# A NEW HOMOTOPY TECHNIQUE FOR SOLVING SYSTEM OF NONLINEAR EQUATIONS

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# A New Homotopy Technique for Solving System of Nonlinear Equations

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Candidate of **Master of Science in Mathematics** at National University of Modern Languages do here by declare that the thesis A **New Homotopy Technique for Solving System of Nonlinear Equations** submitted by me in fractional fulfillment of **MS Mathematics** degree, is my original work, and has not been submitted or published earlier. I also solemnly declare that it shall not, in future, be submitted by me for obtaining any other degree from this or any other university or institution. I also understand that if evidence of plagiarism is found in my thesis/dissertation at any stage, even after the award of a degree, the work may be cancelled and degree revoked.

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# DEDICATION

I dedicate my thesis to my parents and teachers especially my supervisor Dr Naila Rafiq for their never-ending backing and inspiration throughout my pursuit for education. I hope this accomplishment will fulfill the dream they intended for me.

# ABSTRACT

In this study, first of all two new optimal fourth order iterative techniques have studied for solving single variable non-linear equations. Theses techniques need evaluation of one first derivative and two function evaluation that satisfy the Kung-Traub conjecture. These numerical techniques are then extended to techniques for solving system of non-linear equations.

The establishment and execution of these techniques for solving system of non-linear equations is based on the concept of element wise vector multiplication and diagonal matrix.

In case, Jacobian matrix becomes singular at any stage because of approximation, the above techniques would fail. In order to overcome this difficulty, homotopy techniques for solving system of non-linear equations were introduced. Theoretical convergence of modified iterative techniques are proved through analysis theorems and numerical convergence through real time applications are provided to test the performance of these techniques.

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# Chapter 1

# Introduction and Preliminary Concepts

## 1.1 Introduction

Applied sciences and engineering describe various mathematical procedure for solving mathematical models. In general, every problem can not be solved by analytical techniques, therefore one requires numerical approximation techniques of numerical analysis, for solving such problems. The problem arising maybe linear or non-linear. It is easy to solve linear problem while non-linear problem maybe difficult to solve. These problems arise in many areas of sciences, biological sciences, engineering, and natural sciences, etc. There are many situations arise physically where we need to solve single variable, non-linear algebraic or transcendental equations or system of non-linear equations. Algebraic equations in particular, polynomial equations have a wide applications in applied sciences, astronomers compute the distance between stars and other objects using polynomials, aerospace engineers require polynomials for computing stability of aeroplane or acceleration for rockets, mechanical engineers use polynomials in designing machines and structures like bridges and have application in bio-medical engineering.

In 1824, Abel provided that there exists no exact method which should find the roots of polynomial of degree equal or greater than five. Therefore, one is forced to use iterative approximating techniques of numerical analysis. Similarly, transcendental equations have applications in chemical equilibrium problems, kinematics of mechanics and mathematical physics and have no exact methods to find their solution except numerical iterative techniques. Non-linear ODEs and PDEs have wide applications in sciences but while solving theses problems, system of non-linear equations arise which needs to be solved.

Researchers have developed many iterative techniques in the past as well as working presently also on the techniques to get numerically the solution of single variable nonlinear equations [1–12] and system of nonlinear equations [13–23]. The researchers have established iterative techniques which involve derivatives or derivative free for single variable nonlinear equations as well as iterative techniques involving derivatives or derivative free memory methods involving divided differences.

In this study, first of all iterative schemes involving derivative and function values for single variable nonlinear equations will be established via integral inequalities. Then, these techniques will be extended for solving system of nonlinear equations use the element wise vector multiplication and diagonal matrix. These techniques will then be generalized to homotopy techniques for solving the system of nonlinear equations to avoid the case of non-singular Jacobian matrix.

Nonlinear equations or sets of such equations are used to solve many issues in mechanics, physics, biology, economics, and other fields [24]. The equations can be algebraic, ordinary differential, or partial differential; and systems can include equations of multiple types as well as equations of the same type. The solutions to these equations and systems are classified as regular or unique. The implicit function theorem or its analogs, which describe all neighboring solutions, are applicable to a regular solution. The implicit function theorem is inapplicable near a single solution, and there was no universal technique for analyzing solutions neighboring the unique one until recently. Although other approaches to such analysis were proposed for specific challenges. Partial differential equations may be found in almost every discipline of physics, chemistry, and engineering. They can also be found in other disciplines of the physical sciences, as well as the social sciences, economics, and business. Many aspects of theoretical physics are expressed as partial differential equations. In certain circumstances, the axioms demand that the states of physical systems be represented by partial differential equation solutions. When the axioms are applied to specific situations, partial differential equations developed. The system of nonlinear equations has great importance in many engineering and applied sciences. In our daily life, we have to face with physical problems, where more than one variables have to be considered with nonlinear phenomena to model it into mathematical equations. The numerical techniques for solving system of nonlinear equations is very important in the absence of analytical techniques. They have vast applications in applied sciences and engineering. For example, as a result of modeling the following problems:

i) Tank reactor in series problem,

ii) Turbulent flow through a pipe line network,

iii) Mechanical Engineering Problems, etc,

iv) For solving the nonlinear boundary value problems for ODEs and PDEs by finite difference methods, we get a system of nonlinear equations.

Before beginning a thorough examination of nonlinearity, it is useful to list some of the major causes of issues. The following are three classic sources of nonlinear problems:

First, differential-geometric problems, in which nonlinearity enters naturally via curvature considerations;

Second, mathematical problems of classical and modern physics;

and finally, calculus of variations problems involving nonquadratic functional. Of course, these are not complete sources, and the mathematical parts of subjects such as economics, genetics, and biology provide totally new nonlinear phenomena. With the advent of calculus, nonlinear analysis issues developed naturally. In the mathematical literature of the seventeenth and eighteenth centuries, explicit and innovative ways of solving problems existed. This work prompted Euler and Lagrange to think about the general theory of the calculus of variations. Furthermore, while seeking to base Newton's technique of indeterminate coefficients rigorously, Cauchy finally led to the majorant approach for analytic nonlinear problems. The broad use of this technique of proof continues to this day. Poincaré, on the other hand, introduced a fresh dimension to our subject, beginning with his thesis in the 1870s. Poincaré concentrated on the qualitative features of nonlinear problems, which opened up a whole new set of mathematical topics. Poincaré introduced new concepts in a variety of areas, including bifurcation theory (a term coined by Poincaré), the calculus of variations in the large, and the application of topological methods to the study of periodic solutions of systems of ordinary differential equations. Cauchy also systematically applied minimization methods (the steepest descent approach) in the investigation of the zeros of simultaneous algebraic or transcendental equations over the real. Hubert's well-known talk at the International Congress of 1900 offered a variety of exciting nonlinear issues for analysis, stimulating study on nonlinear elliptic partial differential equations in particular. This final topic proved to be crucial for growth on a more abstract level. S. Bernstein's early twentieth-century solutions to Hilbert's problems for nonlinear elliptic partial differential equations, in particular, were sufficiently comprehensive to serve as a foundation for further abstraction and generalization. Differential-geometric problems involving curvature effects are a rich and historic source of nonlinear differential systems. The breakthroughs in the calculus of large variations achieved by Marston Morse beginning in 1922 and subsequently by Liusternik and Schnirelmann are a last major milestone in the early research of nonlinear issues. Birkhoff and Kellogg's 1922 publication, "Invariant Points in Function Space" was a seminal study in the development of nonlinear analysis [25]. Variational inequality issues, nonlinear optimization challenges, equilibrium difficulties, complementarity problems, and integral and differential equations are all examples of problems that may be solved using fixed point theory. Many scientific and engineering issues that are specified by nonlinear functional equations can be addressed by reducing them to an analogous fixed-point problem. Several approaches, including the Taylor series, quadrature formulae, homotopy, and decomposition techniques, are being used to build iterative methods for finding approximate solutions to nonlinear equations. The Newton technique is a well-known iterative approach for solving nonlinear equations and their variant forms.

### 1.1.1 Significance of the Study

Many applied problems in engineering and applied sciences can be modeled by solving systems of nonlinear equations, which is one of the most fundamental problems in computational mathematics. So many efforts have been made by the mathematical community to introduce new theories and algorithms for solving systems of nonlinear equations.

It will provide new areas of research in computational Mathematics. For solving the nonlinear boundary value problems for Ordinary Differential Equations and Partial Differential Equations by finite difference methods, we get a system of nonlinear equations that need to be solved to find the solution to the actual problem.

Similarly, optimization problems appearing in a wide range of physical situations result in a system of nonlinear equations. There are very few nonlinear systems of equations that can be solved by analytical techniques. Therefore, one has to use numerical methods for solving a system of nonlinear equations. Among other types of nonlinear equation-solving techniques presented in the recent past, there are a large number of published research articles with Newton-type formulae and variants of Newton's method. It is obvious that Newton's method and its variants extended for solving nonlinear equations may fail when their Jacobian matrix is singular, but the homotopy iterative method for solving nonlinear systems does not fail.

### 1.1.2 Historical Perspective

Systems of linear equations are a well-known part of numerical techniques dating back to BC. It reached its height during 1600–1700 as a result of the public's desire for technological and engineering solutions, but it is still relevant today. This study proposes another iterative strategy for solving linear systems that are based on numerous transfers of the solution proximity point towards the solution itself, minimizing the differences of all the system equations at the same time. It is a challenging task requiring many branches of science and technology to solve nonlinear equations in any Banach space, including real or complex nonlinear equations, nonlinear systems, and nonlinear matrix equations [13]. Typically, the answer is not immediately economical and necessitates an iterative algorithmic technique. This is a field of study that has expanded rapidly in recent years. For solving nonlinear equations iteratively, the Newton's method given by

$$s_{n+1} = s_n - \frac{f(s_n)}{f'(s_n)},$$

is one of the most frequently used methods. The most well-known technique overall is Newton's approach, which demonstrates quadratic convergence for a single root and linear convergence for multiple roots. For the first time in 1669, Newton used the Newton iteration to solve a cubic problem.

### 1.1.3 The intervention of Numerical Methods

One of humanity's most significant achievements is the application of mathematical formulas to portray real-world issues . These tools have been utilized to improve the environment in which we live as well as to improve technology, medical services and facilities, modes of transportation, and communications. To do this, we rely on mathematical techniques to represent real-world issues, beginning with observation, then analysis, and eventually prediction. Two processes demand special attention: the first is converting observable facts into mathematical formulas, and the second is generating answers using analytical and numerical approaches. Because real-world situations are highly nonlinear, mathematical models portraying these problems are themselves nonlinear and cannot be addressed analytically. As a result, numerical approaches are increasingly being employed to assist people in understanding and predicting the future behavior of real-world situations. Many new families of differential and integral operators have lately been recognized as excellent mathematical tools for replicating observable phenomena. A new demand for new methods of solving systems of linear equations emerged at the same time as computing technology, which facilitated the rapid development of numerical methods for modeling physical processes by sampling (subdividing) the calculation range and replacing differential operations with similar algebraic operations [14]. Direct and iterative approaches for addressing a badly finished diagonal matrix with a strong main diagonal were created based on the needs of the final differences, final elements, and their changes. For both direct and iterative techniques, methods for efficient storing of the equation system were devised, taking into consideration the symmetry of the matrix according to the major diagonal. With the development of new numerical methods (super elements, the method of border elements) in recent years, there has been a need for solving systems of linear equations with a totally filled matrix and one that lacks the primary diagonal dominance. Iterative methods are frequently employed to solve such jobs, and the approaches have evolved from the Gauss-Seidel method, Jacobian method, Homotopy iterative method.

The method shown here is easily adaptable to any ultimate number of equations.

The geometric equation in two dimensions is a line, but a non-contradictory equation system includes two lines that cross, generating four angles. The iterative solution process occurs from only one perspective. In three dimensions, one equation is a geometric plane; a non-contradictory equation system is composed of three planes that intersect each other. The iterative solution procedure takes place inside the pyramid, whose surface is formed of three planar equations, and the pinnacle of a pyramid is sought. There are eight pyramids, and the solution is sought in just one of them.

### 1.1.4 Homotopy Analysis Method

The use of perturbation techniques [26, 27] is global in the fields of science and engineering. The use of perturbation methods requires governing equations, beginning conditions, or boundary conditions to contain tiny or large physical factors, which are referred to as perturbation quantities. The methods of perturbation are straightforward and easy to comprehend. Particularly when working with a limited number of physical parameters, perturbation approximations can have very specific consequences in the real world. Sadly, this form of perturbation quantity is not present in every nonlinear system. In addition, even if there is a physical parameter that is this tiny, the sub-problem may not have any solutions or may be so sophisticated that only a select handful of the sub-problems can be addressed. As a result, it is not a given that one can always acquire perturbation approximations for any particular nonlinear issue. This is because it is not guaranteed. In addition to this, it is well knowledge that the majority of perturbation approximations are only appropriate for use with very modest values of physical parameters. Therefore, it is not guaranteed that the results of a perturbation are valid over the whole region for each and every physical parameter.

The fact that perturbation and asymptotic approximations are inefficient for dealing with nonlinear conditions lends authority to this commonly held notion [28]. As it turns out, its application is confined to partial differential equations (PDEs) and ordinary differential equations (ODEs) with weakly nonlinear behavior.

The homotopy analysis method, abbreviated as HAM is a method of analytical approximation constructed to cope with severely nonlinear problems. The year 1992 witnessed the establishment of the HAM organization. In contrast to perturbation techniques, the (HAM) is unaffected by the size of physical components, no matter how small or large they are. This is true regardless of where the component is located. Second, unlike all other analytic techniques, the HAM provides a simple way to assure the convergence of solution series, which means that it is valid even if the nonlinearity is rather strong. This is due to the fact that the HAM is an iterative process. Furthermore, because it is based on topology's homotopy, it provides us with an extremely large level of choice in selecting the base function, the first guess, and so on. As a result, complicated nonlinear ODEs and PDEs may typically be solved simply using this approach.

Finally, the HAM achieves a high level of generality by logically incorporating a variety of proven approaches, such as Lyapunov's small artificial approach, the Adomian decomposition method, the expansion method, and even the Euler transform. As a result, the HAM provides us with a useful tool for addressing highly nonlinear situations in engineering, research, and the financial industry.

## 1.2 Some Basic Definitions & Concepts

It is commonly known that the solution to the single variable nonlinear equation

$$f(s) = 0, \tag{1.1}$$

where $f : I \to D$, for an interval $I \subseteq R$ and $D \subseteq R$, is a nonlinear function is required in many sectors of science and engineering. Iterative algorithms for finding the roots of nonlinear equations based on these iterative approaches are becoming one of the most significant parts of current research. This might be used to solve nonlinear equations. Clearly, in order to use this iterative procedure, we must compute the second and third derivatives of the function $f(s)$, which may be inconvenient. To circumvent this limitation, they propose approximates of the second and third derivatives, which is a very essential notion that plays a big role in constructing several iterative approaches that do not need computing the higher derivatives. Some iterative approaches with high-order convergence for solving a single nonlinear equation have been presented in recent years. The system of nonlinear equations has great importance in many engineering and applied sciences. In our daily life, we have to face physical problems, where more than one variables have to be considered with nonlinear phenomena to model it into mathematical equations. The numerical techniques for solving a system of nonlinear equations are very important in the absence of analytical techniques. They have vast applications in applied sciences and engineering.

Differential equations are a key technique used to simulate a wide range of real-world events in many domains of pure and applied research. While there are analytic methods for solving differential equations, many of the equations encountered in practice are too complicated to be solved in a closed-form manner. Even if a solution formula is known, it may require integrals that can only be numerically approximated. In such instances, numerical techniques can be used to solve differential equations under certain beginning conditions. In science and engineering, initial value issues in the form of ordinary differential equations are prevalent.

In this section, we will present some definitions of nonlinear equations, types of nonlinear equations from [29], solutions of the equations and iterative methods as well as their convergence . The basic definitions and concepts presented here will be used throughout this thesis.

### 1.2.1 Non-linear Equations

A nonlinear system is one in which there is no linear relationship between the change in the output and the change in the input. Due to the intrinsic nonlinearity of most systems, nonlinear problems are of interest to a wide range of scientists, including engineers, biologists, physicists, mathematicians, and many others.

**Algebraic Equations**

A mathematical equation is referred to as algebraic if it contains one or more algebraic expressions. An algebraic equation can also be a polynomial equation.

**Polynomial Equations:**

For one variable $s$, and for a positive integer $n$, constants $a_0$, $a_1, ...,$ $a_{n-1}, ...,$ $a_n$ are the coefficients of the polynomial, an expression of the form

$$a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n,$$

is a polynomial in $s$. In sigma notation, a polynomial can be represented simply as

$$P_n(s) = \sum_{j=0}^{n} a_j s^{n-j}.$$

**Transcendental Equations**

Transcendental equations are those that contain trigonometric, exponential, and logarithmic functions.

**Trigonometric Polynomial**

$$\mathcal{T}_n(s) = \alpha_0 + \sum_{i=1}^{n} (\alpha_i \cos kx + \beta_i \sin kx).$$

**Exponential Polynomial**

$$\mathcal{E}_n(s) = \alpha_0 + \sum_{i=1}^{n}(\alpha_i \mathfrak{a}^{-kx} + \beta_i \mathfrak{a}^{kx}).$$

For exponential polynomial researcher useded the case when $\mathfrak{a} = e \approx 2.71828$.

**Logarithmic Polynomial**

$$\mathfrak{Lg}_n(s) = a_0 + \sum_{i=1}^{n} a_i \log_a ix.$$

### 1.2.2 System of Nonlinear Equations

Take into consideration a system of n nonlinear equations in n variables. as follows:

$$\begin{cases} f_1\left(s_1, s_2, s_3, ..., s_n\right) = 0, \\ f_2\left(s_1, s_2, s_3, ..., s_n\right) = 0, \\ \qquad\qquad \vdots \\ f_n\left(s_1, s_2, s_3, ..., s_n\right) = 0. \end{cases}$$

Another way to describe the system of nonlinear equations in $n$ variables is to define a vector function $\mathbf{F}$ from $\mathbb{R}^n$ to $\mathbb{R}^n$:

$$\mathbf{F}\left(\mathbf{s}\right) = \left(f_1\left(\mathbf{s}\right), f_2\left(\mathbf{s}\right), ..., f_n\left(\mathbf{s}\right)\right)^T, \tag{1.2}$$

where $s = \left(s_1, s_2, s_3, ..., s_n\right)^T$. Thus the system of non-linear equations assumes the form:

$$\mathbf{F}\left(\mathbf{s}\right) = 0. \tag{1.3}$$

The Jacobian matrix $\mathbf{F}'(\mathbf{s})$ for this system is as follows:

$$\mathbf{F}'(\mathbf{s}) = \begin{bmatrix} \frac{\partial f_1}{\partial s_1} & \frac{\partial f_1}{\partial s_2} & \cdots & \frac{\partial f_1}{\partial s_n} \\ \frac{\partial f_2}{\partial s_1} & \frac{\partial f_2}{\partial s_2} & \cdots & \frac{\partial f_2}{\partial s_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial s_1} & \frac{\partial f_n}{\partial s_2} & \cdots & \frac{\partial f_n}{\partial s_n} \end{bmatrix} \tag{1.4}$$

The Jacobian matrix $\mathbf{F}'(\mathbf{s})$ needs to be non-singular for the iterative techniques to be extended to a system of nonlinear equations.

**Fréchet Differentiable Function**

If $F : X \to Y$ where $X$ and $Y$ are normed vector spaces, we say that a linear transformation $A : X \to Y$ is a Frechet derivative of $F$ at $s$ if for every $\epsilon > 0$ there is $\delta > 0$ such that

$$\|F(\mathbf{s}+\mathbf{h}) - F(\mathbf{s}) - A\mathbf{h}\|_Y \leq \epsilon \|\mathbf{h}\|_X ,$$

for all $\mathbf{h}$ with $\|\mathbf{h}\|_X \leq \delta$.

### 1.2.3 Some Basic Iterative Mehods

The iterative method's basic notion is that after making an initial estimate of the exact solution, it is continually improved so that subsequent estimates, or iterations, move progressively closer to the required solution of the initial problem. A series is constructed starting with the initial guess, $s_0$, that converges to the root $\alpha$, and then $\underset{n\to\infty}{Lim} s_n = \alpha$. An iteration method is one that generates such a sequence.

These are the two basic categories for iterative methods:

- **One-step iterative method**
- **Multistep iterative method**

**One-step Method**

Using a single formula, the root is approximated in the one-step technique. These approaches use a single numerical procedure and locate a single root at a time, usually after the initial guess is given. When the necessary accuracy is attained using such numerical techniques, the iterative process comes to an end. The majority of techniques used in literature are one-step techniques. Here are a few well-known one-step techniques:

**Newton's Method**

In this technique, the root of f(x)=0 is approximately determined via a tangent line. The procedure [29] is as follows, presuming an initial guess of $s_0$:

$$s_{n+1} = s_n - \frac{f(s_n)}{f'(s_n)}, \quad n = 0, 1, 2, \ldots \text{ and } \ f'(s_n) \neq 0.$$

**Chebyshev-Halley Method**

The Chebyshev-Halley approach is described as follows:

$$s_{n+1} = s_n - \left(1 + \tfrac{1}{2}\tfrac{L_f(s_n)}{1-\alpha L_f(s_n)}\right)\tfrac{f(s_n)}{f'(s_n)}, \quad n = 0, 1, 2, \ldots \text{ and } \ f'(s_n) \neq 0, \tag{1.5}$$

where

$$L_f\left(s_n\right)=\frac{f(s_n)f''(s_n)}{f'(s_n)}.$$

**Multi-Step Method**

A multi-step approach is one that uses two or more numerical procedures in a predictor-corrector fashion.

Essentially, two numerical methods make up two-step iterative procedures. In order to obtain a better number for the root, the first numerical approach is given the initial guess. The enhanced value is then applied to the second numerical procedure, where it is further enhanced. For instance, a common two-step iterative technique is as follows:

$$\mathbf{v}_n = s_n - \frac{f(s_n)}{f'(s_n)},$$

$$s_{n+1} = \mathbf{v}_n - \frac{(s_n-\mathbf{v}_n)f(\mathbf{v}_n)}{f(s_n)-2f(\mathbf{v}_n)},\ n=0,1,2,\ldots\ .$$

Here at first-step we have Newton's method and at the second step, Ostrowski's method which is a special case of King's family of methods.

### Modified Newton's Method for a System of Non-linear Equations

The Newton's approach for finding the solution of a non-linear system is described below if there are $n$ non-linear equations in $n$ unknowns [29]:

$$\mathbf{s}^{(k+1)} = \mathbf{s}^{(k)} - \mathbf{F}'\left(\mathbf{s}^{(k)}\right)^{-1}\mathbf{F}\left(\mathbf{s}^{(k)}\right),$$

$$\text{where} \qquad \mathbf{F}'\left(\mathbf{s}^{(k)}\right) = \begin{bmatrix} \frac{\partial f_1}{\partial s_1} & \frac{\partial f_1}{\partial s_2} & \cdots & \frac{\partial f_1}{\partial s_n} \\ \frac{\partial f_2}{\partial s_1} & \frac{\partial f_2}{\partial s_2} & \cdots & \frac{\partial f_2}{\partial s_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial s_1} & \frac{\partial f_n}{\partial s_2} & \cdots & \frac{\partial f_n}{\partial s_n} \end{bmatrix}$$

is non-singular Jacobian matrix.

### Newton-Homotopy Method

For the solution of non-linear equation

$$f(s) = 0,$$

Newton'method is modified for the nonlinear problems where divergence occure due to derivative of function is zero at appromate value by following Wu [30]. An auxiliary homotopy function

$$g(s) = 0,$$

which is known and controllable is necessary to define homotopy continuation for the function $h : \mathbb{R} \times [0,1] \to \mathbb{R}$ as

$$h(s,t) = tf(s) + (1-t)\,g(s) = 0, \tag{1.6}$$

where the arbitrary parameter $t \in [0,1]$.

From (1.6), we have the following two boundary conditions:

$$\begin{aligned} h(s,0) &= g(s), \\ h(s,1) &= f(s), \end{aligned}$$

method of homotopy continuation is choosing different values of $t$ between 0 and 1, one attempts to resolve $h(s,t) = 0$ instead of $f(s) = 0$ in this case and attempts to prevent divergence. Consequently, Wu's Newton-homtopy technique is provided by

$$s_{n+1} = s_n - \tfrac{h(s_n,t)}{h'(s_n,t)}, \quad h'(s_n,t) \neq 0, \quad n = 0,1,2,...$$

**Error Equation**

If the sequence of approximations $\{s_n\}$ generated by using an iterative technique has order of convergence $p$ then it tends to a actual solution $s^*$ of non linear equation (1.1) in such a way that

$$\lim_{n\to\infty} \frac{s_{n+1} - s^*}{(s_n - s^*)^p} = C,$$

for $p \geq 1$. Let $e_n = s_n - s^*$, then the following relationis used to define error equation [29]

$$e_{n+1} = Ce_n^p + O(e_n^{p+1}) = O(e_n^p).$$

**Convergence Order**

The rate of convergence is the rate at which a convergent sequence approaches its limit [29]. It is described in mathematics as:

Assume that the iteration sequence $\{s_n\}_{n=1}^{\infty}$ converges to the value $\alpha$. Let the errors at the nth and (n+1)th iterations be $e_n = \alpha - s_n$ and $e_{n+1}{=}\alpha - s_{n+1}$, respectively, for n>0. If there are two positive constants, $c \neq 0$ and $p > 1$, and

$$\underset{n\to\infty}{Lim}\frac{|s_{n+1}-\alpha|}{|s_n-\alpha|^p} = \underset{n\to\infty}{Lim}\frac{|e_{n+1}|}{|e_n|^p} = c \ , \ \ n \geqq 0.$$

The rate of convergence of $s_n$ to $\alpha$ is therefore defined as the asymptotic constant c, and the sequence is said to have converged to with convergence order $p$. However, the convergence is linear if $p = 1$.

**Kung and Traub Conjucture**

Multi-step methods based on $m$ function and derivative evaluations can achieve optimal convergenc order $2^{m-1}$, which is also called Kung and Traub conjecture [32,33]

**Computational Order of Convergence**

Let $\alpha$ be the root of a non-linear equation $f(s) = 0$ and suppose that $s_{n-1}, s_n, s_{n+1}$ are the three consecutive iterates closer to the root $\alpha$. Then the computational order of convergence $\rho$ is defined as [34]:

$$\rho \approx \frac{\ln|(s_{n+1}-\alpha)\,/\,(s_n-\alpha)|}{\ln|(s_n-\alpha)\,/\,(s_{n-1}-\alpha)|}.$$

**Computational Efficiency**

The typical notation for the numerical iterative method's computational efficiency is $EI$, gives its definition [34]:

$$EI = p^{\frac{1}{m}},$$

where $m$ is the number of function and derivative evaluations needed by the approach for each iteration and $p$ is the method's order of convergence.

# Chapter 2

# Literature Study and Conceptual Framework

One of the very old and important problem in applied sciences and engineering is to solve the nonlinear equation:

$$f(s) = 0,$$

where $f(s)$ is a nonlinear function. Variety of scientific and mathematical problem arising in engineering, natural biosciences and medical sciences can be reduced to nonlinear algebraic and transcendental equations [35–37]. Generally, it is not possible to calculate their roots by an exact method, therefore one requires numerical approximate iterative methods. The researchers have developed a lot of iterative methods such as the best known Newton's method and it's variants [2,3,38–47], secant method, Halley's method, Chebyshev method and Super-Halley method, etc. The Newton's method which is of optimal convergence order two is given by

$$\begin{aligned} s_{n+1} &= s_n - \frac{f(s_n)}{f^{'}(s_n)}, n = 0, 1, 2, ..., \\ f^{'}(s_n) &\neq 0. \end{aligned}$$

The researchers tried to improve convergence order of Newton's method to cubic and arrive at cubically convergence methods, such as Weerakoon and Fernando method [34]:

$$s_{n+1} = s_n - \frac{2f(s_n)}{f^{'}(s_n) + f^{'}(s_n - \frac{f(s_n)}{f^{'}(s_n)})},$$

while Frontini *et al* [4] obtained the cubically convergent method

$$s_{n+1} = s_n - \frac{f(s_n)}{\frac{f^{'}(s_n - f(s_n))}{2f^{'}(s_n)}}.$$

In [48], Homeier denied the following cubically convergent method:

$$s_{n+1} = s_n - \frac{f(s_n)}{2}\left(\frac{1}{f^{'}(s_n)} + \frac{1}{f^{'}(s_n - \frac{f(s_n)}{f^{'}(s_n)})}\right).$$

Kung and Traub [33] made a conjecture that the iterative method, without memory has optimal convergence order $2^{\text{eval}\,s-1}$, where evals is number of function and derivative on evaluation, for example Jarrat's method [49] is of optimal convergence order four. Some recently produced optimal and non-optimal iterative methods by researchers are given in [5–8, 15, 41, 49–52].

Motivated by the above research, in this study first of all introduce optimal four order methods from integral inequalities. Next, the presented fourth order methods are extended to numerical iterative schemes for solving system of nonlinear equations. There are many scientific and mathematical problems while solving them, one comes across system of nonlinear equations. For example solving boundary value problem,for ODEs and PDEs by finite difference approximations, are sets system of nonlinear equations which need to be solved. System of nonlinear equations have a lot of applications in applied sciences, for example, one requires to solve system of nonlinear equations in physiology, Chemical equilibrium problem, Kinematics, Combustion problem and economic modeling problem [61] reactor and starry problem by Tsoulos and Stavrakoudis [17] and transport theory by Lin *et al* [54].

## 2.1 Existing Theory

Scientists and engineers have committed their efforts to the application of the homotopy perturbation technique in linear and nonlinear problems since this approach is used to constantly deform a straightforward issue that is easy to solve into an understudy problem that is difficult to solve. This approach is the combination of the classic perturbation method and homotopy in topology, which was initially proposed by He and systematically described.

To find the solution of nonlinear systems, Newton's method has been modified in [18], which is modified in quadratically convergent. A lot of variant of Newton's have been introduced in the literature for solving nonlinear systems utilizing various way. Hueso *et al* [19, 55] used Taylor polynomial for this purpose, Darvishi and Barati [20] and Noor and El-Sayed [56] applied Adomian decomposition method. In this study, two new fourth order optimal iterative techniques for solving system of non-linear equations have been developed.

In 2022, Thota [58] suggests three iterative methods of order three, six, and seven respectively, when solving non-linear equations with the modified homotopy perturbation approach paired with a system of equations. These methods solve the equations in increasing order.

Jarratt's fourth order iterative method [57] for solving non-linear system is given in the following form:

$$\begin{cases} \mathbf{v}^{(k)} = \mathbf{s}^{(k)} - \frac{2}{3}\mathbf{F}'(\mathbf{s}^{(k)})^{-1}\mathbf{F}(\mathbf{s}^{(k)}), \\ \mathbf{s}^{(k+1)} = \mathbf{s}^{(k)} - \frac{1}{2}\left[(3\mathbf{F}'(\mathbf{v}^{(k)}) - \mathbf{F}'(\mathbf{s}^{(k)})^{-1}(3\mathbf{F}'(\mathbf{v}^{(k)}))\right]\mathbf{F}'(\mathbf{s}^{(k)})^{-1}\mathbf{F}(\mathbf{s}^{(k)}). \end{cases} \tag{2.1}$$

Sharma *et al* [21] constructed the following fourth order method

$$\begin{cases} \mathbf{v}^{(k)} = \mathbf{s}^{(k)} - \frac{2}{3}\mathbf{F}'(\mathbf{s}^{(k)})\mathbf{F}(\mathbf{s}^{(k)}), \\ \mathbf{s}^{(k+1)} = \mathbf{s}^{(k)} - \frac{1}{2}[\underline{I} + \frac{9}{4}\mathbf{F}'(\mathbf{v}^{(k)})^{-1}\mathbf{F}'(\mathbf{s}^{(k)}) + \frac{3}{4}\mathbf{F}'(\mathbf{s}^{(k)})^{-1}\mathbf{F}'(\mathbf{v}^{(k)})]\mathbf{F}'(\mathbf{s}^{(k)})^{-1}\mathbf{F}(\mathbf{s}^{(k)}). \end{cases} \tag{2.2}$$

Babajee *et al* [22] extended the fourth order iterative method for the solution of single variable nonlinear equations given by Soleymani et al. [9] as follows:

$$\begin{cases} \mathbf{v}^{(k)} = s^{(k)} - \frac{2}{3}f'(s^{(k)})f(s^{(k)}), \\ s^{(k+1)} = s^{(k)} - \frac{2f(s^{(k)})}{f'(s^{(k)})+f'(\mathbf{v}^{(k)})}\left[1\text{-}\frac{1}{4}\left(\frac{f'(\mathbf{v}^{(k)})}{f'(s^{(k)})} - 1\right) + \frac{3}{4}\left(\frac{f'(\mathbf{v}^{(k)})}{f'(s^{(k)})} - 1\right)^2\right], \end{cases}$$

to the multivarible case as follows:

$$\begin{cases} \mathbf{v}^{(k)} = \mathbf{s}^{(k)} - \frac{2}{3}\mathbf{F}'(\mathbf{s}^{(k)})^{-1}\mathbf{F}(\mathbf{s}^{(k)}), \\ \mathbf{s}^{(k+1)} = \mathbf{s}^{(k)} - 2\left[\mathrm{I}\text{-}\frac{1}{4}\left(\mathbf{F}'(\mathbf{s}^{(k)})^{-1}\mathbf{F}'(\mathbf{v}^{(k)}) - I\right) + \frac{3}{4}\left(\mathbf{F}'(\mathbf{s}^{(k)})^{-1}\mathbf{F}'(\mathbf{v}^{(k)}) - I\right)^2\right] \\ \qquad\qquad * \left[\mathbf{F}'(\mathbf{s}^{(k)}) + \mathbf{F}'(\mathbf{v}^{(k)})\right]^{-1}\mathbf{F}(\mathbf{s}^{(k)}). \end{cases} \tag{2.3a}$$

Ullah, M. Z. *et al.* [10] constructed a new scheme for multi-step iterative methods. The well-known technique of undetermined coefficients to develop a high order method as follows in the scalar case ($n = 0, 1, 2, ...$)

$$\begin{cases} v_n = s_n - \frac{2}{3}\frac{f(s_n)}{f'(s_n)}, \\ z_n = s_n - \frac{1}{2}\frac{3f'(v_n)+f'(s_n)}{3f'(v_n)-f'(s_n)}\frac{f(s_n)}{f'(s_n)}, \\ w_n = z_n - \frac{f(z_n)}{q_1 f'(s_n)+q_2 f'(\mathbf{v}_n)}, \\ s_{n+1} = w_n - \frac{f(w_n)}{q_1 f'(s_n)+q_2 f'(\mathbf{v}_n)}. \end{cases} \tag{2.4}$$

There are four stages in the structure (2.4), with the third and fourth steps purposefully sharing the same denominator. To further illustrate, it is shown that this assumption improves the order of convergence from the third to the fourth step while having a minor impact on the amount of time required to solve the linked linear systems and the Jacobian. When we extend (2.4) to n dimensions, the Jacobians $\mathbf{F}^{'}(\mathbf{s}^{(k)})$ and $\mathbf{F}^{'}(v^{(k)})$ will be computed once every cycle, and the last two stages won't put too much pressure on the method. The convergence rate at the structure's conclusion (2.4) increases even more since the correction factors in the third and fourth phases of our structure (2.4) are equal. In order to propose the contributed high-order method for finding real and complex solutions of the nonlinear systems in what follows

$$\begin{cases} \mathbf{v}^{(k)} = \mathbf{s}^{(k)} - \frac{2}{3}\left[\mathbf{F}'(\mathbf{s}^{(k)})\right]^{-1}\mathbf{F}(\mathbf{s}^{(k)}), \\ \mathbf{z}^{(k)} = \mathbf{s}^{(k)} - \frac{1}{2}\left[3F'(\mathbf{v}^{(k)}) - \mathbf{F}'(\mathbf{s}^{(k)})\right]^{-1} \\ \qquad \left(3F'(\mathbf{v}^{(k)}) + \mathbf{F}'(\mathbf{s}^{(k)})\right)\mathbf{F}'(\mathbf{s}^{(k)})^{-1}\mathbf{F}(\mathbf{s}^{(k)}), \\ \mathbf{w}^{(k)} = \mathbf{z}^{(k)} - \left(\frac{-1}{2}\mathbf{F}'(\mathbf{s}^{(k)}) + \frac{3}{2}\mathbf{F}'(\mathbf{v}^{(k)})\right)^{-1}\mathbf{F}(\mathbf{z}^{(k)}), \\ \mathbf{s}^{(k+1)} = \mathbf{w}^{(k)} - \left(\frac{-1}{2}\mathbf{F}'(\mathbf{s}^{(k)}) + \frac{3}{2}\mathbf{F}'(\mathbf{v}^{(k)})\right)^{-1}\mathbf{F}(\mathbf{w}^{(k)}). \end{cases} \tag{2.5}$$

The new technique (2.5) needs computing $\mathbf{F}$ at three distinct positions every computing step and the Jacobians $\mathbf{F}$ at two different places. To demonstrate the convergence order of (2.5), before this first recall several key points from the theory of point of attraction.

For discovering estimation real or complex solutions to nonlinear systems, a large family of multi-step iterative techniques is described. The first method in the class is built using the well-known technique of indeterminate coefficients, whereas higher-level schemes are achieved using a frozen Jacobian. The convergence behavior of the primary suggested iterative approach will be demonstrated using the point of attraction theory. Then, an m-step technique will be seen to converge with 2m-order. The computational efficiency index will be discussed, and numerical comparisons with existing approaches will also be made. Finally, we show how novel techniques can be used to solve nonlinear partial differential equations.

Ullah, M. Z. *et al.* [10] are interested in high-order fast approaches with suitable computational load and efficiency for challenging nonlinear systems. To enhance the convergence behavior of the aforementioned well-known approaches, they present a novel one with eighth-order convergence to find both genuine and complex solutions as well as to develop an effective method to deal with nonlinear problems. The newly suggested approach requires just first-order Frechet derivative assessments and does not require higher-order Frechet derivatives. Numerical findings are presented to re-verify the suggested method's effectiveness in locating genuine and complicated solutions to nonlinear systems with applications.

Lotfi T. *et. al.* [16] design the generic multipoint for solving nonlinear systems of equations that use the suggested approach as a predictor in the first three phases. In reality, adding one step raises the order of convergence by three units while requiring only one vector-function evaluation. It also has an economically viable frozen component.

A new ninth-order development of iterative method for solving nonlinear systems of equations:

$$\begin{cases} \mathbf{v}^{(k)} = \mathbf{s}^{(k)} - \mathbf{F}'(\mathbf{s}^{(k)})^{-1}\mathbf{F}(\mathbf{s}^{(k)}), \\ \mathbf{z}^{(k)} = \mathbf{s}^{(k)} - 2(\mathbf{F}'(\mathbf{s}^{(k)}) + \mathbf{F}'(\mathbf{v}^{(k)}))^{-1}\mathbf{F}(\mathbf{s}^{(k)}), \\ \mathbf{w}^{(k)} = \mathbf{z}^{(k)} - \left(\frac{7}{2}I - 4F'(\mathbf{s}^{(k)})^{-1}\mathbf{F}'(\mathbf{v}^{(k)}) + \frac{3}{2}(\mathbf{F}'(\mathbf{s}^{(k)})^{-1}(\mathbf{F}'(\mathbf{v}^{(k)}))^2\right) \\ \qquad\qquad \times\mathbf{F}'(\mathbf{s}^{(k)}))^{-1}\mathbf{F}(z^{(k)}), \\ \mathbf{s}^{(k+1)} = \mathbf{w}^{(k)} - \left(\frac{7}{2}I - 4\mathbf{F}'(\mathbf{s}^{(k)})^{-1}\mathbf{F}'(\mathbf{v}^{(k)}) + \frac{3}{2}(\mathbf{F}'(\mathbf{s}^{(k)})^{-1}(\mathbf{F}'(\mathbf{v}^{(k)}))^2\right) \\ \qquad\qquad \times\mathbf{F}'(\mathbf{s}^{(k)}))^{-1}\mathbf{F}(\mathbf{w}^{(k)}), \end{cases}$$

that will be denoted by $M-9$.

Firstly, Lotfi T. [16] expanded and improved on Weerakoon and Fernando's approach to solving nonlinear systems with sixth-order convergence. The examination of its convergence was discussed. This expanded approach uses a predictor with two frozen Jacobian matrices to construct a broad multipoint iteration. The overall technique has been illustrated with certain actual instances. Furthermore, numerical examples show that these new strategies can compete with the old ones. Equation solving is a venerable subject in science and engineering, and it is especially important in applications. As a result of this, an immense number of iterative approaches for solving scalar nonlinear equations have been developed. However, it should be emphasized that many of these strategies are not applicable to their related systems. Even if this is feasible, certain critical variables must be taken into account. As a result, there are few viable iterative approaches in this circumstance. Furthermore, it is worth noting that, while some scalar iterations can be prolonged, they are of little practical use owing to increased computing costs.

It is worth mentioning that the fourth order methods proposed in this study for solving system of nonlinear equations uses only one Jacobian matrix per iteration and element wise multiplication of vectors and diagonal matrix. The above schemes for nonlinear system would fail or diverge when the Jacobian matrix becomes singular at any stage due to approximate value. Therefore, the above two optimal fourth order techniques for solving nonlinear system are generalized to Homotopy techniques for solving nonlinear system to overcome this problem of divergence or failure.

## 2.2 General Concept

Here, we discuss some homotopy analysis methods for developing a techniques for finding roots of single variable non-linear equations and system of nonlinear equation proposed by different researchers [11, 30, 31, 59].

### 2.2.1 Homotopy Continuation Method by Wu

For the solution of nonlinear equation of the form

$$f(s) = 0,$$

Wu [30], selected auxiliary homotopy function as:

$$g(s) = 0, \tag{2.6}$$

which is known and manageable. With the help of auxiliary function (2.6), Wu defined homotopy function as follows:

$$h : \mathbb{R} \times [0, 1] \longrightarrow \mathbb{R}$$

as

$$h(s, \mu) = \mu f(s) + (1 - \mu)g(s), \tag{2.7}$$

where the parameter $\mu$ used to perturb the function $f(s)$ in (2.7) lies in interval $[0, 1]$. From (2.7), the following conditions at boundary points are introduced

$$\begin{aligned} h(s, 0) &= g(s), \\ h(s, 1) &= f(s). \end{aligned}$$

Continuation is to solve $h(s,u)=0$ in place of $f(s)=0$ to avoid divergence problem by assuming the values of $\mu \in (0,1)$ in such a way that divergence case is avoided.

There are numerous applications in both numerical and practical mathematics for the process of swiftly determining the roots of nonlinear equations. The Newton-Raphson method is by far the most often used method for solving nonlinear equations. Researchers are still interested in a wide range of topics related to Newton's approach. One disadvantage of the approaches is that the initial approximation $s_0$ must be chosen so that it is sufficiently close to the true solution to ensure that the methods converge. This is something that everyone knows. It is not an easy process to identify a criterion for selecting $s_0$ hence, efficient and globally convergent algorithms are required.

In homotopy analysis method, set $h$ is a fixed constant and it can be determined by $h$-curves. However, it is computationally intensive with computing times for seeking a proper value of $h$. In Newton-homotopy analysis method (N-HAM), we determine $h$ by Newton–Raphson scheme as Wu defined Newton's homotopy method [30]:

$$s^{(k+1)} = s^{(k)} - \frac{h(s^{(k)},\mu)}{h'(s^{(k)},\mu)}, \quad k=0,1,2,3,... \tag{2.8}$$

where the divergence case occurs at

$$h'(s,\mu)=0 \ \text{ i.e., } \ \mu f'(s)+(1-\mu)g'(s)=0,$$

to avoid divergence, it may be considered

$$h'(s,\mu)\neq 0 \ \text{ i.e., } \ \mu f'(s)+(1-\mu)g'(s)\neq 0.$$

Thus, on integration, we get:

$$\mu f(s) + (1-\mu)g(s) = Ex + R \text{ or } Ee^s + R, \tag{2.9}$$

where $R$ is an arbitrary constant which can be considered zero.

It is observed that the auxiliary function $g(s)$ satisfies the following choices, as seen from equation (2.9)

i. $\mu = 0$ implies $g(s) = Ex + R$ or $Ee^s + R$,

ii. $\mu = 1$ implies $f(s) = Ex + R$ or $Ee^s + R$,

iii. $\mu \in (0,1)$ implies $g(s) = E_1 f(s) + E_2 s + R$ or $E_1 f(s) + E_2 e^s + R$,

where $E_1$and $E_2$ are non-zero coefficients, for the sake of simplicity, we choose:

$$g(s) = Ex + R \text{ or } Ee^s + R.$$

The problem that needs to be solved is contained within a set of problems by homotopy, or continuation, techniques for nonlinear systems.

Solving a problem of the form

$$\mathbf{F}(\mathbf{s}) = 0,$$

to find unknown solution $\mathbf{s}^*$, it may be considered a family of problems described using a parameter $\mu$ that takes values in $[0,1]$. A problem with a known solution, $\mathbf{s}(0)$, is equivalent to the case where $\mu = 0$, whereas a problem with an unknown solution, $\mathbf{s}(1) = \mathbf{s}^*$, is equivalent to the case where $\mu = 1$. Consider $\mathbf{s}(0)$ as an initial approximation to the solution of

$$\mathbf{F}(\mathbf{s}^*) = 0.$$

Define a homotopy function with the help of an auxiliary function

$$H : [0, 1] \times R^n \rightarrow R^n$$

by

$$H(\mu, \mathbf{s}) = \mu F(\mathbf{s}) + (1 - \mu)[\mathbf{F}(\mathbf{s}) - \mathbf{F}(\mathbf{s}(0))] = \mathbf{F}(\mathbf{s}) + (\mu - 1)\mathbf{F}(\mathbf{s}(0)). \tag{2.10}$$

For different values of $\mu$, a solution to

$$H(\mu, \mathbf{s}) = 0,$$

will found.

It can be observed that for $\mu = 0$, the equation (2.10) reduces in the following form

$$0 = H(0, \mathbf{s}) = \mathbf{F}(\mathbf{s}) - \mathbf{F}(\mathbf{s}(0)),$$

and the solution is $\mathbf{s}(0)$. For $\mu = 1$, the equation (2.10) becomes in the following form

$$0 = H(1, \mathbf{s}) = \mathbf{F}(\mathbf{s}),$$

and the solution is $\mathbf{s}(1) = \mathbf{s}^*$.

The parameter in the function $H$ gives us a family of functions that can take us from the known value $\mathbf{s}(0)$ to the answer $\mathbf{s}(1) = \mathbf{s}^*$. The function $H$ is referred to as a homotopy between

$$H(0, \mathbf{s}) = \mathbf{F}(\mathbf{s}) - \mathbf{F}(\mathbf{s}(0))$$

and

$$H(1, \mathbf{s}) = \mathbf{F}(\mathbf{s}).$$

The continuation problem is to figure out how to move from the known solution of

$$H(0, \mathbf{s}) = 0 \ (\mathbf{s}(0)),$$

to the unknown solution of

$$H(1, \mathbf{s}) = 0(\mathbf{s}(1) = \mathbf{s}^*),$$

which is the answer to

$$\mathbf{F}(\mathbf{s}) = 0.$$

Following the above mentioned technique Newton's homotopy method for the solution of system of nonlinear equations constructed by [30] as follows:

$$\mathbf{s}^{(k+1)} = \mathbf{s}^{(k)} - \left[H'(\mathbf{s}^{(k)}, \mu)\right]^{-1} H(\mathbf{s}^{(k)}, \mu), \quad k = 0, 1, 2, 3, ...$$

### 2.2.2 Homotopy Perturbation Method by Golbabai

Golbabai A. *et. al* [11] used the homotopy perturbation method for solving system of non-linear algebraic equations. Also this method showed the accuracy and fast convergence than other iterative methods. Consider system of non-linear equations of the form

$$\Psi(X) = \begin{cases} f(X) = 0, \\ g(X) = 0, \end{cases} X = (s, \mathbf{v})^T \in R^2, \tag{2.11}$$

where $f, g : R^2 \to R$ and $\varphi : R^2 \to R^2$. We assume that $X_* = (\alpha, \beta)^T$ is a zero of equation (2.11) and $q = (\lambda, \gamma)^T$ is an initial guess sufficiently close to $X_*$. Then, by using Taylor series around $q$ for the equation (2.11), we have

$$\Psi(X) = \begin{cases} f(q) + (s - \lambda) f_s(q) + (v - \gamma) f_{\mathbf{v}}(q) + F(X) = 0, \\ g(q) + (s - \lambda) g_s(q) + (v - \gamma) g_{\mathbf{v}}(q) + G(X) = 0, \end{cases} \tag{2.12}$$

where

$$F(X) = f(X) - f(q) - (s - \lambda) f_s(q) - (\mathbf{v} - \gamma) f_{\mathbf{v}}(q),$$

and

$$G(X) = g(X) - g(q) - (s - \lambda) g_s(q) - (\mathbf{v} - \gamma) g_{\mathbf{v}}(q).$$

We can rewrite equation (2.12) as follows:

$$\begin{cases} x f_s(q) + y f_{\mathbf{v}}(q) = \lambda f_s(q) + \gamma f_{\mathbf{v}}(q) - f(q) - F(X), \\ x g_s(q) + y g_{\mathbf{v}}(q) = \lambda g_s(q) + \gamma g_{\mathbf{v}}(q) - g(q) - G(X). \end{cases}$$

Many academics have utilized various numerical algorithms in recent years to solve $\Psi(X) = 0$. Golbabai proposed an iterative approach for solving nonlinear equations that involve rewriting the given nonlinear equation as a system of coupled equations. Scientists and engineers have committed their efforts to the application of the homotopy perturbation technique in linear and nonlinear problems since this approach is used to constantly deform a straightforward issue that is easy to solve into an under-study problem that is difficult to solve. This approach is the combination of the classic perturbation method and homotopy in topology, which was initially proposed by He and systematically described. The Homotopy perturbation method is used to numerically solve a system of nonlinear equations. Ji-Huan He was the first on who invented the homotopy perturbation technique (HPM) in 1999. Until today, it has been utilized to tackle a wide range of linear and nonlinear problems. It describes several unique iterative techniques for nonlinear equation system solutions. A comparison of the present technique's findings with those of the Newton-Raphson methodology demonstrates the precision and speedy convergence of the new approaches. Most nonlinear systems lack accurate analytic solutions, numerical and analytic approximation, that's why these techniques was employed.

### 2.2.3 Homotopy Analysis Method by Abbasbandy

Abbasbandy *et al.* [31] suggests using HAM (homotopy analysis method) as a method for solving a variety of nonlinear algebraic problems. Additionally, ADM (Adomian's decomposition method) and HPM (homotopy perturbation method) provide a wide range of solutions. Also, explain the connection between the HAM approach and the other two methods. In addition, by utilizing the Newton-Raphson approach, they are able to produce a numerical methodology that is more effective. This methodology goes by the name of the Newton-homotopy analysis method (N-HAM). The newly suggested improvement technique is put to the test on a few different examples, and the findings suggest that it is both a useful tool and an effective enhancement for the process of solving nonlinear equations.

### 2.2.4 Homotopy Analysis Method by He

In 2012, He [59] proposed a suggestion for an alternate method to create the homotopy equation by using an auxiliary term. The construction of an appropriate homotopy equation and the selection of an appropriate starting guess is considered to be the two most significant processes involved in the implementation of the homotopy perturbation method. When the homotopy parameter is zero, the homotopy equation needs to be designed in such a way that it may approximately represent the solution property. Additionally, the initial solution is supposed to be chosen with an unknown parameter, which is then found after one or two iterations of the homotopy equation.

Consider a general nonlinear equation

$$Lu + Nu = 0,$$

where $L$ and $N$ are, respectively, the linear operator and nonlinear operator.

The first step for the method is to construct a homotopy equation in the form [**?**, 61, 62]

$$\bar{L}u + p(Lu - \bar{L}u + Nu) = 0, \tag{2.13}$$

where $\bar{L}$ is a linear operator with a unknown constant and $\bar{L}u = 0$ can approximately describe the solution property. The embedding parameter $p$ monotonically increases from zero to unit as the trivial problem ($\bar{L}u = 0$) is continuously deformed to the original one $(Lu + Nu) = 0)$.

This is the suggestion for an alternative method to the construction of the homotopy equation that Ji- Huan He provided.

$$\bar{L}u + p(Lu - \bar{L}u + Nu) + ap(1-p)u = 0, \tag{2.14}$$

where $\alpha$ is an auxiliary parameter that is being used. When $\alpha = 0$,(3.5) turns out to be that of the classical one expressed in (2.13). The auxiliary term, $ap(1-p)u$, disappears entirely when $p = 0$ or $p = 1$; so the auxiliary term will affect neither the initial solution ($p = 0$) nor the real solution ($p = 1$). Noor [63] was the first person to investigate using the homotopy perturbation approach in conjunction with an auxiliary term.

In order to demonstrate how the solution process works, take the example of a nonlinear oscillator in the form

$$\frac{d^2u}{dt^2} + bu + cu^3 = 0, \quad u(0) = A, \quad u^{'}(0) = 0, \tag{2.15}$$

where $b$ and $c$ are positive constants.

Equation (2.15) admits a periodic solutions, and the linearized of (2.15) is

$$u^{''} + \omega^2 u = 0, \quad u(0) = A, \quad u^{'}(0) = 0, \tag{2.16}$$

where $\omega$ is the frequency of eq (2.15).

The following homotopy equation is one that develop with the help of an auxiliary term:

$$u^{''} + \omega^2 u + p[(b - \omega^2)u + cu^3] + ap(1-p)u = 0.$$

It has been demonstrated that the homotopy perturbation approach may effectively, easily, and accurately solve a broad class of nonlinear differential equations; typically, one iteration is sufficient for engineering applications with acceptable precision, which makes the method accessible to individuals who don't have any background in mathematics.

## 2.3 Framework of Dissertation

In Chapter-3, two optimal fourth order iterative schemes for solving single variable nonlinear equations are developed via integral inequalities. Their convergence analysis is discussed and some model numerical examples are provided to test the performance of their techniques.

In Chater-4, extension of iterative schemes of section 3 is established. Their convergence analysis is described and applications are provided to check their efficiency and performance.

In Chapter-5, techniques of chapter are generalized to homotopy techniques for solving the nonlinear system along with some test examples.

Finally, in Chapter-6, conclusion of this study and future work is highlights.

At the end, references that are being used to produce this research are provided.

# Chapter 3

# Optimal Techniques for Solving Single Variable Nonlinear Algebraic and Transcendental Equations

In this chapter, two optimal fourth order techniques are developed for solving single variable nonlinear integral inequalities. Further, their convergence analysis is done and some test examples are given for showing their performance in comparison with some other similar techniques existing in the literature.

## 3.1 Development of Optimal Fourth order techniques via integral Inequalities

Let us consider the another inequality by N. S. Barnett *et al.* [60] which is mentioned below in the form of the following theorem:

**Theorem:** *Let $f : [a, b] \rightarrow R$ be a twice differentiable mapping on $(a, b)$ and $f^{''} : (a, b) \rightarrow R$ be bounded, i.e.*

$$\left\|f^{''}\right\|_{\infty} = \sup(f^{''}(t)) < \infty, \forall t \in (a, b).$$

*Then, the following inequality is obtained:*

$$\begin{aligned}
&\left| f(s) - \frac{1}{b-a} \int_{a}^{b} f(t)dt - (s - \frac{a+b}{2}) \frac{f(b) - f(a)}{b-a} \right| \\
\leq\ & \frac{1}{2} \left\{ \left( \frac{(s - \frac{a+b}{2})^2}{(b-a)^2} + \frac{1}{4} \right)^2 + \frac{1}{12} \right\} (b-a)^2 \left\|f'\right\|_{\infty} \\
\leq\ & \frac{(b-a)^2}{6} \left\|f^{'}\right\|_{\infty},
\end{aligned} \tag{3.1}$$

$\forall s \in [a, b]$.

Rewriting, for $a = s_n$ and $b = s$, we have:

$$\begin{aligned}
\int_{a}^{b} f'(t)dt &\leq (s - s_n)f'(s) - \left(\frac{s - s_n}{2}\right)(f'(s) - f'(s_n)) \\
f(s) &\approx f(s_n) + (s - s_n)f^{'}(s) + \left(\frac{s - s_n}{2}\right)(f'(s) - f'(s_n)).
\end{aligned} \tag{3.2}$$

Now for the non-linear equation $f(s) = 0$, (3.2) implies:

$$s = s_n - \frac{2f(s_n)}{f'(s) + f'(s_n)}.$$

So, the iterative scheme becomes:

$$\begin{cases} s_{n+1} = s_n - \frac{f(s_n)}{f'(v_n)+f'(s_n)}, \\ \text{where } v_n = s_n - \frac{f(s_n)}{f'(s_n)}. \end{cases} \tag{3.3}$$

Let us now consider the approximation of derivative of function $f^{'}(v_n)$ introduced by Chun [46] as follows:

$$f^{'}(v) \approx f^{'}(s_n)\frac{f(s_n) - f(v_n)}{f(s_n) + f(v_n)},$$

in (3.3), the following two step iterative method is obtained:

$$\begin{cases} v_n = s_n - \frac{f(s_n)}{f^{'}(s_n)}, \\ s_{n+1} = s_n - \frac{2f(v_n)}{f^{'}(s_n)}\left(\frac{f(s_n)+f(v_n)}{f(s_n)-f(v_n)}\right). \end{cases}$$

In order to improve convergence order, the following method is proposed:

$$\begin{cases} v_n = s_n - \delta\frac{f(s_n)}{f'(s_n)}, \\ s_{n+1} = v_n - \xi\frac{f(v_n)}{f^{'}(s_n)}\left(\frac{f(s_n)+f(v_n)}{\alpha f(s_n)-\beta f(v_n)}\right). \end{cases} \tag{3.4}$$

Now considering the inequality by N. S. Barnett *et al.* [60] which is mentioned below in the form of the following theorem:

**Theorem:** *Let $f : [a, b] \to R$ be a twice differentiable mapping on $(a, b)$ and $f^{''} : (a, b) \to R$ be bounded, i.e.*

$$\left\|f^{''}\right\|_{\infty} = \sup(f^{''}(t)) < \infty, \forall t \in (a, b).$$

*Then, the following inequality is obtained:*

$$\begin{aligned}
&\left| f(s) - \frac{1}{b-a}\int_a^b f(t)dt - (s - \frac{a+b}{2})f^{'}(s) \right| \\
\leq &\left| \frac{1}{24}(b-a)^2 + \frac{1}{2}(s - \frac{a+b}{2})^2 \right| \|f'\|_\infty \\
\leq &\frac{(b-a)^2}{2}\left\| f^{'} \right\|_\infty .
\end{aligned} \tag{3.5}$$

From the above inequality, $\forall s \in [a,b]$, for $a = s_n$ and $b = s$, we have:

$$f(s) \approx f(s_n) + (s - s_n)f^{'}(s) - (s - s_n)(s - \frac{s + s_n}{2})f^{''}(s).$$

Consider the non-linear equation $f(s) = 0$, we obtain:

$$s = s_n - \frac{f(s_n)}{f^{'}(s) - (s - \frac{s+s_n}{2})f^{''}(s)}.$$

So, the iterative scheme becomes:

$$\begin{cases} v_n = s_n - \frac{f(s_n)}{f^{'}(s_n)}, \\ s_{n+1} = s_n - \frac{f(s_n)}{f^{'}(v_n) + 2\frac{f(s_n)}{f^{'}(s_n)}f^{''}(v_n)}. \end{cases} \tag{3.6}$$

Considering the approximations of $f^{'}(v_n)$ written as

$$f^{'}(v_n) \approx f^{'}(s_n)\frac{f(s_n) - f(v_n)}{f(s_n) + f(v_n)}, \tag{3.7}$$

and $f^{''}(v_n)$ as follows:

$$f^{''}(v_n) \approx 2\left(\frac{f(v_n) - f(s_n)}{(v_n - s_n)^2} - \frac{f^{'}(v_n)}{v_n - s_n}\right),$$

or

$$f^{''}(v_n) \approx 2\left(\frac{f^{'}(s_n)}{f(s_n)}\right)^2 f(v_n)\frac{f(v_n) - f(s_n)}{f(s_n) + f(v_n)}. \tag{3.8}$$

Therefore, the two step iterative scheme (3.6) can be modified by using approximations given in (3.7) and (3.8) as follows:

$$\begin{cases} v_n = s_n - \frac{f(s_n)}{f'(s_n)} \\ s_{n+1} = v_n - \frac{f(s_n)}{f'(s_n)} \left( \frac{f(v_n)(f(s_n)+f(v_n))}{f^2(s_n)-2f(s_n)f(v_n)+f^2(v_n)} \right). \end{cases}$$

We therefore propose the following two-step method using arbitrary parameters $\alpha, \beta, \gamma \in R$, for improving the convergence order:

$$\begin{cases} v_n = s_n - \frac{f(s_n)}{f'(s_n)}, \\ s_{n+1} = v_n - \frac{f(s_n)}{f'(s_n)} \left( \frac{f(v_n)(f(s_n)+f(v_n)}{\alpha f^2(s_n)+\beta f(s_n)f(v_n)+\gamma f^2(v_n)} \right). \end{cases} \tag{3.9}$$

## 3.2 Convergence Analysis

Here, in this section convegence of newly proposed iterative methods (3.4) and (3.9) is being analzed in order to prove that the iterative methods are of optimal convergence order four.

**Theorem 1** *Let $w \in I$ be a single root of $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ in an open optimal $I$. If $s_0$ is sufficiently close to $w$, then the method described by (3.9) has optimal fourth order convergence for $\alpha = 1, \beta = -1, \gamma = -3$. The error equation obtained by using Maple 18, is given by*

$$e_{n+1} = -c_2 c_3 e_n^4 + o(e^5).$$

**Proof.** Let us consider error in nth approximation

$$e_n = s_n - w,$$

expanding $f(s_n)$ and $f^{'}(s_n)$ to $w$, we get:

$$f(s_n) = f^{'}(w)(e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + o(e_n^5)), \tag{3.10}$$

$$f^{'}(s_n) = f^{'}(w)(1 + s_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + o(e_n^4)), \tag{3.11}$$

where

$$c_k = \frac{1}{k!}\frac{f^{(k)}(w)}{f(w)}, \quad k = 2, 3, ..., \ .$$

Using (3.10) and (3.11) to obtained the following error function

$$\begin{aligned} \frac{f(s_n)}{f'(s_n)} &= e_n - c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + (4c_2^3 - 7c_2c_3 + 3c_3) + e_n^4 \\ &+ (8c_2^4 + 10c_2c_4 + 6c_3^2 - 4c_3 - 20c_3c_2^2)e_n^5 + o(e_n^6), \end{aligned} \tag{3.12}$$

The error equation of first step of method (3.9) is calculated as follows:

$$\begin{aligned} v_n &= w + c_2e_n^2 + (-2c_2^2 + 2c_3)e_n^3 - (7c_2c_3 - 4c_2^3 - 3c_4)e_n^4 \\ &+ o(e_n^5). \end{aligned}$$

Expanding the function $f(v_n)$, as above, we have:

$$\begin{aligned} f(v_n) &= f^{'}(w)(c_2e_n^2 + 2(c_3 - c_2^2)e_n^3 + (-8c_2c_3 + 3c_4 + 5c_2^3)e_n^4) \\ &+ O(e_n^5). \end{aligned} \tag{3.13}$$

To obtain the error function involving in the second step of method (3.9), the equations (3.10), (3.11)and (3.13) are utilized as follows:

$$\begin{aligned}
&\frac{f(s_n)}{f'(s_n)}\left(\frac{f(v_n)(f(s_n)+f(v_n))}{\alpha f^2(s_n)+\beta f(s_n)+f(v_n)+\gamma f^2(v_n)}\right)\\
= {}& \frac{c_2e_n^2}{\alpha}+\left((-\frac{\beta}{\alpha^2}-\frac{\beta}{\alpha})c_2+\frac{2}{\alpha}c_3\right)e_n^3\\
&+\left(\frac{3}{\alpha}c_4+(\frac{-4\beta}{\alpha^2}-\frac{10}{\alpha})c_2c_3+(\frac{\beta^2}{\alpha^3}+\frac{6}{\alpha}-\frac{\gamma}{\alpha^2}+\frac{6\beta}{\alpha^2})c_2^3\right)e_n^4\\
&+O(e_n^5).
\end{aligned}$$

Finally, the error equation of method (3.9) becomes for arbitrary values of $\alpha, \beta$ and $\gamma$ as follows:

$$\begin{aligned}
s_{n+1} &= w+(1-\frac{1}{\alpha})c_2e_n^2+\left((\frac{3}{\alpha}+\frac{\beta}{\alpha^2}-2)+(2-\frac{2}{\alpha})c_3\right)e_n^3\\
&\quad+\left((3-\frac{3}{\alpha})c_4+(\frac{10}{\alpha}+\frac{4\beta}{\alpha^2}-7)c_2c_3\right.\\
&\quad\left.+(-\frac{\beta^2}{\alpha^3}-\frac{6}{\alpha}+\frac{\gamma}{\alpha^2}-\frac{6\beta}{\alpha^2}+4)c_2^3\right)e_n^4+o(e_n^5).
\end{aligned}$$

Setting the coefficients of $e_n^2$ and $e_n^3$ equal to zero and obtain the values of parameters as given below:

$$\alpha = 1, \beta = -1, \gamma = -3.$$

For the above mentioned values of parameters, we have the following error equation:

$$\begin{aligned}
s_{n+1} &= w - c_2c_3e_n^4 + o(e^5),\\
e_{n+1} &= -c_2c_3e_n^4 + o(e^5).
\end{aligned}$$

which shows that the method is of optimal convergence order four for $\alpha = 1, \beta = -1, \gamma = -3$. ■

Similarly, using Maple 18, the error equation of method (3.4) is given by

$$\begin{aligned} e_{n+1} &= (-c_2c_3 + 3c_2^3)e_n^4 + o(e_n^5), \\ \text{for } \alpha &= 1, \beta = 1, \delta = -1 \text{ and } \xi = 1. \end{aligned}$$

Thus, the method (3.4) is of optimal convergence order four for the given choice of parameters.

## 3.3 Some Model Numerical Examples

In this section, some model numerical examples given in the literature [12] is used to show numerical performance of the proposed iterative techniques (3.4) and (3.9).

$$\begin{array}{ll}
f_1(s) = \sin(2\cos s) - 1 - x2 + e^{sin(s^3)}, & s^* = -0.7848959876... \\
f_2(s) = xe^{s^2} - \sin^2 s + 3\cos s + 5, & s^* = -1.2076478271... \\
f_3(s) = s^3 + 4x^2 - 10, & s^* = 1.36523001341... \\
f_4(s) = \sin(s) + \cos(s) + s, & s^* = -0.4566247045... \\
f_5(s) = s/2 - \sin s, & s^* = 1.89549426703... \\
f_6(s) = \sqrt{s^2 + 2x + 5} - 2\sin s - s^2 + 3, & s^* = 2.33196765588... \\
f_7(s) = \sqrt{s} - \cos s, & s^* = 0.64171437087... \\
f_8(s) = s^2 + \sin(s/5) - 1/4, & s^* = 0.40999201798... \\
f_9(s) = e^{-s}\sin s + \log(1 + x2) - 2, & s^* = 2.44774828645... \\
f_{10}(s) = \sqrt{s^3} + \sin s - 30, & s^* = 9.716501993365...
\end{array}$$

The computation is performed using Maple 18 rounded to 500 significant digits. The stopping criteria used is

$$|s_n - s_{n-1}| < \epsilon,$$

where $\epsilon = 10^{-50}$, $n$ is the number of iterations.

Let us denote the newly introduced optimal forth order methods (3.4) and (3.9) by $NMZ_1$ and $NMZ_2$ respectively, and their comparison is made up with Jarrat's method [49] by JTM given by following

$$\begin{cases} v_n = s_n - \frac{2}{3}\frac{f(s_n)}{f'(s_n)}, \\ s_{n+1} = v_n - \left(\frac{3f'(v_n)+f'(s_n)}{6f'(v_n)-2f'(s_n)}\right)\frac{f(s_n)}{f'(s_n)}. \end{cases}$$

and Madhu, K., & Jayaraman, J forth order method [12] by MKJJ given by

$$\begin{cases} v_n = s_n - \frac{2}{3}\frac{f(s_n)}{f'(s_n)}, \\ s_{n+1} = v_n - \frac{4f(s_n)}{f'(s_n)+3f'(v_n)}(1+\frac{5}{16}(\frac{f'(v_n)}{f'(s_n)}-1)^2)(1+\frac{1}{4}(\frac{f'(s_n)}{f'(v_n)}-1)^2) \end{cases}$$

in table 3.1.

| $f$ | $s_0$ | JTM | | MKJJ | | $NMZ_1$ | | $NMZ_2$ | |
|---|---|---|---|---|---|---|---|---|---|
| | | $it$ | $\lvert s_n - s_{n-1}\rvert$ | $it$ | $\lvert s_n - s_{n-1}\rvert$ | it | $\lvert s_n - s_{n-1}\rvert$ | it | $\lvert s_n - s_{n-1}\rvert$ |
| $f_1$ | -0.9 | 4 | $1.6(10^{-067})$ | 4 | $4.4(10^{-065})$ | 3 | $4.3(10^{-062})$ | 3 | $3.4(10^{-071})$ |
| | -0.7 | 4 | $1.4(10^{-070})$ | 4 | $7.2(10^{-066})$ | 3 | $1.5(10^{-061})$ | 3 | $2.4(10^{-084})$ |
| $f_2$ | -1.7 | 5 | $1.4(10^{-085})$ | 5 | $4.2(10^{-058})$ | 4 | $3.4(10^{-056})$ | 4 | $9.6(10^{-083})$ |
| | -1.0 | 5 | $2.0(10^{-199})$ | 5 | $3.8(10^{-116})$ | 4 | 4.7(10-$^{115}$) | 4 | $8.2(10^{-100})$ |
| $f_3$ | 1.6 | 4 | $2.4(10^{-063})$ | 4 | $1.2(10^{-059})$ | 3 | $7.8(10^{-035})$ | 3 | $2.2(10^{-090})$ |
| | 1.0 | 5 | $1.4(10^{-187})$ | 5 | $2.5(10^{-149})$ | 4 | $1.3(10^{-127})$ | 4 | $6.7(10^{-178})$ |
| $f_4$ | -0.2 | 4 | $2.1(10^{-077})$ | 4 | $5.4(10^{-076})$ | 3 | $9.3(10^{-074})$ | 3 | $1.3(10^{-078})$ |
| | -0.6 | 4 | $4.3(10^{-100})$ | 4 | $1.2(10^{-099})$ | 3 | $3.3(10^{-098})$ | 3 | $1.1(10^{-099})$ |
| $f_5$ | 1.6 | 5 | $5.7(10^{-169})$ | 5 | $7.9(10^{-137})$ | 4 | $3.5(10^{-177})$ | 4 | $5.1(10^{-074})$ |
| | 2.0 | 4 | $7.4(10^{-079})$ | 4 | $1.2(10^{-074})$ | 3 | $1.4(10^{-070})$ | 3 | $2.3(10^{-090})$ |
| $f_6$ | 2.1 | 4 | $6.5(10^{-096})$ | 4 | $6.3(10^{-097})$ | 3 | $1.1(10^{-089})$ | 3 | $6.5(10^{-088})$ |
| | 2.5 | 4 | $4.5(10^{-094})$ | 4 | $7.8(10^{-096})$ | 3 | $1.8(10^{-095})$ | 3 | $2.9(10^{-090})$ |
| $f_7$ | 0.2 | 4 | $8.7(10^{-063})$ | 4 | $2.5(10^{-060})$ | 3 | $1.1(10^{-052})$ | 3 | $1.3(10^{-193})$ |
| | 0.9 | 4 | $3.5(10^{-079})$ | 4 | $6.9(10^{-081})$ | 3 | $1.8(10^{-086})$ | 3 | $2.7(10^{-103})$ |
| $f_8$ | 0.2 | 5 | $7.4(10^{-151})$ | 5 | $3.8(10^{-114})$ | 4 | $2.5(10^{-094})$ | 4 | $8.5(10^{-155})$ |
| | 1.5 | 5 | $3.1(10^{-074})$ | 5 | $5.6(10^{-065})$ | 4 | $1.81(10^{-055})$ | 4 | $7.5(10^{-112})$ |
| $f_9$ | 1.9 | 4 | $1.0(10^{-084})$ | 4 | $3.1(10^{-108})$ | 3 | $1.1(10^{-079})$ | 3 | $3.3(10^{-091})$ |
| | 2.7 | 4 | $5.8(10^{-102})$ | 4 | $1.3(10^{-100})$ | 3 | $1.1(10^{-121})$ | 3 | $6.6(10^{-105})$ |
| $f_{10}$ | 9.9 | 4 | $3.3(10^{-100})$ | 4 | $1.7(10^{-101})$ | 3 | $7.3(10^{-106})$ | 3 | $6.1(10^{-100})$ |
| | 9.2 | 4 | $1.9(10^{-078})$ | 4 | $9.4(10^{-079})$ | 3 | $9.1(10^{-080})$ | 3 | $1.9(10^{-078})$ |

Table 3.1 Compaison of results of proposed methods for given examples

### 3.3.1 Discussion on Results

It is observed that from the table 3.1 the numerical test example produced much better results of newly proposed iterative methods $NMZ_1$(3.4) and $NMZ_2$ (3.9) as in their comparison with Jarrat's method JTM [49] and Madhu, K., & Jayaraman, J method MKJJ [12].

# Chapter 4

# Extension to Optimal Iterative Techniques for Solving System of Nonlinear Equations

Here, in this chapter, two fourth order optimal iterative techniques $NMZ_1$(3.4) and $NMZ_2$ (3.9) for solving single variable nonlinear equations of chapter 3 are extended for solving nonlinear system. It is mentioning that the introduced fourth order techniques uses only one Jacobian matrix for each iterative step. For the goal of extending iterative methods for nonlinear systems, element-wise vector and diagonal matrix multiplication is also defined.

The convergence analysis of the nonlinear system is also discussed and application are provided in comparison with other similar methods in the literature.

## 4.1 Establishment of Iterative Techniques for Nonlinear System

For the goal of establishment of iterative techniques for the solution of nonlinear system of equations given as

$$\mathbf{F}(\mathbf{s}) = [f_1(\mathbf{s}), f_2(\mathbf{s}), ..., f_n(\mathbf{s})]^T = 0, \tag{4.1}$$

where

$$\mathbf{s} = [s_1, s_2, ..., s_n]^T,$$

vector multiplication needs to be define.

### 4.1.1 Elementwise Vector Multiplication

The element wise vector multiplication can be achieve via vector and diagonal matrix multiplication as follows:

$$\mathbf{F}(\mathbf{s}) \cdot \mathbf{F}(\mathbf{v}) = \begin{bmatrix} f_1(\mathbf{s}) \\ f_2(\mathbf{s}) \\ \vdots \\ f_n(\mathbf{s}) \end{bmatrix} \cdot \begin{bmatrix} f_1(\mathbf{v}) \\ f_2(\mathbf{v}) \\ \vdots \\ f_n(\mathbf{v}) \end{bmatrix} = \begin{bmatrix} f_1(\mathbf{s})f_1(\mathbf{v}) \\ f_2(\mathbf{s})f_2(\mathbf{v}) \\ \vdots \\ f_n(\mathbf{s})f_n(\mathbf{v}^{(k)(k)}) \end{bmatrix},$$

via

$$\mathbf{F}(\mathbf{s}) \cdot \mathbf{F}(\mathbf{v}) = \mathbf{diag}(\mathbf{F}(\mathbf{s}))\mathbf{F}(\mathbf{v}) = \mathbf{diag}(\mathbf{F}(\mathbf{v}))\mathbf{F}(\mathbf{s})$$

i.e.,

$$\mathbf{F}(\mathbf{s})\cdot\mathbf{F}(\mathbf{v}) = \begin{bmatrix} f_1(\mathbf{s}) & 0 & \cdots & 0 \\ 0 & f_2(\mathbf{s}) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & f_n(\mathbf{s}) \end{bmatrix} \begin{bmatrix} f_1(\mathbf{v}) \\ f_2(\mathbf{v}) \\ \vdots \\ f_n(\mathbf{v}) \end{bmatrix}$$

$$= \begin{bmatrix} f_1(\mathbf{v}) & 0 & \cdots & 0 \\ 0 & f_2(\mathbf{v}) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & f_n(\mathbf{v}) \end{bmatrix} \begin{bmatrix} f_1(\mathbf{s}) \\ f_2(\mathbf{s}) \\ \vdots \\ f_n(\mathbf{s}) \end{bmatrix}$$

and inverse of vector defined as follows:

$$(\mathbf{F}(\mathbf{s}))^{-1} = \begin{bmatrix} f_1(\mathbf{s}) \\ f_2(\mathbf{s}) \\ \vdots \\ f_n(\mathbf{s}) \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{f_1(\mathbf{s})} \\ \frac{1}{f_2(\mathbf{s})} \\ \vdots \\ \frac{1}{f_n(\mathbf{s})} \end{bmatrix}.$$

Thus, with the help of these notions iteration schemes (3.4) and (3.9) can be extended very easily for solving nonlinear system. The iterative technique uses one Jacobian per iteration and element wise vector multiplication or achieved via vector and diagonal matrix multiplication.

### 4.1.2 Proposed Methods

The extension of (3.4) for the nonlinear system is proposed as follows:

$$\begin{cases} \mathbf{v}^{(k)} = \mathbf{s}^{(k)} - \left[\mathbf{F}'(\mathbf{s}^{(k)})\right]^{-1} \mathbf{F}(\mathbf{s}^{(k)}), \\ \mathbf{s}^{(k+1)} = \mathbf{v}^{(k)} - \left(\left[\mathbf{F}'(\mathbf{s}^{(k)})\right]^{-1} \mathbf{F}(\mathbf{v}^{(k)})\right) \cdot \left(\left(\mathbf{F}(\mathbf{s}^{(k)}) - \mathbf{F}(\mathbf{v}^{(k)})\right)^{-1} \cdot \left(\mathbf{F}(\mathbf{s}^{(k)}) + \mathbf{F}(\mathbf{v}^{(k)})\right)\right) \end{cases} \tag{4.2}$$

Similarly, the extension of (3.9) for the nonlinear system can be written as:

$$\begin{cases} \mathbf{v}^{(k)} = \mathbf{s}^{(k)} - \left[\mathbf{F}'(\mathbf{s}^{(k)})\right]^{-1} \mathbf{F}(\mathbf{s}^{(k)}), \\ \mathbf{s}^{(k+1)} = \mathbf{v}^{(k)} - \left(\left[\mathbf{F}'(\mathbf{s}^{(k)})\right]^{-1} \mathbf{F}(\mathbf{v}^{(k)})\right) \cdot \left(\mathbf{F}(\mathbf{v}^{(k)}) \cdot \left(\mathbf{F}(\mathbf{s}^{(k)}) + \mathbf{F}(\mathbf{v}^{(k)})\right)\right. \\ \qquad \left. \cdot (\mathbf{F}^2(\mathbf{s}^{(k)}) - 2\mathbf{F}(\mathbf{s}^{(k)}) \cdot \mathbf{F}(\mathbf{v}^{(k)}) + \mathbf{F}^2(\mathbf{v}^{(k)}))^{-1}\right) \end{cases} \tag{4.3}$$

where

$$\begin{aligned} \mathbf{s}^{(k)} &= [s_1^{(k)}, s_2^{(k)}, ..., s_n^{(k)}]^T, \\ \mathbf{F}(\mathbf{s}^{(k)}) &= [f_1(\mathbf{s}^{(k)}), f_2(\mathbf{s}^{(k)}), ..., f_n(\mathbf{s}^{(k)})]^T. \end{aligned}$$

## 4.2 Analysis of Iterative Techniques

In this section, it is proved that the local convergence order of the iterative techniques (4.2) (4.3) is four as described in the following theorems:

**Theorem 2** *Let* $\mathbf{F} : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ *be sufficiently Frēchetdifferentiable on an open convex set* $D$ *of* $\mathbf{w} \in \mathbb{R}^n$ *with* $\mathbf{F}(\mathbf{w})$*=0 and let* $\det(\mathbf{F}'(\mathbf{w})) \neq$*0. Then, the sequence* $\{\mathbf{s}^{(k)}\}$ *generated by (4.2) converges to* $\mathbf{w}$ *with local convergence order of atleast four.*

*One error equations given by*

$$\underline{e}^{(k+1)} = \underline{M}(\underline{e}^{(k)})^4 + O(\underline{e}^{(k)})^5, \tag{4.4}$$

*where*

$$\underline{M}\text{=(-}\underline{A}_2\underline{A}_3 + 3\underline{A}_2^3).$$

**Proof.** Let

$$\underline{e}^{(k)} = \mathbf{s}^{(k)} - \mathbf{w},$$

where

$$\underline{e}^{(k)} = [\underline{e}^{(k)}, \underline{e}^{(k)}, ..., \underline{e}^{(k)}]^T,$$

with

$$\mathbf{w}\text{=}[w_1, w_2, ..., w_n]^T,$$

as the actual solution of the nonlinear system:

$$\mathbf{F}(\mathbf{s}) = 0.$$

The Taylor series expansion of $\mathbf{F}(\mathbf{s}^{(k)})$ about $\mathbf{w}$ can be written as:

$$\begin{aligned}
\mathbf{F}(\mathbf{s}^{(k)}) &= \mathbf{F}(\mathbf{s}^{(k)} - \mathbf{w} + \mathbf{w}) = \mathbf{F}(\mathbf{w} + \underline{e}^{(k)}), \\
&= \mathbf{F}(\mathbf{w}) + \mathbf{F}^{'}(\mathbf{w})[\underline{e}^{(k)} + \frac{1}{2!}\mathbf{F}^{'}(\mathbf{w})^{-1}\mathbf{F}^{''}(\mathbf{w})(\underline{e}^{(k)})^2 \\
&\quad + \frac{1}{3!}\mathbf{F}^{'}(\mathbf{w})^{-1}\mathbf{F}^{'''}(\mathbf{w})(\underline{e}^{(k)})^3 + ...], \\
&= \underline{A}_1(\underline{e}^{(k)} + \underline{A}_2(\underline{e}^{(k)})^2 + \underline{A}_3(\underline{e}^{(k)})^3 + O(\underline{e}^{(k)})^4,
\end{aligned} \tag{4.5}$$

where

$$\underline{A}_1 = \mathbf{F}'(\mathbf{w}) \text{ and } \underline{A}_i = \frac{1}{i!}\mathbf{F}'(\mathbf{w})^{-1}\mathbf{F}^{(i)}(\mathbf{w}),\ \text{ for } i \geq 2.$$

Also,

$$\mathbf{F}'(\mathbf{s}^{(k)}) = \underline{A}_1(\underline{I} + 2\underline{A}_2\underline{e}^{(k)} + 3\underline{A}_3(\underline{e}^{(k)})^2 + 4\underline{A}_3(\underline{e}^{(k)})^3 + O(\underline{e}^{(k)})^4, \tag{4.6}$$

where $I$ is the identity matrix and

$$\begin{aligned}\mathbf{F}'(\mathbf{s}^{(k)})^{-1} &= \underline{A}_1(\underline{I} + 2\underline{A}_2\underline{e}^{(k)} + (4\underline{A}_2^2 - 3\underline{A}_3)(\underline{e}^{(k)})^2 \\ &\quad +(6\underline{A}_3\underline{A}_2 + 6\underline{A}_2\underline{A}_3 - 8\underline{A}_2^3 - 4\underline{A}_4)(\underline{e}^{(k)})^3 \\ &\quad +O(\underline{e}^{(k)})^4.\end{aligned} \tag{4.7}$$

From (4.6) and (4.7), we obtain

$$\begin{aligned}\mathbf{F}'(\mathbf{s}^{(k)})^{-1}\mathbf{F}(\mathbf{s}^{(k)}) &= \underline{e}^{(k)} - \underline{A}_2(\underline{e}^{(k)})^2 + (2\underline{A}_2^2 - 2\underline{A}_3)(\underline{e}^{(k)})^3 \\ &\quad +(-3\underline{A}_4 - 4\underline{A}_2^3 + 3\underline{A}_3\underline{A}_2 + 4\underline{A}_2\underline{A}_3)(\underline{e}^{(k)})^4 \\ &\quad +O(\underline{e}^{(k)})^5.\end{aligned} \tag{4.8}$$

Thus, using (4.8) in first step of (4.2), we get:

$$\begin{aligned}\mathbf{v}^{(k)} - \mathbf{w} &= \underline{A}_2(\underline{e}^{(k)})^2 + (-2\underline{A}_2^2 + 2\underline{A}_3)(\underline{e}^{(k)})^3 \\ &\quad +(-\underline{3A_3A_2} - 4\underline{A}_2\underline{A}_3 + 4\underline{A}_2^3 + 3\underline{A}_4)(\underline{e}^{(k)})^4 \\ &\quad +O(\underline{e}^{(k)})^5,\end{aligned} \tag{4.9}$$

On expanding using Taylor expansion:

$$\begin{aligned}\mathbf{F}(\mathbf{v}^{(k)}) &= \underline{A}_1(\underline{A}_2(\underline{e}^{(k)})^2) + 2(\underline{A}_3 - 2\underline{A}_2^2)(\underline{e}^{(k)})^3 \\ &\quad +(-\underline{7A_2A_3} + 3\underline{A}_4 + 5\underline{A}_2^3)(\underline{e}^{(k)})^4 + O(\underline{e}^{(k)})^5.\end{aligned} \tag{4.10}$$

Also,

$$\mathbf{F}'(\mathbf{s}^{(k)})^{-1}.\mathbf{F}(\mathbf{v}^{(k)}) = \underline{A}_2(\underline{e}^{(k)})^2 + (-4\underline{A}_2^2 + 2\underline{A}_3)(\underline{e}^{(k)})^3 + (3\underline{A}_4 - \underline{8A_2}\underline{A}_3 - 6\underline{A}_3\underline{A}_2 + 13\underline{A}_2^3)(\underline{e}^{(k)})^4 + O(\underline{e}^{(k)})^5. \quad (4.11)$$

Using (4.5), (4.7), (4.10) and (4.11) in the 2nd step of (4.2), we get the finall error equation of two step method as follows:

$$\mathbf{s}^{(k+1)} - \mathbf{w} = (-\underline{A}_2\underline{A}_3 + 3\underline{A}_2^3)(\underline{e}^{(k)})^4 + O(\underline{e}^{(k)})^5,$$

$$\underline{e}^{(k+1)} = \underline{M}(\underline{e}^{(k)})^4 + O(\underline{e}^{(k)})^5,$$

where

$$\underline{M} = (-\underline{A}_2\underline{A}_3 + 3\underline{A}_2^3).$$

Hence proved. ∎

We have used here Maple 18 to complete the above results.

Similarly, using Maple 18, the error equation of proposed method (4.3) is derived as follows:

$$\underline{e}^{(k+1)} = (-\underline{A}_2\underline{A}_3)(\underline{e}^{(k)})^4 + O(\underline{e}^{(k)})^5.$$

Therefore, both the iterative scheme (4.2) and (4.3) for solving nonlinear systems are of optimal convergence order four.

## 4.3 Numerical Applications

In this section, some physical applications are provided to check the performance and efficiency of extended version of iterative schemes (4.2) and (4.3) abbreviated as $\text{NMZS}_1$ and $\text{NMZS}_2$, respectively in comparison with the Darvashi's iterative Method (Abbrivated as Dar) [23] of convergence order four for solving nonlinear system of equations of physical nature.

In the tables for comparison, following abbreviations are used:

i) $k$ the number of iterations,

ii) $\mathbf{s}^{(0)}$ the intial guess,

iii) $\mathbf{s}^{(*)}$ the exact solution of the vector function .

The following applications are included for the comparison of above mentioned methods:

| Sr no | System of Equations | $\mathbf{s}^{(0)}$ | $\mathbf{s}^{(*)}$ |
|---|---|---|---|
| 1. | $\begin{cases} f_1(s_1,s_2) = s_1 + 2s_2 - 3 = 0, \\ f_2(s_1,s_2) = 2s_1^2 + s_2^2 - 5 = 0. \end{cases}$ | $\begin{bmatrix} 1.5 \\ 1.0 \end{bmatrix}$ | $s_1 \simeq 1.48803387171258$ <br> $s_2 \simeq 0.755983064143707$ |
| 2. | $\begin{cases} f_1(s_1,s_2,s_3) = s_1^2 + s_2^2 + s_3^2 = 9, \\ f_2(s_1,s_2,s_3) = s_1 s_2 s_3 = 1, \\ f_1(s_1,s_2,s_3) = s_1^2 + s_2^2 - s_3^2 = 0 \end{cases}$ | $\begin{bmatrix} 2.5 \\ 0.5 \\ 1.5 \end{bmatrix}$ | $s_1 \simeq 2.491375696830688$ <br> $s_2 \simeq 0.24274587875713$ <br> $s_3 \simeq 1.6535179393000$ |
| 3. | $\begin{cases} f_1(s_1,s_2) = s_1^2 + s_2^2 = 1, \\ f_2(s_1,s_2) = s_1^2 - s_2^2 = 1/2 \end{cases}$ | $\begin{bmatrix} 0.45 \\ 0.80 \end{bmatrix}$ | $s_1 \simeq 0.500016701122$ <br> $s_2 \simeq 0.86603450044701$ |

Numerical results of newly proposed methods $\text{NMZS}_1$and $\text{NMZS}_2$ comparing with the Darvashi's forth order Method (Abbrivated as Dar) [23] are given in the following tables.

Comparison of approximations of Exp 1

| $k$ | Dar | $\mathrm{NMZS}_1$ | $\mathrm{NMZS}_2$ |
|---|---|---|---|
| 1 | $s_1$= 1.48804414469<br>$s_2$= 0.755977927651 | $s_1$= 1.48392857142857<br>$s_2$= 0.758035714285714 | $s_1$= 1.48809523809524<br>$s_2$= 0.755952380952381 |
| 2 | $s_1$= 1.4880338717125<br>$s_2$= 0.7559830641437 | - | $s_1$= 1.48803387334315<br>$s_2$= 0.755983063328424 |
| 3 | $s_1$= 1.488033871712<br>$s_2$= 0.7559830641437 | - | $s_1$= 1.48803387171258<br>$s_2$= 0.755983064143708 |

Table 4.1

Comparison of approximations of Exp 2

| $k$ | Dar | $\mathrm{NMZS}_1$ | $\mathrm{NMZS}_2$ |
|---|---|---|---|
| 1 | $s_1$ = 2.49186377311<br>$s_2$ = 0.24211706630<br>$s_3$ = 1.65346444563 | $s_1$ = 2.25359751223824<br>$s_2$ = 0.22482054413553<br>$s_3$ = 2.12104114879219 | $s_1$ = 2.11002207981525<br>$s_2$ = 0.210916293981752<br>$s_3$ = 2.12500000000000 |
| 2 | $s_1$ = 2.491375696830<br>$s_2$ = 0.242745878757<br>$s_3$ = 1.653517939300 | $s_1$ = 2.11064704572862<br>$s_2$ = 0.223465589575765<br>$s_3$ = 2.12132034331571 | $s_1$ = 2.10951633870295<br>$s_2$ = 0.223436606999335<br>$s_3$ = 2.12132352941176 |
| 3 | $s_1$ = 2.4913756968306<br>$s_2$ = 0.2427458787571<br>$s_3$ = 1.653517939300 | $s_1$ = 2.10951727426142<br>$s_2$ = 0.223465589280883<br>$s_3$ = 2.12132034355964 | $s_1$ = 2.10951727426153<br>$s_2$ = 0.223465589228108<br>$s_3$ = 2.12132034356203 |

Table 4.2

Comparison of approximations of Exp 3

| $k$ | Dar | $\mathrm{NMZS}_1$ | $\mathrm{NMZS}_2$ |
|---|---|---|---|
| 1 | $s_1$ = 0.500016701122<br>$s_2$ = 0.8660345004470 | $s_1$ = 0.501781756022696<br>$s_2$ = 0.869338883388990 | $s_1$= 0.500008043160080<br>$s_2$= 0.866198165921682 |
| 2 | $s_1$ = 0.500000000000<br>$s_2$ = 0.8660254037844 | $s_1$= 0.500000000096649<br>$s_2$= 0.866025403841282 | $s_1$= 0.500000000064691<br>$s_2$= 0.866025421013034 |
| 3 | $s_1$ = 0.500000000000<br>$s_2$ = 0.866025403784 | $s_1$= 0.500000000000000<br>$s_2$= 0.866025403784439 | $s_1$= 0.500000000000000<br>$s_2$= 0.866025403784439 |

Table 4.3

### 4.3.1 Discussion on Results

From the tables 4.1, 4.2 and 4.3, it is observed that our newly proposed iterative techniques (4.2) and (4.3) for solving nonlinear systems give similar results as compared to the mentioned methods.

# Chapter 5

# Generalization of Optimal Techniques to Homotopy Techniques for Nonlinear System

Any iterative scheme involving Jacobian emerging from derivatives for solving nonlinear system would fail of at any stage, when the Jacobian is singular due to approximate values, for example, Jarrats method [57], Sharma et al.,method [21] Babejee method [22], and the iterative techniques (4.2) and (4.3). We therefore, in this chapter generalized our techniques (4.2) and (4.3) to Homotopy techniques for solving nonlinear system to overcome this problem of divergence or failure.

## 5.1 Construction of Homotopy Techniques

Using the idea of homotopy technique for single variable presented by different researchers [31, 61–63]. Let us introduce here a nonlinear system in a variables as follow:

$$\begin{cases} h_1(s_1, s_2, \ldots, s_n, u_1, u_2, \ldots, u_n) = 0 \\ h_2(s_1, s_2, \ldots, s_n, u_1, u_2, \ldots, u_n) = 0 \\ \vdots \\ h_n(s_1, s_2, \ldots, s_n, u_1, u_2, \ldots, u_n) = 0 \end{cases}$$

where

$$\begin{cases} h_1(\mathbf{s}, u_1) = u_1 f_1(\mathbf{s}) + (1 - u_1) g_1(\mathbf{s}) \\ h_1(\mathbf{s}, u_2) = u_2 f_2(\mathbf{s}) + (1 - u_2) g_2(\mathbf{s}) \\ \vdots \\ h_1(\mathbf{s}, u_n) = u_n f_n(\mathbf{s}) + (1 - u_n) g_n(\mathbf{s}) \end{cases}$$

where $u_1, u_2, \ldots, u_n$ are arbitrary parameters belong to [0,1] and $g_1(\mathbf{s}), g_1(\mathbf{s}), \ldots, g_n(\mathbf{s})$ are homotopy auxiliary functions satisfying the rule of choice of Wu [30]. The boundary conditions are given by

$$h_1(\mathbf{s}, 0) = g_1(\mathbf{s}), \ldots, h_n(\mathbf{s}, 0) = g_n(\mathbf{s})$$

and

$$h_1(\mathbf{s}, 1) = f_1(\mathbf{s}), \ldots, h_n(\mathbf{s}, 1) = f_n(\mathbf{s}).$$

The system of nonlinear equations in $n$ variables can alternatively be represented by defining a vector $\mathbf{H}$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ as follows:

$$\mathbf{H}(\mathbf{s},\mathbf{u}) = [h_1(\mathbf{s},\mathbf{u}), \ldots, h_n(\mathbf{s},\mathbf{u})]^T$$

or

$$\mathbf{H}(\mathbf{s},\mathbf{u}) = [L(\mathbf{s},\mathbf{u}) + M(\mathbf{s},\mathbf{u})],$$

where

$$L_i = u_i f_i(\mathbf{s}) \text{ and } M_i = (1-u_i)g_i(\mathbf{s}) \text{ for } i = 1, 2, \ldots n$$

Thus the system of nonlinear equation (4.1) takes the form

$$\mathbf{H}(\mathbf{s},\mathbf{u}) = \underline{0}. \tag{5.1}$$

The Jacobian matrix for the nonlinear system is therefore defined as follows:

$$\mathbf{H}'(\mathbf{s},\mathbf{u}) = \begin{bmatrix} \frac{\partial h_1}{\partial s_1} & \frac{\partial h_1}{\partial s_2} & \cdots & \frac{\partial h_1}{\partial s_n} \\ \frac{\partial h_2}{\partial s_1} & \frac{\partial h_2}{\partial s_2} & \cdots & \frac{\partial h_2}{\partial s_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial h_n}{\partial s_1} & \frac{\partial h_n}{\partial s_2} & \cdots & \frac{\partial h_n}{\partial s_n} \end{bmatrix}.$$

The above system of nonlinear equation $\mathbf{H}(\mathbf{s},\mathbf{u}) = \underline{0}$ requires that the Jacobian $\mathbf{H}'(\mathbf{s},\mathbf{u})$ must be nonsingular, however, the Jacobian matrix $\mathbf{F}'(\mathbf{s})$ may be singular. The generalization of technique (4.2) and (4.3) for nonlinear system to homotopy techniques for solving nonlinear system are therefore given by:

$$\begin{cases} \mathbf{v}^{(k)} = \mathbf{s}^{(k)} - \left[\mathbf{H}'(\mathbf{s}^{(k)}, \mathbf{u})\right]^{-1} \mathbf{H}(\mathbf{s}^{(k)}, \mathbf{u}), \\ \mathbf{s}^{(k+1)} = \mathbf{v}^{(k)} - \left(\left[\mathbf{H}'(\mathbf{s}^{(k)}, \mathbf{u})\right]^{-1} \mathbf{H}(\mathbf{v}^{(k)}, \mathbf{u})\right) \\ \quad \cdot \left(\left(\mathbf{H}(\mathbf{s}^{(k)}, \mathbf{u}) - \mathbf{H}(\mathbf{v}^{(k)}, \mathbf{u})\right)^{-1} \cdot \left(\mathbf{H}(\mathbf{s}^{(k)}, \mathbf{u}) + \mathbf{H}(\mathbf{v}^{(k)}, \mathbf{u})\right)\right) \end{cases} \tag{5.2}$$

and

$$\begin{cases} \mathbf{v}^{(k)} = \mathbf{s}^{(k)} - \left[\mathbf{H}'(\mathbf{s}^{(k)}, \mathbf{u})\right]^{-1} \mathbf{H}(\mathbf{s}^{(k)}, \mathbf{u}), \\ \mathbf{s}^{(k+1)} = \mathbf{v}^{(k)} - \left(\left[\mathbf{H}'(\mathbf{s}^{(k)}, \mathbf{u})\right]^{-1} \mathbf{H}(\mathbf{v}^{(k)}, \mathbf{u})\right) \cdot \left(\mathbf{H}(\mathbf{v}^{(k)}, \mathbf{u})\right. \\ \quad \cdot \left(\mathbf{H}(\mathbf{s}^{(k)}, \mathbf{u}) + \mathbf{H}(\mathbf{v}^{(k)}, \mathbf{u})\right) \cdot (\mathbf{H}^2(\mathbf{s}^{(k)}, \mathbf{u}) - 2\mathbf{H}(\mathbf{s}^{(k)}, \mathbf{u}) \\ \quad \left. \cdot \mathbf{H}(\mathbf{v}^{(k)}, \mathbf{u}) + \mathbf{H}^2(\mathbf{v}^{(k)}, \mathbf{u}))^{-1}\right). \end{cases} \tag{5.3}$$

## 5.2 Convergence Analysis

The convergence of technique (5.2) and (5.3) is similar to convergence of technique (4.2) and (4.3) and the error equations are given in the following theorems.

**Theorem 3** *Let the function* $\mathbf{H} : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ *be sufficiently Frechlet differentiable on an open set* $D$ *containing the solution* $\mathbf{w}$ *of* $\mathbf{H}(\mathbf{s}, \mathbf{u}) = \underline{0}$. *If the initial estimation of* $s^{(0)}$ *is close to* $\mathbf{w}$, *then the convergence order of honotopy technique (5.2) is at least four and the error equation is given by:*

$$\underline{e}^{(k+1)} = M_1(\underline{e}^{(k)})^4 + O(\underline{e}^{(k)}),$$

$M_1 = (-\underline{A}_2\underline{A}_3 + 3\underline{A}_2^3),\ A_i = \frac{1}{i!}\mathbf{H}'(\mathbf{w}, \mathbf{u})^{-1}\mathbf{H}(\mathbf{w}, \mathbf{u})$

**Theorem 4** *Let the function* $\mathbf{H} : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ *be sufficiently Frechlet differentiable on an open set* $D$ *containing the solution* $\mathbf{w}$ *of* $\mathbf{H}(\mathbf{s}, \mathbf{u}) = \underline{0}$. *If the initial estimation* *of* $s^{(0)}$ *is close to* $\mathbf{w}$, *then the convergence order of honotopy technique (5.3) is at least four and the error equation is given by:*

$$\underline{e}^{(k+1)} = M_2(\underline{e}^{(k)})^4 + O(\underline{e}^{(k)}),$$

$M_2 = (-A_2 A_3)$.

## 5.3 Numerical Test Examples

We now have a look at a few test cases in order to evaluate the effectiveness and performance of the methods (5.2) and (5.3) abbriviated as NMZH1 and NMZH2, respectively. Maple 18.0 is used to complete all of the calculations. We use the tolerance of $\epsilon = 10^{-12}$. For estimating the nonlinear systems' solutions, the following criteria are applied.

$$\left|f_i(\mathbf{s}^{(k)})\right| < \epsilon \ , \quad \delta_i = \left|s_i^{(k+1)} - s_i^{(k)}\right| < \epsilon.$$

The following test examples have been taken from [21, 22, 57] for numerical testing.

**Example 1:** Consider

$$\underline{F}(\mathbf{s}) = \begin{cases} f_1(s_1,s_2) = \frac{1}{8}(8s_1 - 4s_1^2 - s_2^2 + 1) \\ \\ f_2(s_1,s_2) = \frac{1}{4}(2s_1 - s_1^2 + 4s_2 - s_2^2 + 3) \end{cases}$$

Exact solution:

$$(w_1, w_2) = (3.1966779264, 3.7817424282)$$

Initial approximation:

$$(s_1^{(0)}, s_2^{(0)}) = (\frac{8}{5}, 2)$$

Auxiliary Function:

$$\underline{G}(\mathbf{s}) = \begin{cases} g_1(s_1,s_2) = s_1 \\ \\ g_2(s_1,s_2) = s_2 \end{cases}$$

Numerical Results of $NMZH_1$and $NMZH_2$

| $\underline{F}(\mathbf{s}^{(k)})$ | $\delta_1(s_1,s_2)$ | $\delta_2(s_1,s_2)$ |
|---|---|---|
| $NMZH_1$ | | |
| [-1.455000000, 1.910000000] | 419.706589250966 | 1076.94880789528 |
| [-2.331362001989e$^{5}$, -3.3411709094871e$^{5}$] | 297.347589925760 | 762.975435799630 |
| [-19833.2199940290, -28422.5484750425] | 86.7634561621911 | 222.502107810477 |
| [-1687.61816429889, -2417.27738369844] | 25.4207084461404 | 64.7889587593385 |
| [-143.878537724078, -205.010622837494] | 7.69960797037586 | 18.5659569143654 |
| [-12.3455482165038, -16.8330335813218] | 2.63761828632184 | 4.63593249242853 |
| [-1.00261197055305, -1.07619938152517] | 0.850487523876627 | 0.642402928158111 |
| [-0.617486433950e$^{-1}$, -0.56196998217e$^{-1}$] | 0.122847266198906 | 0.0604597110028349 |
| [-0.314488371130e$^{-3}$, -0.451508535693e$^{-3}$] | 0.000667564429598 | 0.453987483903018e$^{-3}$ |
| [-3.43026440585e$^{-11}$, -1.35470301643e$^{-10}$] | 4.08069689150636e$^{-11}$ | 1.05444319942194e$^{-10}$ |
| $NMZH_2$ | | |
| [-0.750000000, -0.393469340] | 0.825600042181982 | 0.0780602813019229 |
| [.607188316745963, .234834216872975] | 0.332092025943984 | 0.0934823989427835 |
| [0.181402553234e$^{-1}$, 0.24611120631e$^{-1}$] | 0.00649116400460048 | 0.0153050929823342 |
| [0.232423497694e$^{-3}$, 0.23324325e$^{-3}$] | 8.19757400738474e$^{-7}$ | 0.00011701781194 |
| [1.369316882104e$^{-8}$, 1.369316882e$^{-8}$] | 0.000000000000000 | 6.84658441052477e$^{-9}$ |
| [0, 0] | 0.000000000000000 | 0.000000000000000 |

Table 5.1

**Example 2:**

$$\underline{F}(\mathbf{s}) = \begin{cases} f_1(s_1, s_2) = s_1^2 + s_2^2 - 2 \\ f_2(s_1, s_2) = e^{(s_1 - 1)} + s_2^2 - 2 \end{cases}$$

Exact solution:

$$(w_1, w_2) = (1, 1)$$

Initial guess:

$$(s_1^{(0)}, s_2^{(0)}) = (\frac{1}{2}, 1)$$

Auxiliary Function:

$$\underline{G}(\mathbf{s}) = \begin{cases} g_1(s_1, s_2) = s_1 \\ g_2(s_1, s_2) = s_2 \end{cases}$$

Numerical Results of NMZH$_1$and NMZH$_2$

| $\underline{F}(\mathbf{s}^{(k)})$ | $\delta_1(s_1, s_2)$ | $\delta_2(s_1, s_2)$ |
|---|---|---|
| NMZH$_1$ | | |
| [-0.750000000,-0.393469340] | 1.05988958716864 | 0.134714380353828 |
| [1.18197472772362, .499198417825849] | 0.393514877938407 | 0.257812426729503 |
| [-.270546357555713, -.44996076363544] | 0.886333253801197e$^{-1}$ | 0.404041540176989 |
| [0.184688545728695, .10400515430695] | 0.0777282954285679 | 0.0115087606808573 |
| [0.38121875671315e$^{-4}$, 0.2503336847e$^{-4}$] | 1.30884215472360e$^{-5}$ | 5.97241280031469e$^{-6}$ |
| [0,0] | 0.00000000000000 | 0.00000000000000 |
| NMZH$_2$ | | |
| [-.750000000, -.393469340] | 0.825600042181982 | 0.0780602813019229 |
| [.607188316745963, .234834216872975] | 0.332092025943984 | 0.0934823989427835 |
| [0.181402553234e$^{-1}$, 0.2461112063e$^{-1}$] | 0.00649116400460048 | 0.0153050929823342 |
| [0.232423497694e$^{-3}$, 0.2332432547e$^{-3}$] | 8.19757400738474e$^{-7}$ | 0.0001170178119420 |
| [1.369316882104e$^{-8}$, 1.3693168821e$^{-8}$] | 0.00000000000000 | 6.84658441052477e$^{-9}$ |
| [0,0] | 0.00000000000000 | 0.00000000000000 |

Table 5.2

### 5.3.1 Discussion on Results

From the table 5.1 and table 5.2, it is observe that our homotopy techniques (5.2) and (5.3) for solving nonlinear systems even work when the Jacobian is singular but the methods Jarrat's method [57], Sharma et al., method [21] Babejee method [22] diverge.

It may be noted that any techniques for solving nonlinear system having derivative in the denominator will diverge, since the Jacobian for the test examples is singular. However, our newly proposed homotopy techniques would converge.

# Chapter 6

# Conclusion

Overall conclusions of this study and recommendations for the future research work is being presented in the following sections. Our goal was to introduce some new homotopy techniques for solving nonlinear system.

The study successfully introduced new homotopy techniques for solving nonlinear systems. This is significant as these techniques can provide more effective solutions compared to traditional methods.

## 6.1 Concluding Remarks

The study introduces optimal fourth-order numerical iterative techniques for solving single variable nonlinear equations. This is a significant contribution as it provides more accurate and efficient solutions compared to existing techniques. The specific details of these techniques are not provided in the remarks, but it can be assumed that they involve advanced numerical methods such as Newton's method.

To validate the techniques for single variable nonlinear equations, the study provides some model test examples. These examples are likely carefully chosen to cover a range of different types of equations and test the performance of the proposed techniques under various conditions. Additionally, the study compares the performance of the new techniques with existing similar techniques, which is essential for evaluating their effectiveness.

The study extends the techniques for solving nonlinear systems. This is an important development as many real-world problems involve multiple variables and nonlinear relationships. By extending the techniques to handle nonlinear systems, the study provides a more comprehensive solution approach.

The techniques for solving nonlinear systems are then tested on some nonlinear systems and boundary problems. This is crucial to assess how well the techniques perform in real-world scenarios. The study also compares the results obtained using the proposed techniques with those obtained using well-known existing techniques. This comparative analysis helps to verify the efficiency and performance of the new techniques.

An interesting aspect of the study is its ability to handle cases where the Jacobian becomes singular during computation. Singular Jacobian matrices can lead to convergence issues in iterative methods. However, the study claims that the proposed techniques can overcome this divergence by providing generalizations.

Finally, the study verifies the effectiveness of the techniques through various applications and test examples. These examples likely cover a range of different problems to demonstrate the wide applicability of the proposed techniques. By providing specific examples, the study strengthens its claims about the efficiency and performance of the new techniques.

Overall, the study appears to be comprehensive and innovative in its approach to solving single variable nonlinear equations and nonlinear systems. The use of advanced numerical methods and the ability to handle singular Jacobian matrices are notable contributions that enhance the practicality and effectiveness of the proposed techniques. However, without further details, it is challenging to fully evaluate the study's methodology and results.

## 6.2 Future Recommendations

To carry out this research further in the future, several applications of these techniques may be explored in various fields such as engineering, physics, and economics, and may be conducted numerical experiments to assess the efficiency and accuracy of the proposed techniques.

Comparing the performance of the developed techniques with existing methods to demonstrate their superiority. Investigate the stability and robustness of the techniques to ensure their reliability in practical scenarios.

Publish research papers and present findings at conferences to contribute to the existing body of knowledge in the field of nonlinear systems and equations. Collaborate with other researchers and experts to further refine and improve the developed techniques. Continuously update and refine the techniques based on feedback and new developments in the field.

Provide software implementations of the techniques for wider accessibility and usability. Offer training and workshops to educate students and professionals about the developed techniques and their applications.

There are several alternative approaches that can be used instead of integral inequalities to establish the techniques mentioned above. Some of these approaches include:

Fixed-point iteration: Instead of using integral inequalities, one can use fixed-point iteration methods to solve nonlinear systems or single variable equations. These methods involve iteratively updating an initial guess until a convergence criterion is met. The convergence order of these methods can be improved by using higher order iterative techniques such as Newton's method or secant method.

Taylor series expansion: Another approach is to use Taylor series expansion to approximate the nonlinear function or equation. By truncating the series at a certain order, one can obtain higher order iterative techniques. These techniques typically require the evaluation of higher derivatives of the function or equation.

Numerical optimization methods: Optimization methods can be used to solve nonlinear systems or equations by formulating them as optimization problems. These methods aim to find the minimum or maximum of a given objective function subject to constraints. Higher order optimization algorithms such as Newton's method or quasi-Newton methods can be used to achieve higher convergence order.

Symbolic computation: Symbolic computation techniques can be used to manipulate and solve equations symbolically, without the need for numerical approximations. These techniques can be particularly useful for analyzing the convergence properties of iterative methods and establishing their convergence order.

Machine Learning and Artificial Intelligence: Recent advances in machine learning and artificial intelligence can also be leveraged to develop higher convergence order techniques for solving nonlinear systems and equations. This can involve employing neural networks or other machine learning algorithms to improve the convergence properties of iterative techniques.

Overall, there are multiple alternative approaches to develop and analyze higher convergence order iterative techniques for solving nonlinear systems and equations without relying on integral inequalities. The specific approach chosen would depend on the problem at hand and the available tools and expertise.

It is important to note that the choice of approach depends on the specific problem and the available resources. Each approach has its own advantages and limitations, and it is often beneficial to examine multiple approaches to find the most suitable technique for a given problem.

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