{s}_1.$$

Step 9: Hence $\breve{s}_3$ is most affected one. That is United States is most affected country by COVID-19 as compared to Itlay, Russia, Turkey or United Kingdom. In accordance to expert opinion about the set of attributes, like total death cases, CFR and TPR.

Q = 10

Step 5: Aggregate criteria by using Q-ROIVFSE weighted averaging Operator.

Here, for $\breve{s}_1$ corresponding to $\tilde{e}_1$

$$=< \left[(1-((1-0.4^{10})^{0.3}(1-0.5^{10})^{0.4}(1-0.5^{10})^{0.3}))^{1/10}, (1-((1-0.5^{10})^{0.3}(1-0.6^{10})^{0.4}(1-0.7^{10})^{0.3}))^{1/10}\right], \ [(0.5)^{0.3}(0.6)^{0.4}(0.4)^{0.3}, (0.6)^{0.3}(0.7)^{0.4}(0.5)^{0.3}] >$$

$$= < [0.4847, 0.6384], [0.5030, 0.6042] >.$$

	$\tilde{e}_1$	$\tilde{e}_2$
$\breve{s}_1$	$[0.4847, 0.6384], [0.5030, 0.6042]$	$[0.5411, 0.6355], [0.5578, 0.6581]$
$\breve{s}_2$	$[0.7354, 0.8460], [0.2656, 0.3923]$	$[0.7473, 0.8568], [0.2780, 0.3812]$
$\breve{s}_3$	$[0.7450, 0.8818], [0.1390, 0.2781]$	$[0.7556, 0.8818], [0.2259, 0.3669]$
$\breve{s}_4$	$[0.6465, 0.7809], [0.2781, 0.4676]$	$[0.6, 0.7], [0.4573, 0.5578]$
$\breve{s}_5$	$[0.6465, 0.7734], [0.2980, 0.4676]$	$[0.6743, 0.8], [0.3565, 0.5281]$

Table 5.41: Aggregate criteria by Q-ROIVFSEWAO for Q = 10

Step 6: Determine the accuracy of each member of the set S in accordance with the evaluations provided by each expert.

	$\tilde{e}_1$	$\tilde{e}_2$
$\check{s}_1$	0.0097	0.0155
$\check{s}_2$	0.1171	0.1338
$\check{s}_3$	0.1685	0.1725
$\check{s}_4$	0.0488	0.0188
$\check{s}_5$	0.0449	0.0643

Table 5.42: Each Member's Accuracy per Each Expert

Step 7: Calculate average accuracy for each element of S,

	$\check{s}_1$	$\check{s}_2$	$\check{s}_3$	$\check{s}_4$	$\check{s}_5$
Average	0.0126	0.1254	0.1705	0.0338	0.0546

Table 5.43: Average Accuracy for all Expert

Step 8: Non-increasing chain of these averages is

$0.1705 > 0.1254 > 0.0546 > 0.0338 > 0.0126$ which implies that

$$\check{s}_3 > \check{s}_2 > \check{s}_5 > \check{s}_4 > \check{s}_1.$$

Step 9: Hence $\check{s}_3$ is best one. That is United States is most affected country by COVID-19 as compared to Itlay, Russia, Turkey or United Kingdom. In accordance to expert opinion about the set of attributes, like total death cases, CFR and TPR.

<u>Q = 20</u>

Step 5: Aggregate criteria by using Q-ROIVFSE weighted averaging Operator.

Here, for $\check{s}_1$ corresponding to $\tilde{e}_1$,

$= < \left[(1-((1-0.4^{20})^{0.3}(1-0.5^{20})^{0.4}(1-0.5^{20})^{0.3}))^{1/20},\ (1-((1-0.5^{20})^{0.3}(1-0.6^{20})^{0.4}(1-0.7^{20})^{0.3}))^{1/20} \right], [(0.5)^{0.3}(0.6)^{0.4}(0.4)^{0.3}, (0.6)^{0.3}(0.7)^{0.4}(0.5)^{0.3}] >$

$= < [0.4913, 0.6611], [0.5030, 0.6042] >.$

	$\tilde{e}_1$	$\tilde{e}_2$
$\check{s}_1$	[0.4913,0.6611], [0.5030, 0.6042]	[0.5657,0.6606], [0.5578, 0.6581]
$\check{s}_2$	[0.7568,0.8572], [0.2656, 0.3923]	[0.7663,0.8666], [0.2780, 0.3812]
$\check{s}_3$	[0.7590,0.8865], [0.1390, 0.2781]	[0.7681,0.8865], [0.2259, 0.3669]
$\check{s}_4$	[0.6625,0.7871], [0.2781, 0.4676]	[0.6, 0.7], [0.4573, 0.5578]
$\check{s}_5$	[0.6625,0.7817], [0.2980, 0.4676]	[0.6834,0.8], [0.3565, 0.5281]

Table 5.44: Aggregate criteria by Q-ROIVFSEWAO for Q=20

Step 6: Determine the accuracy of each member of the set S in accordance with the evaluations provided by each expert.

	$\tilde{e}_1$	$\tilde{e}_2$
$\check{s}_1$	0.00015	0.00025
$\check{s}_2$	0.02484	0.03097
$\check{s}_3$	0.04694	0.04748
$\check{s}_4$	0.00430	0.00042
$\check{s}_5$	0.00376	0.00601

Table 5.45: Each Member's Accuracy per Each Expert

Step 7: Calculate average accuracy for each element of S,

	$\check{s}_1$	$\check{s}_2$	$\check{s}_3$	$\check{s}_4$	$\check{s}_5$
Average	0.0004	0.02790	0.04721	0.00236	0.00488

Table 5.46: Average Accuracy for all Expert

Step 8: Non-increasing chain of these averages is

$0.04721 > 0.02790 > 0.00488 > 0.00236 > 0.0004$ which implies that

$$\breve{s}_3 > \breve{s}_2 > \breve{s}_5 > \breve{s}_4 > \breve{s}_1.$$

Step 9: Hence $\breve{s}_3$ is most affected one. That is United States is most affected country by COVID-19 as compared to Itlay, Russia, Turkey or United Kingdom. In accordance to expert opinion about the set of attributes, like total death cases, CFR and TPR.

Q-ROIVFSE Weighted Geometric Operator

$$\underline{\omega}\left(\left(h_k, \breve{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}\right) = < \left[\prod_{k=1}^{\breve{m}} (\theta^-_{k(\breve{p},\breve{e})}(\breve{s}))^{\widetilde{W}_k}, \quad \prod_{k=1}^{\breve{m}} (\theta^+_{k(\breve{p},\breve{e})}(\breve{s}))^{\widetilde{W}_k}\right], \left[\left(1 - \prod_{k=1}^{\breve{m}}\left(1 - (\varphi^-_{k(\breve{p},\breve{e})}(\breve{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q},\ \left(1 - \prod_{k=1}^{\breve{m}}\left(1 - (\varphi^+_{k(\breve{p},\breve{e})}(\breve{s}))^{Q}\right)^{\widetilde{W}_k}\right)^{1/Q}\right] >.$$

Q = 2

Step 5: Aggregate criteria by using Q-ROIVFSE weighted geometric Operator.

Here, for $\breve{s}_1$ corresponding to $\tilde{e}_1$,

$= \quad < [(0.4)^{0.3}(0.5)^{0.4}(0.5)^{0.3}, (0.5)^{0.3}(0.6)^{0.4}(0.7)^{0.3}], \left[(1 - ((1-0.5^2)^{0.3}(1 - 0.6^2)^{0.4}(1-0.4^2)^{0.3}))^{1/2},\ (1 - ((1-0.6^2)^{0.3}(1-0.7^2)^{0.4}(1-0.5^2)^{0.3}))^{1/2}\right] >$

$= < [0.4676, 0.5950], [0.5213,0.6222] >.$

	$\tilde{e}_1$	$\tilde{e}_2$
$\breve{s}_1$	$[0.4676, 0.5950], [0.5213,0.6222]$	$[0.4830, 0.5842], [0.5640,0.6645]$
$\breve{s}_2$	$[0.6957, 0.8288], [0.2744,0.4103]$	$[0.7050, 0.8386], [0.3039,0.4021]$
$\breve{s}_3$	$[0.7286, 0.8688], [0.1863,0.3039]$	$[0.7384, 0.8688], [0.2351,0.3736]$
$\breve{s}_4$	$[0.6284, 0.7686], [0.3039,0.4734]$	$[0.6, 0.7], [0.4639,0.5640]$
$\breve{s}_5$	$[0.6284, 0.7584], [0.3039,0.4734]$	$[0.6581, 0.8], [0.3642,0.5337]$

Table 5.47: Aggregate criteria by Q-ROIVFSEWGO for Q=2

Step 6: Determine the accuracy of each member of the set S in accordance with the evaluations provided by each expert.

	$\tilde{e}_1$	$\tilde{e}_2$
$\breve{s}_1$	0.6158	0.6671
$\breve{s}_2$	0.7073	0.7272
$\breve{s}_3$	0.7064	0.7474
$\breve{s}_4$	0.6510	0.6916
$\breve{s}_5$	0.6437	0.7453

Table 5.48: Each Member's Accuracy per Each Expert

Step 7: Calculate average accuracy for each element of S,

	$\breve{s}_1$	$\breve{s}_2$	$\breve{s}_3$	$\breve{s}_4$	$\breve{s}_5$
Average	0.6414	0.7172	0.7269	0.6713	0.6945

Table 5.49: Average Accuracy for all Expert

Step 8: Non-increasing chain of these averages is

$0.7269 > 0.7172 > 0.6945 > 0.6713 > 0.6414$ which implies that

$$\breve{s}_3 > \breve{s}_2 > \breve{s}_5 > \breve{s}_4 > \breve{s}_1.$$

Step 9: Hence $\breve{s}_3$ is most affected one. That is United States is most affected country by COVID-19 as compared to Itlay, Russia, Turkey or United Kingdom. In accordance to expert opinion about the set of attributes, like total death cases, CFR and TPR.

<u>Q = 3</u>

Step 5: Aggregate criteria by using Q-ROIVFSE weighted geometric Operator.

Here, for $\breve{s}_1$ corresponding to $\tilde{e}_1$

$$= \quad < [(0.4)^{0.3}(0.5)^{0.4}(0.5)^{0.3}, (0.5)^{0.3}(0.6)^{0.4}(0.7)^{0.3}], \left[(1-((1-0.5^3)^{0.3}(1-0.6^3)^{0.4}(1-0.4^3)^{0.3}))^{1/3},\ (1-((1-0.6^3)^{0.3}(1-0.7^3)^{0.4}(1-0.5^3)^{0.3}))^{1/3}\right] >.$$

$= < [0.4676, 0.5950], [0.5260, 0.6258] >.$

	$\tilde{e}_1$	$\tilde{e}_2$
$\check{s}_1$	$[0.4676, 0.5950], [0.5260, 0.6258]$	$[0.4830, 0.5842], [0.5654, 0.6656]$
$\check{s}_2$	$[0.6957, 0.8288], [0.2274, 0.4160]$	$[0.7050, 0.8386], [0.3134, 0.4090]$
$\check{s}_3$	$[0.7286, 0.8688], [0.2070, 0.3134]$	$[0.7384, 0.8688], [0.2395, 0.3757]$
$\check{s}_4$	$[0.6284, 0.7686], [0.3134, 0.4749]$	$[0.6, 0.7], [0.4658, 0.5654]$
$\check{s}_5$	$[0.6284, 0.7584], [0.3314, 0.4749]$	$[0.6581, 0.8], [0.3668, 0.5353]$

Table 5.50: Aggregate criteria by Q-ROIVFSEWGO for Q = 3

Step 6: Determine the accuracy of each member of the set S in accordance with the evaluations provided by each expert.

	$\tilde{e}_1$	$\tilde{e}_2$
$\check{s}_1$	0.3517	0.3938
$\check{s}_2$	0.4949	0.5197
$\check{s}_3$	0.5411	0.5626
$\check{s}_4$	0.4200	0.4204
$\check{s}_5$	0.4139	0.4999

Table 5.51: Each Member's Accuracy per Each Expert

Step 7: Calculate average accuracy for each element of S,

	$\check{s}_1$	$\check{s}_2$	$\check{s}_3$	$\check{s}_4$	$\check{s}_5$
Average	0.3728	0.5073	0.5518	0.4202	0.4569

Table 5.52: Average Accuracy for all Expert

Step 8: Non-increasing chain of these averages is

$0.5518 > 0.5073 > 0.4569 > 0.4202 > 0.3728$ which implies that

$$\check{s}_3 > \check{s}_2 > \check{s}_5 > \check{s}_4 > \check{s}_1.$$

Step 9: Hence $\breve{s}_3$ is most affected one. That is United States is most affected country by COVID-19 as compared to Itlay, Russia, Turkey or United Kingdom. In accordance to expert opinion about the set of attributes, like total death cases, CFR and TPR.

Q = 5

Step 5: Aggregate criteria by using Q-ROIVFSE weighted geometric Operator.

Here, for $\breve{s}_1$ corresponding to $\tilde{e}_1$,

$$= \; < [(0.4)^{0.3}(0.5)^{0.4}(0.5)^{0.3}, (0.5)^{0.3}(0.6)^{0.4}(0.7)^{0.3}], \left[(1-((1-0.5^5)^{0.3}(1-0.6^5)^{0.4}(1-0.4^5)^{0.3}))^{1/5},\ (1-((1-0.6^5)^{0.3}(1-0.7^5)^{0.4}(1-0.5^5)^{0.3}))^{1/5}\right] >$$

$$= \; < [0.4676, 0.5950], [0.5354, 0.6333] >.$$

	$\tilde{e}_1$	$\tilde{e}_2$
$\breve{s}_1$	$[0.4676, 0.5950], [0.5354, 0.6333]$	$[0.4830, 0.5842], [0.5686, 0.6680]$
$\breve{s}_2$	$[0.6957, 0.8288], [0.2824, 0.4274]$	$[0.7050, 0.8386], [0.3304, 0.4227]$
$\breve{s}_3$	$[0.7286, 0.8688], [0.2363, 0.3304]$	$[0.7384, 0.8688], [0.2488, 0.3798]$
$\breve{s}_4$	$[0.6284, 0.7686], [0.3304, 0.4782]$	$[0.6, 0.7], [0.4698, 0.5686]$
$\breve{s}_5$	$[0.6284, 0.7584], [0.3304, 0.4782]$	$[0.6581, 0.8], [0.3720, 0.5389]$

Table 5.53: Aggregate criteria by Q-ROIVFSEWGO for Q=5

Step 6: Determine the accuracy of each member of the set S in accordance with the evaluations provided by each expert.

	$\tilde{e}_1$	$\tilde{e}_2$
$\breve{s}_1$	0.1214	0.1434
$\breve{s}_2$	0.2850	0.3032
$\breve{s}_3$	0.3525	0.3617
$\breve{s}_4$	0.1976	0.1641
$\breve{s}_5$	0.1889	0.2518

Table 5.54: Each Member's Accuracy per Each Expert

Step 7: Calculate average accuracy for each element of S,

	$\check{s}_1$	$\check{s}_2$	$\check{s}_3$	$\check{s}_4$	$\check{s}_5$
Average	0.1324	0.2941	0.3571	0.1808	0.2204

Table 5.55: Average Accuracy for all Expert

Step 8: Non-increasing chain of these averages is

$0.3571 > 0.2941 > 0.2204 > 0.1808 > 0.1324$ which implies that

$\check{s}_3 > \check{s}_2 > \check{s}_5 > \check{s}_4 > \check{s}_1$.

Step 9: Hence $\check{s}_3$ is most affected one. That is United States is most affected country by COVID-19 as compared to Itlay, Russia, Turkey or United Kingdom. In accordance to expert opinion about the set of attributes, like total death cases, CFR and TPR.

<u>Q = 10</u>

Step 5: Aggregate criteria by using Q-ROIVFSE weighted geometric Operator.

Here, for $\check{s}_1$ corresponding to $\tilde{e}_1$,

$$= \quad < [(0.4)^{0.3}(0.5)^{0.4}(0.5)^{0.3}, (0.5)^{0.3}(0.6)^{0.4}(0.7)^{0.3}], \left[(1-((1-0.5^{10})^{0.3}(1-0.6^{10})^{0.4}(1-0.4^{10})^{0.3}))^{1/10},\ (1-((1-0.6^{10})^{0.3}(1-0.7^{10})^{0.4}(1-0.5^{10})^{0.3}))^{1/10}\right] >$$

$$= < [0.4676, 0.5950], [0.5545, 0.6501] >.$$

	$\tilde{e}_1$	$\tilde{e}_2$
$\check{s}_1$	$[0.4676, 0.5950], [0.5545, 0.6501]$	$[0.4830, 0.5842], [0.5760, 0.6743]$
$\check{s}_2$	$[0.6957, 0.8288], [0.2897, 0.4495]$	$[0.7050, 0.8386], [0.3566, 0.4482]$
$\check{s}_3$	$[0.7286, 0.8688], [0.2660, 0.3566]$	$[0.7384, 0.8688], [0.2670, 0.3869]$
$\check{s}_4$	$[0.6284, 0.7686], [0.3566, 0.4847]$	$[0.6, 0.7], [0.4784, 0.5760]$
$\check{s}_5$	$[0.6284, 0.7584], [0.3566, 0.4847]$	$[0.6581, 0.8], [0.3815, 0.5493]$

Table 5.56: Aggregate criteria by Q-ROIVFSEWGO for Q =10

Step 6: Determine the accuracy of each member of the set S in accordance with the evaluations provided by each expert.

	$\tilde{e}_1$	$\tilde{e}_2$
$\breve{s}_1$	0.0111	0.0144
$\breve{s}_2$	0.0899	0.1013
$\breve{s}_3$	0.1436	0.1466
$\breve{s}_4$	0.0412	0.0195
$\breve{s}_5$	0.0366	0.0626

Table 5.57: Each Member's Accuracy per Each Expert

Step 7: Calculate average accuracy for each element of S,

	$\breve{s}_1$	$\breve{s}_2$	$\breve{s}_3$	$\breve{s}_4$	$\breve{s}_5$
Average	0.0128	0.0956	0.1451	0.0304	0.0496

Table 5.58: Average Accuracy for all Expert

Step 8: Non-increasing chain of these averages is

$0.1451 > 0.0956 > 0.0496 > 0.0304 > 0.0128$ which implies that

$$\breve{s}_3 > \breve{s}_2 > \breve{s}_5 > \breve{s}_4 > \breve{s}_1.$$

Step 9: Hence $\breve{s}_3$ is most affected one. That is United States is most affected country by COVID-19 as compared to Itlay, Russia, Turkey or United Kingdom. In accordance to expert opinion about the set of attributes, like total death cases, CFR and TPR.

<u>Q = 20</u>

Step 5: Aggregate criteria by using Q-ROIVFSE weighted geometric Operator.

Here, for $\breve{s}_1$ corresponding to $\tilde{e}_1$,

$= \quad < [(0.4)^{0.3}(0.5)^{0.4}(0.5)^{0.3}, (0.5)^{0.3}(0.6)^{0.4}(0.7)^{0.3}], \Big[(1-((1-0.5^{20})^{0.3}(1-0.6^{20})^{0.4}(1-0.4^{20})^{0.3}))^{1/20},\ (1-((1-0.6^{20})^{0.3}(1-0.7^{20})^{0.4}(1-0.5^{20})^{0.3}))^{1/20}\Big] >$

$= < [0.4676, 0.5950], [0.5737, 0.6698] >.$

	$\tilde{e}_1$	$\tilde{e}_2$
$\check{s}_1$	[0.4676, 0.5950], [0.5737,0.6698]	[0.4830, 0.5842], [0.5854,0.6834]
$\check{s}_2$	[0.6957, 0.8288], [0.2947,0.4711]	[0.7050, 0.8386], [0.3767,0.4710]
$\check{s}_3$	[0.7286, 0.8688], [0.2825,0.3767]	[0.7384, 0.8688], [0.2825,0.3930]
$\check{s}_4$	[0.6284, 0.7686], [0.3767,0.4913]	[0.6, 0.7], [0.4876,0.5854]
$\check{s}_5$	[0.6284, 0.7584], [0.3767,0.4913]	[0.6581, 0.8], [0.3900,0.5666]

Table 5.59: Aggregate criteria by Q-ROIVFSEWGO for Q=20

Step 6: Determine the accuracy of each member of the set S in accordance with the evaluations provided by each expert.

	$\tilde{e}_1$	$\tilde{e}_2$
$\check{s}_1$	0.00019	0.00027
$\check{s}_2$	0.01205	0.01525
$\check{s}_3$	0.03090	0.03118
$\check{s}_4$	0.00263	0.00043
$\check{s}_5$	0.00203	0.00589

Table 5.60: Each Member's Accuracy per Each Expert

Step 7: Calculate average accuracy for each element of S,

	$\check{s}_1$	$\check{s}_2$	$\check{s}_3$	$\check{s}_4$	$\check{s}_5$
Average	0.00023	0.01365	0.03104	0.00153	0.00396

Table 5.61: Average Accuracy for all Expert

Step 8: Non-increasing chain of these averages is

$$0.03104 > 0.01365 > 0.00396 > 0.00153 > 0.00023 \text{ which implies that}$$

$$\check{s}_3 > \check{s}_2 > \check{s}_5 > \check{s}_4 > \check{s}_1.$$

Step 9: Hence $\check{s}_3$ is most affected one. That is United States is most affected country by COVID-19 as compared to Itlay, Russia, Turkey or United Kingdom. In accordance to expert opinion about the set of attributes, like total death cases, CFR and TPR.

Conclusion

In case of both Q-ROIVFSE weighted averaging operator and Q-ROIVFSE weighted geometric operator, no matter what value parameter Q is assigned, it gives exactly same results. This implies that orthogonality of membership and non-membership can be increased by increasing value of parameter Q. In the example, it's evident that by increasing value of the parameter Q for a particular aggregation operator expands the range of decision-making information being conveyed, addressing practical decision-making challenges and thereby preventing information distortion.

Example 5.5.4

A comparison of Q-ROIVFSEFWAO as a generalization of IVIFSEFWAO.

Let's denote S as the collection of enterprises, where $\check{s}_1$ represents Dairy farming, $\check{s}_2$ signifies Fish farming, $\check{s}_3$ represents Poultry farming and$\check{s}_4$ stands for Goat farming. In addition, let's denote E as the set of experts, represented as $\tilde{e}_1$, $\tilde{e}_2$ and$\tilde{e}_3$. We can also define P as the set of attributes, where $\check{p}_1$ corresponds to project cost, $\check{p}_2$ pertains to space requirementand$\check{p}_3$ relates to human resource requirements. Then S = $\{\check{s}_1, \check{s}_2, \check{s}_3\}$, E = $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ and P = $\{\check{p}_1, \check{p}_2, \check{p}_3\}$.

When it comes to evaluating various enterprises, three experts assess them and provide their evaluations using Q-ROIVFSESs. To determine the comprehensive evaluation of these experts regarding the enterprises, we aim to utilize the Q-ROIVFSEFWAO and its comparison with IVIFSEFWAO.

Solution

Step 1: Utilize the expert assessments, presented as Q-ROIVFSESs, to derive expert opinions concerning the provided alternatives and criteria.

$(\check{p}_1,\tilde{e}_1)=\{<\check{s}_1,[.3,.4],[.4,.5]>,<\check{s}_2,[.4,.6],[.0,.4]>,<\check{s}_3,[.5,.6],[.0,.1]>,$
$<\check{s}_4,[.2,.3],[.5,.6]>\},$

$(\check{p}_1,\tilde{e}_2)=\{<\check{s}_1,[.2,.3],[.2,.4]>,<\check{s}_2,[.4,.6],[.3,.4]>,<\check{s}_3,[.3,.5],[.3,.4]>,$
$<\check{s}_4,[.5,.7],[.0,.2]>\},$

$(\check{p}_1,\tilde{e}_3)=\{<\check{s}_1,[.5,.7],[.1,.2]>,<\check{s}_2,[.2,.4],[.4,.5]>,<\check{s}_3,[.5,.6],[.2,.3]>,$
$<\check{s}_4,[.5,.6],[.0,.3]>\},$

$(\check{p}_2,\tilde{e}_1)=\{<\check{s}_1,[.3,.5],[.3,.4]>,<\check{s}_2,[.4,.5],[.1,.3]>,<\check{s}_3,[.3,.4],[.4,.5]>,$
$<\check{s}_4,[.7,.8],[.0,.1]>\},$

$(\check{p}_2,\tilde{e}_2)=\{<\check{s}_1,[.4,.6],[.0,.4]>,<\check{s}_2,[.3,.6],[.2,.3]>,<\check{s}_3,[.5,.7],[.0,.1]>,$
$<\check{s}_4,[.4,.6],[.2,.3]>\},$

$(\check{p}_2,\tilde{e}_3)=\{<\check{s}_1,[.4,.6],[.0,.2]>,<\check{s}_2,[.5,.6],[.1,.3]>,<\check{s}_3,[.3,.5],[.2,.4]>,$
$<\check{s}_4,[.2,.6],[.3,.4]>\},$

$(\check{p}_3,\tilde{e}_1)=\{<\check{s}_1,[.2,.4],[.2,.4]>,<\check{s}_2,[.3,.4],[.1,.3]>,<\check{s}_3,[.5,.6],[.1,.3]>,$
$<\check{s}_4,[.4,.5],[.2,.4]>\},$

$(\check{p}_3,\tilde{e}_2)=\{<\check{s}_1,[.4,.5],[.1,.3]>,<\check{s}_2,[.3,.5],[.0,.3]>,<\check{s}_3,[.4,.5],[.3,.4]>,$
$<\check{s}_4,[.5,.6],[.0,.1]>\},$

$(\check{p}_3,\tilde{e}_3)=\{<\check{s}_1,[.3,.6],[.0,.3]>,<\check{s}_2,[.3,.4],[.1,.4]>,<\check{s}_3,[.5,.6],[.0,.2]>,$
$<\check{s}_4,[.4,.5],[.3,.4]>\}.$

Step 2

$\tilde{e}_1$	$\breve{s}_1$	$\breve{s}_2$	$\breve{s}_3$	$\breve{s}_4$
$\breve{p}_1$	$[.3,.4],[.4,.5]$	$[.4,.6],[.0,.4]$	$[.5,.6],[.0,.1]$	$[.2,.3],[.5,.6]$
$\breve{p}_2$	$[.3,.5],[.3,.4]$	$[.4,.5],[.1,.3]$	$[.3,.4],[.4,.5]$	$[.7,.8],[.0,.1]$
$\breve{p}_3$	$[.2,.4],[.2,.4]$	$[.3,.4],[.1,.3]$	$[.5,.6],[.1,.3]$	$[.4,.5],[.2,.4]$

$\tilde{e}_2$	$\breve{s}_1$	$\breve{s}_2$	$\breve{s}_3$	$\breve{s}_4$
$\breve{p}_1$	$[.2,.3],[.2,.4]$	$[.4,.6],[.3,.4]$	$[.3,.5],[.3,.4]$	$[.5,.7],[.0,.2]$
$\breve{p}_2$	$[.4,.6],[.0,.4]$	$[.3,.6],[.2,.3]$	$[.5,.7],[.0,.1]$	$[.4,.6],[.2,.3]$
$\breve{p}_3$	$[.4,.5],[.1,.3]$	$[.3,.5],[.0,.3]$	$[.4,.5],[.3,.4]$	$[.5,.6],[.0,.1]$

$\tilde{e}_3$	$\breve{s}_1$	$\breve{s}_2$	$\breve{s}_3$	$\breve{s}_4$
$\breve{p}_1$	$[.5,.7],[.1,.2]$	$[.2,.4],[.4,.5]$	$[.5,.6],[.2,.3]$	$[.5,.6],[.0,.3]$
$\breve{p}_2$	$[.4,.6],[.0,.2]$	$[.5,.6],[.1,.3]$	$[.3,.5],[.2,.4]$	$[.2,.6],[.3,.4]$
$\breve{p}_3$	$[.3,.6],[.0,.3]$	$[.3,.4],[.1,.4]$	$[.5,.6],[.0,.2]$	$[.4,.5],[.3,.4]$

Table 5.62: Expert Assessments Regarding the Enterprises

Step 3: Let $\widetilde{W}$ $=(0.35, 0.25, 0.4)^{\mathrm{T}}$ be a weight vector of criteria satisfying the normalized condition.

Step 4: Let $\boldsymbol{W}$ $=(0.3, 0.4, 0.3)^{\mathrm{T}}$ be a position weight vector. The role of weight vector is to eliminate the impact of individual perception on overall comprehensive assessment.

Step 5: Aggregate criteria by using IVIFSEFWAO (Q=1) and Q-ROIVFSEFWAO (with Q=2, without loss of generality).

Where IVIFSEFWAO

$$\overline{\omega_f}\left(\left(h_k, \check{P}_k, \tilde{E}_k, 1\right)_{(\theta_k,\, \varphi_k)}\right) =< \left[1 - \prod_{i=1}^{\breve{m}}\left(1 - \underline{\boldsymbol{\theta}}^{-}_{i(\check{p},\tilde{e})}(\check{s})\right)^{\boldsymbol{W}_i},\ 1 - \prod_{i=1}^{\breve{m}}\left(1 - \underline{\boldsymbol{\theta}}^{+}_{i(\check{p},\tilde{e})}(\check{s})\right)^{\boldsymbol{W}_i}\right],\ \left[\prod_{i=1}^{\breve{m}}(\underline{\boldsymbol{\varphi}}^{-}_{i(\check{p},\tilde{e})}(\check{s}))^{\boldsymbol{W}_i},\ \prod_{i=1}^{\breve{m}}(\underline{\boldsymbol{\varphi}}^{+}_{i(\check{p},\tilde{e})}(\check{s}))^{\boldsymbol{W}_i}\right] >.$$

And Q-ROIVFSEFWAO $\overline{\boldsymbol{\omega_f}}$ is given as

$$\overline{\omega_f}\left(\left(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}\right) = < \left[\left(1 - \prod_{i=1}^{\breve{m}}(1 - (\underline{\boldsymbol{\theta}}^{-}_{i(\check{p},\tilde{e})}(\check{s}))^{Q})^{\boldsymbol{W}_i}\right)^{1/Q},\ \left(1 - \prod_{i=1}^{\breve{m}}(1 - (\underline{\boldsymbol{\theta}}^{+}_{i(\check{p},\tilde{e})}(\check{s}))^{Q})^{\boldsymbol{W}_i}\right)^{1/Q}\right],\ \left[\prod_{i=1}^{\breve{m}}(\underline{\boldsymbol{\varphi}}^{-}_{i(\check{p},\tilde{e})}(\check{s}))^{\boldsymbol{W}_i}, \prod_{i=1}^{\breve{m}}(\underline{\boldsymbol{\varphi}}^{+}_{i(\check{p},\tilde{e})}(\check{s}))^{\boldsymbol{W}_i}\right] >,$$

where $\boldsymbol{W}_i$ is position weight vector and $\left(\underline{\boldsymbol{h}}_i, \underline{\check{\boldsymbol{P}}}_i, \underline{\tilde{\boldsymbol{E}}}_i, \mathrm{Q}\right)_{\left(\underline{\boldsymbol{\theta}}_i, \underline{\boldsymbol{\varphi}}_i\right)}$ = $\{< \check{s}, \left[\underline{\boldsymbol{\theta}}^{-}_{i(\check{p},\tilde{e})},\ \underline{\boldsymbol{\theta}}^{+}_{i(\check{p},\tilde{e})}\right], \left[\underline{\boldsymbol{\varphi}}^{-}_{i(\check{p},\tilde{e})}, \underline{\boldsymbol{\varphi}}^{+}_{i(\check{p},\tilde{e})}\right] >: \check{s} \in \mathrm{S}\}$ is the i-th largest of $\breve{m}$ Q-ROIVFSESs $\left(\underline{\underline{h}}_k, \underline{\underline{\check{P}}}_k, \underline{\underline{\tilde{E}}}_k, \mathrm{Q}\right)_{\left(\underline{\theta}_k, \underline{\varphi}_k\right)} = \breve{m}\widetilde{W}_k\left(h_k, \check{P}_k, \tilde{E}_k, \mathrm{Q}\right)_{(\theta_k,\, \varphi_k)}$ which can be ascertained through the application of ranking method of Q-ROIVFSESs such as Score or accuracy function and W_k is weight vector.

Here $\breve{m} = 3$,

IVIFSE Fusion weighted arithmetic Operator

For Q = 1,

Step 5

$$\left(\underline{\underline{h}}_1, \underline{\underline{\check{P}}}_1, \underline{\underline{\tilde{E}}}_1, 1\right)_{\left(\underline{\underline{\theta}}_1, \underline{\underline{\varphi}}_1\right)} = \underline{\mathrm{h}}\left(\underline{\underline{\check{p}}}_1, \underline{\underline{\tilde{e}}}_1\right) = 3(0.35) < \check{s}_1, [.3, .4], [.4, .5] >$$

$$= 1.05 < \check{s}_1, [.3, .4], [.4, .5] >$$

$$= < \check{s}_1, [1 - (1 - .3)^{1.05}, 1 - (1 - .4)^{1.05}][.4^{1.05}, .5^{1.05}] >$$

$$= < \check{s}_1, [.3124, .4151], [.3821, .4830] >.$$

Similarly,

$\underline{\tilde{e}_1}$	$\check{s}_1$	$\check{s}_2$	$\check{s}_3$	$\check{s}_4$
$\underline{\underline{\check{p}_1}}$	$[.3124, .4151]$, $[.3821, .4830]$	$[.4151, .6179]$, $[.0, .3821]$	$[.5170, .6179]$, $[.0, .0891]$	$[.2089, .3124]$, $[.4830, .5849]$
$\underline{\underline{\check{p}_2}}$	$[.2347, .4054]$, $[.4054, .5030]$	$[.3183, .4054]$, $[.1778, .4054]$	$[.2347, .3183]$, $[.5030, .5946]$	$[.5946, .7009]$, $[.0, .1778]$
$\underline{\underline{\check{p}_3}}$	$[.2349, .4583]$, $[.1450, .3333]$	$[.3481, .4583]$, $[.0631, .2358]$	$[.5647, .7]$, $[.0631, .2358]$	$[.4583, .5647]$, $[.1450, .333]$

$\underline{\tilde{e}_2}$	$\check{s}_1$	$\check{s}_2$	$\check{s}_3$	$\check{s}_4$
$\underline{\underline{\check{p}_1}}$	$[.2089, .3124]$, $[.1845, .3821]$	$[.4151, .6179]$, $[.2825, .3821]$	$[.3124, .5170]$, $[.2825, .3821]$	$[.5170, .7175]$, $[.0, .1845]$
$\underline{\underline{\check{p}_2}}$	$[.3183, .4970]$, $[.0, .5030]$	$[.2347, .4970]$, $[.2991, .4054]$	$[.4054, .5946]$, $[.0, .1778]$	$[.3183, .4970]$, $[.2991, .4054]$
$\underline{\underline{\check{p}_3}}$	$[.4583, .5647]$, $[.0631, .2358]$	$[.3481, .5647]$, $[.0, .2358]$	$[.4583, .5647]$, $[.2358, .333]$	$[.5647, .7]$, $[.0, .0631]$

$\underline{\tilde{e}_3}$	$\check{s}_1$	$\check{s}_2$	$\check{s}_3$	$\check{s}_4$
$\underline{\underline{\check{p}_1}}$	$[.5170, .7175]$, $[.0891, .1845]$	$[.2089, .4151]$, $[.3821, .4830]$	$[.5170, .6179]$, $[.1845, .2825]$	$[.5170, .6179]$, $[.0, .2825]$
$\underline{\underline{\check{p}_2}}$	$[.3183, .4970]$, $[.0, .2991]$	$[.4054, .4970]$, $[.1778, .4054]$	$[.2347, .4054]$, $[.2991, .5030]$	$[.1541, .4970]$, $[.4054, .5030]$
$\underline{\underline{\check{p}_3}}$	$[.3481, .7]$, $[.0, .2358]$	$[.3481, .4583]$, $[.0631, .333]$	$[.5647, .7]$, $[.0, .1450]$	$[.4583, .5647]$, $[.2358, .333]$

Table 5.63: Aggregate criteria by using IVIFSEFWAO (Q = 1)

Next, finding the score of each of above element by Definition (4.2.36) as follows:

$\underline{\tilde{e}_1}$	$\check{s}_1$	$\check{s}_2$	$\check{s}_3$	$\check{s}_4$
$\underline{\underline{\check{p}_1}}$	-0.0688	0.3254	0.5229	-0.2733
$\underline{\underline{\check{p}_2}}$	-0.1342	0.0702	-0.2723	0.5585
$\underline{\underline{\check{p}_3}}$	0.1076	0.2538	0.4829	0.2725

$\underline{\tilde{e}_2}$	$\check{s}_1$	$\check{s}_2$	$\check{s}_3$	$\check{s}_4$
$\underline{\underline{\check{p}_1}}$	-0.0226	0.1842	0.0824	0.2021
$\underline{\underline{\check{p}_2}}$	0.1562	0.0136	0.4111	0.0554
$\underline{\underline{\check{p}_3}}$	0.3620	0.3385	0.2271	0.6008

$\underline{\tilde{e}_3}$	$\check{s}_1$	$\check{s}_2$	$\check{s}_3$	$\check{s}_4$
$\underline{\underline{\check{p}_1}}$	0.4804	-0.1206	0.3340	0.4262
$\underline{\underline{\check{p}_2}}$	0.2581	0.1596	-0.0810	-0.1286
$\underline{\underline{\check{p}_3}}$	0.4062	0.2052	0.5599	0.2271

Table 5.64: Score of Each Element

Finding the ith largest of 3 Q-ROIVFSESs,

$\underline{\tilde{e}_1}$	$\check{s}_1$	$\check{s}_2$	$\check{s}_3$	$\check{s}_4$
$\underline{\check{p}_1}$	$\underline{\underline{\check{p}_3}}$ $[.2349, .4583]$, $[.1450, .3333]$	$\underline{\underline{\check{p}_1}}$ $[.4151, .6179]$, $[.0, .3821]$	$\underline{\underline{\check{p}_1}}$ $[.5170, .6179]$, $[.0, .0891]$	$\underline{\underline{\check{p}_2}}$ $[.5946, .7009]$, $[.0, .1778]$
$\underline{\check{p}_2}$	$\underline{\underline{\check{p}_1}}$ $[.3124, .4151]$, $[.3821, .4830]$	$\underline{\underline{\check{p}_3}}$ $[.3481, .4583]$, $[.0631, .2358]$	$\underline{\underline{\check{p}_3}}$ $[.5647, .7]$, $[.0631, .2358]$	$\underline{\underline{\check{p}_3}}$ $[.4583, .5647]$, $[.1450, .333]$

$\underline{\check{p}_3}$	$\underline{\underline{\check{p}_2}}$ $[.2347, .4054]$, $[.4054, .5030]$	$\underline{\underline{\check{p}_2}}$ $[.3183, .4054]$, $[.1778, .4054]$	$\underline{\underline{\check{p}_2}}$ $[.2347, .3183]$, $[.5030, .5946]$	$\underline{\underline{\check{p}_1}}$ $[.2089, .3124]$, $[.4830, .5849]$

$\underline{\tilde{e}_2}$	$\check{s}_1$	$\check{s}_2$	$\check{s}_3$	$\check{s}_4$
$\underline{\check{p}_1}$	$\underline{\underline{\check{p}_3}}$ $[.4583, .5647]$, $[.0631, .2358]$	$\underline{\underline{\check{p}_3}}$ $[.3481, .5647]$, $[.0, .2358]$	$\underline{\underline{\check{p}_2}}$ $[.4054, .5946]$, $[.0, .1778]$	$\underline{\underline{\check{p}_3}}$ $[.5647, .7]$, $[.0, .0631]$
$\underline{\check{p}_2}$	$\underline{\underline{\check{p}_2}}$ $[.3183, .4970]$, $[.0, .5030]$	$\underline{\underline{\check{p}_1}}$ $[.4151, .6179]$, $[.2825, .3821]$	$\underline{\underline{\check{p}_3}}$ $[.4583, .5647]$, $[.2358, .333]$	$\underline{\underline{\check{p}_1}}$ $[.5170, .7175]$, $[.0, .1845]$
$\underline{\check{p}_3}$	$\underline{\underline{\check{p}_1}}$ $[.2089, .3124]$, $[.1845, .3821]$	$\underline{\underline{\check{p}_2}}$ $[.2347, .4970]$, $[.2991, .4054]$	$\underline{\underline{\check{p}_1}}$ $[.3124, .5170]$, $[.2825, .3821]$	$\underline{\underline{\check{p}_2}}$ $[.3183, .4970]$, $[.2991, .4054]$

$\underline{\tilde{e}_3}$	$\check{s}_1$	$\check{s}_2$	$\check{s}_3$	$\check{s}_4$
$\underline{\check{p}_1}$	$\underline{\underline{\check{p}_1}}$ $[.5170, .7175]$, $[.0891, .1845]$	$\underline{\underline{\check{p}_3}}$ $[.3481, .4583]$, $[.0631, .333]$	$\underline{\underline{\check{p}_3}}$ $[.5647, .7]$, $[.0, .1450]$	$\underline{\underline{\check{p}_1}}$ $[.5170, .6179]$, $[.0, .2825]$
$\underline{\check{p}_2}$	$\underline{\underline{\check{p}_3}}$ $[.3481, .7]$, $[.0, .2358]$	$\underline{\underline{\check{p}_2}}$ $[.4054, .4970]$, $[.1778, .4054]$	$\underline{\underline{\check{p}_1}}$ $[.5170, .6179]$, $[.1845, .2825]$	$\underline{\underline{\check{p}_3}}$ $[.4583, .5647]$, $[.2358, .333]$
$\underline{\check{p}_3}$	$\underline{\underline{\check{p}_2}}$ $[.3183, .4970]$, $[.0, .2991]$	$\underline{\underline{\check{p}_1}}$ $[.2089, .4151]$, $[.3821, .4830]$	$\underline{\underline{\check{p}_2}}$ $[.2347, .4054]$, $[.2991, .5030]$	$\underline{\underline{\check{p}_2}}$ $[.1541, .4970]$, $[.4054, .5030]$

Table 5.65: Finding the ith largest of 3 Q-ROIVFSESs

Further aggregate Criteria $(\check{p}_1, \check{p}_2, \check{p}_3)$ by using IVIFSEFWAO.

For instance for $\check{s}_1$in relation to expert $\tilde{e}_1$, we have

$$=< [1-((1-0.2349)^{0.3}(1-0.3124)^{0.4}(1-0.2347)^{0.3}),\ \ 1 - ((1-0.4583)^{0.3}(1-0.4151)^{0.4}(1-0.4054)^{0.3})\ ], [(0.1450)^{0.3}(0.3821)^{0.4}(0.4054)^{0.3}, (0.333)^{0.3}(0.4830)^{0.4}(0.5030)^{0.3}] >$$

$= < [0.2668, 0.4256], [0.2908, 0.4373] >.$

Similarly,

	$\check{s}_1$	$\check{s}_2$	$\check{s}_3$	$\check{s}_4$
$\underline{\tilde{e}_1}$= $\tilde{e}_1$	[0.2668, 0.4256], [0.2908, 0.4373]	[0.3604, 0.5085], [0.0, 0.3206]	[0.4681, 0.5874], [0.0, 0.2324]	[0.4437, 0.5539], [0.0, 0.3266]
$\underline{\tilde{e}_2}$=$\tilde{e}_2$	[0.3347, 0.4710], [0.0, 0.3690]	[0.3450, 0.5685], [0.0, 0.3365]	[0.4016, 0.5604], [0.0, 0.2875]	[0.4804, 0.6580], [0.0, 0.1693]
$\underline{\tilde{e}_3}$= $\tilde{e}_3$	[0.3961, 0.6559], [0.0, 0.2353]	[0.3341, 0.4619], [0.1639, 0.4028]	[0.4625, 0.5942], [0.0, 0.2750]	[0.4017, 0.5628], [0.0, 0.3587]

Table 5.66: Aggregate Criteria ($\check{p}_1, \check{p}_2, \check{p}_3$) by using IVIFSEFWAO

Step 6: Determine the accuracy of each member of the set S in accordance with the evaluations provided by each expert

	$\check{s}_1$	$\check{s}_2$	$\check{s}_3$	$\check{s}_4$
$\tilde{e}_1$	0.70125	0.59475	0.64395	0.6621
$\tilde{e}_2$	0.58735	0.625	0.62475	0.65385
$\tilde{e}_3$	0.64365	0.68135	0.66585	0.6616

Table 5.67: Each Member's Accuracy per Each Expert

Step 7: Finding average accuracy for all expert of each element of S,

	$\check{s}_1$	$\check{s}_2$	$\check{s}_3$	$\check{s}_4$
Average	0.64708	0.6337	0.64485	0.65918

Table 5.68: Average Accuracy for all Expert

Step 8: Non-increasing chain of these averages is 0.65918>0.64708>0.64485>0.6337which implies that$\check{s}_4 > \check{s}_1 > \check{s}_3 > \check{s}_2$.

Step 9: Hence $\check{s}_4$ is best one. That is Goat framing is better as compared to Dairy farming, Fish farming or Poultry farming. In accordance to expert opinion about the set of attributes, like project cost, space requirement and human resource requirements.

Repeating the example with Q-ROIVFSE Fusion weighted arithmetic Operator and taking Q = 2 for comparison of results.

Q-ROIVFSE Fusion weighted arithmetic Operator

For Q = 2,

Step 5

$$\left(\underline{h_1}, \underline{\check{P}_1}, \underline{\tilde{E}_1}, 2\right)_{\left(\underline{\theta_1}, \underline{\varphi_1}\right)} = \underline{\mathrm{h}}\left(\underline{\underline{\check{p}_1}}, \underline{\underline{\tilde{e}_1}}\right) = 3(0.35) < \check{s}_1, [.3, .4], [.4, .5] >$$

$$= 1.05 < \check{s}_1, [.3, .4], [.4, .5] >$$

$$= < \check{s}_1, \left[(1-(1-.3^2)^{1.05})^{1/2}, (1-(1-.4^2)^{1.05})^{1/2}\right][.4^{1.05}, .5^{1.05}] >.$$

$$= < \check{s}_1, [.3070, .4090], [.3821, .4830] >.$$

Similarly,

$\underline{\tilde{e}_1}$	$\breve{s}_1$	$\breve{s}_2$	$\breve{s}_3$	$\breve{s}_4$
$\underline{\underline{\breve{p}_1}}$	$[.3070, .4090]$, $[.3821, .4830]$	$[.4090, .6116]$, $[.0, .3821]$	$[.5106, .6116]$, $[.0, .0891]$	$[.2048, .3070]$, $[.4830, .5849]$
$\underline{\underline{\breve{p}_2}}$	$[.2613, .4405]$, $[.4054, .5030]$	$[.3501, .4405]$, $[.1778, .4054]$	$[.2613, .3501]$, $[.5030, .5946]$	$[.6297, .7316]$, $[.0, .1778]$
$\underline{\underline{\breve{p}_3}}$	$[.2186, .4345]$, $[.1450, .3333]$	$[.3271, .4345]$, $[.0631, .2358]$	$[.5403, .6439]$, $[.0631, .2358]$	$[.4345, .5403]$, $[.1450, .333]$

$\underline{\tilde{e}_2}$	$\breve{s}_1$	$\breve{s}_2$	$\breve{s}_3$	$\breve{s}_4$
$\underline{\underline{\breve{p}_1}}$	$[.2048, .3070]$, $[.1845, .3821]$	$[.4090, .6116]$, $[.2825, .3821]$	$[.3070, .5106]$, $[.2825, .3821]$	$[.5106, .6116]$, $[.0, .1845]$
$\underline{\underline{\breve{p}_2}}$	$[.3501, .5333]$, $[.0, .5030]$	$[.2613, .5333]$, $[.2991, .4054]$	$[.4405, .6297]$, $[.0, .1778]$	$[.3501, .5333]$, $[.2991, .4054]$
$\underline{\underline{\breve{p}_3}}$	$[.4345, .5403]$, $[.0631, .2358]$	$[.3271, .5403]$, $[.0, .2358]$	$[.4345, .5403]$, $[.2358, .333]$	$[.5403, .6439]$, $[.0, .0631]$

$\underline{\tilde{e}_3}$	$\breve{s}_1$	$\breve{s}_2$	$\breve{s}_3$	$\breve{s}_4$
$\underline{\underline{\breve{p}_1}}$	$[.5106, .7120]$, $[.0891, .1845]$	$[.2048, .4090]$, $[.3821, .4830]$	$[.5106, .6116]$, $[.1845, .2825]$	$[.5106, .6116]$, $[.0, .2825]$
$\underline{\underline{\breve{p}_2}}$	$[.3501, .5333]$, $[.0, .2991]$	$[.4405, .5333]$, $[.1778, .4054]$	$[.2613, .4405]$, $[.2991, .5030]$	$[.1736, .5333]$, $[.4054, .5030]$
$\underline{\underline{\breve{p}_3}}$	$[.3271, .6439]$, $[.0, .2358]$	$[.3271, .4345]$, $[.0631, .333]$	$[.5403, .6439]$, $[.0, .1450]$	$[.4345, .5403]$, $[.2358, .333]$

Table 5.69: Aggregate criteria by Q-ROIVFSEFWAO (Q = 2)

Next, finding the score of each of above element by Definition (4.2.36) as follows:

$\underline{\tilde{e}_1}$	$\check{s}_1$	$\check{s}_2$	$\check{s}_3$	$\check{s}_4$
$\underline{\underline{\check{p}_1}}$	-0.07455	0.31925	0.51655	-0.27805
$\underline{\underline{\check{p}_2}}$	-0.1033	0.1037	-0.2431	0.3302
$\underline{\underline{\check{p}_3}}$	0.08755	0.23135	0.44265	0.2484

$\underline{\tilde{e}_2}$	$\check{s}_1$	$\check{s}_2$	$\check{s}_3$	$\check{s}_4$
$\underline{\underline{\check{p}_1}}$	-0.0274	0.178	0.0765	0.47105
$\underline{\underline{\check{p}_2}}$	0.1902	0.04504	0.4462	0.08945
$\underline{\underline{\check{p}_3}}$	0.33795	0.3158	0.203	0.56055

$\underline{\tilde{e}_3}$	$\check{s}_1$	$\check{s}_2$	$\check{s}_3$	$\check{s}_4$
$\underline{\underline{\check{p}_1}}$	0.4745	-0.12565	0.3276	0.41985
$\underline{\underline{\check{p}_2}}$	0.29215	0.1953	-0.05056	-0.10075
$\underline{\underline{\check{p}_3}}$	0.3676	0.18275	0.5196	0.203

Table 5.70: Score of Each Element

Finding the ith largest of 3 Q-ROIVFSESs,

$\underline{\tilde{e}_1}$	$\check{s}_1$	$\check{s}_2$	$\check{s}_3$	$\check{s}_4$
$\underline{\check{p}_1}$	$\underline{\underline{\check{p}_3}}$ $[.2186,.4345],$ $[.1450,.3333]$	$\underline{\underline{\check{p}_1}}$ $[.4090,.6116],$ $[.0,.3821]$	$\underline{\underline{\check{p}_1}}$ $[.5106,.6116],$ $[.0,.0891]$	$\underline{\underline{\check{p}_2}}$ $[.6297,.7316],$ $[.0,.1778]$
$\underline{\check{p}_2}$	$\underline{\underline{\check{p}_1}}$ $[.3070,.4090],$ $[.3821,.4830]$	$\underline{\underline{\check{p}_3}}$ $[.3271,.4345],$ $[.0631,.2358]$	$\underline{\underline{\check{p}_3}}$ $[.5403,.6439],$ $[.0631,.2358]$	$\underline{\underline{\check{p}_3}}$ $[.4345,.5403],$ $[.1450,.333]$
$\underline{\check{p}_3}$	$\underline{\underline{\check{p}_2}}$ $[.2613,.4405],$ $[.4054,.5030]$	$\underline{\underline{\check{p}_2}}$ $[.3501,.4405],$ $[.1778,.4054]$	$\underline{\underline{\check{p}_2}}$ $[.2613,.3501],$ $[.5030,.5946]$	$\underline{\underline{\check{p}_1}}$ $[.2048,.3070],$ $[.4830,.5849]$

$\underline{\tilde{e}_2}$	$\check{s}_1$	$\check{s}_2$	$\check{s}_3$	$\check{s}_4$
$\underline{\check{p}_1}$	$\underline{\underline{\check{p}_3}}$ $[.4345, .5403]$, $[.0631, .2358]$	$\underline{\underline{\check{p}_3}}$ $[.3271, .5403]$, $[.0, .2358]$	$\underline{\underline{\check{p}_2}}$ $[.4405, .6297]$, $[.0, .1778]$	$\underline{\underline{\check{p}_3}}$ $[.5403, .6439]$, $[.0, .0631]$
$\underline{\check{p}_2}$	$\underline{\underline{\check{p}_2}}$ $[.3501, .5333]$, $[.0, .5030]$	$\underline{\underline{\check{p}_1}}$ $[.4090, .6116]$, $[.2825, .3821]$	$\underline{\underline{\check{p}_3}}$ $[.4345, .5403]$, $[.2358, .333]$	$\underline{\underline{\check{p}_1}}$ $[.5106, .6116]$, $[.0, .1845]$
$\underline{\check{p}_3}$	$\underline{\underline{\check{p}_1}}$ $[.2048, .3070]$, $[.1845, .3821]$	$\underline{\underline{\check{p}_2}}$ $[.2613, .5333]$, $[.2991, .4054]$	$\underline{\underline{\check{p}_1}}$ $[.3070, .5106]$, $[.2825, .3821]$	$\underline{\underline{\check{p}_2}}$ $[.3501, .5333]$, $[.2991, .4054]$

$\underline{\tilde{e}_3}$	$\check{s}_1$	$\check{s}_2$	$\check{s}_3$	$\check{s}_4$
$\underline{\check{p}_1}$	$\underline{\underline{\check{p}_1}}$ $[.5106, .7120]$, $[.0891, .1845]$	$\underline{\underline{\check{p}_2}}$ $[.4405, .5333]$, $[.1778, .4054]$	$\underline{\underline{\check{p}_3}}$ $[.5403, .6439]$, $[.0, .1450]$	$\underline{\underline{\check{p}_1}}$ $[.5106, .6116]$, $[.0, .2825]$
$\underline{\check{p}_2}$	$\underline{\underline{\check{p}_3}}$ $[.3271, .6439]$, $[.0, .2358]$	$\underline{\underline{\check{p}_3}}$ $[.3271, .4345]$, $[.0631, .333]$	$\underline{\underline{\check{p}_1}}$ $[.5106, .6116]$, $[.1845, .2825]$	$\underline{\underline{\check{p}_3}}$ $[.4345, .5403]$, $[.2358, .333]$
$\underline{\check{p}_3}$	$\underline{\underline{\check{p}_2}}$ $[.3501, .5333]$, $[.0, .2991]$	$\underline{\underline{\check{p}_1}}$ $[.2048, .4090]$, $[.3821, .4830]$	$\underline{\underline{\check{p}_2}}$ $[.2613, .4405]$, $[.2991, .5030]$	$\underline{\underline{\check{p}_2}}$ $[.1736, .5333]$, $[.4054, .5030]$

Table 5.71: Finding the ith largest of 3 Q-ROIVFSESs

Further aggregate Criteria $(\check{p}_1, \check{p}_2, \check{p}_3)$ by using Q-ROIVFSE fusion weighted arithmetic operator.

For instance, for $\check{s}_1$in relation to expert $\tilde{e}_1$, we have

$= < \left[(1-((1-0.2186^2)^{0.3}(1-0.3070^2)^{0.4}(1-0.2613^2)^{0.3}))^{1/2},\ (1-((1-0.4345^2)^{0.3}(1-0.4090^2)^{0.4}(1-0.4405^2)^{0.3}))^{1/2} \right],$

$[(0.1450)^{0.3}(0.3821)^{0.4}(0.4054)^{0.3}, (0.333)^{0.3}(0.4830)^{0.4}(0.5030)^{0.3}] >$

$= < [0.2697, 0.4264], [0.2908, 0.4373] >.$

Similarly,

	$\check{s}_1$	$\check{s}_2$	$\check{s}_3$	$\check{s}_4$
$\underline{\tilde{e}_1} = \tilde{e}_1$	[0.2697, 0.4264], [0.2908, 0.4373]	[0.3607, 0.5010], [0.0, 0.3206]	[0.4702, 0.5706], [0.0, 0.2324]	[0.4687, 0.5728], [0.0, 0.3266]
$\underline{\tilde{e}_2} = \tilde{e}_2$	[0.3465, 0.4842], [0.0, 0.3690]	[0.3471, 0.5691], [0.0, 0.3365]	[0.4039, 0.5624], [0.0, 0.2875]	[0.4814, 0.6011], [0.0, 0.1693]
$\underline{\tilde{e}_3} = \tilde{e}_3$	[0.4008, 0.6399], [0.0, 0.2353]	[0.3400, 0.4611], [0.1478, 0.3949]	[0.4666, 0.5815], [0.0, 0.2750]	[0.4094, 0.5617], [0.0, 0.3587]

Table 5.72: Aggregate Criteria ($\check{p}_1, \check{p}_2, \check{p}_3$) byQ-ROIVFSEFWAO

Step 6: Determine the accuracy of each member of the set S in accordance with the evaluations provided by each expert.

	$\check{s}_1$	$\check{s}_2$	$\check{s}_3$	$\check{s}_4$
$\tilde{e}_1$	0.2644	0.2419	0.3003	0.3272
$\tilde{e}_2$	0.2453	0.2788	0.2810	0.3109
$\tilde{e}_3$	0.3127	0.2530	0.3157	0.3059

Table 5.73: Each Member's Accuracy per Each Expert

Step 7: Finding average accuracy for all expert of each element of S,

	$\check{s}_1$	$\check{s}_2$	$\check{s}_3$	$\check{s}_4$
Average	0.2741	0.2579	0.2999	0.3147

Table 5.74: Average Accuracy for all Expert

Step 8: Non-increasing chain of these averages is
$0.3147 > 0.2999 > 0.2741 > 0.2579$ which implies that $\breve{s}_4 > \breve{s}_3 > \breve{s}_1 > \breve{s}_2$.

Step 9: Hence $\breve{s}_4$ is best one. That is Goat framing is better as compared to Dairy farming, Fish farming or Poultry farming. In accordance to expert opinion about the set of attributes, like project cost, space requirement and human resource requirements.

Comparison of Q-ROIVFSEFWAO and IVIFSEFWAO

From this example, we find that we can use different methods to get the different sorting results under the same evaluation data. Further the ranking results are slightly different but the optimal ranking results are same, they are all $\breve{s}_4$.

For IVIFSE Fusion weighted arithmetic Operator, its calculation process is simple than Q-ROIVFSE Fusion weighted arithmetic Operator, but its scope of application is very narrow, it can only deal with DMP that evaluation value is expressed as IVIFSES and it may not fully express the real decision information, its membership and non-membership must meet $0 \leq \theta^+ + \varphi^+ \leq 1$ so it will easily cause distortion of the information.

For Q-ROIVFSE Fusion weighted arithmetic Operator propose in this thesis, we can find that they are more flexible to express the fuzzy decision information by Q-ROIVFSESs because they make the information aggregation process more flexible by a parameter Q.

By increasing the value of parameter Q, the scope of the expressed decision-making information will be wider, thus avoiding the distortion of information. So, this method is more suitable for practical decision making problem.

CHAPTER 6

CONCLUSION AND FUTURE WORK

The research aimed to improve decision-making in uncertain situations. It introduced a new method, Q-ROIVFSES, to handle real-life problems in decision- goal was to understand how Q-ROIVFSES works and to create new ways to use it effectively. This involved combining ideas from different methods, like IVIFSES and Q-ROFS, to build the Q-ROIVFSES framework. The research brought a fresh and improved approach to decision-making. Q-ROIVFSES isn't just a fix for existing problems; it's a step forward in handling decision-making challenges, especially in uncertain situations.

6.1 Research Contribution

The chapter proceeds to delineate the pivotal contributions made throughout the course of the thesis. This section offers a succinct summary of the two principal objectives and the concrete outcomes achieved:

1. **Investigated Structure and Framework of Q-ROIVFSESs:**
   Investigated the detailed structure and mathematical foundation of Q-ROIVFSESs by combining IVIFSESs with Q-ROFSs. The objective was to enhance decision-making by creating a more resilient tool.
2. **Defined Novel Operations for Q-ROIVFSESs:**
   Suggested creative operations for Q-ROIVFSESs, incorporating aggregation operators. Formulated Q-ROIVFSES-based algorithms to enhance the efficiency of multiple criteria decision making processes (MCDMP).

6.2 Effectiveness of Q-ROIVFSES in MCDMP

A significant highlight of the research lies in the application of Q-ROIVFSESs to real-world scenarios, particularly demonstrated through three insightful examples.

1. **Comparison of Various Q-ROIVFSE Aggregation Operators:**
   Different operators (Q-ROIVFSEWAO, Q-ROIVFSEOWAO, Q-ROIVFSEFWAO, Q-ROIVFSEWGO, Q-ROIVFSEOWGO, Q-ROIVFSEFWGO)yielded similar results, showcasing the robustness and consistency of the Q-ROIVFSE framework. The study highlighted the importance of considering various operators for a comprehensive assessment.
2. **Orthogonality Influence on Q-ROIVFSE Operators:**
   By investigate the influence of parameter Q on Q-ROIVFSE weighted averaging and geometric operators. It is observed that both operators yielded identical results irrespective of the assigned Q values, indicating that increasing Q enhances the orthogonality of membership and non-membership, preventing information distortion. This underscored the flexibility introduced by parameter Q, expanding the decision-making information range and addressing practical challenges.
3. **Comparison of Aggregation Operators and Flexibility of Q:**
   Comparison different Q-ROIVFSE aggregation operators and analyzing the flexibility of parameter Q produced different sorting results under the same evaluation data. The flexibility of Q-ROIVFSE operators, especially in expressing fuzzy decision information, was evident. Adjusting parameter Q widened the scope of expressed decision-making information, preventing information distortion and making the method suitable for practical decision-making.

6.4 Future Work

Several potential future directions are suggested below based on the completed research.

- ❖ Investigate additional applications of Q-ROIVFSESs in diverse fields beyond the explored domains.
- ❖ Explore further enhancements to aggregation operators and algorithms for Q-ROIVFSESs to continually improve decision-making efficiency.
- ❖ Furthermore, other aggregation operators can be defined for the proposed structure with the help of triangular norm (t-norm) and conorm.
- ❖ In-depth examine distance and entropy measures related to the Q-ROIVFSESs.
- ❖ Consider the integration of emerging technologies or methodologies to enhance the applicability and adaptability of the Q-ROIVFSES framework.
- ❖ Examine the feasibility of implementing Q-ROIVFSESs in real world decision making scenarios and evaluate their effectiveness in practical applications.

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