$\acute{q}$-Extension of Starlike Functions with respect to Symmetric points Subordinated with $\acute{q}$-sine Function

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Candidate of Master of Science in Mathematics at the National University of Modern Languages do hereby declare that the thesis q́-Extension of Starlike Functions with respect to Symmetric points Subordinated with q́-sine Function submitted by me in partial fulfillment of MSMA degree, is my original work and has not been submitted or published earlier. I also solemnly declare that it shall not, in the future, be submitted by me for obtaining any other degree from this or any other university or institution. I also understand that if evidence of plagiarism is found in my thesis/dissertation at any stage, even after the award of a degree, the work may be canceled and the degree revoked.

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ABSTRACT

Title: q́-Extension of Starlike Functions with respect to Symmetric points Subordinated with q́-sine Function

This thesis aims to introduce and characterize novel subclasses of univalent functions within the open unit disk. The utilization of q-calculus will be employed to establish the q-extension of starlike and convex functions related to symmetric points. Additionally, we will investigate notable properties, including bounds on the coefficients of analytic functions, the Zalcman functional, and the Fekete–Szegő inequality. Furthermore, we will explore upper bounds on Hankel Determinants for functions belonging to these newly defined classes. It will be shown that newly obtained results are advanced as compare to the already derived results by numerous researchers in the field of Geometric Function Theory. The special cases of newly derived results will be presented in the form of corollaries.

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LIST OF SYMBOLS

Ω	-	Open unit disk
$\mathscr{A}$	-	Class of Analytic functions
$\mathscr{S}$	-	Class of Univalent functions
$\mathscr{P}$	-	Class of Carathéodory functions
S^*	-	Class of Starlike functions
C	-	Class of Convex functions
$\prec$	-	Subordination symbol
S_s^*	-	Class of Starlike functions with respect to symmetric points
C_s	-	Class of Convex functions with respect to symmetric points
$S_s^*(\acute{q})$	-	Class of Starlike functions Subordinated with q́-sine function
$S_s^*(\acute{q}-sin)$	-	Class of q́-Starlike functions Subordinated with q́-sine function
$C_s(\acute{q}-sin)$	-	Class of q́-Convex functions Subordinated with q́-sine function
$\mathscr{D}_{\acute{q}}$	-	q-Derivative operator symbol
$\mathscr{H}$	-	Hankel Determinant symbol

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In the name of Allah, the Most Gracious and Most Merciful, I begin this acknowledgment with the verse, " You Alone we Worship; You Alone we ask for Help." I am deeply grateful to Allah, the Most Wise, for His countless blessings that have guided me through this academic journey. I am reminded of the Hadith of the Prophet Muhammad (peace be upon him) who said, "Seek knowledge from the cradle to the grave."

I would like to express my deep appreciation to my family for their unwavering support, to my teachers for their invaluable guidance, for it is through your collective encouragement and inspiration that I stand here today, humbly acknowledging the blessings and opportunities Allah has bestowed upon me.

May Allah accept our efforts and guide us on the path of righteousness and wisdom.

DEDICATION

I dedicate this thesis to my parents, whose boundless love and sacrifices, and to my teachers, whose guidance and wisdom, have illuminated my path and made this achievement possible.

CHAPTER 1

INTRODUCTION

1.1 Historical background

In contemporary works on the history of mathematics, there is a prevalent theme of contrasting the key pioneers of Geometric Function Theory, namely Cauchy, Riemann, and Weierstrass, see [1]. Remmert's perspective highlights that during the 19th century, these three mathematicians played pivotal roles in shaping the modern theory of complex functions, specifically complex analysis. What distinguishes their contributions are their unique approaches to elucidating the concept of holomorphic functions, with each of them offering distinct perspectives and methodologies in tackling this fundamental aspect of mathematical theory.

Remmert states that Cauchy's function theory is based on his famous integral theorem and the residue concept, which enable Cauchy to attend to holomorphic functions in terms of integral representations. Departing from this perspective, Riemann's geometric point of view allowed him to work with holomorphic functions via mappings between domains in the complex plane. Finally, Remmert reports that Weierstrass theory of complex functions is the theory of holomorphic functions developed locally into convergent power series.

Cauchy the initial contributor among the trio of function theory pioneers mentioned earlier, in complex theory at the age of twenty-five, presented over 200 subsequent papers in this domain, for detail see [2]. He introduced the concept of the definite integral with complex limits and established the Cauchy Integral Theorem. In this work, he explored the power series expansion of an analytic function and introduced the Cauchy Integral Formulas. Starting from 1840, Cauchy received support from fellow French mathematicians to lay the foundations of function theory. Notably, Laurent in 1843, discovered the Laurent expansion of an analytic function near an isolated singularity, see [3]. Liouville formulated various theorems related to elliptic functions, and Puiseux studied in his significant paper on algebraic functions, investigated the behavior of these functions around their branch points, for detail see [4]. The collective findings of these mathematicians were systematically compiled for the first time by Briot and Bouquet in a series of articles, for detail see [5, 6, 7].

Riemann, second prominent figure in the development of function theory, made significant contributions during his time at Göttingen a few decades later. In his well-known dissertation, and his renowned articles on Abelian functions, for detail see [8]. Riemann, following Cauchy's lead from 1851 onward, based his definition of an analytic function on the Cauchy-Riemann differential equations, expressed as $f(x+iy) = u+iv$. Riemann establishes that u and v are potential functions and that a conformal mapping of the x, y-plane to the u, v-plane is effected via the analytic function f. In his exploration, Riemann delved into determining the minimal conditions necessary to define such a function. This investigation led him to formulate the well-known Riemann Mapping Theorem, for detail see [9]. This theorem states that if U is a non-empty simply connected open subset of the complex number plane $\mathbb{C}$ (excluding the entire $\mathbb{C}$ itself), then there exists a bijective holomorphic mapping, with a holomorphic inverse, from U onto the open unit disk. Key elements of Riemann's approach include the Dirichlet Principle and Riemann surfaces. When considering the mathematician who most profoundly influenced Riemann in shaping his function theory, Gauss stands out. Gauss had already grasped essential concepts of function theory, including complex integration, the Cauchy Integral Theorem, and contributed significantly to the theory of conformal mapping.

Weierstrass, the third among the founders of function theory, laid the foundation for his later function theory in three papers. In these works, crucial aspects of his later developments are anticipated. Notably, Weierstrass presented a proof of the Laurent Theorem, preceding Laurent's discovery and independent of Cauchy. Other key contributions in these papers include the formulation of the Cauchy Estimates, the introduction of the concept of uniform convergence, the definition of analytic functions through power series, and the establishment of the principle of analytic continuation. Various authors have delved into the origins of Weierstrassian function theory, see [10].

Geometric Function Theory encompasses various classes and subclasses, with a crucial focus on determining coefficient bounds. Within this framework, functions are categorized into different subfamilies belonging to the normalized analytic functions of class $\mathscr{A}$. A notable theorem in this context is the Bieberbach theorem, originally formulated by the German mathematician Ludwig Bieberbach in 1916. This theorem specifically for the class $\mathscr{S}$, which consists of univalent functions. He calculated the second coefficient $\hat{\alpha}_2$ of functions of class $\mathscr{S}$, which is class of univalent functions. This theorem is the source of Bieberbach's conjecture, which led to significant advancement in the field and was frequently pursued in attempts to prove.

The well-known coefficient conjecture for the function τ, of functions of class $\mathscr{S}$, stated as, if $\tau \in \mathscr{S}$ the coefficients of function τ satisfy this relation $|\check{c}_m| \leq m$ for $m \in \{2,3,4,\dots\}$. He proved that $|\check{c}_2| \leq 2$ with equality if only if the function τ was the Koebe function or a rotation of it. The Bieberbach conjecture is straightforward to state, yet it has long been a challenge for many mathematicians. Numerous people have attempted to solve it in vain but have developed alternate strategies that are now widely utilized in the field.

Mathematicians have made numerous attempts to prove this hypothesis, but it has remained a difficult problem to solve. In 1923, the mathematician Karl Loewner proved that $|\check{c}_3| \leq 3$, see [11]. This proof opened the door for others, to prove this result for the general case. More than 30 years passed, there were no progress, until in 1955, the Bieberbach conjecture was proved the first time for $m = 4$, that is, $|\check{c}_4| \leq 4$, by Gangadharan *et al.* [12].

The general form of the Bieberbach conjecture was successfully proved in 1985 by mathematician Louis de Branges, see [13]. He developed a prolonged, complex, but accurate proof of this conjecture. At an international conference held at Purdue in March 1985, De Branges' achievement was highlighted and a number of fresh research questions and directions were put forth.

The Fekete–Szegő inequality, linked to the Bieberbach conjecture, it is primarily used in complex analysis and concerns the coefficients of a polynomial with certain properties, discovered by Fekete and Szegő in 1933, for detail see [14].

The Fekete-Szegő inequality has several important consequences and applications in complex analysis. For example, it can be used to derive bounds on the coefficients of functions in certain subclasses of analytic functions, such as the class of starlike or convex functions. It is worth noting that the Fekete-Szegő inequality is sharp, meaning that there exist functions for which the inequality becomes an equality. These functions are known as extremal functions and play a significant role in understanding how analytic functions behave in the unit disk.

1.2 Preface

Aim of this thesis is to review and define some sub-classes of analytic function through the application of the subordination concept. It is structured into five chapters, and a brief introduction to each chapter is provided as follows:

In **Chapter 2**, a comprehensive literature review is presented, focusing on key concepts within the classes of Geometric Function Theory. This exploration encompasses the class of analytic functions, the class of Carathéodory functions, and the class of univalent functions, along with the examination of relevant subclasses. These concepts are the foundation of this thesis.

Chapter 3 primarily centers on essential elements of Geometric Function Theory, laying a crucial foundation for the chapters that follow. It commences by exploring the notions of analytic functions and normalized univalent functions within the open unit disk, followed by the definition of several fundamental subclasses of univalent functions. The chapter concludes with the presentation of preliminary lemmas, which will be applied in subsequent chapters. It is noteworthy that this chapter does not introduce any novel findings; rather, it comprehensively cites and acknowledges well-established concepts in the field.

Chapter 4 involves an examination of the category of starlike functions concerning symmetric points and introduces the category of convex functions related to symmetric points. Additionally, certain main results are investigated. It is crucial to emphasize that the review work is properly cited.

Chapter 5 focuses on a specific subclass of univalent functions, namely, the category of starlike functions with respect to symmetric points associated with the q́-sine function. The chapter also deduces established findings for functions within this class. Through corollaries, it is demonstrated that the results newly derived align with those previously established by other researchers.

Chapter 6 introduces two subclasses, namely q́-starlike and q́-convex with respect to symmetric points, associated with the trigonometric q́-sine function. For these defined classes several results are examined. Corollaries are presented to demonstrate the equivalence of the newly obtained results with those previously established by other researchers.

CHAPTER 2

LITERATURE REVIEW

2.1 Overview

The geometric properties of analytic functions are the priority of "Geometric Function Theory," a group of complex analysis. Cauchy, Riemann, and Weierstrass made significant contributions to the foundation of contemporary function theory, see [1]. It was established in the early 1900s and is currently one of the most active areas of research today. The Riemann mapping theorem is a result that Bernhard Riemann presented in 1851, see [9]. The conclusion of this result enables us to use open unit disk $\Omega = \{|\hat{z}| < 1; \hat{z} \in \mathbb{C}\}$ as a domain rather than a complex arbitrary domain.Being the cornerstone of the theory of geometric functions, this theorem is significant.

Geometric Function Theory is categorized into various classes, further divided into subclasses based on the characteristics of their image domains and other geometric properties. One of these classes is represented by normalized analytic functions, symbolized as $\mathscr{A}$.

Functions that are analytic in disk Ω and normalized by these axioms $\tau(0) = 0$, $\tau'(0) = 1$, are part of this classification. If τ and ξ are in class $\mathscr{A}$ of functions, we say that, the function τ is said to be subordinated to function ξ, symbolically express as $\tau \prec \xi$, if $\tau(\hat{z}) = \xi(\omega(\hat{z}))$ where $\omega(\hat{z})$ is analytic in the unit disk, satisfying these two conditions $\omega(0) = 0$ with $|\omega(\hat{z})| \leq 1$, for more details see [15].

The functions that are univalent, normalized by these conditions $\tau(0) = 0$, $\tau'(0) = 1$ and analytic, in an open unit disk, contained in class $\mathscr{S}$ function. If ξ is univalent in open unit disk, and τ is analytic in open unit disk then we have the following equivalence relation of function τ and function ξ, τ is subordinated to function ξ, expressed as, $\tau \prec \xi \iff \tau(0) = \xi(0)$, $\tau(\omega) \subseteq \xi(\omega)$. In 1907, Koebe conducted a study on univalent functions, specifically focusing on the examination of univalent analytic functions within the disk ω, see [16].

The major subdivision of family $\mathscr{S}$ function are S^* (collection of Starlike functions), C (collection of convex functions), K (collection of close-to-convex functions), C^* (collection of quasi-convex functions) for detail see [17]. This classification was started when the attempts to prove the Bieberbach conjecture were made. In 1915, Alexander linked two classes S^* Starlike univalent functions and C convex univalent functions through a connection known as Alexander relation, see [18], that can be stated, if $\tau \in \mathscr{A}$ then $\tau \in C \iff \hat{z}\tau' \in S^*$.

Ma and Minda [19] defined the class of starlike functions by using subordination, and studied classes of starlike functions defined as,

$$S^* = \left\{ \tau \in \mathscr{A} : \frac{\hat{z}\tau'(\hat{z})}{\tau(\hat{z})} \prec \delta(\hat{z}), \quad \hat{z} \in \Omega \right\}.$$

Moreover, the class of convex functions described as,

$$C = \left\{ \tau \in \mathscr{A} : \frac{(\hat{z}\tau'(\hat{z}))'}{\tau'(\hat{z})} \prec \delta(\hat{z}), \quad \hat{z} \in \Omega \right\},$$

where $\delta(\hat{z})$ is

$$\delta(\hat{z}) = \left(-1 + \frac{2}{1 - \hat{z}} \right),$$

satisfy Schwaz function in disk Ω.

Owa *et al.* [20] studied the sub-classes of analytic function, $S^*(\check{a})$ class of starlike function of order $\check{a}$, for $0 \leq \check{a} < 1$, defined as,

$$S^*(\check{a}) = \left\{ \tau \in \mathscr{A} : \mathrm{Re}\left(\frac{\hat{z}\tau'(\hat{z})}{\tau(\hat{z})} \right) > \check{a}, \quad \hat{z} \in \Omega \right\},$$

$C(\check{a})$ class of convex function of order $\check{a}$, for $0 \leq \check{a} < 1$, defined as,

$$C = \left\{ \tau \in \mathscr{A} : \mathrm{Re}\left(\frac{(\hat{z}\tau'(\hat{z}))'}{\tau'(\hat{z})} \right) > \check{a}, \quad \hat{z} \in \Omega \right\}.$$

He found multiple sufficient conditions for the starlikeness and convexity of different analytic functions. Several relations, hadamard products, coefficient estimates, distortion theorems, and covering theorems were investigated by Liu *et al.* [21] for each of their defined classes. In addition, some novel distortion theorems for the Srivastava-Saigo-Owa fractional integral operator were discovered. Some of the results presented in this paper were generalised versions of earlier authors' findings.

Gangadharan *et al.* [22] investigated the radii of convexity and strong starlikeness for classes of analytic functions that are analytic on the disk region Ω. The radius of convexity of order β of uniformly convex and starlikeness functions are computed, as well as the radii of strong starlikeness of certain classes of analytic functions. Ali *et al.* [23] estimated the coefficients of a normalized analytic functions which are analytic in disk Ω, where inverse of the function also exists.

The family of functions S_e^* defined by Mendiratta *et al.* [24], in 2014. The class of starlike function which is subordinated to the exponential function that satisfies Schwarz function, given as,

$$S_e^* = \left\{ \tau \in \mathscr{A} : \frac{\hat{z}\tau'(\hat{z})}{\tau(\hat{z})} \prec e^{\hat{z}}, \quad \hat{z} \in \Omega \right\}.$$

Similarly, the class of convex function which is suordinated to the exponential function, that is C_e ,defined as,

$$C_e = \left\{ \tau \in \mathscr{A} : \frac{(\hat{z}\tau'(\hat{z}))'}{\tau'(\hat{z})} \prec e^{\hat{z}}, \quad \hat{z} \in \Omega \right\}.$$

Concerning univalent functions that are analytic within the disk Ω, various aspects such as structural formulas, inclusion relations, coefficient estimates, growth and distortion results, subordination theorems, and various radii constants have been explored. A more recent study conducted in 2018 by Zhang *et al.* [25], investigated the upper bound for the Third Hankel determinant within the aforementioned class.

The class of starlike functions related to symmetric points whose real portion is positive was first described and investigated by Sakaguchi [26], which described here,

$$S_s^* = \left\{ \tau \in \mathscr{A} : Re\left(\frac{2\hat{z}\tau'(\hat{z})}{\tau(\hat{z}) - \tau(-\hat{z})} \right) > 0, \quad \hat{z} \in \Omega \right\},$$

The category of convex functions and odd functions are included in the class of functions that are univalent starlike related to symmetric points. It was further demonstrated that, like convex functions, the nth coefficient of functions in this class is constrained by 1.

Furthermore, Das and Singh [27] proposed a class of convex functions for symmetric points and discovered that the m coefficient of these functions is constrained by $1/m$ for $m \geq 2$. The Hankel Determinant of order three of Starlike function regard to the symmetric points is $5/2$, while the Hankel Determinant of order three of Convex function with respect to the symmetric points is $19/135$, according to Krishna *et al.* [28], who also demonstrated that these bounds were not sharp. Kumar *et al.* [29] improved the bounds of third Hankel Determinant by using the concepts of subordination.

Motivated by their research objectives, Ganesh *et al.* [30], examined functions characterized by both starlike and convex properties. This study specifically focused on functions associated with symmetric points that are subordinated to exponential functions.

$$\frac{2\hat{z}\tau'(\hat{z})}{\tau(\hat{z})-\tau(-\hat{z})} \prec e^{\hat{z}}, \quad \hat{z} \in \Omega,$$

a category of starlike functions of symmetric points subordinated to exponential function. Whereas, the class of convex functions of symmetric points subordinated to exponential function is,

$$\frac{2[\hat{z}\tau'(\hat{z})]'}{[\tau(\hat{z})-\tau(-\hat{z})]'} \prec e^{\hat{z}}, \quad \hat{z} \in \Omega,$$

Also, the symmetric points connected to exponential functions' starlike and convex functions were studied for a possible upper bound on the third-order Hankel determinant. Third Hankel determinant for the function S_e^* is $|\mathscr{H}_3(1)| \leq 0.618$, and the third Hankel determinant for the function C_e is $|\mathscr{H}_3(1)| \leq 0.0338$.

Shi *et al.* [31] found the bounds of third-order Hankel determinant, for the certain categories of starlike and convex univalent functions associated with exponential functions, in an open unit disk. Recently, Joshi *et al.* [32] determined the third-order Hankel determinant for the starlike functions associated with exponential function, in open unit disk. He obtained a new expression for the fourth coefficient of Carathéodory functions then obtained the sharp bound for third-order Hankel determinant.

In 2019, Cho *et al.* [33] introduced the class of starlike functions subordinated to particular trigonometric function such as sine function, which is defined as,

$$S_{sin}^* = \left\{\tau \in \mathscr{A} : \frac{\hat{z}\tau'(\hat{z})}{\tau(\hat{z})} \prec (1+\sin(\hat{z})), \quad \hat{z} \in \Omega\right\}.$$

Similarly, the class of Convex functions subordinated to particular trigonometric function such as sine function, which is defined as,

$$C_{sin} = \left\{\tau \in \mathscr{A} : \frac{(\hat{z}\tau'(\hat{z}))'}{\tau'(\hat{z})} \prec (1+\sin(\hat{z})), \quad \hat{z} \in \Omega\right\}.$$

Also, researchers looked into the geometric characteristics, starlikeness, and convexity coefficients of functions inside the specified Class. Also, other geometrically defined classes and the Janowski starlike function class' radius were determined.

Kuroki *et al.* [34] derived new estimates for the coefficients of functions of the familiar classes such as Starlike function of order α and convex function of order α in the disk Ω. He concluded that the coefficient estimates of defined functions in each of these defined classes totally depend upon the second coefficient of these functions.

An essential tool for tackling complex and challenging information is quantum theory. It is known as ordinary calculus without notion of limits. This mathematical area is very interesting. Moreover, it is crucial to many areas of physics, including cosmic strings and black holes; for more information, see [35]. The q-calculus and the h-calculus are the 2 different types of quantum calculus. Here, h denotes Planck's constant whereas q denotes quantum. The theory of q-calculus and its applications in a variety of fields have attracted the curiosity of researchers.

Euler historically acquired the fundamental q-calculus formulae in the seventeenth century. Jackson was one of the pioneering scientists to define the theories of q-derivative and q-integral in 1909, for further information, see [36]. Basic classical calculus without the concept of limits is really what q-calculus is, q-calculus is developing quickly due to the variety of applications it has in mathematics, mechanics, and physics. Ernst [37] noted that physicists represent a majority of q-calculus users. The exact solutions to several models were first presented in statistical mechanics by Baxter [38]. Several q-heat and q-wave equations were resolved by Bettaibi and Mezlini [39]. Other writers in the literature have also presented a number of interesting results in this field of study, for more detail see [40, 41].

Many subclasses of the class of analytic functions are investigated and studied in great detail with the help of the operator $\mathscr{D}_q$. Ismail *et al.* [42] transformed the set of starlike functions into a q-analogue known as the set of q-starlike functions to carry out the early work of q-calculus in the field of geometric function theory. In addition to exploring the well-known Fekete-Szegő Inequality, he defined the family of q-starlike functions connected to a certain trigonometric function, such as sine functions. Then, a number of previously established convolution findings were used to demonstrate the required and sufficient requirements for the given class. Other topics included the extreme point theorem, growth and distortion bounds, and starlikeness radii.

Ramachandran *et al.* [43] defined q-starlike and q-convex class which is defined as, a function is said to be q-starlike with respect to symmetric points, $S^*_{q,s}(\phi)$ if,

$$\frac{2\hat{z}\mathscr{D}_q\tau(\hat{z})}{\tau(\hat{z})-\tau(-\hat{z})} \prec \phi(\hat{z}).$$

And a function is said to be q-convex with respect to symmetric points, $C_{q,s}(\phi)$ if,

$$\frac{2\mathscr{D}_q(\hat{z}\mathscr{D}_q\tau(\hat{z}))}{\mathscr{D}_q(\tau(\hat{z})-\tau(-\hat{z}))} \prec \phi(\hat{z}),$$

where $\hat{z} \in \Omega$ and $\phi(\hat{z})$ is a Mobius function.

Different researchers made significant contributions in this direction by observing a number of practical characteristics for a new classification of meromorphic multivalent starlike functions described by a redefined q-linear differential operator, for detail see [44, 45]. Sufficiency characteristics, distortion bounds, coefficient estimates, radius of starlikness, and radius of convexity were among these features.

The Hankel determinant is the determinant of the corresponding Hankel matrix. Pommerenke [46] defined the Hankel determinant for the class of univalent functions, for positive integers n,s that defined below,

$$\mathscr{H}_s(n) = \begin{bmatrix} \breve{c}_n & \breve{c}_{n+1} & \breve{c}_{n+2} & \cdots & \breve{c}_{n+s-1} \\ \breve{c}_{n+1} & \breve{c}_{n+2} & \breve{c}_{n+3} & \cdots & \breve{c}_{n+s} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \breve{c}_{n+s-1} & \breve{c}_{n+s} & \breve{c}_{n+s+1} & \cdots & \breve{c}_{n+2s-2} \end{bmatrix}.$$

Babalola [47] was the first person who studied the upper bound of $\mathscr{H}_3(1)$ for subclasses of univalent functions, where

$$\mathscr{H}_3(1) = \begin{bmatrix} \breve{c}_1 & \breve{c}_2 & \breve{c}_3 \\ \breve{c}_2 & \breve{c}_3 & \breve{c}_4 \\ \breve{c}_3 & \breve{c}_4 & \breve{c}_5 \end{bmatrix}.$$

His work for the well-known classes of starlike and convex functions in the disk Ω. Hankel determinant of starlike function is $|\mathscr{H}_3(1)| \leq 16$, where as Hankel determinant of convex function is $|\mathscr{H}_3(1)| \leq \frac{15}{24}$.

The qth Hankel determinant has been explored by Noonan and Thomas [48]. Janteng *et al.* [49] investigated the Hankel Determinant for the starlike and convex functions. Hankel Determinant, $|\mathscr{H}_2(2)| = |\breve{c}_2\breve{c}_4 - \breve{c}_3^2|$ for the starlike class is $|\breve{c}_2\breve{c}_4 - \breve{c}_3^2| \leq 1$ whereas for the convex class is$|\breve{c}_2\breve{c}_4 - \breve{c}_3^2| \leq \frac{1}{8}$, the obtained results were sharp.

Arif *et al.* [50] investigated the qth Hankel determinant for particular subclasses of analytic functions, in his work he estimated the growth rate of the Hankel determinant of analytic function. This determinant was studied by many authors, that is, Noor [51], Pommerenke

[52] studied the Hankel determinant for univalent functions. In literature, many researchers [53, 54, 55, 56, 57, 58] studied about Hankel determinant.

A widely acknowledged Fekete-Szegö inequlity is $|\breve{c}_3 - \breve{c}_2^2| = |\mathscr{H}_2(1)|$. In general, this is expressed as $|\breve{c}_3 - \lambda \breve{c}_2^2|$ for some λ , where λ may be complex or real. Fekete-Szegö gave complicated inequality, which holds for, $0 \leq \lambda < 1$. Fekete-Szegö problem is all about to find out the best possible constant λ, so that the inequality is less or equal to λ, for every analytic function.

At the end of 1960's, Lawrence Zalcman posed a conjecture that the coefficients of univalent functions in unit disk satisfy the sharp inequality

$$|\breve{c}_n^2 - \breve{c}_{2n-1}| \leq (n-1)^2,$$

where this inequality becomes equality only for the Koebe function and its rotation. This remarkable conjecture implies the Bieberbach conjecture, investigated by many mathematicians, and still remains a very difficult open problem for all $n > 3$; it was proved only in certain special cases. When $n = 2$, above inequality transformed into a known result, that is, the Fekete-Szegő Inequality $|\breve{c}_2^2 - \breve{c}_3| \leq 1$. Many researchers [59, 60, 61] studied Zalcman functional. For detailed studies on Zalcman functionals, see the articles [62, 63, 64].

CHAPTER 3

DEFINITIONS AND PRELIMINARY CONCEPTS

3.1 Overview

The purpose of this chapter is to discuss some important definitions and classical results that will serve as a foundation for subsequent research. The Carathéodory functions and normalised analytic univalent functions will be discussed in thoroughly. Certain special functions, well-known linear operator, and preliminary lemmas will be considered. The most fascinating aspect of complex function theory is probably how geometry and analysis connect with each other

Definition 3.1.1. *[65] A function is holomorphic in domain contain in complex field, if it is differentiable at each point of that domain. A complex valued function $\xi(\hat{z})$ is differentiable at point $\hat{z}_0$ if it has derivative,*

$$\xi'(\hat{z}) = \lim_{\hat{z}\to\hat{z}_0} \frac{\xi(\hat{z}) - \xi(\hat{z}_0)}{\hat{z} - \hat{z}_0},$$

at $\hat{z}_0$, such function ξ is analytic at $\hat{z}_0$ if it is differentiable at every point in its neighborhood.

One of the wonders of complex analysis is that all orders of $\hat{z}_0$ must have derivatives, and that ξ has Taylor series expression,

$$\xi(\hat{z}) = \sum_{k=0}^{\infty} \frac{\xi^k(\hat{z}_0)}{k!}(\hat{z} - \hat{z}_0)^k,$$

3.2 Domain

In Geometric Function Theory, our focus is consistently directed towards a specific domain. A domain is an open connected set. Geometrically, the open unit disk corresponds to a disk centered at the origin with a radius of 1, excluding the boundary of the disk. In other words, it includes all complex numbers inside the disk but does not include the points lying on the circumference.

It is worth noting that the open unit disk is a fundamental concept in complex analysis and is often used in various mathematical and analytical contexts, such as mapping functions, conformal mappings, and complex integration.

Definition 3.2.1. *[65, 66] An open unit disk in the complex plane, refers to a set of complex numbers that lie within a specific region in the complex plane. It is defined as the set of all complex numbers whose distance from the origin is less than 1. In mathematical notation, the open unit disk is represented as,*

$$\Omega = \{|\hat{z}| < 1; \hat{z} \in \mathbb{C}\}.$$

Here, $\hat{z}$ represents a complex number, $\mathbb{C}$ denotes the set of complex numbers, and $|\hat{z}|$ denotes the modulus or absolute value of $\hat{z}$. The inequality $|\hat{z}| < 1$ specifies that the distance between $\hat{z}$ and the origin is less than 1.

3.3 Analytic and Univalent functions

The fundamental principles of the theory of univalent functions lie in the interrelation between geometric functions and analytic structures. In this context, we establish categories for both Analytic and Univalent functions.

Definition 3.3.1. *[67] An analytic function, also known as holomorphic function, is a complex-valued function that is defined and differentiable at every point within a certain region of the complex plane. More formally, a function ξ is said to be analytic in a region if it is differentiable at every point within region.*

An important consequence is that an analytic function has a power series representation. This means that within the region where the function is analytic, it can be expressed as an infinite sum of powers of the variable $\hat{z}$,

$$\xi = \hat{z} + \sum_{m=2}^{\infty} \hat{a}_m \hat{z}^m,$$

where the coefficients can be determined.

Definition 3.3.2. *[68] A function is in class $\mathscr{A}$, the class of Normalized Analytic Function, if it is analytic in open unit disk Ω and normalized by these conditions $\xi(0) = 0$ with $\xi'(0) = 1$.*

$$\xi = \hat{z} + \sum_{m=2}^{\infty} \hat{a}_m \hat{z}^m, \quad \hat{z} \in \Omega.$$

Definition 3.3.3. *[16] A univalent function, also known as a univalent mapping or one-to-one analytic function, is a special type of analytic function that preserves injectivity. Specifically, a function ξ defined on a region in the complex plane is said to be univalent if it maps different complex numbers to different images, meaning that it has no two distinct inputs that are mapped to the same output.*

Let a domain in the complex plane, and let ξ be an analytic function defined on defined domain. The function ξ is said to be univalent in domain if, for any distinct complex numbers $\hat{z}_1$ and $\hat{z}_2$ in domain, the condition $\xi(\hat{z}_1) \neq \xi(\hat{z}_2)$ holds.

In other words, a univalent function is one that is injective or one-to-one within its domain. It does not produce any overlaps or self-intersections when mapping points from its domain to its range.

Univalent functions are of significant interest in complex analysis and geometric function theory. They have important applications in various areas, such as conformal mapping, complex dynamics, and the theory of Riemann surfaces. Univalent functions often exhibit desirable geometric properties, and their study helps understand the intricate behavior of complex mappings and transformations.

Definition 3.3.4. *[16] A function is a member of the class $\mathscr{S}$, which consists of Univalent Functions, if it is both analytic and univalent within the open unit disk Ω. The function must satisfy the normalization conditions $\xi(0) = 0$ and $\xi'(0) = 1$.*

An illustrative example of a function belonging to the class $\mathscr{S}$ is the Koebe Function.

$$\xi = \hat{z} + \sum_{m=2}^{\infty} m\hat{z}^m, \quad \hat{z} \in \Omega.$$

3.4 Carathéodory function

It was found that there exist other functions whose image domains are limited to the open half plane when there are so many complex valued functions whose image domains cover the complete complex plane. The Carathéodory function class, represented by $\mathscr{P}$, is comprised of these types of functions.

Definition 3.4.1. *[68] A function* $p \in \mathscr{P}$ *is analytic in* Ω*, expressed as*

$$p(\hat{z}) = 1 + \sum_{i=1}^{\infty} c_i \hat{z}^i,$$

where $\mathrm{Re}[p(\hat{z})] \geq 0$ *and* $p(0) = 1$.

Mobius function is the most prominent example of a function from this class, which is described as,

$$\mathbf{M} = \left(-1 + \frac{2}{1-\hat{z}}\right).$$

3.5 Certain sub-classes of univalent functions

The study of univalent functions is an ancient yet dynamically evolving domain. Numerous notable advancements have occurred in the last decade to fifteen years. Various subclasses within the category of univalent functions have been introduced, primarily guided by the geometric properties of their image domains. Notably, the classes of Starlike and Convex functions have been defined within this context.

Definition 3.5.1. *[68, 67] A function known as a "starlike function" that projects the disk* Ω *onto a domain* $\mathbf{D}$ *that, in relation to the origin, resembles a starlike domain given in Figure 3.1.* S^* *stands for the subclass of that encompasses all starlike functions. If* $\hat{z} = 0$ *and the linear segment connecting 0 to any other point of the domain* $\mathbf{D}$ *lies wholly within the complex plane, then the domain is starlike with respect to the origin. That is,*

$$\forall \hat{z} \in \mathbf{D}, \ \lambda \hat{z} \in \mathbf{D}$$

where, $0 \leq \lambda \leq 1$*, if* $\hat{z} \in \mathbf{D}$*, it is an essential requirement that all of points of domain be visible from* $\hat{z}$.

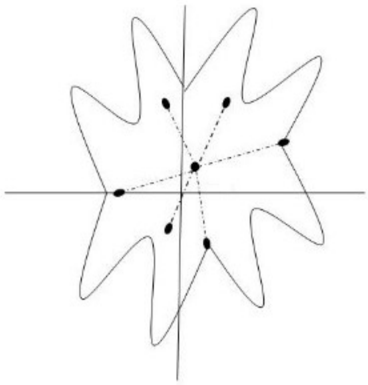

Figure 3.1: Starlike domain

Definition 3.5.2. *[68, 67] A convex function transfers the disk* Ω *onto a domain* $\mathbf{D}$ *that is convex relative to the origin given in Figure 3.2. C stands for the subclass of* $\mathscr{S}$ *that contains all convex functions. If a line segment connecting any two points of a domain* $\mathbf{D}$ *in complex plane lies wholly within that domain, the domain is said to be convex. That is,*

$$[\lambda \hat{z}_1 + (1-\lambda)\hat{z}_2] \in \mathbf{D},$$

where $\hat{z}_1$ *and* $\hat{z}_2$ *both are in* $\mathbf{D}$ *with* $0 \leq \lambda \leq 1$.

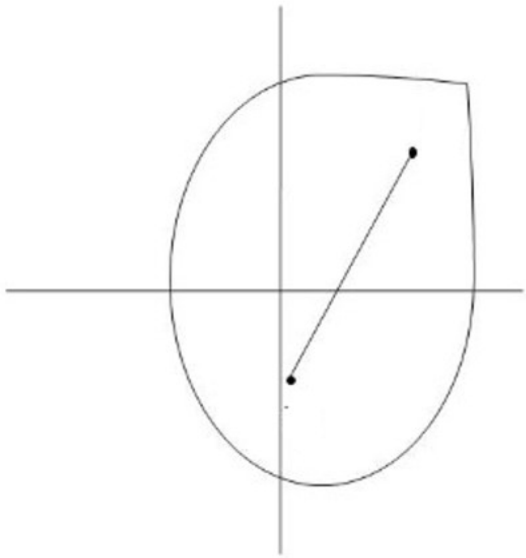

Figure 3.2: Convex domain

There is a beautiful relation between starlike function and convex function, $S^* \subset C$, that is, class of convex function is superset of class of starlike function.

Definition 3.5.3. *[69] A function is considered to be starlike concerning symmetric points, $S_s^*(\phi)$ if,*

$$\frac{2\hat{z}\tau'(\hat{z})}{\tau(\hat{z})-\tau(-\hat{z})} \prec \phi(\hat{z}),$$

where $\hat{z} \in \Omega$ and $\phi(\hat{z})$ is a Mobius function.

Definition 3.5.4. *[69] A function is considered to be convex concerning symmetric points, $C_s(\phi)$ if,*

$$\frac{2(\hat{z}\tau'(\hat{z}))'}{(\tau(\hat{z})-\tau(-\hat{z}))'} \prec \phi(\hat{z}),$$

where $\hat{z} \in \Omega$ and $\phi(\hat{z})$ is a Mobius function.

3.6 Subordination

In geometric function theory, subordination refers to a concept used to study the behavior of analytic functions. Lindelof was the first to present the theory of subordination in 1909. Later developments were made by Littlewood and Rogosinski [15]. The subordination principle is defined by using the Schwarz function.

Definition 3.6.1. *[67] If ξ and ϕ are in class $\mathscr{A}$ of functions, we say that, the function ξ is said to be subordinated to function ϕ, symbolically written as $\xi \prec \phi$, if $\xi(\hat{z}) = \phi(\psi(\hat{z}))$ where $\psi(\hat{z})$ is an analytic function in open unit disk, satisfying these two conditions $\psi(0) = 0$with $|\psi(\hat{z})| \leq 1$.*

3.7 Quantum Calculus

Quantum calculus was initially developed by the American mathematician Jackson in the early 20th century. He was the first to define the derivative and integral operator's q-analog.

Definition 3.7.1. *[70] Quantum calculus, also known as q-calculus or Jackson's q-calculus, is a branch of mathematics that generalizes many concepts from classical calculus by introducing a parameter q.*

The q-derivative operator, often denoted as $\mathscr{D}_q$ is a fundamental concept in q-calculus, a branch of mathematics that generalizes classical calculus by introducing a parameter q.

Definition 3.7.2. *[70] For a differentiable function $\tau(\hat{z})$, q-derivative defined as,*

$$\mathscr{D}_q\tau(\hat{z}) = \frac{\tau(\hat{z})-\tau(\hat{q}z)}{(1-q)\hat{z}}, \quad \hat{z}\neq 0 \quad where 0<q<1.$$

Its Maclaurins series is

$$\mathscr{D}_q\tau(\hat{z}) = \sum_{n=0}^{\infty}[n]_q\check{c}_n\hat{z}^{n-1},$$

where

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & n\in\mathbb{C} \\ \sum_{n=0}^{n-1}q^n, & n\in\mathbb{N}. \end{cases}$$

Definition 3.7.3. *[70] Quantum calculus introduces q-series, which are power series involving q-analogs of the usual calculus operations. These series are used in various areas, including combinatorics (q-binomial theorem) and number theory (q-analog of the partition function).*

Definition 3.7.4. *[43] A function is described as q-starlike with respect to symmetric points, $S^*_{q,s}(\phi)$ if,*

$$\frac{2\hat{z}\mathscr{D}_q\tau(\hat{z})}{\tau(\hat{z})-\tau(-\hat{z})} \prec \phi(\hat{z}),$$

where $\hat{z}\in\Omega$ and $\phi(\hat{z})$ is a Mobius function.

Definition 3.7.5. *[43] A function is described as q-convex with respect to symmetric points, $C_{q,s}(\phi)$ if,*

$$\frac{2\mathscr{D}_q(\hat{z}\mathscr{D}_q\tau(\hat{z}))}{\mathscr{D}_q(\tau(\hat{z})-\tau(-\hat{z}))} \prec \phi(\hat{z}),$$

where $\hat{z}\in\Omega$ and $\phi(\hat{z})$ is a Mobius function.

3.8 Fekete-Szegő Inequality

The Fekete-Szegő inequality has several important consequences and applications in complex analysis. For example, it can be used to derive bounds on the coefficients of functions in certain subclasses of analytic functions, such as the class of starlike or convex functions.

It is worth noting that the Fekete-Szegő inequality is sharp, meaning that there exist functions for which the inequality becomes an equality. These functions are known as extremal functions and play a significant role in understanding the behavior of analytic functions in the unit disk.

Definition 3.8.1. *[71] The Fekete-Szegő inequality is a classical result in complex analysis that provides an upper bound on the absolute value of the determinant of a specific class of analytic functions. Specifically, it applies to functions that are defined in* $\Omega = \{|\hat{z}| \leq 1; \hat{z} \in \mathbb{C}\}$ *and are normalized such that* $\xi(0) = 0$ *and* $\xi'(0) = 1$. *This inequality provides an upper bound on the absolute value of* $\xi(\hat{z})$ *in terms of the modulus* $|\hat{z}|$.

3.9 Hankel Determinant

Definition 3.9.1. *The Hankel determinant is the determinant of the corresponding Hankel matrix. Pommerenke [46] defined the Hankel determinant for the class of univalent functions, for positive integers n,s that defined below,*

$$|\mathscr{H}_n(s)| = \begin{vmatrix} \breve{c}_s & \breve{c}_{s+1} & \breve{c}_{s+2} & \dots & \breve{c}_{s+n-1} \\ \breve{c}_{s+1} & \breve{c}_{s+2} & \breve{c}_{s+3} & \dots & \breve{c}_{s+n} \\ \dots & \dots & \dots & \dots & \dots \\ \breve{c}_{s+n-1} & \breve{c}_{s+n} & \breve{c}_{s+n+1} & \dots & \breve{c}_{s+2n-2} \end{vmatrix}.$$

3.10 Zalcman Functional

Definition 3.10.1. *[72] The well-known Zalcman conjecture, which implies the Bieberbach conjecture, states that the coefficients of univalent functions on the unit disk satisfy the inequality for all* $n > 2$,

$$|\breve{c}_n^2 - \breve{c}_{2n-1}| \leq (n-1)^2,$$

where the equality hold only for the Koebe function.

3.11 Preliminary Lemmas

Here some lemmas, that will be necessary for driving our results in the subsequent chapters, discussed.

Lemma 3.11.1. *[73] Let $p \in \mathscr{P}$ with $p(0) = 1$, and analytic in open unit disk then $|p_n| \leq 2$ for $n \in \mathbb{N}$.*

Lemma 3.11.2. *[74] If $p \in \mathscr{P}$ with $p(0) = 1$, and analytic in open unit disk then*

$$|p_2 - \nu p_1^2| \leq 2max\{1, |2\nu - 1|\}.$$

Lemma 3.11.3. *[75] Suppose that $p \in \mathscr{P}$ has the power series then*

$$|\eta p_1^3 - \tau p_1 p_2 + \lambda p_3| \leq 2|\eta| + 2|\tau - 2\eta| + 2|\eta - \tau + \lambda|.$$

Lemma 3.11.4. *[73] Consider $p \in \mathscr{P}$ then*

$$\left|p_{i+j} - \psi p_i p_j\right| \leq 2,\ 0 \leq \psi \leq 1.$$

Lemma 3.11.5. *[76] If $p \in \mathscr{P}$ then*

$$|p_2 - \kappa p_1^2| \leq \begin{cases} -4\kappa + 2, & \kappa \leq 0 \\ 2, & 0 \leq \kappa \leq 1 \\ 4\kappa + 2, & \kappa \geq 1. \end{cases}$$

Lemma 3.11.6. *[77] Consider $\mathrm{a}, \mathrm{b}, \mathrm{c}$ and d satisfy the inequalities $0 < \mathrm{a} < 1$, $0 < \mathrm{d} < 1$ and*

$$8\mathrm{d}(1-\mathrm{d})[(\mathrm{ab}-2\mathrm{c})^2 + (\mathrm{a}(\mathrm{d}+\mathrm{a}) - \mathrm{b})^2] + \mathrm{a}(1-\mathrm{a})(\mathrm{b}-2\mathrm{da})^2 \leq 4\mathrm{a}^2(1-\mathrm{a})^2\mathrm{d}(1-\mathrm{d}).$$

If $p \in \mathscr{P}$ then

$$\left|\mathrm{c}p_1^4 + \mathrm{d}p_2^2 + 2\mathrm{a}p_1p_3 - \frac{3}{2}\mathrm{b}p_1^2p_2 - p_4\right| \leq 2.$$

CHAPTER 4

CLASSES OF STARLIKE AND CONVEX FUNCTIONS WITH RESPECT TO SYMMETRIC POINTS SUBORDINATED WITH SINE FUNCTION

4.1 Introduction

This chapter aims to explore several foundational and classical results that serve as cornerstones for subsequent research. The section begins by reviewing Starlike functions and introducing a novel class termed Convex functions. These categories are established in connection with symmetric points that are related to the sine function. Additionally, several key findings, including the coefficient bounds, the well-known Fekete–Szegő inequality, Zalcman functional and the Hankel Determinants will examined.

The category of Starlike functions associated with symmetric points related to the trigonometric sine function is introduced by Khan *et al.* [78], which is defined as,

Definition 4.1.1. *A function $\xi \in \mathscr{A}$ is in S_s^* then*

$$\frac{2\hat{z}\xi'(\hat{z})}{\xi(\hat{z}) - \xi(-\hat{z})} \prec 1 + \sin(\hat{z}),$$

for all $\hat{z} \in \Omega$.

Now, the category of Convex functions associated with symmetric points related to the trigonometric sine function, C_s, defined as.

Definition 4.1.2. *A function $\xi \in \mathscr{A}$, is in C_s then*

$$\frac{2[\hat{z}\xi'(\hat{z})]'}{(\xi(\hat{z})-\xi(-\hat{z}))'} \prec 1+\sin(\hat{z}),$$

for all $\hat{z} \in \Omega$.

4.2 Coefficient Inequalities

The following result is related to the class S_s^*.

Theorem 4.2.1. *If $\xi(\hat{z}) \in S_s^*$ then*

$$|\hat{\alpha}_2| \leq \frac{1}{2}, \ |\hat{\alpha}_3| \leq \frac{1}{2}, \ |\hat{\alpha}_4| \leq \frac{1}{4}, \ |\hat{\alpha}_5| \leq \frac{3}{4}.$$

Proof. By definition

$$\frac{2\hat{z}\xi'(\hat{z})}{\xi(\hat{z})-\xi(-\hat{z})} \prec 1+\sin(\hat{z})$$

Since $\xi \in S_s^*$, using subordination principle, we have

$$\frac{2\hat{z}\xi'(\hat{z})}{\xi(\hat{z})-\xi(-\hat{z})} = 1+\sin(\varpi(\hat{z})). \tag{4.1}$$

Let us define the function

$$p(\hat{z}) = \frac{1+\varpi(\hat{z})}{1-\varpi(\hat{z})} = 1+p_1\hat{z}+p_2\hat{z}^2+p_3\hat{z}^3+p_4\hat{z}^4+..., \tag{4.2}$$

where $p(\hat{z})$ is analytic in Ω with $p(0)=1$. This implies that

$$\varpi(\hat{z}) = \frac{p(\hat{z})-1}{p(\hat{z})+1}, \tag{4.3}$$

after simplification we get,

$$\varpi(\hat{z}) = \left(\frac{p_1}{2}\right)\hat{z}+\left(\frac{p_2}{2}-\frac{p_1^2}{4}\right)\hat{z}^2+\left(\frac{p_3}{2}+\frac{p_1^3}{8}-\frac{p_1p_2}{2}\right)\hat{z}^3+ \left(\frac{3p_1^2p_2}{8}-\frac{p_1p_3}{2}-\frac{p_1^4}{16}-\frac{p_2^2}{4}+\frac{p_4}{2}\right)\hat{z}^4+... \tag{4.4}$$

As we know,

$$\sin(\varpi(\hat{z})) = \varpi(\hat{z})-\frac{(\varpi(\hat{z}))^3}{3!}+\frac{(\varpi(\hat{z}))^5}{5!}+...,$$

So, we have

$$\sin(\varpi(\hat{z})) = \left(\frac{p_1}{2}\right)\hat{z} + \left(\frac{p_2}{2} - \frac{p_1^2}{4}\right)\hat{z}^2 + \left(\frac{p_3}{2} + \frac{5p_1^3}{48} - \frac{p_1p_2}{2}\right)\hat{z}^3 + \left(\frac{5p_1^2p_2}{16} - \frac{p_1p_3}{2} - \frac{p_1^4}{32} - \frac{p_2^2}{4} + \frac{p_4}{2}\right)\hat{z}^4 + \ldots \quad (4.5)$$

Now, we get

$$\frac{2\hat{z}\xi'(\hat{z})}{\xi(\hat{z}) - \xi(-\hat{z})} = 1 + (2\hat{\alpha}_2)\hat{z} + (2\hat{\alpha}_3)\hat{z}^2 + (4\hat{\alpha}_4 - 2\hat{\alpha}_2\hat{\alpha}_3)\hat{z}^3 + (4\hat{\alpha}_5 - 2\hat{\alpha}_3^2)\hat{z}^4 + \ldots$$

On substituting values in (4.1) we get,

$$1 + (2\hat{\alpha}_2)\hat{z} + (2\hat{\alpha}_3)\hat{z}^2 + (4\hat{\alpha}_4 - 2\hat{\alpha}_2\hat{\alpha}_3)\hat{z}^3 + (4\hat{\alpha}_5 - 2\hat{\alpha}_3^2)\hat{z}^4 + \ldots = 1 + \left(\frac{p_1}{2}\right)\hat{z} + \left(\frac{p_2}{2} - \frac{p_1^2}{4}\right)\hat{z}^2 + \left(\frac{p_3}{2} + \frac{5p_1^3}{48} - \frac{p_1p_2}{2}\right)\hat{z}^3 + \left(\frac{5p_1^2p_2}{16} - \frac{p_1p_3}{2} - \frac{p_1^4}{32} - \frac{p_2^2}{4} + \frac{p_4}{2}\right)\hat{z}^4 + \ldots$$

On comparing both sides of above equation, we obtain

$$2\hat{\alpha}_2 = \frac{p_1}{2}, \quad (4.6)$$

$$2\hat{\alpha}_3 = \frac{p_2}{2} - \frac{p_1^2}{4}, \quad (4.7)$$

$$4\hat{\alpha}_4 - 2\hat{\alpha}_2\hat{\alpha}_3 = \frac{p_3}{2} + \frac{5p_1^3}{48} - \frac{p_1p_2}{2}, \quad (4.8)$$

$$4\hat{\alpha}_5 - 2\hat{\alpha}_3^2 = \frac{5p_1^2p_2}{16} - \frac{p_1p_3}{2} - \frac{p_1^4}{32} - \frac{p_2^2}{4} + \frac{p_4}{2}, \quad (4.9)$$

Solving (4.6) for coefficient $\hat{\alpha}_2$ we get,

$$\hat{\alpha}_2 = \frac{p_1}{4}. \quad (4.10)$$

On solving (4.7) we get $\hat{\alpha}_3$,

$$\hat{\alpha}_3 = \frac{p_2}{4} - \frac{p_1^2}{8}. \quad (4.11)$$

By solving (4.8) we determine $\hat{\alpha}_4$

$$4\hat{\alpha}_4 - 2\hat{\alpha}_2\hat{\alpha}_3 = \frac{p_3}{2} + \frac{5p_1^3}{48} - \frac{p_1p_2}{2},$$

$$4\hat{\alpha}_4 = 2\hat{\alpha}_2\hat{\alpha}_3 + \frac{p_3}{2} + \frac{5p_1^3}{48} - \frac{p_1p_2}{2},$$

this implies that,

$$\hat{\alpha}_4 = \frac{p_3}{8} + \frac{p_1^3}{96} - \frac{3p_1p_2}{32}. \quad (4.12)$$

Now for $\hat{\alpha}_5$, solving (4.9),

$$4\hat{\alpha}_5 - 2\hat{\alpha}_3^2 = \frac{5p_1^2p_2}{16} - \frac{p_1p_3}{2} - \frac{p_1^4}{32} - \frac{p_2^2}{4} + \frac{p_4}{2},$$

$$4\hat{\alpha}_5 = 2\hat{\alpha}_3^2 + \frac{5p_1^2p_2}{16} - \frac{p_1p_3}{2} - \frac{p_1^4}{32} - \frac{p_2^2}{4} + \frac{p_4}{2},$$

which implies that,

$$\hat{\alpha}_5 = \frac{3p_1^2p_2}{64} - \frac{p_1p_3}{8} - \frac{p_2^2}{32} + \frac{p_4}{8}. \tag{4.13}$$

Next, to find out the absolute values of coefficients, (4.10) gives,

$$\hat{\alpha}_2 = \frac{p_1}{4},$$

using Lemma 3.11.1, we get

$$|\hat{\alpha}_2| \leq \frac{1}{2}. \tag{4.14}$$

From (4.11), we have

$$\hat{\alpha}_3 = \frac{p_2}{4} - \frac{p_1^2}{8},$$

applying Lemma 3.11.2, we get

$$|\hat{\alpha}_3| \leq \frac{1}{4}\left|p_2 - \frac{p_1^2}{2}\right|,$$

$$|\hat{\alpha}_3| \leq \frac{2}{4} max\{1, 0\},$$

$$|\hat{\alpha}_3| \leq \frac{1}{2}. \tag{4.15}$$

Solving (4.12), we get,

$$\hat{\alpha}_4 = \frac{p_3}{8} + \frac{p_1^3}{96} - \frac{3p_1p_2}{32}.$$

An application of Lemma 3.11.3, we get

$$|\hat{\alpha}_4| \leq \frac{1}{48} + \frac{14}{96} + \frac{1}{12} = \frac{1}{4},$$

$$|\hat{\alpha}_4| \leq \frac{1}{4}. \tag{4.16}$$

Solving (4.13), we obtain

$$\hat{\alpha}_5 = \frac{3p_1^2p_2}{64} - \frac{p_1p_3}{8} - \frac{p_2^2}{32} + \frac{p_4}{8},$$

$$|\hat{\alpha}_5| \leq \frac{3}{64}|p_1|^2|p_2| + \frac{1}{32}|p_2|^2 + \frac{1}{8}|p_4 - p_1p_3|,$$

here we are using Lemma 3.11.1 and Lemma 3.11.4, then

$$|\hat{\alpha}_5| \leq \frac{3}{8} + \frac{1}{8} + \frac{1}{4} = \frac{3}{4},$$

$$|\hat{\alpha}_5| \leq \frac{3}{4}, \tag{4.17}$$

which is required. □

Now, the results for the class C_s will investigated.

Theorem 4.2.2. *If* $\xi(\hat{z}) \in C_s$ *then*

$$|\hat{\alpha}_2| \leq \frac{1}{4}, \ |\hat{\alpha}_3| \leq \frac{1}{6}, \ |\hat{\alpha}_4| \leq \frac{1}{16}, \ |\hat{\alpha}_5| \leq \frac{3}{20}.$$

Proof. By definition

$$\frac{2[\hat{z}\xi'(\hat{z})]'}{(\xi(\hat{z}) - \xi(-\hat{z}))'} \prec 1 + \sin(\hat{z})$$

Since $\xi \in C_s$, using subordination principle, we have

$$\frac{2[\hat{z}\xi'(\hat{z})]'}{(\xi(\hat{z}) - \xi(-\hat{z}))'} = 1 + \sin(\varpi(\hat{z})). \tag{4.18}$$

Let us define the function

$$p(\hat{z}) = \frac{1+\varpi(\hat{z})}{1-\varpi(\hat{z})} = 1 + p_1\hat{z} + p_2\hat{z}^2 + p_3\hat{z}^3 + p_4\hat{z}^4 + ..., \tag{4.19}$$

where $p(\hat{z})$ is analytic in Ω with $p(0) = 1$. This implies that

$$\varpi(\hat{z}) = \frac{p(\hat{z}) - 1}{p(\hat{z}) + 1}, \tag{4.20}$$

after simplification we get,

$$\varpi(\hat{z}) = \left(\frac{p_1}{2}\right)\hat{z} + \left(\frac{p_2}{2} - \frac{p_1^2}{4}\right)\hat{z}^2 + \left(\frac{p_3}{2} + \frac{p_1^3}{8} - \frac{p_1 p_2}{2}\right)\hat{z}^3 + \left(\frac{3p_1^2 p_2}{8} - \frac{p_1 p_3}{2} - \frac{p_1^4}{16} - \frac{p_2^2}{4} + \frac{p_4}{2}\right)\hat{z}^4 + ... \tag{4.21}$$

As we know that,

$$\sin(\varpi(\hat{z})) = \varpi(\hat{z}) - \frac{(\varpi(\hat{z}))^3}{3!} + \frac{(\varpi(\hat{z}))^5}{5!} + ...,$$

So, we have

$$\sin(\varpi(\hat{z})) = \left(\frac{p_1}{2}\right)\hat{z} + \left(\frac{p_2}{2} - \frac{p_1^2}{4}\right)\hat{z}^2 + \left(\frac{p_3}{2} + \frac{5p_1^3}{48} - \frac{p_1 p_2}{2}\right)\hat{z}^3 + \left(\frac{5p_1^2 p_2}{16} - \frac{p_1 p_3}{2} - \frac{p_1^4}{32} - \frac{p_2^2}{4} + \frac{p_4}{2}\right)\hat{z}^4 + ... \tag{4.22}$$

Now, we get

$$\frac{2[\hat{z}\xi'(\hat{z})]'}{(\xi(\hat{z}) - \xi(-\hat{z}))'} = 1 + 4\hat{\alpha}_2\hat{z} + 6\hat{\alpha}_3\hat{z}^2 + (16\hat{\alpha}_4 - 12\hat{\alpha}_2\hat{\alpha}_3)\hat{z}^3 + (20\hat{\alpha}_5 - 18\hat{\alpha}_3^2)\hat{z}^4 + ... \tag{4.23}$$

substituting values in (4.18), we obtain,

$$1+4\hat{\alpha}_2\hat{z}+6\hat{\alpha}_3\hat{z}^2+(16\hat{\alpha}_4-12\hat{\alpha}_2\hat{\alpha}_3)\hat{z}^3+(20\hat{\alpha}_5-18\hat{\alpha}_3^2)\hat{z}^4+\ldots=1+\left(\frac{p_1}{2}\right)\hat{z}+\left(\frac{p_2}{2}-\frac{p_1^2}{4}\right)\hat{z}^2+$$
$$\left(\frac{p_3}{2}+\frac{5p_1^3}{48}-\frac{p_1p_2}{2}\right)\hat{z}^3+\left(\frac{5p_1^2p_2}{16}-\frac{p_1p_3}{2}-\frac{p_1^4}{32}-\frac{p_2^2}{4}+\frac{p_4}{2}\right)\hat{z}^4+\ldots$$

On comparing similar powers of $\hat{z}$ we get,

$$4\hat{\alpha}_2=\frac{p_1}{2}, \tag{4.24}$$
$$6\hat{\alpha}_3=\frac{p_2}{2}-\frac{p_1^2}{4}, \tag{4.25}$$
$$16\hat{\alpha}_4-12\hat{\alpha}_2\hat{\alpha}_3=\frac{p_3}{2}+\frac{5p_1^3}{48}-\frac{p_1p_2}{2}, \tag{4.26}$$
$$20\hat{\alpha}_5-18\hat{\alpha}_3^2=\frac{5p_1^2p_2}{16}-\frac{p_1p_3}{2}-\frac{p_1^4}{32}-\frac{p_2^2}{4}+\frac{p_4}{2}, \tag{4.27}$$

On solving (4.24) to find $\hat{\alpha}_2$, we get

$$4\hat{\alpha}_2=\frac{p_1}{2},$$
$$\hat{\alpha}_2=\frac{p_1}{8}. \tag{4.28}$$

Now, to find out $\hat{\alpha}_3$, solving (4.25)

$$6\hat{\alpha}_3=\frac{p_2}{2}-\frac{p_1^2}{4},$$

$$\hat{\alpha}_3=\frac{p_2}{12}-\frac{p_1^2}{24}. \tag{4.29}$$

On solving (4.26) to get $\hat{\alpha}_4$, we have

$$16\hat{\alpha}_4-12\hat{\alpha}_2\hat{\alpha}_3=\frac{p_3}{2}+\frac{5p_1^3}{48}-\frac{p_1p_2}{2},$$

$$16\hat{\alpha}_4=12\hat{\alpha}_2\hat{\alpha}_3+\frac{p_3}{2}+\frac{5p_1^3}{48}-\frac{p_1p_2}{2}.$$

By substituting values of $\hat{\alpha}_2$ and $\hat{\alpha}_3$ from (4.28) and (4.29)

$$16\hat{\alpha}_4=12\left(\frac{p_1}{8}\right)\left(\frac{p_2}{12}-\frac{p_1^2}{24}\right)+\frac{p_3}{2}+\frac{5p_1^3}{48}-\frac{p_1p_2}{2},$$

this implies that

$$\hat{\alpha}_4=\frac{p_1^3}{384}-\frac{3p_1p_2}{128}+\frac{p_3}{32}. \tag{4.30}$$

By solving (4.27) to obtain $\hat{\alpha}_5$

$$20\hat{\alpha}_5-18\hat{\alpha}_3^2=\frac{5p_1^2p_2}{16}-\frac{p_1p_3}{2}-\frac{p_1^4}{32}-\frac{p_2^2}{4}+\frac{p_4}{2},$$

$$20\hat{\alpha}_5 = 18\hat{\alpha}_3^2 + \frac{5p_1^2p_2}{16} - \frac{p_1p_3}{2} - \frac{p_1^4}{32} - \frac{p_2^2}{4} + \frac{p_4}{2}.$$

On substituting value of $\hat{\alpha}_3$ from (4.29), we get

$$20\hat{\alpha}_5 = 18\left(\frac{p_2}{12} - \frac{p_1^2}{24}\right)^2 + \frac{5p_1^2p_2}{16} - \frac{p_1p_3}{2} - \frac{p_1^4}{32} - \frac{p_2^2}{4} + \frac{p_4}{2}.$$

After simplification we get,

$$\hat{\alpha}_5 = \frac{p_4}{40} - \frac{p_1p_3}{40} + \frac{3p_1^2p_2}{320} - \frac{p_2^2}{160}. \tag{4.31}$$

Now,from (4.28) we have

$$\hat{\alpha}_2 = \frac{p_1}{8},$$

we apply Lemma 3.11.1, we obtain

$$|\hat{\alpha}_2| = \left|\frac{p_1}{8}\right| \leq \frac{|p_1|}{8} \leq \frac{1}{4},$$

$$|\hat{\alpha}_2| \leq \frac{1}{4}. \tag{4.32}$$

On solving (4.29),

$$\hat{\alpha}_3 = \frac{p_2}{12} - \frac{p_1^2}{24}.$$

An application of Lemma 3.11.2 leads us to

$$|\hat{\alpha}_3| = \left|\frac{p_2}{12} - \frac{p_1^2}{24}\right| = \frac{1}{12}\left|p_2 - \frac{p_1^2}{2}\right|,$$

$$|\hat{\alpha}_3| \leq \frac{2}{12} max\{1,0\} = \frac{1}{6},$$

$$|\hat{\alpha}_3| \leq \frac{1}{6}. \tag{4.33}$$

From (4.30) we have

$$\hat{\alpha}_4 = \frac{p_1^3}{384} - \frac{3p_1p_2}{128} + \frac{p_3}{32},$$

By applying Lemma 3.11.3,we get

$$|\hat{\alpha}_4| = \left|\frac{p_1^3}{384} - \frac{3p_1p_2}{128} + \frac{p_3}{32}\right|,$$

$$\left|\frac{p_1^3}{384} - \frac{3p_1p_2}{128} + \frac{p_3}{32}\right| \leq 2\left|\frac{1}{384}\right| + 2\left|\frac{1}{128} - \frac{2}{384}\right| + 2\left|\frac{1}{384} - \frac{1}{128} + \frac{1}{32}\right| = \frac{1}{16}$$

$$|\hat{\alpha}_4| \leq \frac{1}{16}. \tag{4.34}$$

On solving (4.31) for $\hat{\alpha}_5$

$$\hat{\alpha}_5 = \frac{p_4}{40} - \frac{p_1p_3}{40} + \frac{3p_1^2p_2}{320} - \frac{p_2^2}{160}.$$

By applying Lemma 3.11.1 and Lemma 3.11.4, we get

$$|\hat{\alpha}_5| = \left|\frac{p_4}{40} - \frac{p_1p_3}{40} + \frac{3p_1^2p_2}{320} - \frac{p_2^2}{160}\right| \leq \frac{1}{40}|p_4 - p_1p_3| + \left|\frac{3p_1^2p_2}{320} - \frac{p_2^2}{160}\right|,$$

$$|\hat{\alpha}_5| \leq \frac{1}{40}|p_4 - p_1p_3| + \frac{3}{320}|p_1|^2|p_2| - \frac{1}{160}|p_2|^2 = \frac{1}{20} + \frac{3}{20} + \frac{1}{40} = \frac{3}{20},$$

$$|\hat{\alpha}_5| \leq \frac{3}{20}. \tag{4.35}$$

□

Hence, proof is completed.

4.3 Fekete–Szegő Inequality

The following Fekete–Szegő Inequality related to the class S_s^*, starlike functions.

Theorem 4.3.1. *If $\xi(\hat{z}) \in S_s^*$ then $|\hat{\alpha}_3 - \hat{\alpha}_2^2| \leq \frac{1}{2}$.*

Proof. From (4.10) and (4.11) we obtain

$$|\hat{\alpha}_3 - \hat{\alpha}_2^2| = \left|\frac{p_2}{4} - \frac{p_1^2}{8} - \frac{p_1^2}{16}\right| = \frac{1}{4}\left|p_2 - \frac{3}{4}p_1^2\right|,$$

using Lemma 3.11.2, we have

$$|\hat{\alpha}_3 - \hat{\alpha}_2^2| \leq \frac{1}{4}(2)$$

thus,

$$|\hat{\alpha}_3 - \hat{\alpha}_2^2| \leq \frac{1}{2}.$$

Hence, proof is completed. □

Now, the result for the corresponding class C_s will determined.

Theorem 4.3.2. *If $\xi(\hat{z}) \in C_s$ then $|\hat{\alpha}_3 - \hat{\alpha}_2^2| \leq \frac{1}{6}$.*

Proof. From (4.28) and (4.29) we obtain

$$|\hat{\alpha}_3-\hat{\alpha}_2^2| = \left|\frac{p_2}{12}-\frac{p_1^2}{24}-\frac{p_1^2}{64}\right| = \frac{1}{12}\left|p_2-\frac{11}{16}p_1^2\right|,$$

applying Lemma 3.11.5,we get,

$$|\hat{\alpha}_3-\hat{\alpha}_2^2| \leq \frac{1}{12}(2) = \frac{1}{6},$$

consequently, we have

$$|\hat{\alpha}_3-\hat{\alpha}_2^2| \leq \frac{1}{6},$$

which is the required result. □

4.4 Hankel Determinants

The following results are related to the class S_s^*.

Theorem 4.4.1. *If $\xi(\hat{z}) \in S_s^*$ then $|\hat{\alpha}_2\hat{\alpha}_3-\hat{\alpha}_4| \leq \frac{1}{4}$.*

Proof. From (4.10), (4.11) and (4.12), we obtain

$$|\hat{\alpha}_2\hat{\alpha}_3-\hat{\alpha}_4| = \left|\frac{p_1}{4}\left(\frac{p_2}{4}-\frac{p_1^2}{8}\right)-\left(\frac{p_3}{8}-\frac{3p_1p_2}{32}+\frac{p_1^3}{96}\right)\right| = \left|\frac{p_1^3}{24}-\frac{5p_1p_2}{32}+\frac{p_3}{8}\right|,$$

using lemma 3.11.3 we get,

$$\left|\frac{p_1^3}{24}-\frac{5p_1p_2}{32}+\frac{p_3}{8}\right| \leq 2\left|\frac{1}{24}\right|+2\left|\frac{5}{32}-\frac{2}{24}\right|+2\left|\frac{1}{24}-\frac{5}{32}+\frac{1}{8}\right| = \frac{1}{4},$$

thus, we obtain

$$|\hat{\alpha}_2\hat{\alpha}_3-\hat{\alpha}_4| \leq \frac{1}{4},$$

This completes the proof. □

Theorem 4.4.2. *If $\xi(\hat{z}) \in S_s^*$ then $|\hat{\alpha}_2\hat{\alpha}_4-\hat{\alpha}_3^2| \leq \frac{11}{16}$.*

Proof. From (4.10), (4.11) and (4.12), we obtain

$$|\hat{\alpha}_2\hat{\alpha}_4-\hat{\alpha}_3^2| = \left|\frac{p_1}{4}\left(\frac{p_3}{8}-\frac{3p_1p_2}{32}+\frac{p_1^3}{96}\right)-\left(\frac{p_2}{4}-\frac{p_1^2}{8}\right)^2\right|$$

$$|\hat{\alpha}_2\hat{\alpha}_4-\hat{\alpha}_3^2| = \left|\frac{p_1p_3}{32}+\frac{5p_1^2p_2}{128}-\frac{p_2^2}{16}-\frac{5p_1^4}{384}\right|,$$

$$|\hat{\alpha}_2\hat{\alpha}_4-\hat{\alpha}_3^2|=\left|\frac{5p_1^2}{128}\left(p_2-\frac{p_1^2}{3}\right)+\frac{p_1p_3}{32}-\frac{p_2^2}{16}\right|.$$

After applying Lemma 3.11.1 and Lemma 3.11.2we get

$$|\hat{\alpha}_2\hat{\alpha}_4-\hat{\alpha}_3^2|\leq\frac{40}{128}+\frac{4}{32}+\frac{4}{16}=\frac{11}{16},$$

$$|\hat{\alpha}_2\hat{\alpha}_4-\hat{\alpha}_3^2|\leq\frac{11}{16},$$

which is the required result. □

Theorem 4.4.3. *If $\xi(\hat{z})\in S_s^*$ then $|\mathscr{H}_1(3)|\leq\frac{25}{32}$.*

Proof. Hankel Determinant of order 3 defined as;

$$\mathscr{H}_3(1)=\hat{\alpha}_5(\hat{\alpha}_3-\hat{\alpha}_2^2)-\hat{\alpha}_4(\hat{\alpha}_4-\hat{\alpha}_2\hat{\alpha}_3)+\hat{\alpha}_3(\hat{\alpha}_2\hat{\alpha}_4-\hat{\alpha}_3^2),$$

Taking modulus on both sides and applying triangular inequality, we have

$$|\mathscr{H}_3(1)|\leq|\hat{\alpha}_5||\hat{\alpha}_3-\hat{\alpha}_2^2|+|\hat{\alpha}_4||\hat{\alpha}_4-\hat{\alpha}_2\hat{\alpha}_3|+|\hat{\alpha}_3||\hat{\alpha}_2\hat{\alpha}_4-\hat{\alpha}_3^2|.$$

On substituting values from Theorem 4.2.1, Theorem 4.3.1, Theorem 4.4.1 and Theorem 4.4.2

$$|\mathscr{H}_3(1)|\leq\frac{1}{2}\left(\frac{11}{16}\right)+\frac{1}{4}\left(\frac{1}{4}\right)+\frac{3}{4}\left(\frac{1}{2}\right)=\frac{25}{32},$$

hence, we get

$$|\mathscr{H}_3(1)|\leq\frac{25}{32}\approx 0.78125.$$

This completes the proof. □

Now, the following results belong to the convex class, C_s.

Theorem 4.4.4. *If $\xi(\hat{z})\in C_s$ then $|\hat{\alpha}_2\hat{\alpha}_3-\hat{\alpha}_4|\leq\frac{1}{16}$.*

Proof. From (4.28), (4.29) and (4.30) we obtain,

$$|\hat{\alpha}_2\hat{\alpha}_3-\hat{\alpha}_4|=\left|\frac{p_1}{8}\left(\frac{p_2}{12}-\frac{p_1^2}{24}\right)-\left(\frac{p_1^3}{384}-\frac{3p_1p_2}{128}+\frac{p_3}{32}\right)\right|=\left|\frac{p_1^3}{128}-\frac{13p_1p_2}{384}+\frac{p_3}{32}\right|,$$

using Lemma 3.11.3 we get,

$$\left|\frac{p_1^3}{128}-\frac{13p_1p_2}{384}+\frac{p_3}{32}\right|\leq 2\left|\frac{1}{128}\right|+2\left|\frac{13}{384}-\frac{2}{128}\right|+2\left|\frac{1}{128}-\frac{13}{384}+\frac{1}{32}\right|=\frac{1}{16},$$

therefore, we get

$$|\hat{\alpha}_2\hat{\alpha}_3-\hat{\alpha}_4|\leq\frac{1}{16}.$$

□

Theorem 4.4.5. *If $\xi(\hat{z}) \in C_s$ then $|\hat{\alpha}_2\hat{\alpha}_4 - \hat{\alpha}_3^2| \leq \frac{29}{384}$.*

Proof. From (4.28), (4.29) and (4.30), we obtain

$$|\hat{\alpha}_2\hat{\alpha}_4 - \hat{\alpha}_3^2| = \left|\frac{p_1}{8}\left(\frac{p_1^3}{384} - \frac{3p_1p_2}{128} + \frac{p_3}{32}\right) - \left(\frac{p_2}{12} - \frac{p_1^2}{24}\right)^2\right|,$$

$$|\hat{\alpha}_2\hat{\alpha}_4 - \hat{\alpha}_3^2| = \left|\frac{p_1p_3}{256} + \frac{37p_1^2p_2}{9216} - \frac{p_2^2}{144} - \frac{13p_1^4}{9216}\right|,$$

$$|\hat{\alpha}_2\hat{\alpha}_4 - \hat{\alpha}_3^2| = \left|\frac{37p_1^2}{9216}\left(p_2 - \frac{13p_1^2}{37}\right) + \frac{p_1p_3}{256} - \frac{p_2^2}{144}\right|.$$

After applying Lemma 3.11.1 and Lemma 3.11.2, we get

$$|\hat{\alpha}_2\hat{\alpha}_4 - \hat{a}_3^2| \leq \frac{37}{1152} + \frac{1}{36} + \frac{1}{64} = \frac{29}{384},$$

$$|\hat{\alpha}_2\hat{\alpha}_4 - \hat{\alpha}_3^2| \leq \frac{29}{384},$$

which is the required result. □

Theorem 4.4.6. *If $\xi(\hat{z}) \in C_s$ then $|\mathscr{H}_3(1)| \leq \frac{239}{5760}$.*

Proof. Hankel Determinant of order 3 defined as;

$$\mathscr{H}_3(1) = \hat{\alpha}_5(\hat{\alpha}_3 - \hat{\alpha}_2^2) - \hat{\alpha}_4(\hat{\alpha}_4 - \hat{\alpha}_2\hat{\alpha}_3) + \hat{\alpha}_3(\hat{\alpha}_2\hat{\alpha}_4 - \hat{\alpha}_3^2),$$

Taking modulus on both sides and applying triangular inequality, we have

$$|\mathscr{H}_3(1)| \leq |\hat{\alpha}_5||\hat{\alpha}_3 - \hat{\alpha}_2^2| + |\hat{\alpha}_4||\hat{\alpha}_4 - \hat{\alpha}_2\hat{\alpha}_3| + |\hat{\alpha}_3||\hat{\alpha}_2\hat{\alpha}_4 - \hat{\alpha}_3^2|.$$

On substituting values from Theorem 4.2.2, Theorem 4.3.2, Theorem 4.4.4 and Theorem 4.4.5 in above inequality,

$$|\mathscr{H}_3(1)| \leq \frac{1}{6}\left(\frac{29}{384}\right) + \frac{1}{16}\left(\frac{1}{16}\right) + \frac{3}{20}\left(\frac{1}{6}\right) = \frac{239}{5760},$$

$$|\mathscr{H}_3(1)| \leq \frac{239}{5760} \approx 0.04149.$$

This completes the proof. □

4.5 Zalcman Functional

The following result is related to the starlike class.

Theorem 4.5.1. *If* $\xi(\hat{z}) \in S_s^*$ *then* $|\hat{\alpha}_3^2 - \hat{\alpha}_5| \leq \frac{1}{4}$.

Proof. From (4.11) and (4.13), we have

$$|\hat{\alpha}_3^2 - \hat{\alpha}_5| = \left| \left(\frac{p_2}{4} - \frac{p_1^2}{8} \right)^2 - \left(\frac{3p_1^2 p_2}{64} - \frac{p_1 p_3}{8} - \frac{p_2^2}{32} + \frac{p_4}{8} \right) \right|,$$

on rearranging above equation we obtain

$$|\hat{\alpha}_3^2 - \hat{\alpha}_5| = \frac{1}{8} \left| \frac{p_1^4}{8} + \frac{3p_2^2}{4} - \frac{7p_1^2 p_2}{8} + 2\left(\frac{1}{2}\right) p_1 p_3 - p_4 \right|.$$

Using Lemma 3.11.6, we get

$$|\hat{\alpha}_3^2 - \hat{\alpha}_5| \leq \frac{1}{8}(2) = \frac{1}{4},$$

$$|\hat{\alpha}_3^2 - \hat{\alpha}_5| \leq \frac{1}{4} \approx 0.25,$$

this completes the proof. □

Now, the following result belongs to C_s class.

Theorem 4.5.2. *If* $\xi(\hat{z}) \in C_s$ *then* $|\hat{\alpha}_3^2 - \hat{\alpha}_5| \leq \frac{1}{20}$.

Proof. From (4.29) and (4.31), we get

$$|\hat{\alpha}_3^2 - \hat{\alpha}_5| = \left| \left(\frac{p_2}{12} - \frac{p_1^2}{24} \right)^2 - \left(\frac{p_4}{40} - \frac{p_1 p_3}{40} + \frac{3p_1^2 p_2}{320} - \frac{p_2^2}{160} \right) \right|,$$

on rearranging above equation we obtain

$$|\hat{\alpha}_3^2 - \hat{\alpha}_5| = \frac{1}{40} \left| \frac{5p_1^4}{72} + \frac{19p_2^2}{36} - \frac{47p_1^2 p_2}{72} + 2\left(\frac{1}{2}\right) p_1 p_3 - p_4 \right|.$$

Using Lemma 3.11.6, we get

$$|\hat{\alpha}_3^2 - \hat{\alpha}_5| \leq \frac{1}{40}(2) = \frac{1}{20},$$

$$|\hat{\alpha}_3^2 - \hat{\alpha}_5| \leq \frac{1}{20} \approx 0.05,$$

which is the required result. □

4.6 Summary

In this chapter, the category of starlike functions related to symmetric points was studied, that was defined by Khan *et al.* [78]. Additionally, a new category, namely the class of convex functions related to symmetric points, was introduced. For both of these classes, various results concerning coefficient bounds, the Fekete–Szegő inequality, the Zalcman functional, and Hankel determinants were investigated.

CHAPTER 5

CLASS OF STARLIKE FUNCTIONS WITH RESPECT TO SYMMETRIC POINTS SUBORDINATED WITH ǵ-SINE FUNCTION

5.1 Introduction

This chapter introduces a new subclass of analytic functions, the class of starlike functions related to symmetric points. This subclass is closely related to a q-series representation of trigonometric sine functions. Throughout this chapter, a number of fundamental findings such that coefficients inequalities, Fekete–Szegő Inequality, Zalcman functional and Hankel Determinants will examined.

Definition 5.1.1. *A function $\xi \in \mathscr{A}$, is in $S_s^*(\acute{\mathfrak{q}})$ then*

$$\frac{2\hat{z}\xi'(\hat{z})}{\xi(\hat{z})-\xi(-\hat{z})} \prec 1+\sin_{\acute{\mathfrak{q}}}(\hat{z}),$$

for all $\hat{z} \in \Omega$.

5.2 Coefficients Inequalities

Theorem 5.2.1. *If* $\xi(\hat{z}) \in S_s^*(\acute{\mathfrak{q}})$ *then*

$$|\hat{\alpha}_2| \leq \frac{1}{2}, \ |\hat{\alpha}_3| \leq \frac{1}{2},$$

$$|\hat{\alpha}_4| \leq \frac{1}{32}\left|1 - \frac{2}{[3]!_{\acute{\mathfrak{q}}}}\right| + \frac{1}{8}\left|1 + \frac{1}{[3]!_{\acute{\mathfrak{q}}}}\right| + \frac{1}{32}\left|3 - \frac{2}{[3]!_{\acute{\mathfrak{q}}}}\right|,$$

$$|\hat{\alpha}_5| \leq \frac{3}{8} + \frac{1}{2}\left|1 - \frac{3}{2[3]!_{\acute{\mathfrak{q}}}}\right| + \frac{1}{8}\left|1 - \frac{6}{[3]!_{\acute{\mathfrak{q}}}}\right|.$$

Proof. By definition

$$\frac{2\hat{z}\xi'(\hat{z})}{\xi(\hat{z}) - \xi(-\hat{z})} \prec 1 + \sin_{\acute{\mathfrak{q}}}(\hat{z}).$$

Since $\xi \in S_s^*$, using subordination principle, we have

$$\frac{2\hat{z}\xi'(\hat{z})}{\xi(\hat{z}) - \xi(-\hat{z})} = 1 + \sin_{\acute{\mathfrak{q}}}(\varpi(\hat{z})). \tag{5.1}$$

Let us define the function

$$p(\hat{z}) = \frac{1 + \varpi(\hat{z})}{1 - \varpi(\hat{z})} = 1 + p_1\hat{z} + p_2\hat{z}^2 + p_3\hat{z}^3 + p_4\hat{z}^4 + ..., \tag{5.2}$$

where $p(\hat{z})$ is analytic in Ω with $p(0) = 1$. This implies that

$$\varpi(\hat{z}) = \frac{p(\hat{z}) - 1}{p(\hat{z}) + 1}, \tag{5.3}$$

after simplification we get,

$$\varpi(\hat{z}) = \left(\frac{p_1}{2}\right)\hat{z} + \left(\frac{p_2}{2} - \frac{p_1^2}{4}\right)\hat{z}^2 + \left(\frac{p_3}{2} + \frac{p_1^3}{8} - \frac{p_1p_2}{2}\right)\hat{z}^3 + \left(\frac{3p_1^2p_2}{8} - \frac{p_1p_3}{2} - \frac{p_1^4}{16} - \frac{p_2^2}{4} + \frac{p_4}{2}\right)\hat{z}^4 + ... \tag{5.4}$$

As we know that,

$$\sin_{\acute{\mathfrak{q}}}(\varpi(\hat{z})) = \varpi(\hat{z}) - \frac{(\varpi(\hat{z}))^3}{[3]!_{\acute{\mathfrak{q}}}} + \frac{(\varpi(\hat{z}))^5}{[5]!_{\acute{\mathfrak{q}}}} + ...,$$

So, we get

$$\sin_{\acute{\mathfrak{q}}}(\varpi(\hat{z})) = \left(\frac{p_1}{2}\right)\hat{z} + \left(\frac{p_2}{2} - \frac{p_1^2}{4}\right)\hat{z}^2 + \left(\frac{p_3}{2} + \frac{p_1^3}{8} - \frac{p_1p_2}{2} - \frac{p_1^3}{8[3]!_{\acute{\mathfrak{q}}}}\right)\hat{z}^3 + \left(\frac{3p_1^2p_2}{8} - \frac{p_1p_3}{2} - \frac{p_1^4}{16} - \frac{p_2^2}{4} + \frac{p_4}{2} - \frac{1}{[3]!_{\acute{\mathfrak{q}}}}\left(\frac{3p_1^2p_2}{8} - \frac{3p_1^4}{16}\right)\right)\hat{z}^4 + ... \tag{5.5}$$

Now,

$$\frac{2\hat{z}\xi'(\hat{z})}{\xi(\hat{z})-\xi(-\hat{z})}=1+(2\hat{\alpha}_2)\hat{z}+(2\hat{\alpha}_3)\hat{z}^2+(4\hat{\alpha}_4-2\hat{\alpha}_2\hat{\alpha}_3)\hat{z}^3+(4\hat{\alpha}_5-2\hat{\alpha}_3^2)\hat{z}^4+...$$

On substituting values in (5.1) we get,

$$1+(2\hat{\alpha}_2)\hat{z}+(2\hat{\alpha}_3)\hat{z}^2+(4\hat{\alpha}_4-2\hat{\alpha}_2\hat{\alpha}_3)\hat{z}^3+(4\hat{\alpha}_5-2\hat{\alpha}_3^2)\hat{z}^4+...=1+\left(\frac{p_1}{2}\right)\hat{z}+\left(\frac{p_2}{2}-\frac{p_1^2}{4}\right)\hat{z}^2+$$
$$\left(\frac{p_3}{2}+\frac{p_1^3}{8}-\frac{p_1p_2}{2}-\frac{p_1^3}{8[3]!_{\acute{q}}}\right)\hat{z}^3+\left(\frac{3p_1^2p_2}{8}-\frac{p_1p_3}{2}-\frac{p_1^4}{16}-\frac{p_2^2}{4}+\frac{p_4}{2}-\frac{1}{[3]!_{\acute{q}}}\left(\frac{3p_1^2p_2}{8}-\frac{3p_1^4}{16}\right)\right)\hat{z}^4+...$$

On comparing both sides of above equation, we obtain

$$2\hat{\alpha}_2=\frac{p_1}{2}, \tag{5.6}$$
$$2\hat{\alpha}_3=\frac{p_2}{2}-\frac{p_1^2}{4}, \tag{5.7}$$
$$4\hat{\alpha}_4-2\hat{\alpha}_2\hat{\alpha}_3=\frac{p_3}{2}+\frac{p_1^3}{8}-\frac{p_1p_2}{2}-\frac{p_1^3}{8[3]!_{\acute{q}}}, \tag{5.8}$$
$$4\hat{\alpha}_5-2\hat{\alpha}_3^2=\frac{3p_1^2p_2}{8}-\frac{p_1p_3}{2}-\frac{p_1^4}{16}-\frac{p_2^2}{4}+\frac{p_4}{2}-\frac{1}{[3]!_{\acute{q}}}\left(\frac{3p_1^2p_2}{8}-\frac{3p_1^4}{16}\right), \tag{5.9}$$

On solving (5.6), we get

$$\hat{\alpha}_2=\frac{p_1}{4}. \tag{5.10}$$

By solving (5.7), we get

$$\hat{\alpha}_3=\frac{p_2}{4}-\frac{p_1^2}{8}. \tag{5.11}$$

From (5.8), we have

$$4\hat{\alpha}_4-2\hat{\alpha}_2\hat{\alpha}_3=\frac{p_3}{2}+\frac{p_1^3}{8}-\frac{p_1p_2}{2}-\frac{p_1^3}{8[3]!_{\acute{q}}},$$
$$4\hat{\alpha}_4=2\hat{\alpha}_2\hat{\alpha}_3+\frac{p_3}{2}+\frac{p_1^3}{8}-\frac{p_1p_2}{2}-\frac{p_1^3}{8[3]!_{\acute{q}}},$$

this implies that,

$$\hat{\alpha}_4=\frac{p_3}{8}-\frac{3p_1p_2}{32}+\left(\frac{1}{64}-\frac{1}{32[3]!_{\acute{q}}}\right)p_1^3. \tag{5.12}$$

Now solving (5.9), we get

$$4\hat{\alpha}_5-2\hat{\alpha}_3^2=\frac{3p_1^2p_2}{8}-\frac{p_1p_3}{2}-\frac{p_1^4}{16}-\frac{p_2^2}{4}+\frac{p_4}{2}-\frac{1}{[3]!_{\acute{q}}}\left(\frac{3p_1^2p_2}{8}-\frac{3p_1^4}{16}\right),$$

$$4\hat{\alpha}_5=2\hat{\alpha}_3^2+\frac{3p_1^2p_2}{8}-\frac{p_1p_3}{2}-\frac{p_1^4}{16}-\frac{p_2^2}{4}+\frac{p_4}{2}-\frac{1}{[3]!_{\acute{q}}}\left(\frac{3p_1^2p_2}{8}-\frac{3p_1^4}{16}\right),$$

we get,

$$\hat{\alpha}_5 = \frac{p_4}{8} - \frac{p_1p_3}{8} - \frac{p_2^2}{32} + \left(\frac{3}{64[3]!_{\acute{q}}} - \frac{1}{128}\right)p_1^4 + \left(\frac{1}{16} - \frac{3}{32[3]!_{\acute{q}}}\right)p_1^2p_2. \tag{5.13}$$

Next, to find out the values of coefficients (5.10) gives,

$$\hat{\alpha}_2 = \frac{p_1}{4},$$

Here by using Lemma 3.11.1, to find $\hat{\alpha}_2$, we get

$$|\hat{\alpha}_2| \leq \frac{1}{2}. \tag{5.14}$$

From (5.11), we have

$$\hat{\alpha}_3 = \frac{p_2}{4} - \frac{p_1^2}{8}.$$

To solve this equation for the coefficient $\hat{\alpha}_3$, we applying Lemma 3.11.2

$$|\hat{\alpha}_3| \leq \frac{1}{4}\left|p_2 - \frac{p_1^2}{2}\right|$$

$$|\hat{\alpha}_3| \leq \frac{2}{4}max\{1,0\}$$

$$|\hat{\alpha}_3| \leq \frac{1}{2}. \tag{5.15}$$

Now, on solving (5.12), we get

$$\hat{\alpha}_4 = \frac{p_3}{8} - \frac{3p_1p_2}{32} + \left(\frac{1}{64} - \frac{1}{32[3]!_{\acute{q}}}\right)p_1^3,$$

applying Lemma 3.11.3 in above equation, we get

$$|\hat{\alpha}_4| \leq 2\left|\frac{1}{64} - \frac{1}{32[3]!_{\acute{q}}}\right| + 2\left|\frac{3}{32} - 2\left(\frac{1}{64} - \frac{1}{32[3]!_{\acute{q}}}\right)\right| + 2\left|\frac{1}{64} - \frac{1}{32[3]!_{\acute{q}}} - \frac{3}{32} + \frac{1}{8}\right|,$$

$$|\hat{\alpha}_4| \leq \frac{1}{32}\left|1 - \frac{2}{[3]!_{\acute{q}}}\right| + \frac{1}{8}\left|1 + \frac{1}{[3]!_{\acute{q}}}\right| + \frac{1}{32}\left|3 - \frac{2}{[3]!_{\acute{q}}}\right|,$$

this implies that

$$|\hat{\alpha}_4| \leq \frac{1}{32}\left|1 - \frac{2}{[3]!_{\acute{q}}}\right| + \frac{1}{8}\left|1 + \frac{1}{[3]!_{\acute{q}}}\right| + \frac{1}{32}\left|3 - \frac{2}{[3]!_{\acute{q}}}\right|. \tag{5.16}$$

Next, solving (5.13) to find the value of $\hat{\alpha}_5$

$$\hat{\alpha}_5 = \frac{p_4}{8} - \frac{p_1p_3}{8} - \frac{p_2^2}{32} + \left(\frac{3}{64[3]!_{\acute{q}}} - \frac{1}{128}\right)p_1^4 + \left(\frac{1}{16} - \frac{3}{32[3]!_{\acute{q}}}\right)p_1^2p_2,$$

$$|\hat{\alpha}_5| \leq \frac{1}{8}|p_4 - p_1p_3| + \frac{1}{32}|p_2|^2 + \left|\frac{3}{64[3]!_{\acute{q}}} + \frac{1}{128}\right||p_1|^4 + \left|\frac{1}{16} - \frac{3}{32[3]!_{\acute{q}}}\right||p_1|^2|p_2|,$$

here applying Lemma 3.11.1 and Lemma 3.11.4, this leads us to the required result

$$|\hat{\alpha}_5| \leq \frac{3}{8} + \frac{1}{2}\left|1 - \frac{3}{2[3]!_{\acute{q}}}\right| + \frac{1}{8}\left|1 - \frac{6}{[3]!_{\acute{q}}}\right|. \tag{5.17}$$

□

If $\acute{q} \to 1^-$ in the above result, this leads us the already proved result for class S_s^* by Khan *et al.* [78], as shown in following corollary.

Corollary 5.2.1.1. *If $\xi(\hat{z}) \in S_s^*$ then*

$$|\hat{\alpha}_2| \leq \frac{1}{2}, \ |\hat{\alpha}_3| \leq \frac{1}{2}, \ |\hat{\alpha}_4| \leq \frac{1}{4}, \ |\hat{\alpha}_5| \leq \frac{3}{4}.$$

5.3 Fekete–Szegő Inequality

Theorem 5.3.1. *If $\xi(\hat{z}) \in S_s^*(\mathfrak{q})$ then $|\hat{\alpha}_3 - \hat{\alpha}_2^2| \leq \frac{1}{2}$.*

Proof. From (5.10) and (5.11), we obtain

$$|\hat{\alpha}_3 - \hat{\alpha}_2^2| = \left|\frac{p_2}{4} - \frac{p_1^2}{8} - \frac{p_1^2}{16}\right| = \frac{1}{4}\left|p_2 - \frac{3}{4}p_1^2\right|.$$

An application of Lemma 3.11.2 gives us

$$|\hat{\alpha}_3 - \hat{\alpha}_2^2| \leq \frac{1}{4}(2)$$

$$|\hat{\alpha}_3 - \hat{\alpha}_2^2| \leq \frac{1}{2}.$$

This completes the proof. □

5.4 Hankel Determinants

Theorem 5.4.1. *If $\xi(\hat{z}) \in S_s^*(\acute{q})$ then*

$$|\hat{\alpha}_2\hat{\alpha}_3 - \hat{\alpha}_4| \leq \frac{1}{32}\left|3 - \frac{2}{[3]!_{\acute{q}}}\right| + \frac{1}{8}\left|1 + \frac{1}{[3]!_{\acute{q}}}\right| + \frac{1}{32}\left|1 - \frac{2}{[3]!_{\acute{q}}}\right|.$$

Proof. From (5.10), (5.11) and (5.12), we have

$$|\hat{\alpha}_2\hat{\alpha}_3 - \hat{\alpha}_4| = \left|\frac{p_1}{4}\left(\frac{p_2}{4} - \frac{p_1^2}{8}\right) - \left(\frac{p_3}{8} - \frac{3p_1p_2}{32} + \left(\frac{1}{64} - \frac{1}{32[3]!_{\acute{q}}}\right)p_1^3\right)\right|,$$

after simplification we get,

$$|\hat{\alpha}_2\hat{\alpha}_3 - \hat{\alpha}_4| = \left|\left(\frac{3}{64} - \frac{1}{32[3]!_{\acute{q}}}\right)p_1^3 - \frac{5p_1p_2}{32} + \frac{p_3}{8}\right|.$$

Here we applying Lemma 3.11.3, we get

$$\left|\left(\frac{3}{64}-\frac{1}{32[3]!_{\acute{q}}}\right)p_1^3-\frac{5p_1p_2}{32}+\frac{p_3}{8}\right|\leq 2\left|\frac{3}{64}-\frac{1}{32[3]!_{\acute{q}}}\right|+2\left|\frac{5}{32}-2\left(\frac{3}{64}-\frac{1}{32[3]!_{\acute{q}}}\right)\right|$$
$$+2\left|\frac{3}{64}-\frac{1}{32[3]!_{\acute{q}}}-\frac{5}{32}+\frac{1}{8}\right|,$$

consequently, we have

$$|\hat{\alpha}_2\hat{\alpha}_3-\hat{\alpha}_4|\leq\frac{1}{32}\left|3-\frac{2}{[3]!_{\acute{q}}}\right|+\frac{1}{8}\left|1+\frac{1}{[3]!_{\acute{q}}}\right|+\frac{1}{32}\left|1-\frac{2}{[3]!_{\acute{q}}}\right|,$$

which is the required result. □

Now, taking $\acute{q}\to 1^-$ in this theorem, leads us to already derived result in [78], as given in next corollary.

Corollary 5.4.1.1. *If* $\xi(\hat{z})\in S_s^*$ *then* $|\hat{\alpha}_2\hat{\alpha}_3-\hat{\alpha}_4|\leq\frac{1}{4}$.

Theorem 5.4.2. *If* $\xi(\hat{z})\in S_s^*(\acute{q})$ *then*

$$|\hat{\alpha}_2\hat{\alpha}_4-\hat{\alpha}_3^2|\leq\frac{11}{16}.$$

Proof. From (5.10), (5.11) and (5.12), we obtain

$$|\hat{\alpha}_2\hat{\alpha}_4-\hat{\alpha}_3^2|=\left|\frac{p_1}{4}\left(\frac{p_3}{8}-\frac{3p_1p_2}{32}+\left(\frac{1}{64}-\frac{1}{32[3]!_{\acute{q}}}\right)p_1^3\right)-\left(\frac{p_2}{4}-\frac{p_1^2}{8}\right)^2\right|,$$

$$|\hat{\alpha}_2\hat{\alpha}_4-\hat{\alpha}_3^2|=\left|\frac{p_1p_3}{32}-\frac{p_2^2}{16}+\frac{5p_1^2p_2}{128}-\left(\frac{3}{256}+\frac{1}{128[3]!_{\acute{q}}}\right)p_1^4\right|,$$

$$|\hat{\alpha}_2\hat{\alpha}_4-\hat{\alpha}_3^2|\leq\frac{5}{128}|p_1|^2\left|p_2-\left(\frac{3}{10}+\frac{1}{5[3]!_{\acute{q}}}\right)p_1^2\right|+\frac{|p_1||p_3|}{32}+\frac{|p_2|^2}{16},$$

after applying Lemma 3.11.1 and Lemma 3.11.2 , we get

$$|\hat{\alpha}_2\hat{\alpha}_4-\hat{\alpha}_3^2|\leq\frac{40}{128}+\frac{4}{32}+\frac{4}{16}=\frac{11}{16},$$

$$|\hat{\alpha}_2\hat{\alpha}_4-\hat{\alpha}_3^2|\leq\frac{11}{16}.$$

This completes the proof. □

Now, considering $\acute{q}\to 1^-$ in this theorem, results leads us to the proved result, see [78].

Theorem 5.4.3. *If* $\xi(\hat{z})\in S_s^*(\acute{q})$ *then*

$$|\mathscr{H}_3(1)|\leq\frac{17}{32}+\frac{1}{4}\left|1-\frac{3}{2[3]!_{\acute{q}}}\right|+\frac{1}{16}\left|1-\frac{6}{[3]!_{\acute{q}}}\right|+\left[\frac{1}{32}\left|1-\frac{2}{[3]!_{\acute{q}}}\right|+\frac{1}{8}\left|1+\frac{1}{[3]!_{\acute{q}}}\right|+\frac{1}{32}\left|3-\frac{2}{[3]!_{\acute{q}}}\right|\right]$$
$$\left[\frac{1}{32}\left|1-\frac{2}{[3]!_{\acute{q}}}\right|+\frac{1}{8}\left|1+\frac{1}{[3]!_{\acute{q}}}\right|+\frac{1}{32}\left|3-\frac{2}{[3]!_{\acute{q}}}\right|\right]$$

Proof. Hankel Determinant of order 3 defined as;

$$\mathscr{H}_3(1) = \hat{\alpha}_5(\hat{\alpha}_3 - \hat{\alpha}_2^2) - \hat{\alpha}_4(\hat{\alpha}_4 - \hat{\alpha}_2\hat{\alpha}_3) + \hat{\alpha}_3(\hat{\alpha}_2\hat{\alpha}_4 - \hat{\alpha}_3^2),$$

Taking modulus on both sides and applying triangular inequality, we have

$$|\mathscr{H}_3(1)| \leq |\hat{\alpha}_5||\hat{\alpha}_3 - \hat{\alpha}_2^2| + |\hat{\alpha}_4||\hat{\alpha}_4 - \hat{\alpha}_2\hat{\alpha}_3| + |\hat{\alpha}_3||\hat{\alpha}_2\hat{\alpha}_4 - \hat{\alpha}_3^2|.$$

On substituting values of Theorem 5.2.1, Theorem 5.3.1, Theorem 5.4.1 and Theorem 5.4.2, we get

$$\begin{aligned}|\mathscr{H}_3(1)| \leq \frac{1}{2}\left(\frac{11}{16}\right) &+ \left[\frac{1}{32}\left|1 - \frac{2}{[3]!_{\acute{q}}}\right| + \frac{1}{8}\left|1 + \frac{1}{[3]!_{\acute{q}}}\right| + \frac{1}{32}\left|3 - \frac{2}{[3]!_{\acute{q}}}\right|\right] \\ &\left[\frac{1}{32}\left|1 - \frac{2}{[3]!_{\acute{q}}}\right| + \frac{1}{8}\left|1 + \frac{1}{[3]!_{\acute{q}}}\right| + \frac{1}{32}\left|3 - \frac{2}{[3]!_{\acute{q}}}\right|\right] \\ &+ \frac{1}{2}\left[\frac{3}{8} + 8\left|\frac{1}{16} - \frac{3}{32[3]!_{\acute{q}}}\right| + 16\left|\frac{3}{64[3]!_{\acute{q}}} - \frac{1}{128}\right|\right],\end{aligned}$$

$$\begin{aligned}|\mathscr{H}_3(1)| \leq \frac{17}{32} + \frac{1}{4}\left|1 - \frac{3}{2[3]!_{\acute{q}}}\right| + \frac{1}{16}\left|1 - \frac{6}{[3]!_{\acute{q}}}\right| &+ \left[\frac{1}{32}\left|1 - \frac{2}{[3]!_{\acute{q}}}\right| + \frac{1}{8}\left|1 + \frac{1}{[3]!_{\acute{q}}}\right| + \frac{1}{32}\left|3 - \frac{2}{[3]!_{\acute{q}}}\right|\right] \\ &\left[\frac{1}{32}\left|1 - \frac{2}{[3]!_{\acute{q}}}\right| + \frac{1}{8}\left|1 + \frac{1}{[3]!_{\acute{q}}}\right| + \frac{1}{32}\left|3 - \frac{2}{[3]!_{\acute{q}}}\right|\right],\end{aligned}$$

which is the required result. □

Now, taking $\acute{q} \to 1^-$, we get the known result as shown below.

Corollary 5.4.3.1. *Consider* $\xi(\hat{z}) \in S_s^*$ *then* $|\mathscr{H}_3(1)| \leq \frac{25}{32}$.

5.5 Zalcman Functional

Theorem 5.5.1. *If* $\xi(\hat{z}) \in S_s^*(\acute{q})$ *then*

$$|\hat{\alpha}_3^2 - \hat{\alpha}_5| \leq \frac{1}{4}.$$

Proof. From (5.11) and (5.13), we have

$$|\hat{\alpha}_3^2 - \hat{\alpha}_5| = \left| \left(\frac{p_2}{4} - \frac{p_1^2}{8} \right)^2 - \left(\frac{p_4}{8} - \frac{p_1 p_3}{8} - \frac{p_2^2}{32} + \left(\frac{3}{64[3]!_{\acute{q}}} - \frac{1}{128} \right) p_1^4 + \left(\frac{1}{16} - \frac{3}{32[3]!_{\acute{q}}} \right) p_1^2 p_2 \right) \right|.$$

On solving this equation we obtain

$$|\hat{\alpha}_3^2 - \hat{\alpha}_5| = \frac{1}{8} \left| \left(\frac{3}{16} - \frac{3}{8[3]!_{\acute{q}}} \right) p_1^4 + \frac{3p_2^2}{4} - \left(1 - \frac{3}{4[3]!_{\acute{q}}} \right) p_1^2 p_2 + 2\left(\frac{1}{2} \right) p_1 p_3 - p_4 \right|.$$

Here applying Lemma 3.11.6, this leads us to the required result

$$|\hat{\alpha}_3^2 - \hat{\alpha}_5| \leq \frac{1}{4} \approx 0.25.$$

□

Now, taking $\acute{q} \to 1^-$, we get the known result as shown below.

Corollary 5.5.1.1. *Consider* $\xi(\hat{z}) \in S_s^*$ *then* $|\hat{\alpha}_3^2 - \hat{\alpha}_5| \leq \frac{1}{4}$.

5.6 Summary

In this chapter, a category of starlike functions related to symmetric points associated with the $\acute{q}$-sine function was defined. For this class few results coefficient bounds, Fekete–Szegő inequality, Hankel determinants and Zalcman functional were investigated. Additionally, this study introduces several corollaries that reveal a remarkable alignment with the results previously investigated by Khan *et al.* [78] when the limit $\acute{q} \to 1^-$ applied.

CHAPTER 6

q́-EXTENSION OF STARLIKE AND CONVEX FUNCTIONS WITH RESPECT TO SYMMETRIC POINTS SUBORDINATED WITH q́-SINE FUNCTION

6.1 Introduction

The purpose of this chapter is to define certain new classes of univalent functions. These classes are q-extensions of starlike and convex functions related to the q-series of particular trigonometric function that is sine function. Here $\check{q}_o \in (0,1)$. This chapter includes some important results.

The class of q́-starlike functions with respect to symmetric points related to q́-sine function is defined as under.

Definition 6.1.1. *A function $\xi \in \mathscr{A}$, is in $S_s^*(\acute{q} - sin)$ then*

$$\frac{2\hat{z}\mathscr{D}_{\acute{q}}(\xi(\hat{z}))}{\xi(\hat{z}) - \xi(-\hat{z})} \prec 1 + \sin_{\acute{q}}(\hat{z}),$$

for all $\hat{z} \in \Omega$.

The class of q́-convex functions with respect to symmetric points related to q́-sine function is defined as under.

Definition 6.1.2. *A function $\xi \in \mathscr{A}$, is in $C_s(\acute{q} - sin)$ then*

$$\frac{2\mathscr{D}_{\acute{q}}[\hat{z}\mathscr{D}_{\acute{q}}(\xi(\hat{z}))]}{\mathscr{D}_{\acute{q}}(\xi(\hat{z}) - \xi(-\hat{z}))} \prec 1 + \sin_{\acute{q}}(\hat{z}),$$

for all $\hat{z} \in \Omega$.

6.2 Coefficients Inequalities

The following result is related to the $\acute{\mathfrak{q}}$-starlike class.

Theorem 6.2.1. *If* $\xi \in S_s^*(\acute{\mathfrak{q}} - sin)$ *then*

$$|\hat{\alpha}_2| = \frac{1}{(1+\check{\mathfrak{q}}_o)}, \qquad |\hat{\alpha}_3| = \frac{1}{\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o)},$$

$$|\hat{\alpha}_4| = \frac{1}{(1+\check{\mathfrak{q}}_o+\check{\mathfrak{q}}_o^2+\check{\mathfrak{q}}_o^3)}\left[\left|\frac{1}{4}-\frac{1}{4[3]!_{\check{\mathfrak{q}}_o}}-\frac{1}{4\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o)}\right|+\left|\frac{1}{2}+\frac{1}{2[3]!_{\check{\mathfrak{q}}_o}}\right|+\left|\frac{1}{4}-\frac{1}{4[3]!_{\check{\mathfrak{q}}_o}}+\frac{1}{4\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o)}\right|\right],$$

$$|\hat{\alpha}_5| = \frac{1}{(1+\check{\mathfrak{q}}_o+\check{\mathfrak{q}}_o^2+\check{\mathfrak{q}}_o^3)}\left[1+\left|3-\frac{2}{\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o)}-\frac{3}{[3]!_{\check{\mathfrak{q}}_o}}\right|+\left|1-\frac{1}{\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o)}\right|+\left|\frac{1}{\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o)}+\frac{3}{[3]!_{\check{\mathfrak{q}}_o}}-1\right|\right].$$

Proof. By definition

$$\frac{2\hat{z}\mathscr{D}_{\acute{\mathfrak{q}}}(\xi(\hat{z}))}{\xi(\hat{z})-\xi(-\hat{z})} \prec 1+\sin_{\acute{\mathfrak{q}}}(\hat{z})$$

Since $\xi \in S_s^*(\acute{\mathfrak{q}} - sin)$, using subordination principle, we have

$$\frac{2\hat{z}\mathscr{D}_{\acute{\mathfrak{q}}}(\xi(\hat{z}))}{\xi(\hat{z})-\xi(-\hat{z})} = 1+\sin_{\acute{\mathfrak{q}}}(\varpi(\hat{z})). \tag{6.1}$$

Let us define the function

$$p(\hat{z}) = \frac{1+\varpi(\hat{z})}{1-\varpi(\hat{z})} = 1+p_1\hat{z}+p_2\hat{z}^2+p_3\hat{z}^3+p_4\hat{z}^4+..., \tag{6.2}$$

where $p(\hat{z})$ is analytic in Ω with $p(0)=1$. This implies that

$$\varpi(\hat{z}) = \frac{p(\hat{z})-1}{p(\hat{z})+1}, \tag{6.3}$$

after simplification we get,

$$\varpi(\hat{z}) = \left(\frac{p_1}{2}\right)\hat{z}+\left(\frac{p_2}{2}-\frac{p_1^2}{4}\right)\hat{z}^2+\left(\frac{p_3}{2}+\frac{p_1^3}{8}-\frac{p_1p_2}{2}\right)\hat{z}^3 + \left(\frac{3p_1^2p_2}{8}-\frac{p_1p_3}{2}-\frac{p_1^4}{16}-\frac{p_2^2}{4}+\frac{p_4}{2}\right)\hat{z}^4+... \tag{6.4}$$

As we know that,

$$\sin_{\acute{\mathfrak{q}}}(\varpi(\hat{z})) = \varpi(\hat{z})-\frac{(\varpi(\hat{z}))^3}{[3]!_{\check{\mathfrak{q}}_o}}+\frac{(\varpi(\hat{z}))^5}{[5]!_{\check{\mathfrak{q}}_o}}+...,$$

So, we have

$$\sin_{\mathfrak{q}}(\varpi(\hat{z})) = \left(\frac{p_1}{2}\right)\hat{z} + \left(\frac{p_2}{2} - \frac{p_1^2}{4}\right)\hat{z}^2 + \left(\frac{p_3}{2} + \frac{p_1^3}{8} - \frac{p_1 p_2}{2} - \frac{p_1^3}{8[3]!_{\check{\mathfrak{q}}_o}}\right)\hat{z}^3 + \left(\frac{3p_1^2 p_2}{8} - \frac{p_1 p_3}{2} - \frac{p_1^4}{16} - \frac{p_2^2}{4} + \frac{p_4}{2} - \frac{1}{[3]!_{\check{\mathfrak{q}}_o}}\left(\frac{3p_1^2 p_2}{8} - \frac{3p_1^4}{16}\right)\right)\hat{z}^4 + \dots \tag{6.5}$$

Now, we get

$$\xi(\hat{z}) = \hat{z} + \hat{\alpha}_2 \hat{z}^2 + \hat{\alpha}_3 \hat{z}^3 + \hat{\alpha}_4 \hat{z}^4 + \dots,$$

and,

$$\mathscr{D}_{\mathfrak{q}}(\xi(\hat{z})) = \frac{\xi(\hat{z}) - \xi(\check{\mathfrak{q}}_o \hat{z})}{(1-\check{\mathfrak{q}}_o)\hat{z}},$$

after simplification we get,

$$\frac{2\hat{z}\mathscr{D}_{\mathfrak{q}}(\xi(\hat{z}))}{\xi(\hat{z}) - \xi(-\hat{z})} = 1 + \hat{\alpha}_2(1+\check{\mathfrak{q}}_o)\hat{z} + \hat{\alpha}_3\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o)\hat{z}^2 + [\hat{\alpha}_4(1+\check{\mathfrak{q}}_o+\check{\mathfrak{q}}_o^2+\check{\mathfrak{q}}_o^3) - \hat{\alpha}_2\hat{\alpha}_3(1+\check{\mathfrak{q}}_o)]\hat{z}^3 + [\hat{\alpha}_5\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o+\check{\mathfrak{q}}_o^2+\check{\mathfrak{q}}_o^3) - \hat{\alpha}_3^2\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o)]\hat{z}^4 + \dots$$

On substituting values in (6.1), we get

$$1 + \hat{\alpha}_2(1+\check{\mathfrak{q}}_o)\hat{z} + \hat{\alpha}_3\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o)\hat{z}^2 + [\hat{\alpha}_4(1+\check{\mathfrak{q}}_o+\check{\mathfrak{q}}_o^2+\check{\mathfrak{q}}_o^3) - \hat{\alpha}_2\hat{\alpha}_3(1+\check{\mathfrak{q}}_o)]\hat{z}^3 + [\hat{\alpha}_5\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o+\check{\mathfrak{q}}_o^2+\check{\mathfrak{q}}_o^3) - \hat{\alpha}_3^2\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o)]\hat{z}^4 + \dots = 1 + \left(\frac{p_1}{2}\right)\hat{z} + \left(\frac{p_2}{2} - \frac{p_1^2}{4}\right)\hat{z}^2 + \left(\frac{p_3}{2} + \frac{p_1^3}{8} - \frac{p_1 p_2}{2} - \frac{p_1^3}{8[3]!_{\check{\mathfrak{q}}_o}}\right)\hat{z}^3 + \left(\frac{3p_1^2 p_2}{8} - \frac{p_1 p_3}{2} - \frac{p_1^4}{16} - \frac{p_2^2}{4} + \frac{p_4}{2} - \frac{1}{[3]!_{\check{\mathfrak{q}}_o}}\left(\frac{3p_1^2 p_2}{8} - \frac{3p_1^4}{16}\right)\right)\hat{z}^4 + \dots$$

On comparing both sides of above equation for similar powers of $\hat{z}$, we get following equations,

$$\hat{\alpha}_2(1+\check{\mathfrak{q}}_o) = \frac{p_1}{2}, \tag{6.6}$$

$$\hat{\alpha}_3\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o) = \frac{p_2}{2} - \frac{p_1^2}{4}, \tag{6.7}$$

$$\hat{\alpha}_4(1+\check{\mathfrak{q}}_o+\check{\mathfrak{q}}_o^2+\check{\mathfrak{q}}_o^3) - \hat{\alpha}_2\hat{\alpha}_3(1+\check{\mathfrak{q}}_o) = \frac{p_3}{2} + \frac{p_1^3}{8} - \frac{p_1 p_2}{2} - \frac{p_1^3}{8[3]!_{\check{\mathfrak{q}}_o}}, \tag{6.8}$$

$$\hat{\alpha}_5\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o+\check{\mathfrak{q}}_o^2+\check{\mathfrak{q}}_o^3) - \hat{\alpha}_3^2\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o) = \frac{3p_1^2 p_2}{8} - \frac{p_1 p_3}{2} - \frac{p_1^4}{16} - \frac{p_2^2}{4} + \frac{p_4}{2} - \frac{1}{[3]!_{\check{\mathfrak{q}}_o}}\left(\frac{3p_1^2 p_2}{8} - \frac{3p_1^4}{16}\right), \tag{6.9}$$

On solving (6.6), we get

$$\hat{\alpha}_2 = \frac{p_1}{2(1+\check{\mathfrak{q}}_o)}. \tag{6.10}$$

On solving (6.7), we get

$$\hat{\alpha}_3 = \frac{1}{\check{\text{q}}_o(1+\check{\text{q}}_o)}\left(\frac{p_2}{2} - \frac{p_1^2}{4}\right). \tag{6.11}$$

Now, solving (6.8), we have

$$\hat{\alpha}_4(1+\check{\text{q}}_o+\check{\text{q}}_o^2+\check{\text{q}}_o^3) - \hat{\alpha}_2\hat{\alpha}_3(1+\check{\text{q}}_o) = \frac{p_3}{2}+\frac{p_1^3}{8}-\frac{p_1p_2}{2}-\frac{p_1^3}{8[3]!_{\check{\text{q}}_o}},$$

$$\hat{\alpha}_4(1+\check{\text{q}}_o+\check{\text{q}}_o^2+\check{\text{q}}_o^3) = \hat{\alpha}_2\hat{\alpha}_3(1+\check{\text{q}}_o) + \frac{p_3}{2}+\frac{p_1^3}{8}-\frac{p_1p_2}{2}-\frac{p_1^3}{8[3]!_{\check{\text{q}}_o}}.$$

On substituting values of $\hat{\alpha}_2$ and $\hat{\alpha}_3$, we get

$$\hat{\alpha}_4(1+\check{\text{q}}_o+\check{\text{q}}_o^2+\check{\text{q}}_o^3) = (1+\check{\text{q}}_o)\left[\frac{p_1}{2(1+\check{\text{q}}_o)}\right]\left[\frac{1}{\check{\text{q}}_o(1+\check{\text{q}}_o)}\left(\frac{p_2}{2}-\frac{p_1^2}{4}\right)\right]+\frac{p_3}{2}+\frac{p_1^3}{8}- \frac{p_1p_2}{2}-\frac{p_1^3}{8[3]!_{\check{\text{q}}_o}},$$

this implies that,

$$\hat{\alpha}_4 = \frac{1}{(1+\check{\text{q}}_o+\check{\text{q}}_o^2+\check{\text{q}}_o^3)}\left[\left(\frac{1}{8}-\frac{1}{8[3]!_{\check{\text{q}}_o}}-\frac{1}{8\check{\text{q}}_o(1+\check{\text{q}}_o)}\right)p_1^3-\left(\frac{1}{2}-\frac{1}{4\check{\text{q}}_o(1+\check{\text{q}}_o)}\right)p_1p_2+\frac{p_3}{2}\right]. \tag{6.12}$$

Now, solving (6.9), we obtain

$$\check{\text{q}}_o(1+\check{\text{q}}_o+\check{\text{q}}_o^2+\check{\text{q}}_o^3)\hat{\alpha}_5 - \check{\text{q}}_o(1+\check{\text{q}}_o)\hat{\alpha}_3^2 = \frac{3p_1^2p_2}{8}-\frac{p_1p_3}{2}-\frac{p_1^4}{16}-\frac{p_2^2}{4}+\frac{p_4}{2}-\frac{1}{[3]!_{\check{\text{q}}_o}}\left(\frac{3p_1^2p_2}{8}-\frac{3p_1^4}{16}\right),$$

$$\check{\text{q}}_o(1+\check{\text{q}}_o+\check{\text{q}}_o^2+\check{\text{q}}_o^3)\hat{\alpha}_5 = \check{\text{q}}_o(1+\check{\text{q}}_o)\hat{\alpha}_3^2+\frac{3p_1^2p_2}{8}-\frac{p_1p_3}{2}-\frac{p_1^4}{16}-\frac{p_2^2}{4}+\frac{p_4}{2} -\frac{1}{[3]!_{\check{\text{q}}_o}}\left(\frac{3p_1^2p_2}{8}-\frac{3p_1^4}{16}\right),$$

substituting value of $\hat{\alpha}_3$, we get

$$\check{\text{q}}_o(1+\check{\text{q}}_o+\check{\text{q}}_o^2+\check{\text{q}}_o^3)\hat{\alpha}_5 = \check{\text{q}}_o(1+\check{\text{q}}_o)\left[\frac{1}{\check{\text{q}}_o(1+\check{\text{q}}_o)}\left(\frac{p_2}{2}-\frac{p_1^2}{4}\right)\right]^2+\frac{3p_1^2p_2}{8}-\frac{p_1p_3}{2}-\frac{p_1^4}{16}- \frac{p_2^2}{4}+\frac{p_4}{2}-\frac{1}{[3]!_{\check{\text{q}}_o}}\left(\frac{3p_1^2p_2}{8}-\frac{3p_1^4}{16}\right),$$

after simplification, we have

$$\hat{\alpha}_5 = \frac{1}{\check{\text{q}}_o(1+\check{\text{q}}_o+\check{\text{q}}_o^2+\check{\text{q}}_o^3)}\left[\left(\frac{3}{8}-\frac{1}{4\check{\text{q}}_o(1+\check{\text{q}}_o)}-\frac{3}{8[3]!_{\check{\text{q}}_o}}\right)p_1^2p_2-\left(\frac{1}{4}-\frac{1}{4\check{\text{q}}_o(1+\check{\text{q}}_o)}\right)p_2^2 +\left(\frac{1}{16\check{\text{q}}_o(1+\check{\text{q}}_o)}-\frac{1}{16}+\frac{3}{16[3]!_{\check{\text{q}}_o}}\right)p_1^4+\frac{p_4}{2}-\frac{p_1p_3}{2}\right]. \tag{6.13}$$

Next, to find out the values of coefficients, (6.10) solving for $\hat{\alpha}_2$,

$$\hat{\alpha}_2 = \frac{p_1}{2(1+\check{\mathfrak{q}}_o)}.$$

An application of Lemma 3.11.1 leads us to the result,

$$|\hat{\alpha}_2| \leq \frac{1}{(1+\check{\mathfrak{q}}_o)}. \tag{6.14}$$

From (6.11), we have

$$\hat{\alpha}_3 = \frac{1}{\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o)}\left(\frac{p_2}{2} - \frac{p_1^2}{4}\right).$$

Using Lemma 3.11.2, we obtain

$$|\hat{\alpha}_3| \leq \frac{1}{2\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o)}\left|p_2 - \frac{p_1^2}{2}\right|,$$

$$|\hat{\alpha}_3| \leq \frac{2}{2(1+\check{\mathfrak{q}}_o)} max\{1,0\} = \frac{1}{\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o)},$$

$$|\hat{\alpha}_3| \leq \frac{1}{\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o)}. \tag{6.15}$$

Solving (6.12), we get

$$\hat{\alpha}_4 = \frac{1}{(1+\check{\mathfrak{q}}_o+\check{\mathfrak{q}}_o^2+\check{\mathfrak{q}}_o^3)}\left[\left(\frac{1}{8} - \frac{1}{8[3]!_{\check{\mathfrak{q}}_o}} - \frac{1}{8\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o)}\right)p_1^3 - \left(\frac{1}{2} - \frac{1}{4\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o)}\right)p_1p_2 + \frac{p_3}{2}\right],$$

$$|\hat{\alpha}_4| \leq \frac{1}{(1+\check{\mathfrak{q}}_o+\check{\mathfrak{q}}_o^2+\check{\mathfrak{q}}_o^3)}\left|\left(\frac{1}{8} - \frac{1}{8[3]!_{\check{\mathfrak{q}}_o}} - \frac{1}{8\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o)}\right)p_1^3 - \left(\frac{1}{2} - \frac{1}{4\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o)}\right)p_1p_2 + \frac{p_3}{2}\right|,$$

An application of Lemma 3.11.3 leads us to the result

$$\left|\left(\frac{1}{8} - \frac{1}{8[3]!_{\check{\mathfrak{q}}_o}} - \frac{1}{8\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o)}\right)p_1^3 - \left(\frac{1}{2} - \frac{1}{4\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o)}\right)p_1p_2 + \frac{p_3}{2}\right| \leq 2\left|\frac{1}{8} - \frac{1}{8[3]!_{\check{\mathfrak{q}}_o}} - \frac{1}{8\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o)}\right| + 2\left|\frac{1}{2} - \frac{1}{4\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o)} - 2\left(\frac{1}{8} - \frac{1}{8[3]!_{\check{\mathfrak{q}}_o}} - \frac{1}{8\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o)}\right)\right| + 2\left|\frac{1}{8} - \frac{1}{8[3]!_{\check{\mathfrak{q}}_o}} - \frac{1}{8\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o)} - \left(\frac{1}{2} - \frac{1}{4\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o)}\right) + \frac{1}{2}\right|,$$

$$\left|\left(\frac{1}{8} - \frac{1}{8[3]!_{\check{\mathfrak{q}}_o}} - \frac{1}{8\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o)}\right)p_1^3 - \left(\frac{1}{2} - \frac{1}{4\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o)}\right)p_1p_2 + \frac{p_3}{2}\right| \leq \left|\frac{1}{4} - \frac{1}{4[3]!_{\check{\mathfrak{q}}_o}} - \frac{1}{4\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o)}\right| + \left|\frac{1}{2} + \frac{1}{2[3]!_{\check{\mathfrak{q}}_o}}\right| + \left|\frac{1}{4} - \frac{1}{4[3]!_{\check{\mathfrak{q}}_o}} + \frac{1}{4\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o)}\right|,$$

thus, we have

$$|\hat{\alpha}_4| \leq \frac{1}{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}\left[\left|\frac{1}{4}-\frac{1}{4[3]!_{\check{q}_o}}-\frac{1}{4\check{q}_o(1+\check{q}_o)}\right|+\left|\frac{1}{2}+\frac{1}{2[3]!_{\check{q}_o}}\right|\right.$$
$$\left.+\left|\frac{1}{4}-\frac{1}{4[3]!_{\check{q}_o}}+\frac{1}{4\check{q}_o(1+\check{q}_o)}\right|\right]. \quad (6.16)$$

On solving (6.13), to find $\hat{\alpha}_5$

$$\hat{\alpha}_5 = \frac{1}{\check{q}_o(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}\left[\left(\frac{3}{8}-\frac{1}{4\check{q}_o(1+\check{q}_o)}-\frac{3}{8[3]!_{\check{q}_o}}\right)p_1^2p_2-\left(\frac{1}{4}-\frac{1}{4\check{q}_o(1+\check{q}_o)}\right)p_2^2\right.$$
$$\left.+\left(\frac{1}{16\check{q}_o(1+\check{q}_o)}-\frac{1}{16}+\frac{3}{16[3]!_{\check{q}_o}}\right)p_1^4+\frac{p_4}{2}-\frac{p_1p_3}{2}\right],$$

applying modulus on both sides of above equation, we get

$$|\hat{\alpha}_5| \leq \frac{1}{\check{q}_o(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}\left[\left|\frac{3}{8}-\frac{1}{4\check{q}_o(1+\check{q}_o)}-\frac{3}{8[3]!_{\check{q}_o}}\right||p_1|^2|p_2|+\left|\frac{1}{4}-\frac{1}{4\check{q}_o(1+\check{q}_o)}\right||p_2|^2+\right.$$
$$\left.\left|\frac{1}{16\check{q}_o(1+\check{q}_o)}-\frac{1}{16}+\frac{3}{16[3]!_{\check{q}_o}}\right||p_1|^4+\frac{1}{2}|p_1p_3-p_4|\right].$$

Now, using Lemma 3.11.1 and Lemma 3.11.4, we get

$$|\hat{\alpha}_5| \leq \frac{1}{\check{q}_o(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}\left[\frac{2}{2}+\left|3-\frac{2}{\check{q}_o(1+\check{q}_o)}-\frac{3}{[3]!_{\check{q}_o}}\right|+\left|\frac{\check{q}_o(1+\check{q}_o)-1}{\check{q}_o(1+\check{q}_o)}\right|+\right.$$
$$\left.\left|1-\frac{3}{[3]!_{\check{q}_o}}-\frac{1}{\check{q}_o(1+\check{q}_o)}\right|\right],$$

$$|\hat{\alpha}_5| \leq \frac{1}{\check{q}_o(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}\left[1+\left|3-\frac{2}{\check{q}_o(1+\check{q}_o)}-\frac{3}{[3]!_{\check{q}_o}}\right|+\left|\frac{\check{q}_o(1+\check{q}_o)-1}{\check{q}_o(1+\check{q}_o)}\right|+\right.$$
$$\left.\left|1-\frac{3}{[3]!_{\check{q}_o}}-\frac{1}{\check{q}_o(1+\check{q}_o)}\right|\right]. \quad (6.17)$$

Hence, this completes the proof. □

If $\check{q}_o \to 1^-$ in the above result, this leads us the already proved result for class S_s^* by Khan *et al.* [78], as shown in the following corollary.

Corollary 6.2.1.1. *If $\xi(\hat{z}) \in S_s^*$ then*

$$|\hat{\alpha}_2| \leq \frac{1}{2}, \ |\hat{\alpha}_3| \leq \frac{1}{2}, \ |\hat{\alpha}_4| \leq \frac{1}{4}, \ |\hat{\alpha}_5| \leq \frac{3}{4}.$$

Now, the following result is related to the class of q́-convex functions.

Theorem 6.2.2. *If $\xi \in C_s(\acute{\mathfrak{q}} - sin)$ then*

$$|\hat{\alpha}_2| = \frac{1}{(1+\check{\mathfrak{q}}_o)^2}, \qquad |\hat{\alpha}_3| = \frac{1}{\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o)(1+\check{\mathfrak{q}}_o+\check{\mathfrak{q}}_o^2)},$$

$$|\hat{\alpha}_4| = \frac{1}{(1+\check{\mathfrak{q}}_o+\check{\mathfrak{q}}_o^2+\check{\mathfrak{q}}_o^3)^2}\left[\left|\frac{1}{4} - \frac{1}{4[3]!_{\check{\mathfrak{q}}_o}} - \frac{1}{4\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o)}\right| + \left|\frac{1}{2} + \frac{1}{2[3]!_{\check{\mathfrak{q}}_o}}\right| + \left|\frac{1}{4} - \frac{1}{4[3]!_{\check{\mathfrak{q}}_o}} + \frac{1}{4\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o)}\right|\right],$$

$$|\hat{\alpha}_5| = \frac{1}{\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o+\check{\mathfrak{q}}_o^2+\check{\mathfrak{q}}_o^3)(1+\check{\mathfrak{q}}_o+\check{\mathfrak{q}}_o^2+\check{\mathfrak{q}}_o^3+\check{\mathfrak{q}}_o^4)}\left[1 + \left|3 - \frac{2}{\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o)} - \frac{3}{[3]!_{\check{\mathfrak{q}}_o}}\right| + \left|1 - \frac{1}{\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o)}\right| + \left|\frac{1}{\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o)} + \frac{3}{[3]!_{\check{\mathfrak{q}}_o}} - 1\right|\right].$$

Proof. By definition

$$\frac{2\mathscr{D}_{\acute{\mathfrak{q}}}[\hat{z}\mathscr{D}_{\acute{\mathfrak{q}}}(\xi(\hat{z}))]}{\mathscr{D}_{\acute{\mathfrak{q}}}(\xi(\hat{z}) - \xi(-\hat{z}))} \prec 1 + \sin_{\acute{\mathfrak{q}}}(\hat{z})$$

Since $\xi \in C(\acute{\mathfrak{q}} - sin)$, using subordination principle, we have

$$\frac{2\mathscr{D}_{\acute{\mathfrak{q}}}[\hat{z}\mathscr{D}_{\acute{\mathfrak{q}}}(\xi(\hat{z}))]}{\mathscr{D}_{\acute{\mathfrak{q}}}(\xi(\hat{z}) - \xi(-\hat{z}))} = 1 + \sin_{\acute{\mathfrak{q}}}(\varpi(\hat{z})). \tag{6.18}$$

Let us define the function

$$p(\hat{z}) = \frac{1+\varpi(\hat{z})}{1-\varpi(\hat{z})} = 1 + p_1\hat{z} + p_2\hat{z}^2 + p_3\hat{z}^3 + p_4\hat{z}^4 + ..., \tag{6.19}$$

where $p(\hat{z})$ is analytic in Ω with $p(0) = 1$. This implies that

$$\varpi(\hat{z}) = \frac{p(\hat{z}) - 1}{p(\hat{z}) + 1}, \tag{6.20}$$

after simplification we get,

$$\varpi(\hat{z}) = \left(\frac{p_1}{2}\right)\hat{z} + \left(\frac{p_2}{2} - \frac{p_1^2}{4}\right)\hat{z}^2 + \left(\frac{p_3}{2} + \frac{p_1^3}{8} - \frac{p_1p_2}{2}\right)\hat{z}^3 + \left(\frac{3p_1^2p_2}{8} - \frac{p_1p_3}{2} - \frac{p_1^4}{16} - \frac{p_2^2}{4} + \frac{p_4}{2}\right)\hat{z}^4 + ...$$

As we know that,

$$\sin_{\acute{\mathfrak{q}}}(\varpi(\hat{z})) = \varpi(\hat{z}) - \frac{(\varpi(\hat{z}))^3}{[3]!_{\check{\mathfrak{q}}_o}} + \frac{(\varpi(\hat{z}))^5}{[5]!_{\check{\mathfrak{q}}_o}} + ...,$$

So, we have

$$\sin_{\acute{\mathfrak{q}}}(\varpi(\hat{z})) = \left(\frac{p_1}{2}\right)\hat{z} + \left(\frac{p_2}{2} - \frac{p_1^2}{4}\right)\hat{z}^2 + \left(\frac{p_3}{2} + \frac{p_1^3}{8} - \frac{p_1p_2}{2} - \frac{p_1^3}{8[3]!_{\check{\mathfrak{q}}_o}}\right)\hat{z}^3 + \left(\frac{3p_1^2p_2}{8} - \frac{p_1p_3}{2} - \frac{p_1^4}{16} - \frac{p_2^2}{4} + \frac{p_4}{2} - \frac{1}{[3]!_{\check{\mathfrak{q}}_o}}\left(\frac{3p_1^2p_2}{8} - \frac{3p_1^4}{16}\right)\right)\hat{z}^4 + ...$$

Now, the Analytic function defined as;

$$\xi(\hat{z}) = \hat{z} + \hat{\alpha}_2\hat{z}^2 + \hat{\alpha}_3\hat{z}^3 + \hat{\alpha}_4\hat{z}^4 + ...,$$

and the q-derivative operator of an Analytic function is;

$$\mathscr{D}_{\acute{q}}(\xi(\hat{z})) = \frac{\xi(\hat{z}) - \xi(\check{q}_o\hat{z})}{(1-\check{q}_o)\hat{z}},$$

after simplification we get,

$$\frac{2\mathscr{D}_{\acute{q}}[\hat{z}\mathscr{D}_{\acute{q}}(\xi(\hat{z}))]}{\mathscr{D}_{\acute{q}}(\xi(\hat{z}) - \xi(-\hat{z}))} = 1 + \hat{\alpha}_2(1+\check{q}_o)^2\hat{z} + \hat{\alpha}_3[\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)]\hat{z}^2 + [\hat{\alpha}_4(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2 - \hat{\alpha}_2\hat{\alpha}_3(1+\check{q}_o)^2(1+\check{q}_o+\check{q}_o^2)]\hat{z}^3 + [\hat{\alpha}_5\check{q}_o(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3+\check{q}_o^4) - \hat{\alpha}_3^2\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)^2]\hat{z}^4 + ...$$

On substituting values in (6.18), we get

$$\begin{aligned}
&1 + \hat{\alpha}_2(1+\check{q}_o)^2\hat{z} + \hat{\alpha}_3[\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)]\hat{z}^2 + [\hat{\alpha}_4(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2 - \hat{\alpha}_2\hat{\alpha}_3(1+\check{q}_o)^2(1+\check{q}_o+\check{q}_o^2)]\hat{z}^3 \\
&+ [\hat{\alpha}_5\check{q}_o(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3+\check{q}_o^4) - \hat{\alpha}_3^2\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)^2]\hat{z}^4 + ... \\
&= 1 + \left(\frac{p_1}{2}\right)\hat{z} + \left(\frac{p_2}{2} - \frac{p_1^2}{4}\right)\hat{z}^2 + \left(\frac{p_3}{2} + \frac{p_1^3}{8} - \frac{p_1p_2}{2} - \frac{p_1^3}{8[3]!_{\check{q}_o}}\right)\hat{z}^3 + \\
&\left(\frac{3p_1^2p_2}{8} - \frac{p_1p_3}{2} - \frac{p_1^4}{16} - \frac{p_2^2}{4} + \frac{p_4}{2} - \frac{1}{[3]!_{\check{q}_o}}\left(\frac{3p_1^2p_2}{8} - \frac{3p_1^4}{16}\right)\right)\hat{z}^4 + ...
\end{aligned}$$

On comparing both sides of above equation, we get

$$\hat{\alpha}_2(1+\check{q}_o)^2 = \frac{p_1}{2}, \tag{6.21}$$

$$\hat{\alpha}_3\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2) = \frac{p_2}{2} - \frac{p_1^2}{4}, \tag{6.22}$$

$$\hat{\alpha}_4(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2 - \hat{\alpha}_2\hat{\alpha}_3(1+\check{q}_o)^2(1+\check{q}_o+\check{q}_o^2) = \frac{p_3}{2} + \frac{p_1^3}{8} - \frac{p_1p_2}{2} - \frac{p_1^3}{8[3]!_{\check{q}_o}}, \tag{6.23}$$

$$\begin{aligned}
&\hat{\alpha}_5\check{q}_o(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3+\check{q}_o^4) - \hat{\alpha}_3^2\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)^2 \\
&= \frac{3p_1^2p_2}{8} - \frac{p_1p_3}{2} - \frac{p_1^4}{16} - \frac{p_2^2}{4} + \frac{p_4}{2} - \frac{1}{[3]!_{\check{q}_o}}\left(\frac{3p_1^2p_2}{8} - \frac{3p_1^4}{16}\right),
\end{aligned} \tag{6.24}$$

By solving (6.21), we get

$$\hat{\alpha}_2 = \frac{p_1}{2(1+\check{q}_o)^2}. \tag{6.25}$$

By solving (6.22), we get

$$\hat{\alpha}_3 = \frac{1}{\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)}\left(\frac{p_2}{2} - \frac{p_1^2}{4}\right). \tag{6.26}$$

On solving (6.23), we have

$$\hat{\alpha}_4(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2-\hat{\alpha}_2\hat{\alpha}_3(1+\check{q}_o)^2(1+\check{q}_o+\check{q}_o^2)=\frac{p_3}{2}+\frac{p_1^3}{8}-\frac{p_1p_2}{2}-\frac{p_1^3}{8[3]!_{\check{q}_o}},$$

$$\hat{\alpha}_4(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2=\hat{\alpha}_2\hat{\alpha}_3(1+\check{q}_o)^2(1+\check{q}_o+\check{q}_o^2)+\frac{p_3}{2}+\frac{p_1^3}{8}-\frac{p_1p_2}{2}-\frac{p_1^3}{8[3]!_{\check{q}_o}}.$$

By substituting values of $\hat{\alpha}_2$ and $\hat{\alpha}_3$, we get

$$\hat{\alpha}_4(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2=(1+\check{q}_o)^2(1+\check{q}_o+\check{q}_o^2)\left[\frac{p_1}{2(1+\check{q}_o)^2}\right]\left[\frac{1}{\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)}\left(\frac{p_2}{2}-\frac{p_1^2}{4}\right)\right]+\frac{p_3}{2}+\frac{p_1^3}{8}-\frac{p_1p_2}{2}-\frac{p_1^3}{8[3]!_{\check{q}_o}},$$

after simplification we get,

$$\hat{\alpha}_4=\frac{1}{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}\left[\left(\frac{1}{8}-\frac{1}{8[3]!_{\check{q}_o}}-\frac{1}{8\check{q}_o(1+\check{q}_o)}\right)p_1^3-\left(\frac{1}{2}-\frac{1}{4\check{q}_o(1+\check{q}_o)}\right)p_1p_2+\frac{p_3}{2}\right]. \tag{6.27}$$

On solving (6.24), we obtain

$$\hat{\alpha}_5\check{q}_o(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3+\check{q}_o^4)-\hat{\alpha}_3^2\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)^2=\frac{3p_1^2p_2}{8}-\frac{p_1p_3}{2}-\frac{p_1^4}{16}-\frac{p_2^2}{4}+\frac{p_4}{2}-\frac{1}{[3]!_{\check{q}_o}}\left(\frac{3p_1^2p_2}{8}-\frac{3p_1^4}{16}\right),$$

$$\hat{\alpha}_5\check{q}_o(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3+\check{q}_o^4)=\hat{\alpha}_3^2\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)^2+\frac{3p_1^2p_2}{8}-\frac{p_1p_3}{2}-\frac{p_1^4}{16}-\frac{p_2^2}{4}+\frac{p_4}{2}-\frac{1}{[3]!_{\check{q}_o}}\left(\frac{3p_1^2p_2}{8}-\frac{3p_1^4}{16}\right).$$

On substituting value of $\hat{\alpha}_3$, we get

$$\hat{\alpha}_5\check{q}_o(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3+\check{q}_o^4)=\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)^2\left[\frac{1}{\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)}\left(\frac{p_2}{2}-\frac{p_1^2}{4}\right)\right]^2+\frac{3p_1^2p_2}{8}-\frac{p_1p_3}{2}-\frac{p_1^4}{16}-\frac{p_2^2}{4}+\frac{p_4}{2}-\frac{1}{[3]!_{\check{q}_o}}\left(\frac{3p_1^2p_2}{8}-\frac{3p_1^4}{16}\right),$$

after simplification, we have

$$\hat{\alpha}_5=\frac{1}{\check{q}_o(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3+\check{q}_o^4)}\left[\left(\frac{3}{8}-\frac{1}{4\check{q}_o(1+\check{q}_o)}-\frac{3}{8[3]!_{\check{q}_o}}\right)p_1^2p_2-\left(\frac{1}{4}-\frac{1}{4\check{q}_o(1+\check{q}_o)}\right)p_2^2+\left(\frac{1}{16\check{q}_o(1+\check{q}_o)}-\frac{1}{16}+\frac{3}{16[3]!_{\check{q}_o}}\right)p_1^4+\frac{p_4}{2}-\frac{p_1p_3}{2}\right]. \tag{6.28}$$

Next, to find out the values of coefficients, (6.25) gives,

$$\hat{\alpha}_2=\frac{p_1}{2(1+\check{q}_o)^2}.$$

Using Lemma 3.11.1, we get

$$|\hat{\alpha}_2| \leq \frac{1}{(1+\check{q}_o)^2}. \tag{6.29}$$

From (6.26), we have

$$\hat{\alpha}_3 = \frac{1}{\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)}\left(\frac{p_2}{2}-\frac{p_1^2}{4}\right).$$

An application of Lemma 3.11.2 leads us to the result

$$|\hat{\alpha}_3| \leq \frac{1}{2\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)}\left|p_2-\frac{p_1^2}{2}\right|,$$

$$|\hat{\alpha}_3| \leq \frac{2}{2\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)} max\{1,0\} = \frac{1}{\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)},$$

$$|\hat{\alpha}_3| \leq \frac{1}{\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)}. \tag{6.30}$$

On solving (6.27), we get

$$\hat{\alpha}_4 = \frac{1}{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}\left[\left(\frac{1}{8}-\frac{1}{8[3]!_{\check{q}_o}}-\frac{1}{8\check{q}_o(1+\check{q}_o)}\right)p_1^3-\left(\frac{1}{2}-\frac{1}{4\check{q}_o(1+\check{q}_o)}\right)p_1p_2+\frac{p_3}{2}\right],$$

$$|\hat{\alpha}_4| \leq \frac{1}{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}\left|\left(\frac{1}{8}-\frac{1}{8[3]!_{\check{q}_o}}-\frac{1}{8\check{q}_o(1+\check{q}_o)}\right)p_1^3-\left(\frac{1}{2}-\frac{1}{4\check{q}_o(1+\check{q}_o)}\right)p_1p_2+\frac{p_3}{2}\right|.$$

Here applying Lemma 3.11.3, we have

$$\begin{aligned}\left|\left(\frac{1}{8}-\frac{1}{8[3]!_{\check{q}_o}}-\frac{1}{8\check{q}_o(1+\check{q}_o)}\right)p_1^3-\left(\frac{1}{2}-\frac{1}{4\check{q}_o(1+\check{q}_o)}\right)p_1p_2+\frac{p_3}{2}\right| &\leq 2\left|\frac{1}{8}-\frac{1}{8[3]!_{\check{q}_o}}-\frac{1}{8\check{q}_o(1+\check{q}_o)}\right|+\\ &2\left|\frac{1}{2}-\frac{1}{4\check{q}_o(1+\check{q}_o)}-2\left(\frac{1}{8}-\frac{1}{8[3]!_{\check{q}_o}}-\frac{1}{8\check{q}_o(1+\check{q}_o)}\right)\right|+\\ &2\left|\frac{1}{8}-\frac{1}{8[3]!_{\check{q}_o}}-\frac{1}{8\check{q}_o(1+\check{q}_o)}-\left(\frac{1}{2}-\frac{1}{4\check{q}_o(1+\check{q}_o)}\right)+\frac{1}{2}\right|,\end{aligned}$$

$$\begin{aligned}\left|\left(\frac{1}{8}-\frac{1}{8[3]!_{\check{q}_o}}-\frac{1}{8\check{q}_o(1+\check{q}_o)}\right)p_1^3-\left(\frac{1}{2}-\frac{1}{4\check{q}_o(1+\check{q}_o)}\right)p_1p_2+\frac{p_3}{2}\right| &\leq \left|\frac{1}{4}-\frac{1}{4[3]!_{\check{q}_o}}-\frac{1}{4\check{q}_o(1+\check{q}_o)}\right|+\\ &\left|\frac{1}{2}+\frac{1}{2[3]!_{\check{q}_o}}\right|+\left|\frac{1}{4}-\frac{1}{4[3]!_{\check{q}_o}}+\frac{1}{4\check{q}_o(1+\check{q}_o)}\right|,\end{aligned}$$

thus we get,

$$|\hat{\alpha}_4| \leq \frac{1}{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}\left[\left|\frac{1}{4}-\frac{1}{4[3]!_{\check{q}_o}}-\frac{1}{4\check{q}_o(1+\check{q}_o)}\right|+\left|\frac{1}{2}+\frac{1}{2[3]!_{\check{q}_o}}\right|+\left|\frac{1}{4}-\frac{1}{4[3]!_{\check{q}_o}}+\frac{1}{4\check{q}_o(1+\check{q}_o)}\right|\right]. \tag{6.31}$$

By solving (6.28), we get

$$\hat{\alpha}_5 = \frac{1}{\check{q}_o(1\check{q}_o+\check{q}_o^2+\check{q}_o^3)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3+\check{q}_o^4)}\left[\left(\frac{3}{8}-\frac{1}{4\check{q}_o(1+\check{q}_o)}-\frac{3}{8[3]!_{\check{q}_o}}\right)p_1^2p_2- \left(\frac{1}{4}-\frac{1}{4\check{q}_o(1+\check{q}_o)}\right)p_2^2+\left(\frac{1}{16\check{q}_o(1+\check{q}_o)}-\frac{1}{16}+\frac{3}{16[3]!_{\check{q}_o}}\right)p_1^4+\frac{p_4}{2}-\frac{p_1p_3}{2}\right].$$

Taking modulus on both sides and applying triangular inequality, we get

$$|\hat{\alpha}_5| \leq \frac{1}{\check{q}_o(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3+\check{q}_o^4)}\left[\left|\frac{3}{8}-\frac{1}{4\check{q}_o(1+\check{q}_o)}-\frac{3}{8[3]!_{\check{q}_o}}\right||p_1|^2|p_2|+ \left|\frac{1}{4}-\frac{1}{4\check{q}_o(1+\check{q}_o)}\right||p_2|^2+\left|\frac{1}{16\check{q}_o(1+\check{q}_o)}-\frac{1}{16}+\frac{3}{16[3]!_{\check{q}_o}}\right||p_1|^4+\frac{1}{2}|p_1p_3-p_4|\right].$$

Using Lemma 3.11.1 and Lemma 3.11.4, we get

$$|\hat{\alpha}_5| \leq \frac{1}{\check{q}_o(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3+\check{q}_o^4)}\left[\frac{2}{2}+\left|3-\frac{2}{\check{q}_o(1+\check{q}_o)}-\frac{3}{[3]!_{\check{q}_o}}\right|+ \left|\frac{\check{q}_o(1+\check{q}_o)-1}{\check{q}_o(1+\check{q}_o)}\right|+\left|1-\frac{3}{[3]!_{\check{q}_o}}-\frac{1}{\check{q}_o(1+\check{q}_o)}\right|\right],$$

$$|\hat{\alpha}_5| \leq \frac{1}{\check{q}_o(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3+\check{q}_o^4)}\left[1+\left|3-\frac{2}{\check{q}_o(1+\check{q}_o)}-\frac{3}{[3]!_{\check{q}_o}}\right|+ \left|\frac{\check{q}_o(1+\check{q}_o)-1}{\check{q}_o(1+\check{q}_o)}\right|+\left|1-\frac{3}{[3]!_{\check{q}_o}}-\frac{1}{\check{q}_o(1+\check{q}_o)}\right|\right]. \quad (6.32)$$

This completes the proof. □

If $\check{q}_o \to 1^-$ in the above theorem, this leads us the already proved result for class C_s.

Corollary 6.2.2.1. *If* $\xi(\hat{z}) \in C_s$ *then*

$$|\hat{\alpha}_2| \leq \frac{1}{4},\ |\hat{\alpha}_3| \leq \frac{1}{6},\ |\hat{\alpha}_4| \leq \frac{1}{16},\ |\hat{\alpha}_5| \leq \frac{3}{20}.$$

6.3 Fekete–Szegő Inequality

This inequality will examined for the class $S_s^*(\acute{q}-sin)$.

Theorem 6.3.1. *If* $\xi(\hat{z}) \in S_s^*(\acute{q}-sin)$ *then* $|\hat{\alpha}_3-\hat{\alpha}_2^2| \leq \frac{1}{\check{q}_o(1+\check{q}_o)}$.

Proof. From (6.10) and (6.11), we obtain

$$|\hat{\alpha}_3-\hat{\alpha}_2^2| = \left|\frac{1}{\check{q}_o(1+\check{q}_o)}\left(\frac{p_2}{2}-\frac{p_1^2}{4}\right)-\left(\frac{p_1}{2(1+\check{q}_o)}\right)^2\right| = \frac{1}{2\check{q}_o(1+\check{q}_o)}\left|p_2-\left(\frac{1+2\check{q}_o}{1+\check{q}_o}\right)\frac{p_1^2}{2}\right|,$$

$$|\hat{\alpha}_3-\hat{\alpha}_2^2| = \frac{1}{2\check{q}_o(1+\check{q}_o)}\left|p_2-\left(\frac{1+2\check{q}_o}{1+\check{q}_o}\right)\frac{p_1^2}{2}\right|,$$

$\frac{1+2\check{q}_o}{2(1+\check{q}_o)} \leq 1$ for $\check{q}_o \in (0,1)$. An application of Lemma 3.11.5 leads us to the result required result

$$|\hat{\alpha}_3-\hat{\alpha}_2^2| \leq \frac{1}{2\check{q}_o(1+\check{q}_o)}(2),$$

$$|\hat{\alpha}_3-\hat{\alpha}_2^2| \leq \frac{1}{\check{q}_o(1+\check{q}_o)}.$$

This completes the proof. □

Now, taking $\check{q}_o \to 1^-$, we get the known result as shown in the following corollary.

Corollary 6.3.1.1. *If $\xi(\hat{z}) \in S_s^*$ then $|\hat{\alpha}_3-\hat{\alpha}_2^2| \leq \frac{1}{2}$.*

Here, Fekete–Szegő Inequality will derived for $C_s(\acute{q}-sin)$.

Theorem 6.3.2. *If $\xi(\hat{z}) \in C_s(\acute{q}-sin)$ then $|\hat{\alpha}_3-\hat{\alpha}_2^2| \leq \frac{1}{\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)}$.*

Proof. From (6.25) and (6.26), we have

$$|\hat{\alpha}_3-\hat{\alpha}_2^2| = \left|\frac{1}{\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)}\left(\frac{p_2}{2}-\frac{p_1^2}{4}\right)-\left[\frac{p_1}{2(1+\check{q}_o)^2}\right]^2\right|,$$

$$|\hat{\alpha}_3-\hat{\alpha}_2^2| = \frac{1}{2\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)}\left|p_2-p_1^2\left(\frac{(1+\check{q}_o)^3+\check{q}_o(1+\check{q}_o+\check{q}_o^2)}{2(1+\check{q}_o)^3}\right)\right|,$$

$\frac{(1+\check{q}_o)^3+\check{q}_o(1+\check{q}_o+\check{q}_o^2)}{2(1+\check{q}_o)^3} \leq 1$ for $\check{q}_o \in (0,1)$. An application of Lemma 3.11.5 leads us to the result

$$|\hat{\alpha}_3-\hat{\alpha}_2^2| \leq \frac{1}{2\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)}(2),$$

$$|\hat{\alpha}_3-\hat{\alpha}_2^2| \leq \frac{1}{\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)},$$

Hence, this completes the proof. □

If $\check{q}_o \to 1^-$ in the above theorem, result leads to the proved result for class C_s which is given in next corollary.

Corollary 6.3.2.1. *If $\xi(\hat{z}) \in C_s$ then $|\hat{\alpha}_3-\hat{\alpha}_2^2| \leq \frac{1}{6}$.*

6.4 Hankel Determinants

These results will investigated for the class of q-starlike functions, $S_s^*(\acute{q}-sin)$.

Theorem 6.4.1. *If* $\xi(\hat{z}) \in S_s^*(\acute{q}-sin)$ *then*

$$|\hat{\alpha}_2\hat{\alpha}_3-\hat{\alpha}_4| \leq \frac{1}{2(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}\left|1+\frac{1}{[3]!_{\check{q}_o}}\right|+\frac{1}{4}\left|\frac{1}{\check{q}_o(1+\check{q}_o)^2}+\frac{1}{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}-\frac{1}{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)[3]!_{\check{q}_o}}-\frac{1}{\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}\right|+\frac{1}{4}\left|\frac{1}{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}-\frac{1}{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)[3]!_{\check{q}_o}}-\frac{1}{\check{q}_o(1+\check{q}_o)^2}+\frac{1}{\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}\right|.$$

Proof. From (6.10), (6.11) and (6.12), we have

$$|\hat{\alpha}_2\hat{\alpha}_3-\hat{\alpha}_4| = \left|\left[\frac{p_1}{2(1+\check{q}_o)}\right]\left[\frac{1}{\check{q}_o(1+\check{q}_o)}\left(\frac{p_2}{2}-\frac{p_1^2}{4}\right)\right]-\left[\frac{1}{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}\left(\left(\frac{1}{8}-\frac{1}{8[3]!_{\check{q}_o}}-\frac{1}{8\check{q}_o(1+\check{q}_o)}\right)p_1^3-\left(\frac{1}{2}-\frac{1}{4\check{q}_o(1+\check{q}_o)}\right)p_1p_2+\frac{p_3}{2}\right)\right]\right|,$$

$$|\hat{\alpha}_2\hat{\alpha}_3-\hat{\alpha}_4| = \left|\frac{p_1p_2}{4\check{q}_o(1+\check{q}_o)^2}-\frac{p_1^3}{8\check{q}_o(1+\check{q}_o)^2}-\frac{p_1^3}{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}\left(\frac{1}{8}-\frac{1}{8[3]!_{\check{q}_o}}-\frac{1}{8\check{q}_o(1+\check{q}_o)}\right)+\frac{p_1p_2}{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}\left(\frac{1}{2}-\frac{1}{4\check{q}_o(1+\check{q}_o)}\right)-\frac{p_3}{2(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}\right|,$$

$$|\hat{\alpha}_2\hat{\alpha}_3-\hat{\alpha}_4| = \left|p_1^3\left(\frac{1}{8\check{q}_o(1+\check{q}_o)^2}+\frac{1}{8(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}-\frac{1}{8(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)[3]!_{\check{q}_o}}-\frac{1}{8\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}\right)-p_1p_2\left(\frac{1}{4\check{q}_o(1+\check{q}_o)^2}+\frac{1}{2(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}-\frac{1}{4\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}\right)+\frac{p_3}{2(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}\right|,$$

using Lemma 3.11.3, we get

$$\Bigg|p_1^3\left(\frac{1}{8\check{q}_o(1+\check{q}_o)^2}+\frac{1}{8(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}-\frac{1}{8(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)[3]!_{\check{q}_o}}-\frac{1}{8\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}\right)-p_1p_2\left(\frac{1}{4\check{q}_o(1+\check{q}_o)^2}+\frac{1}{2(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}-\frac{1}{4\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}\right)+\frac{p_3}{2(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}\Bigg|\leq$$

$$\left|\frac{1}{4\check{q}_o(1+\check{q}_o)^2}+\frac{1}{4(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}-\frac{1}{4(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)[3]!_{\check{q}_o}}-\frac{1}{4\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}\right|+$$
$$+\left|\frac{1}{4(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}-\frac{1}{4(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)[3]!_{\check{q}_o}}-\frac{1}{4\check{q}_o(1+\check{q}_o)^2}+\frac{1}{4\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}\right|+$$
$$\left|\frac{1}{2(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}+\frac{1}{2(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)[3]!_{\check{q}_o}}\right|,$$

consequently, we get

$$|\hat{\alpha}_2\hat{\alpha}_3-\hat{\alpha}_4|\leq\frac{1}{2(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}\left|1+\frac{1}{[3]!_{\check{q}_o}}\right|+\frac{1}{4}\left|\frac{1}{\check{q}_o(1+\check{q}_o)^2}+\frac{1}{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}-\frac{1}{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)[3]!_{\check{q}_o}}-\frac{1}{\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}\right|+\frac{1}{4}\left|\frac{1}{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}-\frac{1}{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)[3]!_{\check{q}_o}}-\frac{1}{\check{q}_o(1+\check{q}_o)^2}+\frac{1}{\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}\right|.$$

Which is the required result. □

Now, taking $\check{q}_o\to 1^-$ in the above theorem leads us to already derived result in [78], as given in next corollary.

Corollary 6.4.1.1. *If* $\xi(\hat{z})\in S_s^*$ *then* $|\hat{\alpha}_2\hat{\alpha}_3-\hat{\alpha}_4|\leq\frac{1}{4}$.

Theorem 6.4.2. *If* $\xi(\hat{z})\in S_s^*(\acute{q}-sin)$ *then*

$$|\hat{\alpha}_2\hat{\alpha}_4-\hat{\alpha}_3^2|\leq\frac{1}{(1+\check{q}_o)}\left[\frac{1}{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}+\frac{1}{\check{q}_o^2(1+\check{q}_o)}+\left|\frac{(-2-3\check{q}_o)}{\check{q}_o^2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}\right|\right].$$

Proof. From (6.10), (6.11) and (6.12), we obtain

$$|\hat{\alpha}_2\hat{\alpha}_4-\hat{\alpha}_3^2|=\left|\left[\frac{p_1}{2(1+\check{q}_o)}\right]\left[\frac{1}{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}\left[\left(\frac{1}{8}-\frac{1}{8[3]!_{\check{q}_o}}-\frac{1}{8\check{q}_o(1+\check{q}_o)}\right)p_1^3-\left(\frac{1}{2}-\frac{1}{4\check{q}_o(1+\check{q}_o)}\right)p_1p_2+\frac{p_3}{2}\right]\right]-\left[\frac{1}{\check{q}_o(1+\check{q}_o)}\left(\frac{p_2}{2}-\frac{p_1^2}{4}\right)\right]^2\right|,$$

$$|\hat{\alpha}_2\hat{\alpha}_4-\hat{\alpha}_3^2| = \Bigg|p_1^4\Bigg(\frac{1}{16(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}-\frac{1}{16(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)[3]!_{\check{q}_o}}-$$
$$\frac{1}{16\check{q}_o(1+\check{q}_o)^2(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}-\frac{1}{16\check{q}_o^2(1+\check{q}_o)^2}\Bigg)-$$
$$p_1^2p_2\left(\frac{1}{4(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}-\frac{1}{8\check{q}_o(1+\check{q}_o)^2(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}-\frac{1}{4\check{q}_o^2(1+\check{q}_o)^2}\right)+$$
$$\frac{p_1p_3}{4(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}-\frac{p_2^2}{4\check{q}_o^2(1+\check{q}_o)^2}\Bigg|,$$

$$|\hat{\alpha}_2\hat{\alpha}_4-\hat{\alpha}_3^2| \leq \frac{1}{(1+\check{q}_o)}\Bigg[\frac{|p_1||p_3|}{4(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}+\frac{|p_2|^2}{4\check{q}_o^2(1+\check{q}_o)}+|p_1|^2\Bigg|p_2\Bigg(\frac{1}{4(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}-$$
$$\frac{1}{8\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}-\frac{1}{4\check{q}_o^2(1+\check{q}_o)}\Bigg)-p_1^2\Bigg(\frac{1}{16(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}-\frac{1}{16(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)[3]!_{\check{q}_o}}$$
$$-\frac{1}{16\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}-\frac{1}{16\check{q}_o^2(1+\check{q}_o)}\Bigg)\Bigg|\Bigg].$$

After simplification, we have

$$\Bigg|p_2\left(\frac{1}{4(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}-\frac{1}{8\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}-\frac{1}{4\check{q}_o^2(1+\check{q}_o)}\right)-$$
$$p_1^2\Bigg(\frac{1}{16(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}-\frac{1}{16(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)[3]!_{\check{q}_o}}-\frac{1}{16\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}-$$
$$\frac{1}{16\check{q}_o^2(1+\check{q}_o)}\Bigg)\Bigg| = \Bigg|p_2\left(\frac{(-2-3\check{q}_o)}{8\check{q}_o^2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}\right)-$$
$$p_1^2\left(\frac{\check{q}_o^2(1+\check{q}_o)[3]!_{\check{q}_o}-\check{q}_o^2(1+\check{q}_o)-[3]!_{\check{q}_o}-(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)[3]!_{\check{q}_o}}{16\check{q}_o^2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)[3]!_{\check{q}_o}}\right)\Bigg|,$$

$$\Bigg|p_2\left(\frac{1}{4(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}-\frac{1}{8\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}-\frac{1}{4\check{q}_o^2(1+\check{q}_o)}\right)-$$
$$p_1^2\Bigg(\frac{1}{16(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}-\frac{1}{16(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)[3]!_{\check{q}_o}}-\frac{1}{16\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}-$$
$$\frac{1}{16\check{q}_o^2(1+\check{q}_o)}\Bigg)\Bigg| = \Bigg|\left(\frac{(-2-3\check{q}_o)}{8\check{q}_o^2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}\right)$$
$$\left(p_2-p_1^2\left(\frac{\check{q}_o^2(1+\check{q}_o)[3]!_{\check{q}_o}-\check{q}_o^2(1+\check{q}_o)-[3]!_{\check{q}_o}-(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)[3]!_{\check{q}_o}}{2(-2-3\check{q}_o)[3]!_{\check{q}_o}}\right)\right)\Bigg|,$$

$\kappa = \left(\frac{\check{q}_o^2(1+\check{q}_o)[3]!_{\check{q}_o}-\check{q}_o^2(1+\check{q}_o)-[3]!_{\check{q}_o}-(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)[3]!_{\check{q}_o}}{2(-2-3\check{q}_o)[3]!_{\check{q}_o}}\right) \leq 1$ for $\check{q}_o \in (0,1)$. Using Lemma 3.11.5, we have

$$\Bigg|p_2\left(\frac{1}{4(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}-\frac{1}{8\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}-\frac{1}{4\check{q}_o^2(1+\check{q}_o)}\right)-$$
$$p_1^2\Bigg(\frac{1}{16(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}-\frac{1}{16(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)[3]!_{\check{q}_o}}-\frac{1}{16\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}-$$
$$\frac{1}{16\check{q}_o^2(1+\check{q}_o)}\Bigg)\Bigg| = \Bigg|\left(\frac{(-2-3\check{q}_o)}{8\check{q}_o^2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}\right)\Bigg|(2),$$

Therefore, we get

$$|\hat{\alpha}_2\hat{\alpha}_4-\hat{\alpha}_3^2|\leq \frac{1}{(1+\check{q}_o)}\left[\frac{|p_1||p_3|}{4(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}+\frac{|p_2|^2}{4\check{q}_o^2(1+\check{q}_o)}+ |p_1|^2\left|\frac{(-2-3\check{q}_o)}{4\check{q}_o^2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}\right|\right].$$

Here applying Lemma 3.11.1, we get

$$|\hat{\alpha}_2\hat{\alpha}_4-\hat{\alpha}_3^2|\leq \frac{1}{(1+\check{q}_o)}\left[\frac{4}{4(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}+\frac{4}{4\check{q}_o^2(1+\check{q}_o)}+4\left|\frac{(-2-3\check{q}_o)}{4\check{q}_o^2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}\right|\right],$$

$$|\hat{\alpha}_2\hat{\alpha}_4-\hat{\alpha}_3^2|\leq \frac{1}{(1+\check{q}_o)}\left[\frac{1}{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}+\frac{1}{\check{q}_o^2(1+\check{q}_o)}+\left|\frac{(-2-3\check{q}_o)}{\check{q}_o^2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}\right|\right].$$

This completes the proof. □

Now, considering $\check{q}_o\to 1^-$ in the above theorem, then result is same, see [78].

Corollary 6.4.2.1. *If $\xi(\hat{z})\in S_s^*$ then $|\hat{\alpha}_2\hat{\alpha}_4-\hat{\alpha}_3^2|\leq \frac{11}{16}$.*

Theorem 6.4.3. *If $\xi(\hat{z})\in S_s^*(\acute{q}-sin)$ then*

$$\begin{aligned}|\mathscr{H}_3(1)|\leq{}& \frac{1}{\check{q}_o(1+\check{q}_o)^2}\left[\frac{1}{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}+\frac{1}{\check{q}_o^2(1+\check{q}_o)}+\left|\frac{(-2-3\check{q}_o)}{\check{q}_o^2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}\right|\right]+\\ &\left[\frac{1}{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}\left(\left|\frac{1}{4}-\frac{1}{4[3]!_{\check{q}_o}}-\frac{1}{4\check{q}_o(1+\check{q}_o)}\right|+\left|\frac{1+[3]!_{\check{q}_o}}{2[3]!_{\check{q}_o}}\right|+\left|\frac{1}{4}-\frac{1}{4[3]!_{\check{q}_o}}+\frac{1}{4\check{q}_o(1+\check{q}_o)}\right|\right)\right]\\ &\left[\left|\frac{1}{4\check{q}_o(1+\check{q}_o)^2}+\frac{1}{4(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}-\frac{1}{4(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)[3]!_{\check{q}_o}}-\frac{1}{4\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}\right|+\right.\\ &\left|\frac{1+[3]!_{\check{q}_o}}{2(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}\right|+\left|\frac{1}{4(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}-\frac{1}{4\check{q}_o(1+\check{q}_o)^2}-\frac{1}{4(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)[3]!_{\check{q}_o}}+\right.\\ &\left.\left.\frac{1}{4\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}\right|\right]+\frac{1}{\check{q}_o^2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}\left[1+\left|3-\frac{2}{\check{q}_o(1+\check{q}_o)}-\frac{3}{[3]!_{\check{q}_o}}\right|+\right.\\ &\left.\left|1-\frac{1}{\check{q}_o(1+\check{q}_o)}\right|+\left|\frac{1}{\check{q}_o(1+\check{q}_o)}+\frac{3}{[3]!_{\check{q}_o}}-1\right|\right].\end{aligned}$$

Proof. Hankel Determinant of order 3 defined as;

$$\mathscr{H}_3(1)=\hat{\alpha}_5(\hat{\alpha}_3-\hat{\alpha}_2^2)-\hat{\alpha}_4(\hat{\alpha}_4-\hat{\alpha}_2\hat{\alpha}_3)+\hat{\alpha}_3(\hat{\alpha}_2\hat{\alpha}_4-\hat{\alpha}_3^2),$$

Taking modulus on both sides and applying triangular inequality, we have

$$|\mathscr{H}_3(1)|\leq|\hat{\alpha}_5||\hat{\alpha}_3-\hat{\alpha}_2^2|+|\hat{\alpha}_4||\hat{\alpha}_4-\hat{\alpha}_2\hat{\alpha}_3|+|\hat{\alpha}_3||\hat{\alpha}_2\hat{\alpha}_4-\hat{\alpha}_3^2|,$$

By substituting Theorem 6.2.1, Theorem 6.3.1, Theorem 6.4.1 and Theorem 6.4.2, we get

$$|\mathscr{H}_3(1)| \leq \frac{1}{\check{q}_o(1+\check{q}_o)^2}\left[\frac{1}{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}+\frac{1}{\check{q}_o^2(1+\check{q}_o)}+\left|\frac{(-2-3\check{q}_o)}{\check{q}_o^2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}\right|\right]+$$
$$\left[\frac{1}{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}\left(\left|\frac{1}{4}-\frac{1}{4[3]!_{\check{q}_o}}-\frac{1}{4\check{q}_o(1+\check{q}_o)}\right|+\left|\frac{1+[3]!_{\check{q}_o}}{2[3]!_{\check{q}_o}}\right|+\left|\frac{1}{4}-\frac{1}{4[3]!_{\check{q}_o}}+\frac{1}{4\check{q}_o(1+\check{q}_o)}\right|\right)\right]$$
$$\left[\left|\frac{1}{4\check{q}_o(1+\check{q}_o)^2}+\frac{1}{4(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}-\frac{1}{4(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)[3]!_{\check{q}_o}}-\frac{1}{4\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}\right|+\right.$$
$$\left|\frac{1+[3]!_{\check{q}_o}}{2(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}\right|+\left|\frac{1}{4(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}-\frac{1}{4\check{q}_o(1+\check{q}_o)^2}-\frac{1}{4(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)[3]!_{\check{q}_o}}+\right.$$
$$\left.\left.\frac{1}{4\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}\right|\right]+\frac{1}{\check{q}_o^2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)}\left[1+\left|3-\frac{2}{\check{q}_o(1+\check{q}_o)}-\frac{3}{[3]!_{\check{q}_o}}\right|+\right.$$
$$\left.\left|1-\frac{1}{\check{q}_o(1+\check{q}_o)}\right|+\left|\frac{1}{\check{q}_o(1+\check{q}_o)}+\frac{3}{[3]!_{\check{q}_o}}-1\right|\right].$$

Which is the required result. □

Now, taking $\check{q}_o \to 1^-$, result is known, as shown below.

Corollary 6.4.3.1. *Consider* $\xi(\hat{z}) \in S_s^*$ *then* $|\mathscr{H}_3(1)| \leq \frac{25}{32}$.

Now, the following results will investigating for the class of $\acute{q}$-convex functions, $C_s(\acute{q}-sin)$.

Theorem 6.4.4. *If* $\xi(\hat{z}) \in C_s(\acute{q}-sin)$ *then*

$$|\hat{\alpha}_2\hat{\alpha}_3-\hat{\alpha}_4| \leq \frac{1}{2(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}\left|1+\frac{1}{[3]!_{\check{q}_o}}\right|+\frac{1}{4}\left|\frac{1}{\check{q}_o(1+\check{q}_o)^3(1+\check{q}_o+\check{q}_o^2)}+\frac{1}{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}-\right.$$
$$\left.\frac{1}{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2[3]!_{\check{q}_o}}-\frac{1}{\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}\right|+\frac{1}{4}\left|\frac{1}{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}-\right.$$
$$\left.\frac{1}{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2[3]!_{\check{q}_o}}-\frac{1}{\check{q}_o(1+\check{q}_o)^3(1+\check{q}_o+\check{q}_o^2)}+\frac{1}{\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}\right|.$$

Proof. From (6.25), (6.26) and (6.27), we obtain

$$|\hat{\alpha}_2\hat{\alpha}_3-\hat{\alpha}_4| = \left|\left[\frac{p_1}{2(1+\check{q}_o)^2}\right]\left[\frac{1}{\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)}\left(\frac{p_2}{2}-\frac{p_1^2}{4}\right)\right]-\left[\frac{1}{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}\right.\right.$$
$$\left.\left.\left(\left(\frac{1}{8}-\frac{1}{8[3]!_{\check{q}_o}}-\frac{1}{8\check{q}_o(1+\check{q}_o)}\right)p_1^3-\left(\frac{1}{2}-\frac{1}{4\check{q}_o(1+\check{q}_o)}\right)p_1p_2+\frac{p_3}{2}\right)\right]\right|,$$

$$|\hat{\alpha}_2\hat{\alpha}_3-\hat{\alpha}_4| = \left|\left(\frac{1}{8\check{q}_o(1+\check{q}_o)^3(1+\check{q}_o+\check{q}_o^2)}+\frac{1}{8(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}-\frac{1}{8(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2[3]!_{\check{q}_o}}-\right.\right.$$
$$\left.\frac{1}{8\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}\right)p_1^3-\left(\frac{1}{4\check{q}_o(1+\check{q}_o)^3(1+\check{q}_o+\check{q}_o^2)}+\frac{1}{2(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}-\right.$$
$$\left.\left.\frac{1}{4\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}\right)p_1p_2+\frac{p_3}{2(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}\right|,$$

using Lemma 3.11.3, we obtain

$$\left|\left(\frac{1}{8\check{q}_o(1+\check{q}_o)^3(1+\check{q}_o+\check{q}_o^2)}+\frac{1}{8(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}-\frac{1}{8(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2[3]!_{\check{q}_o}}-\frac{1}{8\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}\right)p_1^3-\left(\frac{1}{4\check{q}_o(1+\check{q}_o)^3(1+\check{q}_o+\check{q}_o^2)}+\frac{1}{2(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}-\frac{1}{4\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}\right)p_1p_2+\frac{p_3}{2(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}\right|\leq\frac{1}{2(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}\left|1+\frac{1}{[3]!_{\check{q}_o}}\right|+\frac{1}{4}\left|\frac{1}{\check{q}_o(1+\check{q}_o)^3(1+\check{q}_o+\check{q}_o^2)}+\frac{1}{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}-\frac{1}{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2[3]!_{\check{q}_o}}-\frac{1}{\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}\right|+\frac{1}{4}\left|\frac{1}{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}-\frac{1}{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2[3]!_{\check{q}_o}}-\frac{1}{\check{q}_o(1+\check{q}_o)^3(1+\check{q}_o+\check{q}_o^2)}+\frac{1}{\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}\right|,$$

consequently, we get

$$|\hat{\alpha}_2\hat{\alpha}_3-\hat{\alpha}_4|\leq\frac{1}{2(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}\left|1+\frac{1}{[3]!_{\check{q}_o}}\right|+\frac{1}{4}\left|\frac{1}{\check{q}_o(1+\check{q}_o)^3(1+\check{q}_o+\check{q}_o^2)}+\frac{1}{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}-\frac{1}{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2[3]!_{\check{q}_o}}-\frac{1}{\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}\right|+\frac{1}{4}\left|\frac{1}{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}-\frac{1}{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2[3]!_{\check{q}_o}}-\frac{1}{\check{q}_o(1+\check{q}_o)^3(1+\check{q}_o+\check{q}_o^2)}+\frac{1}{\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}\right|.$$

This completes the proof. □

On substituting $\check{q}_o \to 1^-$ in this result, obtained result is same as for the class C_s, that shows in following corollary.

Corollary 6.4.4.1. *If $\xi(\hat{z}) \in C_s$ then $|\hat{\alpha}_2\hat{\alpha}_3-\hat{\alpha}_4| \leq \frac{1}{16}$.*

Theorem 6.4.5. *If $\xi(\hat{z}) \in C_s(\acute{q}-sin)$ then*

$$|\hat{\alpha}_2\hat{\alpha}_4-\hat{\alpha}_3^2|\leq\frac{1}{(1+\check{q}_o)^2}\left[\frac{\check{q}_o^2(1+\check{q}_o+\check{q}_o^2)^2+(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}{\check{q}_o^2(1+\check{q}_o+\check{q}_o^2)^2(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}+\left|\frac{2\check{q}_o^2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)^2-\check{q}_o(1+\check{q}_o+\check{q}_o^2)^2-2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}{\check{q}_o^2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)^2(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}\right|\right].$$

Proof. From (6.25), (6.26) and (6.27), we get

$$|\hat{\alpha}_2\hat{\alpha}_4-\hat{\alpha}_3^2|=\left|\left[\frac{p_1}{2(1+\check{q}_o)^2}\right]\left[\frac{1}{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}\left(\left(\frac{1}{8}-\frac{1}{8[3]!_{\check{q}_o}}-\frac{1}{8\check{q}_o(1+\check{q}_o)}\right)p_1^3-\left(\frac{1}{2}-\frac{1}{4\check{q}_o(1+\check{q}_o)}\right)p_1p_2+\frac{p_3}{2}\right)\right]-\left[\frac{1}{\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)}\left(\frac{p_2}{2}-\frac{p_1^2}{4}\right)\right]^2\right|,$$

$$|\hat{\alpha}_2\hat{\alpha}_4-\hat{\alpha}_3^2| = \Bigg|p_1^4\bigg(\frac{1}{16(1+\check{q}_o)^2(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}-\frac{1}{16(1+\check{q}_o)^2(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2[3]!_{\check{q}_o}}-\frac{1}{16\check{q}_o(1+\check{q}_o)^3(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}-\frac{1}{16\check{q}_o^2(1+\check{q}_o)^2(1+\check{q}_o+\check{q}_o^2)^2}\bigg)-p_1^2p_2\bigg(\frac{1}{4(1+\check{q}_o)^2(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}-\frac{1}{8\check{q}_o(1+\check{q}_o)^3(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}-\frac{1}{4\check{q}_o^2(1+\check{q}_o)^2(1+\check{q}_o+\check{q}_o^2)^2}\bigg)+\frac{p_1p_3}{4(1+\check{q}_o)^2(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}-\frac{p_2^2}{4\check{q}_o^2(1+\check{q}_o)^2(1+\check{q}_o+\check{q}_o^2)^2}\Bigg|,$$

$$|\hat{\alpha}_2\hat{\alpha}_4-\hat{\alpha}_3^2| \leq \frac{1}{(1+\check{q}_o)^2}\Bigg[\frac{|p_1||p_3|}{4(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}+\frac{|p_2|^2}{4\check{q}_o^2(1+\check{q}_o+\check{q}_o^2)^2}+|p_1|^2\Bigg|p_2\bigg(\frac{1}{4(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}-\frac{1}{8\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}-\frac{1}{4\check{q}_o^2(1+\check{q}_o+q_o^2)^2}\bigg)-p_1^2\bigg(\frac{1}{16(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}-\frac{1}{16(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2[3]!_{\check{q}_o}}-\frac{1}{16\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}-\frac{1}{16\check{q}_o^2(1+\check{q}_o+\check{q}_o^2)^2}\bigg)\Bigg|\Bigg],$$

$$\Bigg|p_2\bigg(\frac{1}{4(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}-\frac{1}{8\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}-\frac{1}{4\check{q}_o^2(1+\check{q}_o+\check{q}_o^2)^2}\bigg)-p_1^2\bigg(\frac{1}{16(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}-\frac{1}{16(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2[3]!_{\check{q}_o}}-\frac{1}{16\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}-\frac{1}{16\check{q}_o^2(1+\check{q}_o+\check{q}_o^2)^2}\bigg)\Bigg| = \Bigg|\bigg(\frac{2\check{q}_o^2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)^2-\check{q}_o(1+\check{q}_o+\check{q}_o^2)^2-2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}{8\check{q}_o^2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)^2(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}\bigg)\Bigg[p_2-p_1^2\bigg(\frac{\check{q}_o^2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)2[3]!_{\check{q}_o}-\check{q}_o^2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)^2}{2[3]!_{\check{q}_o}(2\check{q}_o^2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)^2-\check{q}_o(1+\check{q}_o+\check{q}_o^2)^2-2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2)}-\frac{(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2[3]!_{\check{q}_o}+\check{q}_o(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2[3]!_{\check{q}_o}}{2[3]!_{\check{q}_o}(2\check{q}_o^2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)^2-\check{q}_o(1+\check{q}_o+\check{q}_o^2)^2-2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2)}\bigg)\Bigg]\Bigg|,$$

$$\kappa = \bigg(\frac{\check{q}_o^2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)2[3]!_{\check{q}_o}-\check{q}_o^2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)^2-(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2[3]!_{\check{q}_o}-\check{q}_o(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2[3]!_{\check{q}_o}}{2[3]!_{\check{q}_o}(2\check{q}_o^2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)^2-\check{q}_o(1+\check{q}_o+\check{q}_o^2)^2-2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2)}\bigg) \leq 1,$$

for $\check{q}_o \in (0,1)$. Using Lemma 3.11.5, we get

$$\Bigg|p_2\bigg(\frac{1}{4(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}-\frac{1}{8\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}-\frac{1}{4\check{q}_o^2(1+\check{q}_o+\check{q}_o^2)^2}\bigg)-p_1^2\bigg(\frac{1}{16(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}-\frac{1}{16(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2[3]!_{\check{q}_o}}-\frac{1}{16\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}-\frac{1}{16\check{q}_o^2(1+\check{q}_o+\check{q}_o^2)^2}\bigg)\Bigg| = \Bigg|\bigg(\frac{2\check{q}_o^2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)^2-\check{q}_o(1+\check{q}_o+\check{q}_o^2)^2-2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}{8\check{q}_o^2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)^2(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}\bigg)\Bigg| \quad (2)$$

thus we get,

$$\Bigg|p_2\left(\frac{1}{4(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}-\frac{1}{8\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}-\frac{1}{4\check{q}_o^2(1+\check{q}_o+\check{q}_o^2)^2}\right)-$$
$$p_1^2\left(\frac{1}{16(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}-\frac{1}{16(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2[3]!_{\check{q}_o}}-\right.$$
$$\left.\frac{1}{16\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}-\frac{1}{16\check{q}_o^2(1+\check{q}_o+\check{q}_o^2)^2}\right)\Bigg|=$$
$$\left|\left(\frac{2\check{q}_o^2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)^2-\check{q}_o(1+\check{q}_o+\check{q}_o^2)^2-2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}{4\check{q}_o^2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)^2(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}\right)\right|.$$

This implies that,

$$|\hat{\alpha}_2\hat{\alpha}_4-\hat{\alpha}_3^2|\leq\frac{1}{(1+\check{q}_o)^2}\left[\frac{|p_1||p_3|}{4(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}+\frac{|p_2|^2}{4\check{q}_o^2(1+\check{q}_o+\check{q}_o^2)^2}+|p_1|^2\right.$$
$$\left.\left|\frac{2\check{q}_o^2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)^2-\check{q}_o(1+\check{q}_o+\check{q}_o^2)^2-2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}{4\check{q}_o^2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)^2(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}\right|\right],$$

applying Lemma 3.11.1, we get

$$|\hat{\alpha}_2\hat{\alpha}_4-\hat{\alpha}_3^2|\leq\frac{1}{(1+\check{q}_o)^2}\left[\frac{4}{4(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}+\frac{4}{4\check{q}_o^2(1+\check{q}_o+\check{q}_o^2)^2}+\right.$$
$$\left.4\left|\frac{2\check{q}_o^2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)^2-\check{q}_o(1+\check{q}_o+\check{q}_o^2)^2-2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}{4\check{q}_o^2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)^2(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}\right|\right],$$

after simplification we get the required result

$$|\hat{\alpha}_2\hat{\alpha}_4-\hat{\alpha}_3^2|\leq\frac{1}{(1+\check{q}_o)^2}\left[\frac{\check{q}_o^2(1+\check{q}_o+\check{q}_o^2)^2+(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}{\check{q}_o^2(1+\check{q}_o+\check{q}_o^2)^2(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}+\right.$$
$$\left.\left|\frac{2\check{q}_o^2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)^2-\check{q}_o(1+\check{q}_o+\check{q}_o^2)^2-2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}{\check{q}_o^2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)^2(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}\right|\right].$$

This completes the proof. □

If we take $\check{q}_o \to 1^-$ then result is same as proved result which is given below, for the class C_s.

Corollary 6.4.5.1. *If* $\xi(\hat{z}) \in C_s$ *then* $|\hat{\alpha}_2\hat{\alpha}_4-\hat{\alpha}_3^2| \leq \frac{29}{384}$.

Theorem 6.4.6. *If* $\xi(\hat{z}) \in C_s(\acute{q} - sin)$ *then*

$$\begin{aligned}
&|\mathscr{H}_3(1)| \leq \frac{1}{\check{q}_o^2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3+\check{q}_o^4)}\Bigg[1+\\
&\left|3-\frac{2}{\check{q}_o(1+\check{q}_o)}-\frac{3}{[3]!_{\check{q}_o}}\right|+\left|\frac{\check{q}_o(1+\acute{q}_o)-1}{\check{q}_o(1+\check{q}_o)}\right|+\left|1-\frac{3}{[3]!_{\check{q}_o}}-\frac{1}{\check{q}_o(1+\check{q}_o)}\right|\Bigg]+\Bigg[\frac{1}{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}\\
&\Bigg[\left|\frac{1}{4}-\frac{1}{4[3]!_{\check{q}_o}}-\frac{1}{4\check{q}_o(1+\check{q}_o)}\right|+\left|\frac{1}{2}+\frac{1}{2[3]!_{\check{q}_o}}\right|+\left|\frac{1}{4}-\frac{1}{4[3]!_{\check{q}_o}}+\frac{1}{4[3]!_{\check{q}_o}}\right|\Bigg]\Bigg]\\
&\Bigg[\frac{1}{2(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}\left|1+\frac{1}{[3]!_{\check{q}_o}}\right|+\frac{1}{4}\Bigg|\frac{1}{\check{q}_o(1+\check{q}_o)^3(1+\check{q}_o+\check{q}_o^2)}+\frac{1}{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}-\\
&\frac{1}{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2[3]!_{\check{q}_o}}-\frac{1}{\acute{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}\Bigg|+\frac{1}{4}\Bigg|\frac{1}{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}-\\
&\frac{1}{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2[3]!_{\check{q}_o}}-\frac{1}{\check{q}_o(1+\check{q}_o)^3(1+\check{q}_o+\check{q}_o^2)}+\frac{1}{\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}\Bigg|\Bigg]+\\
&\frac{1}{\check{q}_o(1+\check{q}_o)^3(1+\check{q}_o+\check{q}_o^2)}\Bigg[\frac{\check{q}_o^2(1+\check{q}_o+\check{q}_o^2)^2+(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}{\check{q}_o^2(1+\check{q}_o+\check{q}_o^2)^2(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}+\\
&\left|\frac{2\check{q}_o^2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)^2-\check{q}_o(1+\check{q}_o+\check{q}_o^2)^2-2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}{\check{q}_o^2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)^2(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}\right|\Bigg].
\end{aligned}$$

Proof. Hankel Determinant of order 3 defined as;

$$\mathscr{H}_3(1) = \hat{\alpha}_5(\hat{\alpha}_3 - \hat{\alpha}_2^2) - \hat{\alpha}_4(\hat{\alpha}_4 - \hat{\alpha}_2\hat{\alpha}_3) + \hat{\alpha}_3(\hat{\alpha}_2\hat{\alpha}_4 - \hat{\alpha}_3^2),$$

Taking modulus on both sides and applying triangular inequality, we have

$$|\mathscr{H}_3(1)| \leq |\hat{\alpha}_5||\hat{\alpha}_3 - \hat{\alpha}_2^2| + |\hat{\alpha}_4||\hat{\alpha}_4 - \hat{\alpha}_2\hat{\alpha}_3| + |\hat{\alpha}_3||\hat{\alpha}_2\hat{\alpha}_4 - \hat{\alpha}_3^2|,$$

using Theorem 6.2.2, Theorem 6.3.2, Theorem 6.4.4 and Theorem 6.4.5, we get

$$\begin{aligned}
&|\mathscr{H}_3(1)| \leq \frac{1}{\check{q}_o^2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3+\check{q}_o^4)}\Bigg[1+\\
&\left|3-\frac{2}{\check{q}_o(1+\check{q}_o)}-\frac{3}{[3]!_{\check{q}_o}}\right|+\left|\frac{\check{q}_o(1+\acute{q}_o)-1}{\check{q}_o(1+\check{q}_o)}\right|+\left|1-\frac{3}{[3]!_{\check{q}_o}}-\frac{1}{\check{q}_o(1+\check{q}_o)}\right|\Bigg]+\Bigg[\frac{1}{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}\\
&\Bigg[\left|\frac{1}{4}-\frac{1}{4[3]!_{\check{q}_o}}-\frac{1}{4\check{q}_o(1+\check{q}_o)}\right|+\left|\frac{1}{2}+\frac{1}{2[3]!_{\check{q}_o}}\right|+\left|\frac{1}{4}-\frac{1}{4[3]!_{\check{q}_o}}+\frac{1}{4[3]!_{\check{q}_o}}\right|\Bigg]\Bigg]\\
&\Bigg[\frac{1}{2(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}\left|1+\frac{1}{[3]!_{\check{q}_o}}\right|+\frac{1}{4}\Bigg|\frac{1}{\check{q}_o(1+\check{q}_o)^3(1+\check{q}_o+\check{q}_o^2)}+\frac{1}{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}-\\
&\frac{1}{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2[3]!_{\check{q}_o}}-\frac{1}{\acute{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}\Bigg|+\frac{1}{4}\Bigg|\frac{1}{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}-\\
&\frac{1}{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2[3]!_{\check{q}_o}}-\frac{1}{\check{q}_o(1+\check{q}_o)^3(1+\check{q}_o+\check{q}_o^2)}+\frac{1}{\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}\Bigg|\Bigg]+\\
&\frac{1}{\check{q}_o(1+\check{q}_o)^3(1+\check{q}_o+\check{q}_o^2)}\Bigg[\frac{\check{q}_o^2(1+\check{q}_o+\check{q}_o^2)^2+(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}{\check{q}_o^2(1+\check{q}_o+\check{q}_o^2)^2(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}+\\
&\left|\frac{2\check{q}_o^2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)^2-\check{q}_o(1+\check{q}_o+\check{q}_o^2)^2-2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}{\check{q}_o^2(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)^2(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)^2}\right|\Bigg].
\end{aligned}$$

which is required result. □

If $\check{\mathfrak{q}}_o \to 1^-$ in the above result, this leads us to the proved result for class C_s that mentioned in the following corollary.

Corollary 6.4.6.1. *If* $\xi(\hat{z}) \in C_s$ *then* $|\mathscr{H}_3(1)| \leq \frac{239}{5760}$.

6.5 Zalcman Functional

The following result will examine for $S_s^*(\acute{\mathfrak{q}} - sin)$ class.

Theorem 6.5.1. *If* $\xi(\hat{z}) \in S_s^*(\acute{\mathfrak{q}} - sin)$ *then*

$$|\hat{\alpha}_3^2 - \hat{\alpha}_5| \leq \frac{1}{\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o+\check{\mathfrak{q}}_o^2+\check{\mathfrak{q}}_o^3)}.$$

Proof. From (6.11) and (6.13), we have

$$|\hat{\alpha}_3^2 - \hat{\alpha}_5| = \left|\left[\frac{1}{\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o)}\left(\frac{p_2}{2}-\frac{p_1^2}{4}\right)\right]^2 - \left[\frac{1}{\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o+\check{\mathfrak{q}}_o^2+\check{\mathfrak{q}}_o^3)}\left(\left(\frac{3}{8}-\frac{1}{4\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o)}-\frac{3}{8[3]!_{\check{\mathfrak{q}}_o}}\right)p_1^2p_2 - \left(\frac{1}{4}-\frac{1}{4\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o)}\right)p_2^2 + \left(\frac{1}{16\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o)}-\frac{1}{16}+\frac{3}{16[3]!_{\check{\mathfrak{q}}_o}}\right)p_1^4 + \frac{p_4}{2}-\frac{p_1p_3}{2}\right)\right]\right|,$$

After simplification, we get

$$|\hat{\alpha}_3^2 - \hat{\alpha}_5| = \frac{1}{2\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o+\check{\mathfrak{q}}_o^2+\check{\mathfrak{q}}_o^3)}\left|p_1^4\left(\frac{(1+\check{\mathfrak{q}}_o+\check{\mathfrak{q}}_o^2+\check{\mathfrak{q}}_o^3)}{8\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o)^2}-\frac{1}{8\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o)}+\frac{1}{8}-\frac{3}{8[3]!_{\check{\mathfrak{q}}_o}}\right)+ p_2^2\left(\frac{(1+\check{\mathfrak{q}}_o+\check{\mathfrak{q}}_o^2+\check{\mathfrak{q}}_o^3)}{2\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o)^2}+\frac{1}{2}-\frac{1}{2\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o)}\right)- p_1^2p_2\left(\frac{(1+\check{\mathfrak{q}}_o+\check{\mathfrak{q}}_o^2+\check{\mathfrak{q}}_o^3)}{2\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o)^2}+\frac{3}{4}-\frac{1}{2\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o)}-\frac{3}{4[3]!_{\check{\mathfrak{q}}_o}}\right)+2\left(\frac{1}{2}\right)p_1p_3 - p_4\right|.$$

Using Lemma 3.11.6, this leads us to the required result

$$|\hat{\alpha}_3^2 - \hat{\alpha}_5| \leq \frac{1}{\check{\mathfrak{q}}_o(1+\check{\mathfrak{q}}_o+\check{\mathfrak{q}}_o^2+\check{\mathfrak{q}}_o^3)}.$$

Hence, proof is complete. □

If $\check{\mathfrak{q}}_o \to 1^-$ in the above theorem, this leads us to the proved result [78] for class S_s^*.

Corollary 6.5.1.1. *If* $\xi(\hat{z}) \in S_s^*$ *then* $|\hat{\alpha}_3^2 - \hat{\alpha}_5| \leq \frac{1}{4}$.

Now, the following result will determine for $C_s(\acute{\mathfrak{q}} - sin)$ class.

Theorem 6.5.2. *If* $\xi(\hat{z}) \in C_s(\acute{\text{q}} - sin)$ *then*

$$|\hat{\alpha}_3^2 - \hat{\alpha}_5| \leq \frac{1}{\check{q}_o(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3+\check{q}_o^4)}.$$

Proof. From (6.26) and (6.28), we have

$$|\hat{\alpha}_3^2-\hat{\alpha}_5| = \Bigg|\left[\frac{1}{\check{q}_o(1+\check{q}_o)(1+\check{q}_o+\check{q}_o^2)}\left(\frac{p_2}{2}-\frac{p_1^2}{4}\right)\right]^2 - \Bigg[\frac{1}{\check{q}_o(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3+\check{q}_o^4)} \left(\left(\frac{3}{8}-\frac{1}{4\check{q}_o(1+\check{q}_o)}-\frac{3}{8[3]!_{\check{q}_o}}\right)p_1^2p_2 - \left(\frac{1}{4}-\frac{1}{4\check{q}_o(1+\check{q}_o)}\right)p_2^2 + \left(\frac{1}{16\check{q}_o(1+\check{q}_o)}-\frac{1}{16}+\frac{3}{16[3]!_{\check{q}_o}}\right)p_1^4+\frac{p_4}{2}-\frac{p_1p_3}{2}\right)\Bigg]\Bigg|,$$

After simplification, we get

$$|\hat{\alpha}_3^2-\hat{\alpha}_5| = \frac{1}{2\check{q}_o(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3+\check{q}_o^4)}\Bigg|p_1^4\left(\frac{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3+\check{q}_o^4)}{8\check{q}_o(1+\check{q}_o)^2(1+\check{q}_o+\check{q}_o^2)^2} - \frac{1}{8\check{q}_o(1+\check{q}_o)}+\frac{1}{8}-\frac{3}{8[3]!_{\check{q}_o}}\right)+p_2^2\left(\frac{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3+\check{q}_o^4)}{2\check{q}_o(1+\check{q}_o)^2(1+\check{q}_o+\check{q}_o^2)^2}+\frac{1}{2}-\frac{1}{2\check{q}_o(1+\check{q}_o)}\right) - p_1^2p_2\left(\frac{(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3+\check{q}_o^4)}{2\check{q}_o(1+\check{q}_o)^2(1+\check{q}_o+\check{q}_o^2)^2}+\frac{3}{4}-\frac{1}{2\check{q}_o(1+\check{q}_o)}-\frac{3}{4[3]!_{\check{q}_o}}\right)+2\left(\frac{1}{2}\right)p_1p_3-p_4\Bigg|.$$

Using Lemma 3.11.6, this leads us to the required result

$$|\hat{\alpha}_3^2 - \hat{\alpha}_5| \leq \frac{1}{\check{q}_o(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3)(1+\check{q}_o+\check{q}_o^2+\check{q}_o^3+\check{q}_o^4)}.$$

This completes the proof. □

If $\check{q}_o \to 1^-$ in the above result, this leads us to the proved result for class C_s that mentioned in the following corollary.

Corollary 6.5.2.1. *If* $\xi(\hat{z}) \in C_s$ *then* $|\hat{\alpha}_3^2 - \hat{\alpha}_5| \leq \frac{1}{20}$.

6.6 Summary

In this chapter two subclasses of univalent function, starlike and convex functions defined. For the newly specified classes, Hankel Determinants, Zalcman functional, Fekete–Szegő inequality, and coefficient estimates determined. This study also defines a few corollaries, which indicate that when limit $\check{q}_o \to 1^-$ is substituted, then the obtained results are same as proved by researchers.

CHAPTER 7

CONCLUSION

This thesis primarily explores the initial coefficient bounds of functions that are analytic, univalent, and normalized within the open unit disk. We began by explaining some fundamental definitions and preliminary results derived from Geometric Function Theory. Our innovative discoveries originate from these fundamental principles, and we also explored recent concepts introduced in Quantum Calculus. Alongside a comprehensive investigation into the applications of the q-derivative operator in Geometric Function Theory, we utilized q-Calculus to establish several novel classes of analytic functions associated with symmetric points.

Our research concentrates on two fundamental categories of univalent functions: starlike functions and convex functions related to symmetric points. We explored the extension of these classes using q-calculus, building upon the prior work by Khan *et al.* [78] on the S_s^* class of starlike functions associated with symmetric points and the C_s class of convex functions associated with symmetric points. We introduced the $S_s^*(\acute{q})$ class, signifying starlike functions concerning symmetric points subordinate to the q́-sine function-an expansion of the original S_s^* class. We presented the q́-extension of these classes by defining the $S_s^*(\acute{q} - sin)$ class for q́-starlike functions and the $C_s(\acute{q} - sin)$ class for q́-convex functions, both subordinate to the q́-sine function. These classes were introduced through the q-derivative operator, and we utilized the subordination technique to investigate their properties.

We have explored various noteworthy properties of functions within our recently introduced classes, including coefficient bounds, the Zalcman functional, and a well-known Fekete–Szegő inequality. Additionally, we have investigated Hankel determinants, both second and third orders, for functions belonging to our newly defined classes. It has been observed that these novel classes represent a refinement compared to existing ones, and the results obtained signify advancements over previously established theorems by numerous researchers in the field of Geometric Function Theory. To validate our findings, we have confirmed them by taking the limit as $\acute{q} \to 1^-$, yielding known results. We expect that this study will contribute significantly to the advancements in Geometric Function Theory. It is pleased to mention that a part of this study has been presented in National Conference on Engineering and Computing-2023.

7.1 Future work

This thesis focuses on two primary categories within univalent function theory: starlike functions associated with symmetric points and convex functions associated with symmetric points, subordinate to a specific trigonometric function, namely the sine function. These classes can be advanced by using the concept of close-to-convexity and results can be determined for the advanced class of $\acute{q}$-quasi convex functions and relationship between these classes and the classes presented in this thesis can be drawn analytically and geometrically.

Appendices

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