# MODIFICATION OF FINITE DIFFERENCE SCHEME FOR THE TIME-FRACTIONAL HYPERBOLIC PROBLEM WITH STABILITY ANALYSIS

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Candidate of **Master of Science in Mathematics (MS)** at the National University of Modern Languages do hereby declare that the thesis **Modification of finite difference scheme for the time-fractional hyperbolic problem with stability analysis** submitted by me in partial fulfillment of MS degree is my original work, and has not been submitted or published earlier. I also solemnly declare that it shall not, in the future, be submitted by me for obtaining any other degree from this or any other university or institution. I also understand that if evidence of plagiarism is found in my thesis/dissertation at any stage, even after the award of a degree, the work may be cancelled and the degree revoked.

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# ABSTRACT

**Title: Modification of finite difference scheme for the time-fractional hyperbolic problem with stability analysis**

In this thesis the modification of finite difference scheme for the time-fractional wave problem with stability analysis for one- and two-dimensional time fractional wave equations (1D-TFWE and 2D-TFWE, respectively) on a finite domain is investigated. In the empire of mathematical physics and engineering, it has been recently discovered that the majority of physical processes give fractional order wave equations when modelled. To examine the techniques for solving fractional order wave equations and turn this scenario into an attractive research project, the precise solutions are crucial. Furthermore, problems in physics, environmental science, biology, and other fields of application have been modeled using fractional wave equations. For the (1D-TFWE) and (2D-TFWE), a Crank-Nicolson difference approximation is proposed. We explored the method's stability and convergence using mathematical induction. Finally, some numerical examples are shown. The numerical result and our theoretical analysis accord quite well.

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# LIST OF ABBREVIATIONS

| | |
|---|---|
| FC | Fractional Calculus |
| FDEs | Fractional Differential Equations |
| IVPs | Initial Value Problems |
| BVPs | Boundary Value Problems |
| PDEs | Partial Differential Equations |
| TFWEs | Time-Fractional Wave Equations |
| CNM | Crank-Nicholson Method |
| FDM | Finite Difference Method |

# LIST OF SYMBOLS

| | |
|---|---|
| $\sum$ | Summation |
| $\partial$ | Partial Differential |
| **Γ** | Gamma Function |
| **D** | Fractional order Differential |
| **q (x,y, t)** | Represents source term |
| **Ω** | Represents boundary conditions |
| **Δ** | Represents change in quantity |

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# DEDICATION

This thesis work is dedicated to my parents and my teachers throughout my education career who have not only loved me unconditionally but whose good examples have taught me to work hard for the things that I aspire to achieve.

# CHAPTER 1

# INTRODUCTION

## 1.1 Overview

The fractional-order partial differential equations have recently gained increased attention, due to their copious applications in numerous arenas of science and engineering. For instance, describe a wide range of physical and chemical processes as well as biological systems, environmental science, and multidisciplinary engineering fields Atanackovic *et al*. [6]. A class of finite difference techniques is used to introduce a numerical analysis for the fractional wave equations. The weighted average approaches for regular (non-fractional) wave equations are extended using these techniques. An approach that was recently proposed and is similar to the traditional John von Neumann stability analysis provides the stability analysis for the suggested methodologies. A straightforward and precise stability criterion that is applicable to various fractional derivative discretization schemes, arbitrary weight factors, and arbitrary orders of the fractional derivative is provided and numerically verified. For clarity, numerical test examples and comparisons have been provided.

The use of partial differential equations (PDEs) as given in Eq. (1.1) mostly in the real-world includes solving physical and other issues involving multiple variables, such as heat or sound wave propagation Atanackovic *et al*. [6], elasticity, and others. Academics have frequently used the finite difference method (FDM), a potent strategy for resolving Time-fractional wave equations (TFWEs), to address a wide range of issues, including the one raised above. The Dong *et al.* [1], introduced an exhibited first-order convergence and unconditional stability. Also, first-order consistency for advection dispersion equations with partial derivative boundary conditions using an implicit finite difference approach. The objective of this work is to get numerical solutions for fractional-order telegraph PDEs with non-local boundary conditions and investigate the stability of the proposed method. Both the Dufort-Frankel difference scheme and the implicit difference method were employed. The authors Sweilam *et al.* [15], developed the finite difference method for resolving the generalised time fractional telegraph problem; they also provided a study of the stability and convergence of the suggested solution. Studying the parameters of the telegraph equation linked to the FDEs involved using the R-L fractional time derivative and an approximate solution based on FDM.

A high-order partial differential system with mixed partial derivatives in terms of time and space, the pseudo-hyperbolic telegraph equation, is a PDE. In the study of longitudinal vibrations, plasma physics, nerve conduction, and response wave, among other physical phenomena, the well-known mathematical physics equation known as the pseudo hyperbolic equation is commonly studied. Existence and originality of numerical solutions Abdulazeez *et al.* [32]. A number of articles have examined various aspects of pseudo-hyperbolic equations, the presence, nonexistence, and uniqueness of solutions, as well as numerical solutions, stability analysis, and the existence of solutions. With pseudo-hyperbolic equations in vibration, convergence analysis for approximative solutions, semi-discrete and completely discrete error estimates, and Cauchy-type issues involving semi-linear higher-order pseudo-hyperbolic equations in terms of both Caputo and Atangana-Baleanu Caputo fractional derivatives, the author was able to achieve approximation. The solution to the pseudo-hyperbolic telegraph PDEs using the Dufort-Frankel difference method approach was used in order to explain PDEs in the sense of the Caputo differential; the Crank-Nicholson finite difference approach was utilised, while the Fourier analysis scheme was used to proposal stability study and certain reproducible kernel functions.

Wave propagation is described by a hyperbolic partial differential equation, and wave propagation has a finite speed as mentioned in Eq. (1.1). It is frequently maintained or creates discontinuities (in the absence of damping) and occurs in various disciplines of science and engineering like electromagnetism, fluid dynamics, acoustics, etc. Consider the following general partial differential equation involving two variables as

$$a\frac{\partial^2 u}{\partial x^2}+b\frac{\partial^2 u}{\partial x\partial y}+c\frac{\partial^2 u}{\partial y^2}+d\frac{\partial u}{\partial x}+e\frac{\partial u}{\partial y}+fu+g(x,y)=0. \tag{1.1}$$

If $b^2-4ac>0$ then Eq. (1.1) is known as the hyperbolic equation. The subsequent wave equation is one of the well-known types of a hyperbolic equation in one dimension,

$$\frac{\partial^2 u}{\partial t^2}=c^2\Delta u(x,t)+\phi(x,t,u), \tag{1.2}$$

where $u_{tt}$, is the divergence of the gradient, $c$ is a constant link to the material elasticity of the string, and $\Delta u(x,t)$ represents the Laplace operator and $\phi(x,t,u)$ is a source term.

Fractional calculus is a branch of mathematics that has an extensive range of applications in science, engineering, and mathematical physics. This field was initiated first time by two well-known scientists Leibniz and Hospital almost 300 years. On September 30, 1695, Hospital received a letter, Leibniz asked: "Is it possible to convert derivatives with integer order to non-integer order?" Then Hospital said, "What if the order is only half?" Leibniz's concept is recognized as the precise age at which fractional calculus was born.

Fractional or non-integer differential equations and fractional calculus Barro *et al.* [31], are fascinating mathematical topics that the scientific world is currently researching. Due to its distinct appearance in mathematics as well as other fields of science and engineering, scientific communities' devotion is rather realistic. The symbolization of those derivatives changes when the order of the derivatives is non-integer or fractional. In several scientific disciplines, the study of fractional order mechanics and fractional calculus has advanced recently, from pure mathematical theories to modeling fractional order physical issues in a variety of engineering and applied science fields, such as ultra-capacitors, beam heating, and the transfer of heat in heterogeneous media, among others. Physical and natural processes can be defined using differential equations, and computing the answers to problems of both integer and non-integer order is worthwhile because it saves both time and money. The fundamental effect of employing fractional differential equations is that they have non-local features, which implies that the dynamical systems' current state and all of their previous states have an impact on their future state. The practical scenarios stated are typically caused by the fact that several physical structures are connected to non-integer order dynamics.

It can be seen that nearly every area of modern engineering and study uses the techniques and tools of fractional calculus (FC). The victory of FC applications is largely attributable to the novel models of fractional order, which are often more accurate as compared to integer-order modeling and have more benefits than the corresponding classical models. All fractional operators can describe the nonlocal and distributed impacts typically seen in technological and natural phenomena because they take into account the entire history of the activity under examination. As a result, fractional calculus offers a fantastic collection of tools for illuminating the genetic and memory properties of many materials and processes. Chan *et al*. [16], covered real-world applications in several scientific and engineering fields. Despite the fact that numerous astonishing discoveries have previously been made and published by scholars in important monographs, this review article's objective is to showcase a few concise descriptions written by well-known specialists in the field of fractional calculus.

According to Delany and Bazley's experimental findings Cai *et al.* [22], both the frequency-dependent characteristic impedance and the acoustic wave's spread coefficient were present. Later, numerous models were created to illustrate the phenomenon without using a physical clarification of the frequency-dependent indices. As a foundation for the continuity equation, state equation, and characteristic impendence, the fractional-order acoustic wave problem was suggested. The order of fractional derivatives, which has a specific physical significance, was created by combining the two distinctive indices. The Herschel model Chen *et al.* [16], has been successfully applied in engineering, and significant influences have been made to studying the properties of transport related to non-Newtonian fluids in shear flow founded on the conventional non-Newtonian constitutive problem. The two classifications of non-Newtonian fluids are time dependent and independent non-Newtonian fluids depending on how viscosity is defined. The history-dependent property that the reversible effect suggests the variation of inner structure has can be successfully described by a time-variant fractional-order non-Newtonian problem. The fact that most non-Newtonian fluids do not have a single constitutive description in empirical models is the other issue with time-independent non-Newtonian fluids. A fractional constitutive equation was presented to address this shortcoming and represent the observed rise of shear stress at varied velocity gradients.

## 1.2 Preliminaries

In this area, we go through various definitions of a fractional derivative, fractional integrals and Caputo derivatives of fractional order. The concept of a derivative, which is employed to specify the rate of change of a particular function before being utilized to create mathematical equations that characterize the behaviour of real-world events, is probably one of the most frequently utilized notions in applied mathematics. However, due to the complexity of real-life situations, this notion was modified to the fractional derivative concept, which was better suited than the local derivative for simulating real-world issues. Also present here some basic definitions and concepts which we will use throughout this dissertation.

### 1.2.1 Definition of Caputo Fractional Derivatives

Caputo derivative of fractional order, first planned by Caputo in 1967. The phenomena that take into account earlier interactions as well as problems with nonlocal features can be explained by the Caputo derivative. In this way, the equation can be thought of as having "memory." The formula for this type of fractional derivative is Cai *et al.* [22].

$$ {}_{0}^{C}D_{x}^{\alpha}u(x)=\frac{1}{\Gamma(n-\alpha)}\int_{0}^{t}\frac{1}{(t-x)^{\alpha-n+1}}u^{(n)}(x)\mathrm{d}x, n-1<\alpha\le n, n\in N. \quad (1.3) $$

Where Γ (·) is the Gamma function and the function *u(x)* under consideration.

### 1.2.2 Finite Difference Scheme

- **The Explicit Method**

The explicit technique uses the system status that is currently known to compute the state of the system in the future. The implicit technique predicts the system status at a later time based on the system significance at present and future times. When a differential equation is included, for instance.

$$ y'=F(y,t), $$

the explicit method expresses it as:

$$ y_{n+1}=y_n+hF(y_n+t_n). $$

In other words, if you are aware of the Position at n, you can determine the Position at $n+1$.

- **The Implicit Method**

The state at $n+1$ is located on the right side of the implicit method as

$$ y_{n+1}=y_n+hF(y_{n+1}+t_{n+1}), $$

the explicit method takes less time to calculate and is simpler to program. However, because of its poor stability, you must pick a step size small sufficient to prevent divergence. On the other hand, if you use the right settings, the implicit technique is highly stable and converges. However, because an equation needs to be solved at each step, the calculation takes a while.

- **Crank-Nicolson method**

The fundamental benefit of the suggested approach is that it provides efficient and straightforward answer to the fractional wave equation, without compromising the precision of the findings. The performance of the approach was more than excellent in terms of calculation accuracy and reliability when the numerical results were compared with the variations between the investigated variables of (implicit and explicit). When compared to the implicit and explicit algorithms, the proposed technique performed significantly better in terms of running times Samarinaz *et al.* [33]. The wave equation and other fractional order partial differential equations can be quantitatively solved using the CNM, which is a finite difference approach. In terms of time, it is a second-order strategy, numerically stable, and may be described as an implicit Runge-Kutta algorithm. Here we consider these approximations as,

$$\frac{\partial^2}{\partial x^2}u = \frac{1}{2}\left[\frac{u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n}}{\Delta x^2} + \frac{u_{i+1,j}^{n+1} - 2u_{i,j}^{n+1} + u_{i-1,j}^{n+1}}{\Delta x^2}\right] + O(\Delta x^2), \tag{1.4}$$

$$\frac{\partial^2}{\partial y^2}u = \frac{1}{2}\left[\frac{u_{i,j-1}^{n} - 2u_{i,j}^{n} + u_{i,j+1}^{n}}{\Delta y^2} + \frac{u_{i,j-1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j+1}^{n+1}}{\Delta y^2}\right] + O(\Delta y^2). \tag{1.5}$$

## 1.3 Problem Statement

In this thesis, we will assume the following time-fractional wave equation arising in space is given as,

$$ {}_0^C D_t^{\alpha} u(x,t) = \Delta u(x,t) + \phi\big(x,t,u(x,t)\big), 1 < \alpha \le 2, t > 0, 0 < x < 1, \tag{1.6}$$

where ${}_0^C D_t^{\alpha}$ represent the fractional order Caputo derivative also $\Delta u(x,t)$ represents the Laplace operator and $\phi\big(x,t,u(x,t)\big)$, is called source term. If value of α is equal to one then this problem converted into standard heat equation and if the value of α is equal to two then this problem changed into ordinary wave equation that’s why I have consider the values of α between one and two. Besides through the subsequent initial conditions and boundary condition are given as:

$$u(x,0) = f(x) \quad, u_t(x,0) = \mathrm{g}(x), \mathrm{a} \le x \le \mathrm{b}, \tag{1.7}$$

$$u(a,t) = h_1(t), u(b,t) = h_2(t), t > 0. \tag{1.8}$$

In the above, Δ are given as,

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{1.9}$$

Respectively, we will extend the finite difference scheme to investigate the physical behaviour besides the parameter $\alpha$ through simulations and examine the convergence, error bound, and stability of the proposed scheme.

## 1.4 Research Questions

After the inclusive literature survey stated above the following research questions or research gaps were established:

- What is the mathematical formulation of the time-fractional wave problem arising in one and two-dimensional environments?
- To examine the behaviour graphically of the hyperbolic equation against the fractional parameter?
- What is the significant influence of fractional-order parameters on the wave problem?
- To explore the error bound, of the proposed discretized scheme theoretically?
- Whether the numerical scheme design for governing problem is stable theoretically?
- To test the suggested discretized scheme's convergence?
- What effect do the boundary conditions of Dirichlet and Neumann have on the numerical solutions?

## 1.5 Aim of the Research

There is no earlier study that offers a numerical and theoretical analysis of the fractional order wave problem for the Crank-Nicolson method. For the fractional order hyperbolic PDEs using the Crank-Nicolson scheme, no one looked at the error limits, stability, or convergence analysis. Because of this, we'll study the fractional order wave equation numerically and provide analysis of the problem's error bound, stability, and convergence.

- This research will address the wave phenomena against the variation of fractional-order parameters.
- This research will offer a comprehensive numerical scheme with convergence and stability analysis which will be a significant contribution to computational mathematics.
- The numerical scheme further uses to analyze the behaviour of a more physical model of fractional order.

## 1.6 Research Objectives

In light of the research questions stated in the above sections, our main objectives of this research work are to disclose the features in the thesis as follows:

- The mathematical formulation of time-fractional wave problems arising in one and two-dimensional environments.
- Comprehensive study of error bound, of the proposed Crank-Nicolson methods.
- Examine how the fractional-order parameter affects the wave problem through various simulations.
- Numerical investigation of the influence of boundary conditions on the wave problem.
- Extend the numerical solutions for the time-fractional wave issue and explore its stability theoretically.
- Investigation convergence of the proposed numerical method.

## 1.7 Thesis Organization

In order to address a range of problems, we used a finite difference technique in this thesis. It has been discovered that the suggested algorithm is quite effective and user-friendly. The accuracy of the offered method is fully supported by computational effort and subsequent numerical findings. The assignment is structured as follows:

The first chapter is distributed into four sections. The first section provides basic definitions of TFWEs; the second section includes various basic fractional calculus preliminaries; the third section includes some definitions of fractional derivatives; and the fourth section discusses the time-fractional wave equation.

Second chapter consists of brief literature survey of fractional hyperbolic differential equations. The study focuses mostly on the wave equation for time fractional order.

Therefore, an analytical analysis of the fractional order wave equation in one and two dimensions has been done. The current work also includes an equation investigation of the Caputo fractional-order derivations. The differential Crank-Nicolson, implicit and explicit finite difference approach has been used to analyze the hyperbolic equation. This study also outlines potential benefits of applying fractional order wave to actual situations.

The importance of one-dimensional hyperbolic problems is discussed in Chapter third along with their analytical analysis. The mathematical modeling, stability analysis, convergence analysis, and error bound explained in this chapter. With the aid of the Tec plot software, numerical results of the fractional-order wave problems are obtained.

In Chapter four, along with their analytical analysis, two-dimensional hyperbolic problems are presented. In this chapter we explain the mathematical modeling, stability analysis, and convergence analysis. The numerical outcomes of the governing equations are derived using the Tec plot software.

An analysis of the numerical results' performance will be presented in chapter five. This chapter will demonstrate and analyses simulation results using the Tec plot programming language in order to evaluate our suggested time-fractional hyperbolic partial differential equation presented in Chapters 3 and 4. This section is divided essentially into two parts. Results analysis, which compares the performance of TFWEs with the precise solution by measuring it according to a number of different criteria, is described.

The thesis will be concluded in chapter six, which will also provide a summary of the contributions and also presented the upcoming directions.

# CHAPTER 2

# LITERATURE REVIEW

## 2.1 Overview

It is observed that modeling of physical procedures utilizing fractional calculus offers numerous advantages and discloses significant information about complex systems that could be limited with the modeling through the use of classical calculus. In this regard, the research community's most popular and difficult topic is the solution to the differential problem of fractional order. Research demonstrates that fractional modeling is a reliable method for predicting the behaviour of any physical system Atanackovic *et al,* [6]. Because of its various advantages and ability to provide a significant deal of information about complex systems, fractional calculus has attracted a lot of attention from academics lately compared to the use of classical calculus. The importance of modeling with fractional differential equations has lately come to light due to the numerous real-world circumstances that call for the use of fractional equations. Numerous disciplines, including signal processing, image theory, economics, biology, mechanics, heat transport, chemistry, and Physics, use fractional equation modeling Zhang *et al.* [25]. Many fractional models indeed lack an analytical solution, which is why many academics have been striving to predict the development of various numerical approaches to come close to a solution. The fractional-order differential equation is a challenging subject for this field of study.

Dong *et al*. discussed the numerical schemes for the first time to find an accurate solution to wave problems. In this article, some remarkable research has been done on the execution of splitting methods Dong *et al.* [1]. An extensive review of the application of a semi-analytical scheme is provided by Jafari *et al.* [2], to look at the solution of fractional-order issues in series. In this research, the authors did not provide the proper information about the stability of the method. Later, an efficient spectral scheme using fractional Legendre functions, the Galerkin approach Frank *et al.* [26], discussed to investigate the numerical solutions of the fractional-order physical model. Some new operational matrices for fractional-order derivatives were investigated there for the first time and the suggested approach is quite practical for resolving issues of this nature. Some more powerful schemes can be found in the Mustafa *et al.* [30], where the researcher extended the numerical, spectral, and semi-analytical schemes to explore the physical behaviour besides the fractional-order parameter in the fractional-order differential problems. Pskhu *et al*. [7], examine the Riemann and Caputo derivatives of the essential solution to the fractional-order wave equation. Fundamental quantum and classical equations are used by Vazquez *et al*. [11], to generate the finite difference equation using the traditional method. Adel *et al*. [3], describe a group of numerical techniques for solving finite wave equations. The Hermite formula-based FDM is a prerequisite for this class of strategies. In specifically, the convergence and stability of the fractional FDM are inspected utilizing the Von Neumann stability analysis. Sun *et al*. [16], study the numerical result of variable-order time-fractional wave equations. The authors investigated the explicit, implicit, and Crank-Nicholson finite difference method for the numerical solution. The Fourier method is used to provide and demonstrate the stability conditions for these three methods. Probabilistic interpretation and numerical results of time-fractional wave problem studied by Iafrate *et* [28]. It is deducted that when the fractional parameter plays a significant role in the Brownian motion phenomena. Barro *et al.* [31], used the Laplace transform and the method of variable separation also applied the Caputo order type derivative to finite wave equations with memory effect. They conclude that using fractional derivatives raises fresh problems for mathematics and engineering.

Recently, Zeng *et al*. [4], consider the comprehensive analysis of the differential mode of fractional order. The equations under study principally involve the time and space derivatives of the dispersion- or wave-related fractional kinetic equations. These numerical methods can be seen as extensions of finite difference methods. The numerical strategy for FDEs also makes use of more established tools, such as the Fourier and von Neumann analytic methods. The analytical study of fractional-order linear electrical model was investigated first time by Zahra *et al*. [5]. This article contains the assessments of the fractional-order and classical electrical models demonstrated employing the Laplace transform and non-standard finite difference technique. Abdulazeez *et al.* [32], presented the explicit finite difference algorithm along the hyperbolic PDEs of fractional order is explained by the fractional derivative in the Caputo sense. To investigate the numerical solution of a first-order FDM is made for partial differential models of hyperbolic telegraph type of fractional order. A judgment among the precise and estimated results is presented to evaluate the suggested approach's precision and efficiency. Finally, the answers both accurate and approximate under various conditions are displayed graphically. Saeed *et al*. [21], deliberated, a novel approach that is grounded on the explicit finite difference estimation is recommended to inspect the study of fractional-order hyperbolic-type PDEs. The authors proved that the suggested numerical strategy for the discussed model is very accurate and effective. The numerical results of the anticipated system are compared to accurate responses and the previous method to show the efficacy of the new approach. The stable difference approach is described for numerically solving the multi-dimensional fractional-order hyperbolic-type equation by Pinar *et al*. [8]. They proved the discussed numerical algorithm is stable. This difference scheme is solved using a modified Gauss elimination scheme for one-dimensional fractional-order hyperbolic-type partial differential equations. Yaseen *et al*. [24], planned a well-organized numerical technique for the precise outcome of a temporal-fractional wave problem using a response term founded on the basis of cubic trigonometric functions. The temporal fractional derivative is calculated using the conventional FDM, while the space-based derivative is discretized using cubic trigonometric B-spline functions. To make sure that errors are not amplified, a stability study of the method is conducted. The numerical results are related to finite difference models built using the Hermite formulation and the radial-basis functions. It is learned that because of its basic interpolation, easy implementation, and low computing cost, our numerical solution outperforms the currently used techniques. The scheme's convergence analysis is also covered. Many definitions of fractional derivatives and integrals (differential integrals) were addressed by Sheng *et al.* [12], they allow for the explicit derivation of formulas and graphs for some particular functions. They also looked at various fractional calculus applications. Odibat *et al.* [3], proposed the consideration is given to the TFWEs. The TFWEs is produced by using a fractional derivative of order $\alpha \in (0,2]$ in the standard wave equation in place of the first-order time derivative. Fractional derivatives are referred regarded as having the Caputo sense. This article uses the Adomian decomposition method to show the analytical results to the fractional wave equations. The explicit solutions to the equations have been provided in closed form using beginning conditions, and their numerical solutions have then been visually displayed. To illustrate how the present technique is applied, four examples are given. The simplicity and efficiency of the current approach are excellent. Danesh *et al.* [13], study classical wave equations are generalized to create spatially fractional order wave equations, which are increasingly employed to represent real-world super diffusive issues in fields like finance and fluid flow. The analytical Adomian's decomposition method (ADM) solutions to the space fractional wave equations are presented in this study. The equations' explicit solutions have been given in closed form using started conditions. To demonstrate the use of the new methodologies, two examples the first a one-dimensional fractional wave problem and the second a two-dimensional fractional hyperbolic differential equation are provided. The simplicity and efficiency of the current approach are excellent. In this article, Humaira Yasmin *et al.* [35], presented the fractional derivatives are used to analyses fractional nonlinear convection, reaction, and wave equations. An efficient method that analysis of this fractional-order suggested model combines the Aboodh transformation with the homotopy perturbation approach. To approximate these derivatives, a modified method called the homotopy perturbation transform method is developed. Both a graphic and a tabular investigation of the analytical simulation are conducted. Rasool shah *et al.* [34], discuss in this article using the novel iterative methodology and the Homotopy perturbation method. The fractional derivatives are labeled using the Caputo sense. The findings obtained with the help of the suggested methodologies can also be achieved at different fractional derivative orders. Due to the fact that we are operating within the realm of fractional calculus, we are able to get initial conditions with fractional exponents when employing the Laplace transform, which are truly both mathematically and practically realistic. Therefore, the majority of academics believed that the Riemann-Liouville definition was more precise. It is important to emphasize that these derivatives cannot be used to simulate real-world situations involving continuous probability distributions, despite the fact that they are strong mathematical tools for doing so. However, because fractional derivatives have a memory quality, modelling biological processes as well as those in mathematics, physics, and engineering has greatly benefited from the theory and applications of fractional calculus. The use of fractional-order derivatives is becoming more prevalent in research focused on simulating real-world issues. Because of this, using fractional-order derivatives to express various types of issues has become increasingly popular over the past few decades in a number of scientific and technical domains. One of the primary justifications for their employment in diverse applications is the memory impact of these derivatives as well as their nonlocal characteristics. FDEs have progressively assumed more significant roles during the last few decades in a variety of disciplines, including physics, biology, mechanics, and chemistry Adel *et al.* [23]. Fractional quantum mechanics was created more recently as a result of the expansion of FDEs' applications to quantum mechanics. Everyone is aware that it is frequently challenging to find analytical remedies for these issues. In light of this, it is crucial and beneficial to employ approximate approaches to discover the approximate solutions to these equations. To solve the different fractional equations FDM, finite element methods, spectral approaches, etc. have all been put forth. This area of study is still evolving at the moment due to the numerous applications it has in areas as diverse such as hydrology, viscoelasticity, and fluid flow. In recent decades, there has been a great deal of interest in developing efficient numerical techniques for substantially approximating FDE solutions. The last few decades have seen a remarkable advancement in spectral approaches. The main benefit of spectral approaches is their ability to produce very accurate results. The four widely used spectrum methods are tau, collocation, spectral element approaches, and spectral methods. It is obvious that the form of initial condition and the kind of differential equation or boundary conditions that regulate it affect the choice of the most effective spectral approach that is recommended for solving such differential equations Baleanu *et al.* [25]. For a variety of physical phenomena, including wave processes and damping laws, PDEs incorporating derivatives of fractional orders have proven to be suitable models. Finance, arterial science, electrochemistry, electromagnetics, and the theory of extremely slow processes are among the other uses. The graphs and tables demonstrate that the fractional-order solutions converge to an integer solution when the fractional orders get closer to the integer order of the problems. Tec plot can be used to display the aforementioned problems in tabular and graphical form. Due to the precision, ease of use, and simplicity of the provided methodologies, they can be used to resolve current non-linear fractional partial differential equations.

A group of fractional PDEs with variable coefficients on a finite domain that have initial and boundary values is studied, along with some useful numerical methods to solve them. We investigate a case where the PDEs may have a fractional spatial derivative with either a left or a right hand. The approaches' stability, consistency, and convergence are discussed. The stability and convergence results in the fractional PDE are used to merge the corresponding conclusions for the classical parabolic and hyperbolic cases into a single condition. Tadjeran *et al.* [4], Researchers in this paper take into account a model of fractional-order nerve impulses. Understanding the answers to this model's problems enables the management of the nerve impulse process. Because of the memory effect, specifically taking into account this model as fractional-order guarantees the ability to analyse in detail. In this situation, we first employ an analytical solution, and then, in order to achieve this answer, we provide numerical solutions by combining two numerical schemes. The walking wave-type solutions to the original problem are then shown. Complex hyperbolic functions, complex trigonometric functions Yavuz *et al.* [29], and algebraic functions are all part of these solutions. Also, a linear stability analysis is carried out, and the absolute error is discovered by contrasting the numerical and analytical results. This essay discusses the variations and resemblances of the mentioned solution approaches in addition to highlighting the model's precise and numerical solutions. As a result, the findings of this research are significant and helpful for both mathematicians and engineers, as well as neuroscientists and physicists.

## 2.2 Research Gap and Conceptual Framework

The current study concentrates on the two-dimensional finite domain TFWEs. For the two-dimensional TFWE, a finite difference approximation is proposed. For obtaining the numerical solution of FDEs, variety of schemes and treatments are investigated. These include the iteration method, the Adams-Bashforth Moulton method, the FDM and homotropy perturbation method. The FDM is one of the most effective strategies used in scientific computing and applied mathematics. Then, the unknown coefficients are discovered in order to declare that the exact solution is approximate with a very small error. For many different forms of FDEs, these numerical techniques are regarded as an excellent tool for numerical solution. Researchers explore the method's stability and convergence utilizing induction in mathematics. Then, a numerical example is given. The numerical result and our theoretical analysis are in perfect agreement. The numerically results are computed by using Tec plot.

In light of the research gap stated in the above sections, our main objectives in the thesis are:

- The mathematical formulation of the problem of time-fractional waves that occurs in one and two dimensions.
- Through several simulations, examine the impact of the fractional-order parameter on the wave problem.
- An analysis using numbers of how boundary conditions affect the wave problem.
- Convergence of the suggested numerical approach is being investigated.
- Analyze the theoretical stability of the time-fractional wave problem and extend the numerical solutions.
- Inclusive study of error bound, of the proposed Crank-Nicolson methods.

# CHAPTER 3

# MODIFICATION OF ONE-DIMENSIONAL HYPERBOLIC PROBLEM

## 3.1 Introduction

In the areas of anomalous wave, viscous-elasticity, control, etc. fractional calculus has attracted a lot of attention. Undoubtedly the most crucial regions of FC application is the anomalous wave model, which is one of those. In the fields of physics, chemistry, and biology, anomalous wave events are frequently encountered. The understanding of integral problems is essential in many areas of applied mathematics sinc e they naturally arise in several applications in engineering and science presented the Bao *et al.* [1]. The wave process in inhomogeneous porous media, also known as time-dependent anomalous wave, can be described using a recently discovered and promising method called the TFWEs model. The development of effective numerical techniques is urgently required to further the research of the characteristics of variable-order time fractional sub wave equation models. FDM approaches Yajun *et al.* [20], have been widely used to construct approximations for partial differential equations regulating wave propagation due to their simplicity, flexibility, and robustness. Even stable and precise systems can display this behaviour, despite the fact that waves in numerical schemes might propagate at different wave speeds than in the actual medium. When creating finite difference schemes in the past, accuracy constraints were applied. We look into how finite difference algorithms are developed and used to get smaller numerical dispersion errors.

Fractional calculus is used in a very large number of scientific and technical problems, and this number is constantly increasing. The fact that fractional derivatives offer a great method for describing memory and hereditary features of diverse materials and processes is one of the key benefits of the fractional calculus. Numerous numerical techniques have been presented for solving FDEs using various types of fractional derivative operators Shen *et al.*

[14]. Recent years have seen an increase in the number of scientists finding that a variety of important dynamical problems exhibit fractional-order behaviour that may alter across time or space. This proves that variable-order calculus is a natural candidate to provide a helpful mathematical framework for the understanding of complex dynamical problems. Variable-order differential operators have been defined in a variety of ways by various authors, each with a unique meaning tailored to achieve the desired outcomes. A variable-order nonlinear fractional wave equation approximation using explicit finite differences was studied by Chen *et al*. [9], for stability and convergence. In order to approximate the variable-order fractional advection-wave equation with a nonlinear source term, The Riemann definition, the Caputo definition, and the requirement that Current formulations of the variable order operator that have been proposed in the literature show that it yields the correct fractional derivative that relates to the argument of the functional order. For variable-order fractional differential equations, analytical solutions are more challenging to find since the variable-exponent kernel of the variable-order operators has received little attention. The growth of numerical methods for the solution of variable-order fractional differential equations is still in its infancy. Sun *et al*. [25], based on the various physical causes that would have driven the variable-order, utilized the Crank-Nicholson scheme to generate the wave curve of the variable-order differential operator model, and created a classification of the variable-order fractional wave models. Many of these authors haven't, however, covered the numerical solutions' convergence and stability. Zhuang *et al.* [5], suggested explicit and implicit Euler approximations. The numerical methods for TFWEs in a finite domain are investigated in this thesis. The Crank-Nicholson scheme is used along with other finite difference schemes. This chapter explain the discretization of one-dimensional TFWEs and check its stability, convergence and error bound.

## 3.2 Mathematical Methodology

This section constructed to elaborate the development of finite difference method using the Crank-Nicholson strategy. In order to investigating the potential solutions, we will follow the following methodology using the finite difference approach also given in diagram 3.1:

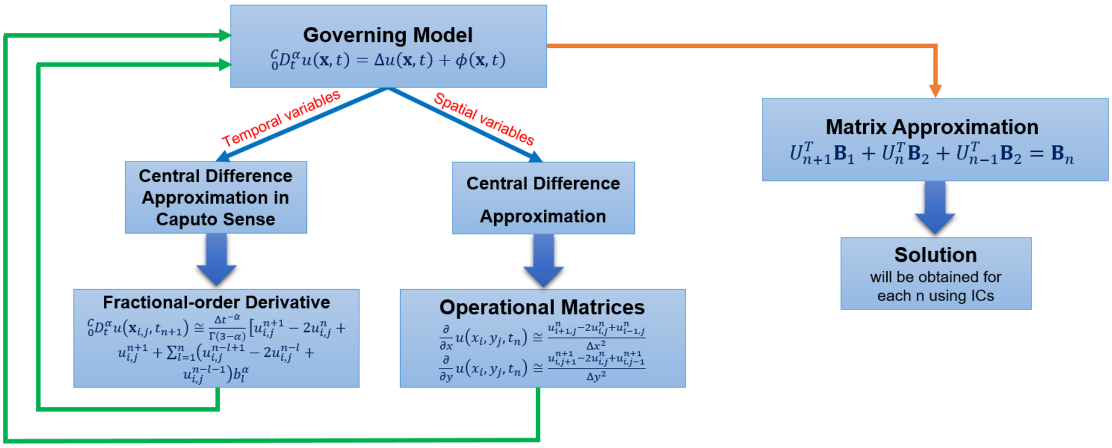


**Diagram 3.1** Working step of the methodology

The fractional order term $D_t^\alpha u(x,t)$ will be approximated by the following formula

$$ {}_0^C D_t^\alpha u(x,t) = \frac{\Delta t^{-\alpha}}{\Gamma(3-\alpha)} \left[ u_i^{n+1} - 2u_i^n + u_i^{n-1} + \sum_{k=1}^{n} \left( u_i^{n-k+1} - 2u_i^{n-k} + u_i^{n-k-1} \right) b_\alpha^k \right] + O(\Delta t^{4-\alpha}). \tag{3.1} $$

where $b_\alpha^k = ((1+k)^{2-\alpha} + k^{2-\alpha})$. Space derivative $u_{xx}$ will be approximated by

$$ \frac{\partial^2}{\partial x^2} u(x,t) = \frac{1}{2} \left[ \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} + \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2} \right] + O(\Delta x^2). \tag{3.2} $$

Similarly, the derivative w.r.t. $y$ can be approximated accordingly. Now, consider one-dimension hyperbolic fractional order partial differential equation

$$ {}_0^C D_t^\alpha u(x,t) = \frac{\partial^2 u}{\partial x^2} + \phi(x,t), a < x < b, 0 < t \le T, 1 < \alpha \le 2, \tag{3.3} $$

Beside through the subsequent initial conditions and boundary condition

$$ u(x,0) = f(x), u_t(x,0) = g(x), a \le x \le b, \tag{3.4} $$

$$ u(a,t) = h_1(t), u(b,t) = h_2(t), t > 0. \tag{3.5} $$

Putting the values from equation (3.1) and (3.2) in equation (3.3) we get;

$$ \begin{aligned} u_i^{n+1} - 2u_i^n + u_i^{n-1} + \sum_{k=1}^{n} \left( u_i^{n-k+1} - 2u_i^{n-k} + u_i^{n-k-1} \right) b_k \\ = r\left( \frac{1}{2} u_{i+1}^{n+1} - u_i^{n+1} + \frac{1}{2} u_{i-1}^{n+1} + \frac{1}{2} u_{i+1}^n - u_i^n + \frac{1}{2} u_{i-1}^n \right) \\ + \frac{1}{2} \Gamma(3-\alpha) \Delta t^\alpha \phi_i^{n+1} + \frac{1}{2} \Gamma(3-\alpha) \Delta t^\alpha \phi_i^n, \end{aligned} \tag{3.6} $$

$$
\begin{aligned}
\frac{-r}{2}u_{i-1}^{n+1} + (1+r)u_i^{n+1} - \frac{r}{2}u_{i+1}^{n+1} \\
&= \frac{r}{2}u_{i-1}^{n} + (2-r)u_i^{n} + \frac{r}{2}u_{i+1}^{n} - u_i^{n-1} \\
&- \sum_{k=1}^{n}\left(u_i^{n-k+1} - 2u_i^{n-k} + u_i^{n-k-1}\right)b_k + \frac{1}{2}\Gamma(3-\alpha)\Delta t^{\alpha}\phi_i^{n+1} \\
&+ \frac{1}{2}\Gamma(3-\alpha)\Delta t^{\alpha}\phi_i^{n}, \;\; 1 \le i \le M-1, 0 \le n \le N-1
\end{aligned} \tag{3.7}
$$

$$
\begin{aligned}
\Delta x = \frac{b-a}{M}, \Delta t = \frac{T}{N}, n = \;, x_i = a + i\Delta x, t_n = n\Delta t, r = \frac{\Gamma(3-\alpha)\Delta t^{\alpha}}{\Delta x^2}, u_i^n \\
= u(x_i, t_n), \phi_i^n = \phi(x_i, t_n), f_i = f(x_i), g_i = g(x_i), \\
h_1^n = h_1(t_n), h_2^n = h_2(t_n), b_k = (1+k)^{2-\alpha} - k^{2-\alpha}.
\end{aligned} \tag{3.8}
$$

The summation term in Eq. (3.7) can be simplified as:

$$
\begin{aligned}
\sum_{k=1}^{n}\left(u_i^{n-k+1} - 2u_i^{n-k} + u_i^{n-k-1}\right)b_k = (u_i^n b_1 - 2u_i^{n-1}b_1 + u_i^{n-2}b_1) \\
+(u_i^{n-1}b_2 - 2u_i^{n-2}b_2 + u_i^{n-3}b_2) + (u_i^{n-2}b_3 - 2u_i^{n-3}b_3 + u_i^{n-4}b_3) \\
+\left(u_i^{n-3}b_4 - 2u_i^{n-4}b_4 + u_i^{n-5}b_4\right) + \cdots + (u_i^3 b_{n-2} - 2u_i^2 b_{n-2} + u_i^1 b_{n-2}) \\
+(u_i^2 b_{n-1} - 2u_i^1 b_{n-1} + u_i^0 b_{n-1}) + (u_i^1 b_n - 2u_i^0 b_n + u_i^{-1} b_n)
\end{aligned}
$$

It can be rewritten as:

$$
\sum_{k=1}^{n}\left(u_i^{n-k+1} - 2u_i^{n-k} + u_i^{n-k-1}\right)b_k = u_i^n b_1 + \sum_{k=1}^{n-1} d_k u_i^{n-k} - 2u_i^0 b_n + u_i^{-1} b_n,
$$

where $(b_{k+1} - 2b_k + b_{k-1}) = d_k$

After rearranging the Eq. (3.7), we obtained the following discretization formula as:

$$
\begin{aligned}
\frac{-r}{2}u_{i-1}^{n+1} + (1+r)u_i^{n+1} - \frac{r}{2}u_{i+1}^{n+1} \\
&= \frac{r}{2}u_{i-1}^{n} + (2-r)u_i^{n} + \frac{r}{2}u_{i+1}^{n} - u_i^{n-1} - u_i^n b_1 - \sum_{k=1}^{n-1} d_k u_i^{n-k} \\
&+ 2u_i^0 b_n - u_i^{-1} b_n + \frac{1}{2}\Gamma(3-\alpha)\Delta t^{\alpha}\phi_i^{n+1} + \frac{1}{2}\Gamma(3-\alpha)\Delta t^{\alpha}\phi_i^{n},
\end{aligned} \tag{3.9}
$$

It can be changed into the following form:

$$
\begin{aligned}
\frac{-r}{2}u_{i-1}^{1} + (1+r)u_i^{1} - \frac{r}{2}u_{i+1}^{1} \\
&= \frac{r}{2}u_{i-1}^{0} + (2-r)u_i^{0} + \frac{r}{2}u_{i+1}^{0} - u_i^{-1} - u_i^0 b_1 + 2u_i^0 b_0 - u_i^{-1} b_0 \\
&+ \frac{1}{2}\Gamma(3-\alpha)\Delta t^{\alpha}\phi_i^{1} + \frac{1}{2}\Gamma(3-\alpha)\Delta t^{\alpha}\phi_i^{0}, \text{for } n = 0,
\end{aligned} \tag{3.10}
$$

$$\frac{-r}{2}u_{i-1}^{n+1}+(1+r)u_i^{n+1}-\frac{r}{2}u_{i+1}^{n+1}$$
$$=\frac{r}{2}u_{i-1}^{n}+(2-r)u_i^{n}+\frac{r}{2}u_{i+1}^{n}-u_i^{n-1}-u_i^{n}b_1-\sum_{k=1}^{n-1}d_k u_i^{n-k}$$
$$+2u_i^0 b_n-u_i^{-1}b_n+\frac{1}{2}\Gamma(3-\alpha)\Delta t^{\alpha}\phi_i^{n+1}+\frac{1}{2}\Gamma(3-\alpha)\Delta t^{\alpha}\phi_i^{n}, \tag{3.11}$$

for $n \geq 1$.

The discretization described above can be recast in the matrix form shown below:

$$\mathbb{A}\vec{\mathbf{U}}^{n+1}=\mathbb{B}\vec{\mathbf{U}}^{n}-\vec{\mathbf{U}}^{n-1}+\frac{1}{2}\left(\vec{\mathbf{f}}^{n}+\vec{\mathbf{f}}^{n+1}\right)+\vec{\mathbf{b}}^{n}+\vec{\mathbf{c}}^{n}\quad \text{for } n=0, \tag{3.12}$$

$$\mathbb{A}\vec{\mathbf{U}}^{n+1}=\mathbb{B}\vec{\mathbf{U}}^{n}-b_1\vec{\mathbf{U}}^{n}-(1-2b_1)\vec{\mathbf{U}}^{n-1}-b_1\vec{\mathbf{U}}^{n-2}+\frac{1}{2}\left(\vec{\mathbf{f}}^{n}+\vec{\mathbf{f}}^{n+1}\right)+\vec{\mathbf{b}}^{n}\quad for\ n=1 \tag{3.13}$$

$$\mathbb{A}\vec{\mathbf{U}}^{n+1}=\mathbb{B}\vec{\mathbf{U}}^{n}-b_1\vec{\mathbf{U}}^{n}-(1-2b_1+b_2)\vec{\mathbf{U}}^{n-1}-b_1\vec{\mathbf{U}}^{n-2}-(b_{n-1}-2b_n)\vec{\mathbf{U}}^{0}-b_n\vec{\mathbf{U}}^{-1}$$
$$+\frac{1}{2}\left(\vec{\mathbf{f}}^{n}+\vec{\mathbf{f}}^{n+1}\right)+\vec{\mathbf{b}}^{n}\quad for\ n=2 \tag{3.14}$$

$$\mathbb{A}\vec{\mathbf{U}}^{n+1}=\mathbb{B}\vec{\mathbf{U}}^{n}-b_1\vec{\mathbf{U}}^{n}-(1-2b_1+b_2)\vec{\mathbf{U}}^{n-1}-\sum_{k=1}^{n-2}(b_{k+2}-2b_{k+1}+b_k)\vec{\mathbf{U}}^{n-k-1}$$
$$-(b_{n-1}-2b_n)\vec{\mathbf{U}}^{0}-b_n\vec{\mathbf{U}}^{-1}+\frac{1}{2}\left(\vec{\mathbf{f}}^{n}+\vec{\mathbf{f}}^{n+1}\right)+\vec{\mathbf{b}}^{n}\quad for\ n\geq 3 \tag{3.15}$$

$$\mathbb{A}=\begin{bmatrix}1+r & \frac{-r}{2} & 0 & \cdots & 0 & 0\\ \frac{-r}{2} & 1+r & \frac{-r}{2} & \cdots & 0 & 0\\ 0 & \frac{-r}{2} & 1+r & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 1+r & \frac{-r}{2}\\ 0 & 0 & 0 & \cdots & \frac{-r}{2} & 1+r\end{bmatrix},$$

$$\mathbb{B}=\begin{bmatrix}2-r & \frac{r}{2} & 0 & \cdots & 0 & 0\\ \frac{r}{2} & 2-r & \frac{r}{2} & \cdots & 0 & 0\\ 0 & \frac{r}{2} & 2-r & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 2-r & \frac{r}{2}\\ 0 & 0 & 0 & \cdots & \frac{r}{2} & 2-r\end{bmatrix},$$

$$\vec{\mathbf{f}}^{n}=\begin{bmatrix}\Gamma(3-\alpha)\Delta t^{\alpha}\phi_1^n\\ \Gamma(3-\alpha)\Delta t^{\alpha}\phi_2^n\\ \Gamma(3-\alpha)\Delta t^{\alpha}\phi_3^n\\ \vdots\\ \Gamma(3-\alpha)\Delta t^{\alpha}\phi_{M-2}^n\\ \Gamma(3-\alpha)\Delta t^{\alpha}\phi_{M-1}^n\end{bmatrix},\vec{\mathbf{U}}^{n}=\begin{bmatrix}u_1^n\\ u_2^n\\ u_3^n\\ \vdots\\ u_{M-2}^n\\ u_{M-1}^n\end{bmatrix},\vec{\mathbf{b}}^{n}=\begin{bmatrix}r\theta u_0^n+r\theta u_0^{n+1}\\ 0\\ 0\\ \vdots\\ 0\\ r\theta u_M^n+r\theta u_M^{n+1}\end{bmatrix},\vec{\mathbf{c}}^{n}=\begin{bmatrix}2\Delta t g_1\\ 2\Delta t g_2\\ 2\Delta t g_3\\ \vdots\\ 2\Delta t g_{M-2}\\ 2\Delta t g_{M-1}\end{bmatrix}$$

## 3.3 Stability Analysis

The roundoff error equation is expressed as follows:

$$-ru_{i-1}^{1}+(1+2r)u_i^1-ru_{i+1}^{1}=ru_{i-1}^{0}+(2-2r)u_i^0+ru_{i+1}^{0}-u_i^{-1}\ \text{ for } n=0, \quad (3.16)$$

$$-ru_{i-1}^{n+1}+(1+2r)u_i^{n+1}-ru_{i+1}^{n+1}=ru_{i-1}^{n}-2ru_i^n+ru_{i+1}^{n}+(2-b_1)u_i^n+\sum_{k=1}^{n-1}u_i^{n-k}d_k+2u_i^0b_n-u_i^{-1}b_n \qquad n\geq 1 \quad (3.17)$$

We suppose that the solution of the above equation has the following form $u_p^k=\delta_k e^{i\beta hp}$, from equation (3.16)

$$-r\delta_1 e^{i\beta h(p-1)}+(1+2r)\delta_1 e^{i\beta hp}-r\delta_1 e^{i\beta h(p+1)}=r\delta_0 e^{i\beta h(p-1)}+(2-2r)\delta_0 e^{i\beta hp}+r\delta_0 e^{i\beta h(p+1)}-\delta_{-1}e^{i\beta hp}, \quad (3.18)$$

$$e^{i\beta hp}\left[-re^{-i\beta h}+(1+2r)-re^{i\beta h}\right]\delta_1=e^{i\beta hp}\left[re^{-i\beta h}+(2-2r)+re^{i\beta h}\right]\delta_0-\delta_{-1}e^{i\beta hp}, \quad (3.19)$$

$$\left[-2r\{\frac{e^{-i\beta h}+e^{i\beta h}}{2}\}+(1+2r)\right]\delta_1=\left[2r\{\frac{e^{-i\beta h}+e^{i\beta h}}{2}\}+(2-2r)\right]\delta_0-\delta_{-1}, \quad (3.20)$$

$$[-2rcos\beta h+1+2r]\delta_1=[2rcos\beta h+2-2r]\delta_0-\delta_{-1}, \quad (3.21)$$

$$[1+2r(1-cos\beta h)]\delta_1=[2-2r(1-cos\beta h)]\delta_0-\delta_{-1}, \quad (3.22)$$

$$\left[1+2r(2sin^2\frac{\beta h}{2})\right]\delta_1=\left[2-2r(2sin^2\frac{\beta h}{2})\right]\delta_0-\delta_{-1}, \quad (3.23)$$

$$\left[1+4rsin^2(\frac{\beta h}{2})\right]\delta_1=\left[2-4rsin^2(\frac{\beta h}{2})\right]\delta_0-\delta_{-1} \qquad \therefore k=0, \quad (3.24)$$

**Lemma: 1** In **[16]** the coefficients r and $d_k$ have the following properties,

r > 0, 0 < $b_k$ < $b_{k-1}$ $\leq 1$ , $\qquad \forall\ k=1,2,\dots N,$

0 < $d_k$ < 1, $\sum_{k=0}^{n-1}d_k=1-b_k$.

**Lemma: 2** In **[16]** assume that $\delta_k(k=1,2\dots,N-1)$ is the solution, then we have

$$|\delta_k|\leq|\delta_0|$$

Proof: we use mathematical induction to achieve the proof method. When k=0, we get

$$\delta_1=\frac{\left[2-4rsin^2(\frac{\beta h}{2})\right]\delta_0-\delta_{-1}}{\left[1+4rsin^2(\frac{\beta h}{2})\right]}=\frac{\left[2-4rsin^2(\frac{\beta h}{2})\right]\delta_0}{\left[1+4rsin^2(\frac{\beta h}{2})\right]}-\frac{\delta_{-1}}{\left[1+4rsin^2\left(\frac{\beta h}{2}\right)\right]}, \quad (3.25)$$

$$|\delta_1|\leq\frac{\left[2-4rsin^2\left(\frac{\beta h}{2}\right)\right]}{\left[1+4rsin^2\left(\frac{\beta h}{2}\right)\right]}|\delta_0|-\frac{1}{\left[1+4rsin^2\left(\frac{\beta h}{2}\right)\right]}|\delta_{-1}|, \quad (3.26)$$

$$|\delta_1|\leq\frac{2-4rsin^2\left(\frac{\beta h}{2}\right)-1}{\left[1+4rsin^2\left(\frac{\beta h}{2}\right)\right]}|\delta_0|, \quad (3.27)$$

$$|\delta_1| \leq \frac{1-4rsin^2\left(\frac{\beta h}{2}\right)}{\left[1+4rsin^2\left(\frac{\beta h}{2}\right)\right]}|\delta_0|, \tag{3.28}$$

$$|\delta_k| \leq |\delta_0|. \tag{3.29}$$

from Equation (3.17)

$$-r\delta_{k+1}e^{i\beta h(p-1)} + (1+2r)\delta_{k+1}e^{i\beta hp} - r\delta_{k+1}e^{i\beta h(p+1)} = r\delta_k e^{i\beta h(p-1)} + (2-b_1-2r)\delta_k e^{i\beta hp} + r\delta_k e^{i\beta h(p+1)} + \sum_{k=1}^{n-1}\delta_{k-j}e^{i\beta hp}d_k + 2b_n\delta_0 e^{i\beta hp} - b_n\delta_{-1}e^{i\beta hp}, \tag{3.30}$$

$$\delta_{k+1}\left[1+2r-re^{i\beta h}-re^{-i\beta h}\right] = \delta_k\left[re^{-i\beta h}+(2-b_1-2r)+re^{i\beta h}\right] + \sum_{k=1}^{n-1}\delta_{k-j}d_k + 2b_n\delta_0 - b_n\delta_{-1}\,. \tag{3.31}$$

After simplifying we have

$$\delta_{k+1}[1+4rsin^2(\tfrac{\beta h}{2})] = \delta_k[2-b_1-4rsin^2(\tfrac{\beta h}{2})] + \sum_{k=1}^{n-1}\delta_{k-j}d_k + 2b_n\delta_0 - b_n\delta_{-1}. \tag{3.32}$$

The above equation can be written as

$$\delta_{k+1} = \frac{\delta_k[2-b_1-4rsin^2(\frac{\beta h}{2})] + \sum_{k=1}^{n-1}\delta_{k-j}d_k + 2b_n\delta_0 - b_n\delta_{-1}}{1+4rsin^2(\frac{\beta h}{2})}, \tag{3.33}$$

next, we let

$|\delta_n| \leq |\delta_0|$ (n=2, 3, …. k),

$$|\delta_{k+1}| \leq \left|\frac{\delta_k[2-b_1-4rsin^2(\frac{\beta h}{2})]+\sum_{k=1}^{n-1}\delta_{k-j}d_k+2b_n\delta_0-b_n\delta_{-1}}{1+4rsin^2(\frac{\beta h}{2})}\right|, \tag{3.34}$$

$$|\delta_{k+1}| \leq \left|\frac{-4rsin^2(\frac{\beta h}{2})\delta_k+\sum_{k=0}^{n-1}\delta_{k-j}d_k+2b_n\delta_0-b_n\delta_{-1}}{1+4rsin^2(\frac{\beta h}{2})}\right|, \tag{3.35}$$

$$|\delta_{k+1}| \leq \frac{4rsin^2\left(\frac{\beta h}{2}\right)|\delta_k|+\sum_{k=0}^{n-1}d_k|\delta_{k-j}|+2b_n|\delta_0|-b_n|\delta_{-1}|}{1+4rsin^2\left(\frac{\beta h}{2}\right)}, \tag{3.36}$$

$$|\delta_{k+1}| \leq \frac{4rsin^2\left(\frac{\beta h}{2}\right)+\sum_{k=0}^{n-1}d_k+2b_n-b_n}{1+4rsin^2(\frac{\beta h}{2})}|\delta_0|, \tag{3.37}$$

$$|\delta_{k+1}| \leq \frac{4rsin^2\left(\frac{\beta h}{2}\right)+\sum_{k=0}^{n-1}d_k+b_n}{1+4rsin^2(\frac{\beta h}{2})}|\delta_0|, \tag{3.38}$$

$$|\delta_{k+1}| \leq \frac{1+4rsin^2\left(\frac{\beta h}{2}\right)}{1+4rsin^2\left(\frac{\beta h}{2}\right)}|\delta_0| \qquad \therefore using\ lemma\ 1(b), \tag{3.39}$$

$$|\delta_{k+1}| \leq |\delta_0| \tag{3.40}$$

Hence completes the proof.

## 3.4 Convergence Analysis

Consider, equation (3.10) and (3.11)

$$-ru_{i-1}^{1}+(1+2r)u_{i}^{1}-ru_{i+1}^{1}=ru_{i-1}^{0}+(2-2r)u_{i}^{0}+ru_{i+1}^{0}-u_{i}^{-1}+\phi_{i} \quad \text{for} \quad n=0, \tag{3.41}$$

$$-ru_{i-1}^{n+1}+(1+2r)u_{i}^{n+1}-ru_{i+1}^{n+1}=ru_{i-1}^{n}-2ru_{i}^{n}+ru_{i+1}^{n}+(2-b_{1})u_{i}^{n}$$
$$+\sum_{k=1}^{n-1}u_{i}^{n-k}d_{k}+2u_{i}^{0}b_{n}-u_{i}^{-1}b_{n}+\phi_{i}^{n} \quad \text{for} \quad n\geq 1. \tag{3.42}$$

Let $u(x_i,t_k)$ , $(\mathrm{i}=\ 1,2,\ldots\ldots,\mathrm{m}-1;\ \mathrm{k}=\ 1,2,\ldots\ldots,\mathrm{n})$ be the precise answer of the TFWE at mesh point $(x_i,t_k)$. Then express as,

$$e_{i}^{k}=u(x_{i},t_{k})-u_{i}^{k}, \quad i=1,2,\ldots\ldots,m-1; \quad k=\ 1,2,\ldots\ldots,n,$$
$$e^{k}=\left(e_{1}^{k},e_{2}^{k},e_{3}^{k},\ldots\ldots,e_{m-1}^{k}\right)^{T},$$

using $e^{0}=0$, therefore we have to

$$-re_{i-1}^{1}+(1+2r)e_{i}^{1}-re_{i+1}^{1}+e_{i}^{-1}=R_{i}^{1} \quad \text{for} \quad \mathrm{n}=0, \tag{3.43}$$

$$-re_{i-1}^{n+1}+(1+2r)e_{i}^{n+1}-re_{i+1}^{n+1}+e_{i}^{-1}b_{n}=re_{i-1}^{n}-2re_{i}^{n}+re_{i+1}^{n}$$
$$+(2-b_{1})e_{i}^{n}+\sum_{k=1}^{n-1}e_{i}^{n-k}d_{k}+R_{i}^{n+1} \quad \text{for} \quad n>0, \tag{3.44}$$

where,

$$R_{i}^{n+1}=u(x_{i},t_{n+1})-2u(x_{i},t_{n})+u(x_{i},t_{n-1})$$
$$+\sum_{k=1}^{n-1}b_{k}\{\,u(x_{i},t_{n-k+1})-2u(x_{i},t_{n-k})+u(x_{i},t_{n-k-1})\}$$
$$-r\{u(x_{i+1},t_{n})-2u(x_{i},t_{n})+u(x_{i-1},t_{n})+u(x_{i+1},t_{n+1})-2u(x_{i},t_{n+1})$$
$$+u(x_{i-1},t_{n+1})\}. \tag{3.45}$$

$$R_{i}^{n+1}=\sum_{k=0}^{n}b_{k}\{\,u(x_{i},t_{n-k+1})-2u(x_{i},t_{n-k})+u(x_{i},t_{n-k-1})\}-\mathrm{r}\{u(x_{i+1},t_{n})-2u(x_{i},t_{n})+u(x_{i-1},t_{n})+u(x_{i+1},t_{n+1})-2u(x_{i},t_{n+1})+u(x_{i-1},t_{n+1})\}\ . \tag{3.45}$$

Consider,

$$D_{t}^{\alpha}u(x,t)+C_{1}(\tau)=\frac{\Delta t^{-\alpha}}{\Gamma(3-\alpha)}\left[\sum_{k=0}^{n}\left(u_{i}^{n-k+1}-2u_{i}^{n-k}+u_{i}^{n-k-1}\right)b_{k}\right], \tag{3.46}$$

and

$$\frac{\partial^{2}}{\partial x^{2}}u(x,t)+C_{2}(\Delta x^{2})=\frac{1}{2\Delta x^{2}}\{u_{i+1}^{n}-2u_{i}^{n}+u_{i-1}^{n}+u_{i+1}^{n+1}-2u_{i}^{n+1}+u_{i-1}^{n+1}\}, \tag{3.47}$$

therefore,

$$R_i^{n+1} = \Delta t^{\alpha}\Gamma(3-\alpha)\left[D_t^{\alpha}u(x,t) - \frac{\partial^2}{\partial x^2}u(x,t)\right] + C_1(\tau^{1+\alpha}) + C_2(\tau^{\alpha}\Delta x^2). \tag{3.48}$$

Also $\left|R_i^{n+1}\right| \le C(\tau^{1+\alpha} + \tau^{\alpha}\Delta x^2),\ \mathrm{i} =\ 1,2,\ldots\ldots,\mathrm{m}-1,\ \mathrm{k} = 0,1,2,\ldots\ldots,\mathrm{n}$

where C is a constant term,

**Proposition**: In **[5]** $\|e^k\|_\infty \le Cb_{k-1}^{-1}(\tau^{1+\alpha} + \tau^{\alpha}\Delta x^2),\ k = 1,2,\ldots\ldots,n$ , where $\|e\|_\infty = \max\limits_{1\le i\le m-1}\left|e_i^k\right|$ and C is a constant value.

**Proof:** Applying the mathematical induction method. For$K = 1$ , let$\|e^1\|_\infty = \left|e_i^1\right| = \max\limits_{1\le i\le m-1}\left|e_i^1\right|$, we have

$$\left|e_i^1\right| \le -r\left|e_{i-1}^1\right| + (1+2r)\left|e_i^1\right| - r\left|e_{i+1}^1\right| + \left|e_i^{-1}\right|, \tag{3.49}$$

$$\left|e_i^1\right| \le \left|-re_{i-1}^1 + (1+2r)e_i^1 - re_{i+1}^1 + e_i^{-1}\right| = R_i^1, \tag{3.50}$$

$$\le Cb_0^{-1}(\tau^{1+\alpha} + \tau^{\alpha}\Delta x^2). \tag{3.51}$$

Suppose that

$$\left\|e^{j-1}\right\|_\infty \le Cb_j^{-1}(\tau^{1+\alpha} + \tau^{\alpha}\Delta x^2), j = 1,2,\ldots,K \textit{ and } \left|e_i^{K+1}\right| = \max\limits_{1\le i\le m-1}\left|e_i^{K+1}\right|. \tag{3.52}$$

Note that $b_j^{-1} \le b_k^{-1}, j = 0,1,2,\ldots k,$ we have

$$\left|e_i^{K+1}\right| \le -r\left|e_{i-1}^{K+1}\right| + (1+2r)\left|e_i^{K+1}\right| - r\left|e_{i+1}^{K+1}\right| + b_n\left|e_i^{-1}\right|, \tag{3.53}$$

$$\left|e_i^{K+1}\right| \le \left|-re_{i-1}^{k+1} + (1+2r)e_i^{k+1} - re_{i+1}^{k+1} + e_i^{-1}b_n\right|, \tag{3.54}$$

$$= \left|re_{i-1}^k - 2re_i^k + re_{i+1}^k + (2-b_1)e_i^k + \sum_{j=1}^{k-1} e_i^{k-j}d_k + R_i^{n+1}\right|, \tag{3.55}$$

$$\le \left|re_{i-1}^k - 2re_i^k + re_{i+1}^k + (2-b_1)e_i^k + \sum_{j=1}^{k-1} e_i^{k-j}d_j\right| + \left|R_i^{n+1}\right|, \tag{3.56}$$

$$\le r\left|e_{i-1}^K\right| + (2-b_1-2r)\left|e_i^K\right| + r\left|e_{i+1}^K\right| + \sum_{j=1}^{k-1} d_j\left|e_i^{K-j}\right| + C(\tau^{1+\alpha} + \tau^{\alpha}\Delta x^2), \tag{3.57}$$

$$\le C_1\|e^k\|_\infty + C_2\|e^k\|_\infty + C_3\|e^k\|_\infty + \sum_{j=1}^{k-1} d_j\left\|e^{k-j}\right\|_\infty + C(\tau^{1+\alpha} + \tau^{\alpha}\Delta x^2), \tag{3.58}$$

$$\le \{\ C_1 + C_2 + C_3 + \sum_{j=1}^{k-1} d_j + b_k\}b_k^{-1}C(\tau^{1+\alpha} + \tau^{\alpha}\Delta x^2), \tag{3.59}$$

$$= b_k^{-1}C(\tau^{1+\alpha} + \tau^{\alpha}\Delta x^2). \tag{3.60}$$

Because

$$\lim_{n\to\infty}\frac{b_k^{-1}}{k^{\alpha}} = \lim_{n\to\infty}\frac{k^{-\alpha}}{(K+1)^{1-\alpha} - k^{1-\alpha}}, \tag{3.61}$$

$$= \lim_{n\to\infty}\frac{k^{-1}}{(1+\frac{1}{k})^{1-\alpha} - 1}, \tag{3.62}$$

$$= \lim_{n\to\infty}\frac{k^{-1}}{(1-\alpha)k^{-1}}, = \frac{1}{1-\alpha}. \tag{3.63}$$

Hence, there is a constant C.

$$\|e^k\|_\infty \leq Ck^\alpha(\tau^{1+\alpha} + \tau^\alpha \Delta x^2). \tag{3.64}$$

Therefore $k\tau \leq T$ is finite, the outcome is as follows.

**Theorem:** In **[5]** let $u_i^k$ be the calculated approximate value of $u(x_i, t_k)$ using the difference approach. In that case, a positive constant C exists such that

$$\left|u_i^k - u(x_i, t_k)\right| \leq C(\tau + \Delta x^2), \;\; \mathrm{i} = 1, 2 \dots \mathrm{m} - 1; \; \mathrm{k} = 1, 2 \dots \mathrm{n}. \tag{3.65}$$

## 3.5 Error Bound

Consider the equation (3.11) we get,

$$+ru_{i-1}^{n+1} - (1+2r)u_i^{n+1} + ru_{i+1}^{n+1} + ru_{i-1}^n + (2 - b_1 - 2r)u_i^n + ru_{i+1}^n + 2u_i^0 b_n - u_i^{-1} b_n + \sum_{k=1}^{n-1} u_i^{n-k} d_k = 0 \,, \tag{3.66}$$

Expand $u_{i-1,j+1}, u_{i,j+1}, u_{i+1,j+1}, u_{i-1,j}, u_{i+1,j}$ and $u_{i,j-k}$ by Taylor's series expansion

$$\begin{aligned}
\mathbb{T}_{i,j} = r\Bigg(&\left((u)_{i,j} - h(u_x)_{i,j} + \tfrac{1}{2!}h^2(u_{xx})_{i,j} - \tfrac{1}{3!}h^3(u_{xxx})_{i,j} + \cdots\right) - \\
&k\left(\left(u_y\right)_{i,j} - h\left(u_{yx}\right)_{i,j} + \tfrac{1}{2!}h^2(u_{yxx})_{i,j} - \tfrac{1}{3!}h^3(u_{yxxx})_{i,j} + \cdots\right) + \\
&\tfrac{1}{2}k^2\left(\left(u_{yy}\right)_{i,j} - h\left(u_{yyx}\right)_{i,j} + \tfrac{1}{2!}h^2\left(u_{yyxx}\right)_{i,j} - \tfrac{1}{3!}h^3\left(u_{yyxxx}\right)_{i,j} + \cdots\right) - \\
&\tfrac{1}{3!}k^3\left(\left(u_{yyy}\right)_{i,j} - h\left(u_{yyyx}\right)_{i,j} + \tfrac{1}{2!}h^2\left(u_{yyyxx}\right)_{i,j} - \tfrac{1}{3!}h^3\left(u_{yyyxxx}\right)_{i,j} + \cdots\right) + \\
&\cdots\Bigg) - (1-2r)\left((u)_{i,j} + k\left(u_y\right)_{i,j} + \tfrac{1}{2!}k^2\left(u_{yy}\right)_{i,j} + \tfrac{1}{3!}k^3\left(u_{yyy}\right)_{i,j} + \cdots\right) + \\
&r\Bigg(\left((u)_{i,j} + h(u_x)_{i,j} + \tfrac{1}{2!}h^2(u_{xx})_{i,j} + \tfrac{1}{3!}h^3(u_{xxx})_{i,j} + \cdots\right) + k\left(\left(u_y\right)_{i,j} + \right. \\
&\left. h\left(u_{yx}\right)_{i,j} + \tfrac{1}{2!}h^2\left(u_{yxx}\right)_{i,j} + \tfrac{1}{3!}h^3\left(u_{yxxx}\right)_{i,j} + \cdots\right) + \tfrac{1}{2}k^2\left(\left(u_{yy}\right)_{i,j} + \right. \\
&\left. h\left(u_{yyx}\right)_{i,j} + \tfrac{1}{2!}h^2\left(u_{yyxx}\right)_{i,j} + \tfrac{1}{3!}h^3\left(u_{yyxxx}\right)_{i,j} + \cdots\right) + \tfrac{1}{3!}k^3\left(\left(u_{yyy}\right)_{i,j} + \right. \\
&\left. h\left(u_{yyyx}\right)_{i,j} + \tfrac{1}{2!}h^2\left(u_{yyyxx}\right)_{i,j} + \tfrac{1}{3!}h^3\left(u_{yyyxxx}\right)_{i,j} + \cdots\right) + \cdots\Bigg) + r\left((u)_{i,j} - \right. \\
&\left. h(u_x)_{i,j} + \tfrac{1}{2!}h^2(u_{xx})_{i,j} - \tfrac{1}{3!}h^3(u_{xxx})_{i,j} + \cdots\right) + (2 - b_1 - 2r)(u)_{i,j} + \\
&r\left((u)_{i,j} + h(u_x)_{i,j} + \tfrac{1}{2!}h^2(u_{xx})_{i,j} + \tfrac{1}{3!}h^3(u_{xxx})_{i,j} + \cdots\right) + 2b_n(u)_{i,0} - \\
&b_n(u)_{i,-1} + \sum_{k=1}^{n-1} d_k\left((u)_{i,j} - nk\left(u_y\right)_{i,j} + \tfrac{1}{2!}(nk)^2\left(u_{yy}\right)_{i,j} - \right. \\
&\left. \tfrac{1}{3!}(nk)^3\left(u_{yyy}\right)_{i,j} + \cdots\right).
\end{aligned} \tag{3.67}$$

After simplification we get,

$$\mathbb{T}_{i,j} = -(u)_{i,j} + b_1(u)_{i,j} - 2b_n(u)_{i,0} + b_n(u)_{i,-1} - 2h^2(u_{xx})_{i,j} - rh^2\left(u_{yxx}\right)_{i,j} - \frac{r}{2!}(kh)^2\left(u_{yyxx}\right)_{i,j} - \frac{r}{3!}k^3h^2\left(u_{yyyxx}\right)_{i,j} + ku_y + \frac{1}{2!}k^2\left(u_{yy}\right)_{i,j} + \frac{1}{3!}k^3\left(u_{yyy}\right)_{i,j} - \sum_{k=1}^{n-1} d_k\left((u)_{i,j} - nk\left(u_y\right)_{i,j} + \frac{1}{2!}(nk)^2\left(u_{yy}\right)_{i,j} - \frac{1}{3!}(nk)^3\left(u_{yyy}\right)_{i,j} + \cdots\right) + O(h^3) + O(kh)^3 + O(k^5h^4) + O(k^4) + O(nk)^4. \quad (3.68)$$

Hence, the order of the Local truncation error at the point $(ih, jk)$ is $O(h^3) + O(kh)^3 + O(k^5h^4) + O(k^4) + O(nk)^4$.

## <u>Algorithm: 1</u>

1: $\mathrm{a, b, T, M, N} \in N$ , $U(x,t)$ $1 < \alpha \leq 2$ (Input data)
2: $f \coloneqq fracdiff(U(x,t), t, \alpha) - \frac{\partial^2}{\partial x^2}U(x,t);$ (Evaluation of function)
3: for i from 0 by 1 while i ≤ M do;
  $x[i] \coloneqq a + \Delta x i;$ end do: (Step size in x direction)
4: for n from 0 by 1 while n ≤ N do;
  $t[n] \coloneqq \Delta t n;$ end do: (Step size in time direction)
5: for i from 0 by 1 while i ≤ M do;
  $u[i,0] \coloneqq eval(U(x,t), [x = x[i], t = 0]);$ (Evaluation of initial conditions)
  $u[i,-1] \coloneqq u[i,1] - \Delta t.eval(U(x,t), [x = x[i], t = 0]);$
  end do:
6: for n from 0 by 1 while n ≤ N do;
  $u[0,n] \coloneqq eval(U(x,t), [x = a, t = t[n]]);$ (Evaluation of boundary conditions)
  $u[M,n] \coloneqq eval(U(x,t), [x = b, t = t[n]]);$
  end do:
7: for n from 0 by 1 while n ≤ N-1 do;
  for i from 0 by 1 while i ≤ M-1 do;
  (Evaluation of equation 3.6)
  end do;
8: Sol[n+1]: =fsolve ({Eq [i1, n] $ i1=1...M-1}); assign(op(sol[n+1]);
  end do:

# CHAPTER 4

# MODIFICATION OF TWO-DIMENSIONAL HYPERBOLIC PROBLEM

In this chapter, the understanding of the fractional differential problem is essential in many areas of applied mathematics since it naturally arises in several applications in engineering and science. Calculus that uses fractions or non-integers agreements with integrals and derivatives of any real or complex order Jahanshahi *et al.* [27]. This topic was the result of a famous scientific debate between L'Hopital and Leibniz in 1695, which many eminent mathematicians, including Euler, Laplace, Abel, Liouville, and Riemann, later explored and expanded. Numerous scientists have studied the topic, not just in mathematics but also in physics and engineering. As a result, in recent years, it has gained attention. The concept of a derivative is widened by fractional calculus in circumstances where the order of the derivative is not an integer. Despite the fact that the notion of fractional derivatives and integrals can be seen as a generalisation of the corresponding conventional ones, it is nevertheless a very strange and challenging subject. Therefore, this mathematical instrument may occasionally be deemed to be far from reality. However, since many physical events are intrinsically described by fractional orders, fractional order calculus is required to fully understand them. It is crucial to remember that there are only two main definitions of the fractional derivative: the first, proposed by Riemann, is the derivative of the convolution of a given function and a power law kernel, and the second, proposed by Caputo, is the convolution of the local derivative of a given function with a power law function. Current discussion among academics in this field has made it difficult to determine which definition is mathematically well-formulated. As a result, many have conducted theoretical and applied investigations to undoubtedly prove that definition. Because the Caputo allows for typical initial circumstances when experimenting with integral transforms, such as the Laplace transform Sousa *et al.* [19], some applied mathematicians have proposed that it is useful for real-world problems. The use of FPDEs in mathematical modelling has also has attracted a lot of interest recently. The numerical solution of FPDEs has only received a few suggestions for algorithms. These techniques include the spectrum approach, variational iteration method, homotopy analysis method, generalised differential transform method, homotopy perturbation method, Jacobi-Tau approximation method, and finite difference method Danesh *et al,* [13]. Integral equation theory and application are significant topics in applied math. Additionally, there is a close connection between differential and integral equations, and various issues that are employed in the modelling of numerous physical and chemical processes can be written in either way. The behaviour of the process under investigation is often considered to depend only on its current state in the mathematical description of a physical process; this assumption is supported by a large class of dynamical systems. When this assumption is not true, however, the use of a classical model in systems analysis and their design may lead to inferior performance. In these circumstances, it is best to keep in mind that the system's behaviour also contains details about the preceding state. Time delay systems are what these are known as. In terms of the values of the function at earlier times, the fractional derivative of an unknown function at a given time is represented. Another modification is to suppose that the order of fractional derivatives and integrals is not constant because it can take on any value. This provides an extension of the conventional fractional calculus known as variable-order FC.

In order to simulate continuum mechanics, wave, heat conduction, geophysics, magnetism, electricity, neutron transport, and many other phenomena, fractional differential problems are required Chen *et al.* [9]. Many boundary-value problems (BVPs) and initial-value problems (IVPs), related to ordinary differential equations and partial differential equations, respectively, can be converted into integral problems. Due to the fact that the solution of inverse BVPs with fractal curves as their domains may be investigated, which is something that the classical calculus is unable to perform, the singular and weakly singular integral issues are of particular significance. We get the fractional-order differential problems of the Abel and other types when we model. These applications of the singular problem Bekiros *et al.* [5], have made the numerical solutions of such problems a popular issue among researchers. Several methods have been developed for this purpose, including finite difference Chen *et al.* [16], Laplace transforms, product integration, collocation type, Homotopy perturbation Yasmin *et al.* [35], semi-analytical, spline Abbas *et al.* [24], etc. The integral modeling issues frequently lack a smooth kernel. Due to the dependence of approximate technique convergence on the smoothness of the integral equation solution, it is difficult to examine the solution and estimate it numerically in this case.

As a result, the standard approaches did not work effectively under these circumstances. FDEs have been used to formulate a significant number of applied problems, and as a result, their solutions have received a great deal of attention. The anomalous wave or dispersion observed in many problems is modelled using fractional space derivatives. Sousa *et al.* [19], several papers on fractional calculus are connected to wave issues. There are more and more publications in the literature that use numerical methods to analyze various fractional wave models. The fractional wave equation explaining super wave has recently become the subject of numerical solutions. Mathematical strategies that do not always require second-order discretization for the fractional derivative to attain second-order precision have been used to produce numerical methods for models with super wave.

The difficult issue for scientists is to create a numerical approach that is quick, precise, and efficient. The following fractional order hyperbolic partial differential model is solved in the chapter using a finite difference scheme and the Crank-Nicolson method.

.

$$ {}_{0}^{C}D_t^{\alpha}u(x,y,t) = \Delta u(x,y,t) + \phi\big(x,y,t,u(x,y,t)\big), 1 < \alpha \leq 2, t > 0, x \in [0,1], \mathrm{y} \in [0,1]. \tag{4.1} $$

Besides through the subsequent initial conditions and boundary condition

$$ u(x,\mathrm{y},0) = \mathrm{f}(x,y), \quad u_t(x,\mathrm{y},0) = \mathrm{g}(x,y), \quad 0 \leq x \leq 1, \quad 0 \leq \mathrm{y} \leq 1, \tag{4.2} $$

$$ u(x,\mathrm{y},t) = \mathrm{h}(x,\mathrm{y},t),\ x \in \partial\Omega \times [0,1],\ \mathrm{y} \in \partial\Omega \times [0,1],\ 0 \leq \mathrm{t} \leq \mathrm{T}, \tag{4.3} $$

${}_{0}^{C}D_t^{\alpha}$, Represent the fractional order Caputo derivative, $\Delta u(x,y,t)$ Symbolize partial derivatives and $\phi\big(x,y,t,u(x,t)\big)$, is called source term.

## 4.1 Mathematical Method

It is aimed to analyze the two-dimensional time-fractional hyperbolic partial differential equations, Moreover, the concentration equation under the effect of source term with initial and boundary value problem. By considering the above assumptions, the governing TFWEs are.

$$ D_t^{\alpha}u(x,y,t) = \frac{\partial^2}{\partial x^2}u(x,y,t) + \frac{\partial^2}{\partial y^2}u(x,y,t) + \phi(x,y,t), \quad 1 < \alpha \leq 2 \quad \text{for } x \in [0,1], \quad y \in [0,1], t > 0\ . \tag{4.4} $$

Since

$$D_t^\alpha u(x,y,t) = \frac{\Delta t^{-\alpha}}{\Gamma(3-\alpha)}\left[u_{i,j}^{n+1} - 2u_{i,j}^{n} + u_{i,j}^{n-1} + \sum_{k=1}^{n}\left(u_{i,j}^{n-k+1} - 2u_{i,j}^{n-k} + u_{i,j}^{n-k-1}\right)\left((1+k)^{2-\alpha} + k^{2-\alpha}\right)\right] + O(\Delta t^{4-\alpha}). \quad (4.5)$$

Using crank-Nicolson scheme

$$\frac{\partial^2}{\partial x^2}u(x,y,t) = \frac{1}{2}\left[\frac{u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n}}{\Delta x^2} + \frac{u_{i+1,j}^{n+1} - 2u_{i,j}^{n+1} + u_{i-1,j}^{n+1}}{\Delta x^2}\right] + O(\Delta x^2), \quad (4.6)$$

$$\frac{\partial^2}{\partial y^2}u(x,y,t) = \frac{1}{2}\left[\frac{u_{i,j-1}^{n} - 2u_{i,j}^{n} + u_{i,j+1}^{n}}{\Delta y^2} + \frac{u_{i,j-1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j+1}^{n+1}}{\Delta y^2}\right] + O(\Delta y^2). \quad (4.7)$$

Putting the values from equation (4.5), (4.6) and (4.7) in equation (4.4) we get

$$\frac{\Delta t^{-\alpha}}{\Gamma(3-\alpha)}\left[u_{i,j}^{n+1} - 2u_{i,j}^{n} + u_{i,j}^{n-1} + \sum_{k=1}^{n}\left(u_{i,j}^{n-k+1} - 2u_{i,j}^{n-k} + u_{i,j}^{n-k-1}\right)\left((1+k)^{2-\alpha} + k^{2-\alpha}\right)\right] = \frac{1}{2\Delta x^2}\left[u_{i+1,j}^{n+1} - 2u_{i,j}^{n+1} + u_{i-1,j}^{n+1} + u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n}\right] + \frac{1}{2\Delta y^2}\left[u_{i,j+1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j-1}^{n+1} + u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}\right] + \phi\left(x_i, y_j, t^n\right). \quad (4.8)$$

Dividing both sides with $\Delta t^{-\alpha}/\Gamma(3-\alpha)$ we have

$$u_{i,j}^{n+1} - 2u_{i,j}^{n} + u_{i,j}^{n-1} + \sum_{k=1}^{n}\left(u_{i,j}^{n-k+1} - 2u_{i,j}^{n-k} + u_{i,j}^{n-k-1}\right)\left((1+k)^{2-\alpha} + k^{2-\alpha}\right) = \frac{\Gamma(3-\alpha)}{2\Delta t^{-\alpha}\Delta x^2}\left[u_{i+1,j}^{n+1} - 2u_{i,j}^{n+1} + u_{i-1,j}^{n+1} + u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n}\right] + \frac{\Gamma(3-\alpha)}{2\Delta t^{-\alpha}\Delta x^2}\left[u_{i,j+1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j-1}^{n+1} + u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}\right] + \frac{\Gamma(3-\alpha)}{\Delta t^{-\alpha}}\phi\left(x_i, y_j, t^n\right). \quad (4.9)$$

Suppose that

$\frac{\Gamma(3-\alpha)}{2\Delta t^{-\alpha}\Delta x^2} = r_1$, $\frac{\Gamma(3-\alpha)}{2\Delta t^{-\alpha}\Delta y^2} = r_2$, $\frac{\Gamma(3-\alpha)}{\Delta t^{-\alpha}}\phi\left(x_i, y_j, t^n\right) = \phi\left(x_i, y_j, t^n\right) = \phi_{i,j}^{n}$,

$\left((1+k)^{2-\alpha} + k^{2-\alpha}\right) = b_k$ ,

$$\begin{aligned} u_{i,j}^{n+1} - 2u_{i,j}^{n} + u_{i,j}^{n-1} + \sum_{k=1}^{n}\left(u_{i,j}^{n-k+1} - 2u_{i,j}^{n-k} + u_{i,j}^{n-k-1}\right) b_k \\ = r_1\left[u_{i+1,j}^{n+1} - 2u_{i,j}^{n+1} + u_{i-1,j}^{n+1} + u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n}\right] \\ + r_2\left[u_{i,j+1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j-1}^{n+1} + u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}\right] + \phi_{i,j}^{n}. \end{aligned} \quad (4.10)$$

After rearranging the above equation,

$$u_{i,j}^{n+1} = 2u_{i,j}^{n} - u_{i,j}^{n-1} + r_1\left[u_{i+1,j}^{n+1} - 2u_{i,j}^{n+1} + u_{i-1,j}^{n+1} + u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n}\right] + r_2\left[u_{i,j+1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j-1}^{n+1} + u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}\right] - \sum_{k=1}^{n}\left(u_{i,j}^{n-k+1} - 2u_{i,j}^{n-k} + u_{i,j}^{n-k-1}\right) b_k + \phi_{i,j}^{n}. \quad (4.11)$$

OR

$$-r_1\left(u_{i+1,j}^{n+1} + u_{i-1,j}^{n+1}\right) + (1 + 2r_1 + 2r_2)u_{i,j}^{n+1} - r_2\left(u_{i,j+1}^{n+1} + u_{i,j-1}^{n+1}\right) = \quad (4.12)$$

$$r_1\left(u_{i+1,j}^{n}+u_{i-1,j}^{n}\right)+(2-2r_1-2r_2)u_{i,j}^{n}+r_2\left(u_{i,j+1}^{n}+u_{i,j-1}^{n}\right)-u_{i,j}^{n-1}-$$
$$\sum_{k=1}^{n}\left(u_{i,j}^{n-k+1}-2u_{i,j}^{n-k}+u_{i,j}^{n-k-1}\right)b_k+\phi_{i,j}^{n},$$

The summation term in Eq. (4.12) can be simplified as:

$$\sum_{k=1}^{n}\left(u_{i,j}^{n-k+1}-2u_{i,j}^{n-k}+u_{i,j}^{n-k-1}\right)b_k=\left(u_{i,j}^{n}b_1-2u_{i,j}^{n-1}b_1+u_{i,j}^{n-2}b_1\right)$$
$$+\left(=u_{i,j}^{n-1}b_2-2u_{i,j}^{n-2}b_2+u_{i,j}^{n-3}b_2\right)+\left(u_{i,j}^{n-2}b_3-2u_{i,j}^{n-3}b_3+u_{i,j}^{n-4}b_3\right)$$
$$+\left(u_{i,j}^{n-3}b_4-2u_{i,j}^{n-4}b_4+u_{i,j}^{n-5}b_4\right)+\cdots+\left(u_{i,j}^{3}b_{n-2}-2u_{i,j}^{2}b_{n-2}+u_{i,j}^{1}b_{n-2}\right)$$
$$+\left(u_{i,j}^{2}b_{n-1}-2u_{i,j}^{1}b_{n-1}+u_{i,j}^{0}b_{n-1}\right)+\left(u_{i,j}^{1}b_n-2u_{i,j}^{0}b_n+u_{i,j}^{-1}b_n\right)$$

It can be rewritten as:

$$\sum_{k=1}^{n}\left(u_{i,j}^{n-k+1}-2u_{i,j}^{n-k}+u_{i,j}^{n-k-1}\right)b_k=u_{i,j}^{n}b_1+\sum_{k=1}^{n-1}d_k u_{i,j}^{n-k}-2u_{i,j}^{0}b_n+u_{i,j}^{-1}b_n,$$

where $(b_{k+1}-2b_k+b_{k-1})=d_k$

$$-r_1\left(u_{i+1,j}^{n+1}+u_{i-1,j}^{n+1}\right)+(1+2r_1+2r_2)u_{i,j}^{n+1}-r_2\left(u_{i,j+1}^{n+1}+u_{i,j-1}^{n+1}\right)=$$
$$r_1\left(u_{i+1,j}^{n}+u_{i-1,j}^{n}\right)+(2-2r_1-2r_2)u_{i,j}^{n}+r_2\left(u_{i,j+1}^{n}+u_{i,j-1}^{n}\right)-u_{i,j}^{n}b_1+ \quad (4.13)$$
$$2u_{i,j}^{0}b_n-u_{i,j}^{-1}b_n+\sum_{k=1}^{n-1}u_{i,j}^{n-k}d_k+\phi_{i,j}^{n} \quad .$$

Hence, discretized form of equation (4.4) is:

$$-r_1\left(u_{i+1,j}^{n+1}+u_{i-1,j}^{n+1}\right)+(1+2r_1+2r_2)u_{i,j}^{n+1}-r_2\left(u_{i,j+1}^{n+1}+u_{i,j-1}^{n+1}\right)=$$
$$r_1\left(u_{i+1,j}^{n}+u_{i-1,j}^{n}\right)+(2-2r_1-2r_2-b_1)u_{i,j}^{n}+r_2\left(u_{i,j+1}^{n}+u_{i,j-1}^{n}\right)+2u_{i,j}^{0}b_n- \quad (4.14)$$
$$u_{i,j}^{-1}b_n+\sum_{k=1}^{n-1}u_{i,j}^{n-k}d_k+\phi_{i,j}^{n}.$$

Transferred equation (4.13) and (4.14) respectively into the following form,

$$-r_1(u_{i-1,j}^{1}+u_{i+1,j}^{1})+(1+2r_1+2r_2)u_{i,j}^{1}-r_2(u_{i,j-1}^{1}+u_{i,j+1}^{1})$$
$$=r_1(u_{i-1,j}^{0}+u_{i+1,j}^{0})+(4-2r_1-2r_2-b_1)u_{i,j}^{0}+r_2(u_{i,j-1}^{0} \quad (4.15)$$
$$+u_{i,j+1}^{0})-u_{i,j}^{-1}+\phi_{i,j} \quad for\ n=0,$$

$$-r_1\left(u_{i+1,j}^{n+1}+u_{i-1,j}^{n+1}\right)+(1+2r_1+2r_2)u_{i,j}^{n+1}-r_2\left(u_{i,j+1}^{n+1}+u_{i,j-1}^{n+1}\right)=$$
$$r_1\left(u_{i+1,j}^{n}+u_{i-1,j}^{n}\right)+(2-2r_1-2r_2-b_1)u_{i,j}^{n}+r_2\left(u_{i,j+1}^{n}+u_{i,j-1}^{n}\right)+2u_{i,j}^{0}b_n- \quad (4.16)$$
$$u_{i,j}^{-1}b_n+\sum_{k=1}^{n-1}u_{i,j}^{n-k}d_k+\phi_{i,j}^{n}\ \ for\ \ n\geq 1,$$
$$where\ \ i=1,2,3,\ldots l\,,j=1,2,3,\ldots m,$$

The discretization described above can be recast in the matrix form shown below:

for $n=0$

$$\mathbb{A}\vec{\mathbf{U}}^{n+1} = \mathbb{B}\vec{\mathbf{U}}^{n} - \vec{\mathbf{U}}^{n-1} + \frac{1}{2}\left(\vec{\mathbf{f}}^{n} + \vec{\mathbf{f}}^{n+1}\right) + \vec{\mathbf{b}}^{n} + \vec{\mathbf{c}}^{n} \tag{4.17}$$

for $n = 1$

$$\mathbb{A}\vec{\mathbf{U}}^{n+1} = \mathbb{B}\vec{\mathbf{U}}^{n} - b_1\vec{\mathbf{U}}^{n} - (1-2b_1)\vec{\mathbf{U}}^{n-1} - b_1\vec{\mathbf{U}}^{n-2} + \frac{1}{2}\left(\vec{\mathbf{f}}^{n} + \vec{\mathbf{f}}^{n+1}\right) + \vec{\mathbf{b}}^{n}$$

for $n = 2$

$$\mathbb{A}\vec{\mathbf{U}}^{n+1} = \mathbb{B}\vec{\mathbf{U}}^{n} - b_1\vec{\mathbf{U}}^{n} - (1-2b_1+b_2)\vec{\mathbf{U}}^{n-1} - b_1\vec{\mathbf{U}}^{n-2} - (b_{n-1}-2b_n)\vec{\mathbf{U}}^{0} - b_n\vec{\mathbf{U}}^{-1} + \frac{1}{2}\left(\vec{\mathbf{f}}^{n} + \vec{\mathbf{f}}^{n+1}\right) + \vec{\mathbf{b}}^{n}$$

for $n \geq 3$

$$\mathbb{A}\vec{\mathbf{U}}^{n+1} = \mathbb{B}\vec{\mathbf{U}}^{n} - b_1\vec{\mathbf{U}}^{n} - (1-2b_1+b_2)\vec{\mathbf{U}}^{n-1} - \sum_{k=1}^{n-2}(b_{k+2}-2b_{k+1}+b_k)\vec{\mathbf{U}}^{n-k-1} - (b_{n-1}-2b_n)\vec{\mathbf{U}}^{0} - b_n\vec{\mathbf{U}}^{-1} + \frac{1}{2}\left(\vec{\mathbf{f}}^{n} + \vec{\mathbf{f}}^{n+1}\right) + \vec{\mathbf{b}}^{n} \tag{4.18}$$

where

$$\mathbb{A} = \begin{bmatrix} 1+2r\theta & -r\theta & 0 & \cdots & 0 & 0 \\ -r\theta & 1+2r\theta & -r\theta & \cdots & 0 & 0 \\ 0 & -r\theta & 1+2r\theta & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1+2r\theta & -r\theta \\ 0 & 0 & 0 & \cdots & -r\theta & 1+2r\theta \end{bmatrix}, \vec{\mathbf{U}}^{n} = \begin{bmatrix} u_{i,1}^{n} \\ u_{i,2}^{n} \\ u_{i,3}^{n} \\ \vdots \\ u_{i,M-2}^{n} \\ u_{i,M-1}^{n} \end{bmatrix},$$

$$\vec{\mathbf{f}}^{n} = \begin{bmatrix} \Gamma(3-\alpha)\Delta t^{\alpha}\phi_{i,1}^{n} \\ \Gamma(3-\alpha)\Delta t^{\alpha}\phi_{i,2}^{n} \\ \Gamma(3-\alpha)\Delta t^{\alpha}\phi_{i,3}^{n} \\ \vdots \\ \Gamma(3-\alpha)\Delta t^{\alpha}\phi_{i,M-2}^{n} \\ \Gamma(3-\alpha)\Delta t^{\alpha}\phi_{i,M-1}^{n} \end{bmatrix}, \vec{\mathbf{b}}^{n} = \begin{bmatrix} r\theta u_{i,0}^{n} + r\theta u_{i,0}^{n+1} \\ 0 \\ 0 \\ \vdots \\ 0 \\ r\theta u_{i,M}^{n} + r\theta u_{i,M}^{n+1} \end{bmatrix}$$

$$\mathbb{B} = \begin{bmatrix} 2-2r(1-\theta) & r(1-\theta) & 0 & \cdots & 0 & 0 \\ r(1-\theta) & 2-2r(1-\theta) & r(1-\theta) & \cdots & 0 & 0 \\ 0 & r(1-\theta) & 2-2r(1-\theta) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2-2r(1-\theta) & r(1-\theta) \\ 0 & 0 & 0 & \cdots & r(1-\theta) & 2-2r(1-\theta) \end{bmatrix},$$

$$\vec{\mathbf{c}}^{n} = \begin{bmatrix} 2\Delta t g_{i,1} \\ 2\Delta t g_{i,2} \\ 2\Delta t g_{i,3} \\ \vdots \\ 2\Delta t g_{i,M-2} \\ 2\Delta t g_{i,M-1} \end{bmatrix}, \vec{\mathbf{U}}^{0} = \begin{bmatrix} f_{i,1} \\ f_{i,2} \\ f_{i,3} \\ \vdots \\ f_{i,M-2} \\ f_{i,M-1} \end{bmatrix}.$$

$\vec{\mathbf{U}}^{-1}$ can be computed as:

$$\vec{\mathbf{U}}^{-1} = \vec{\mathbf{U}}^{1} - \vec{\mathbf{c}}^{n}$$

We can obtain the following results.

**Lemma:** The coefficients $b_n (n = 0,1,2 \ldots)$ fulfill:

1) $b_n > b_{n+1}, n = 0,1,2 \ldots;$
2) $b_0 = 1, b_n > 0, n = 0,1,2 \ldots;$

## 4.2 Stability Analysis

We assume that $\tilde{u}_{i,j}^{k}$ , $(i = 0,1,2 \dots, l\,; j = 0,1,2 \dots, m;\;\; k = 0,1,2 \dots, n;)$ is estimated solution. The error is:

$$\varepsilon_{i,j}^{k} = \tilde{u}_{i,j}^{k} - u_{i,j}^{k}, (i = 0,1,2 \dots, l\,; j = 0,1,2 \dots, m;\;\; k = 0,1,2 \dots, n;) \tag{4.19}$$

Satisfies,

$$-r_1\left(u_{i+1,j}^{n+1} + u_{i-1,j}^{n+1}\right) + (1 + 2r_1 + 2r_2)u_{i,j}^{n+1} - r_2\left(u_{i,j+1}^{n+1} + u_{i,j-1}^{n+1}\right) = r_1\left(u_{i+1,j}^{n} + u_{i-1,j}^{n}\right) + (2 - 2r_1 - 2r_2)u_{i,j}^{n} + r_2\left(u_{i,j+1}^{n} + u_{i,j-1}^{n}\right) - u_{i,j}^{n-1} - \sum_{k=1}^{n}\left(u_{i,j}^{n-k+1} - 2u_{i,j}^{n-k} + u_{i,j}^{n-k-1}\right) b_k + \phi_{i,j}^{n}, \tag{4.20}$$

from equation (4.12)

$$-r_1\left(u_{i+1,j}^{n+1} + u_{i-1,j}^{n+1}\right) + (1 + 2r_1 + 2r_2)u_{i,j}^{n+1} - r_2\left(u_{i,j+1}^{n+1} + u_{i,j-1}^{n+1}\right) = r_1\left(u_{i+1,j}^{n} + u_{i-1,j}^{n}\right) + (2 - 2r_1 - 2r_2 - b_1)u_{i,j}^{n} + r_2\left(u_{i,j+1}^{n} + u_{i,j-1}^{n}\right) + 2u_{i,j}^{0}b_n - u_{i,j}^{-1}b_n + \sum_{k=1}^{n-1} u_{i,j}^{n-k}d_k + \phi_{i,j}^{n} \quad for\;\; n \geq 1, \tag{4.21}$$

from equation (4.14)

$$-r_1(\varepsilon_{i-1,j}^{1} + \varepsilon_{i+1,j}^{1}) + (1 + 2r_1 + 2r_2)\varepsilon_{i,j}^{1} - r_2(\varepsilon_{i,j-1}^{1} + \varepsilon_{i,j+1}^{1}) + \varepsilon_{i,j}^{-1} = r_1(\varepsilon_{i-1,j}^{0} + \varepsilon_{i+1,j}^{0}) + (2 - 2r_1 - 2r_2)\varepsilon_{i,j}^{0} + r_2(\varepsilon_{i,j-1}^{0} + \varepsilon_{i,j+1}^{0}) \qquad for \quad n = 0, \tag{4.22}$$

$$-r_1(\varepsilon_{i-1,j}^{n+1} + \varepsilon_{i+1,j}^{n+1}) + (1 + 2r_1 + 2r_2)\varepsilon_{i,j}^{n+1} - r_2(\varepsilon_{i,j-1}^{n+1} + \varepsilon_{i,j+1}^{n+1}) + b_n\varepsilon_{i,j}^{-1} = r_1(\varepsilon_{i+1,j}^{n} + \varepsilon_{i-1,j}^{n}) + (2 - 2r_1 - 2r_2 - b_1)\varepsilon_{i,j}^{n} + r_2\left(\varepsilon_{i,j+1}^{n} + \varepsilon_{i,j-1}^{n}\right) + 2\varepsilon_{i,j}^{0}b_n + \sum_{k=1}^{n-1} \varepsilon_{i,j}^{n-k}d_k \quad for\, n \geq 1. \tag{4.23}$$

This can be written as:

$$\begin{cases} A\boldsymbol{E}^{1} + \boldsymbol{E}^{-1} = B\boldsymbol{E}^{0} \\ A\boldsymbol{E}^{K+1} + b_n\boldsymbol{E}^{-1} = B'\boldsymbol{E}^{K} - b_1\boldsymbol{E}^{k} + \sum_{k=1}^{n-1} d_k\boldsymbol{E}^{n-k} + 2b_n\boldsymbol{E}^{0} \end{cases} \tag{4.24}$$

where,

$$\boldsymbol{E}^{k} = \begin{bmatrix} \boldsymbol{E}_1^{k} \\ \boldsymbol{E}_2^{k} \\ . \\ . \\ . \\ \boldsymbol{E}_{l-2}^{k} \\ \boldsymbol{E}_{l-1}^{k} \end{bmatrix}, \text{ and } \;\; \boldsymbol{E}_i^{k} = \begin{bmatrix} \varepsilon_{i,1}^{k} \\ \varepsilon_{i,2}^{k} \\ . \\ . \\ . \\ \varepsilon_{i,m-2}^{k} \\ \varepsilon_{i,m-1}^{k} \end{bmatrix}, \qquad i = 1,2,3, \dots l,$$

Therefore, the following conclusion can be supported by mathematical induction.

**Theorem: 1** $\|E^k\|_\infty \le \|E^0\|_\infty, k = 0,1,2,\ldots.$

**Proof:** For K=1,

$$-r_1(\varepsilon_{i-1,j}^1 + \varepsilon_{i+1,j}^1) + (1 + 2r_1 + 2r_2)\varepsilon_{i,j}^1 - r_2(\varepsilon_{i,j-1}^1 + \varepsilon_{i,j+1}^1) + \varepsilon_{i,j}^{-1}$$
$$= r_1(\varepsilon_{i-1,j}^0 + \varepsilon_{i+1,j}^0) + (2 - 2r_1 - 2r_2)\varepsilon_{i,j}^0 + r_2(\varepsilon_{i,j-1}^0 + \varepsilon_{i,j+1}^0) \quad (4.25)$$
$$= B\boldsymbol{E^0}.$$

Let, $|\varepsilon_{p,q}^1| = \max_{1\le i\le l-1; 1\le j\le m-1} |\varepsilon_{i,j}^1|$, we have

$$|\varepsilon_{p,q}^1| = -r_1(|\varepsilon_{p,q}^1| + |\varepsilon_{p,q}^1|) + (1 + 2r_1 + 2r_2)|\varepsilon_{p,q}^1| - r_2(|\varepsilon_{p,q}^1| + |\varepsilon_{p,q}^1|) + |\varepsilon_{p,q}^{-1}|, \quad (4.26)$$

$$\le -r_1(|\varepsilon_{p+1,q}^1| + |\varepsilon_{p-1,q}^1|) + (1 + 2r_1 + 2r_2)|\varepsilon_{p,q}^1| - r_2(|\varepsilon_{p,q+1}^1| + |\varepsilon_{p,q-1}^1|) + |\varepsilon_{p,q}^{-1}|, \quad (4.27)$$

$$\le |-r_1(\varepsilon_{p+1,q}^1 + \varepsilon_{p-1,q}^1) + (1 + 2r_1 + 2r_2)\varepsilon_{p,q}^1 - r_2(\varepsilon_{p,q+1}^1 + \varepsilon_{p,q-1}^1) + \varepsilon_{p,q}^{-1}|, \quad (4.28)$$

$$|B\boldsymbol{E^0}| \le \|E^0\|_\infty, \quad (4.29)$$

also, $\|\boldsymbol{E^1}\|_\infty \le \|\boldsymbol{E^0}\|_\infty$. (4.30)

Suppose that $\|\boldsymbol{E^s}\|_\infty \le \|\boldsymbol{E^0}\|_\infty, \boldsymbol{s} = \boldsymbol{1}, \boldsymbol{2}, \ldots \boldsymbol{k}$. Let $|\varepsilon_{p,q}^{k+1}| = \max_{1\le i\le l-1; 1\le j\le m-1} |\varepsilon_{i,j}^{k+1}|$,

$$|\varepsilon_{p,q}^{k+1}| = -r_1(|\varepsilon_{p,q}^{k+1}| + |\varepsilon_{p,q}^{k+1}|) + (1 + 2r_1 + 2r_2)|\varepsilon_{p,q}^{k+1}| - r_2(|\varepsilon_{p,q}^{k+1}| + |\varepsilon_{p,q}^{k+1}|) + b_n|\varepsilon_{p,q}^{-1}|, \quad (4.31)$$

$$\le -r_1(|\varepsilon_{p+1,q}^{k+1}| + |\varepsilon_{p-1,q}^{k+1}|) + (1 + 2r_1 + 2r_2)|\varepsilon_{p,q}^{k+1}| - r_2(|\varepsilon_{p,q+1}^{k+1}| + |\varepsilon_{p,q-1}^{k+1}|) + b_n|\varepsilon_{p,q}^{-1}|, \quad (4.32)$$

$$\le |-r_1(\varepsilon_{p+1,q}^{k+1} + \varepsilon_{p-1,q}^{k+1}) + (1 + 2r_1 + 2r_2)\varepsilon_{p,q}^{k+1} - r_2(\varepsilon_{p,q+1}^{k+1} + \varepsilon_{p,q-1}^{k+1}) + b_n\varepsilon_{p,q}^{-1}|, \quad (4.33)$$

$$= \left| r_1(\varepsilon_{p+1,q}^k + \varepsilon_{p-1,q}^k) + (2 - 2r_1 - 2r_2 - b_1)\varepsilon_{p,q}^k + r_2(\varepsilon_{p,q+1}^k + \varepsilon_{p,q-1}^k) + 2\varepsilon_{p,q}^0 b_k + \sum_{s=1}^{k-1} \varepsilon_{p,q}^{k-s} d_s \right|, \quad (4.34)$$

$$\le r_1(|\varepsilon_{p+1,q}^k| + |\varepsilon_{p-1,q}^k|) + (2 - 2r_1 - 2r_2 - b_1)|\varepsilon_{p,q}^k| + r_2(|\varepsilon_{p,q+1}^k| + |\varepsilon_{p,q-1}^k|) + 2b_k|\varepsilon_{p,q}^0| + \sum_{s=1}^{k-1} d_s|\varepsilon_{p,q}^{k-s}|, \quad (4.35)$$

$$\le r_1(\|\boldsymbol{E^k}\|_\infty + \|\boldsymbol{E^k}\|_\infty) + (2 - 2r_1 - 2r_2 - b_1)\|\boldsymbol{E^k}\|_\infty + r_2\left(\|\boldsymbol{E^k}\|_\infty + \right. \quad (4.36)$$

$\|\boldsymbol{E^k}\|_\infty\Big) + 2b_k\|\boldsymbol{E^k}\|_\infty + \sum_{s=1}^{k-1} d_s\|\boldsymbol{E^k}\|_\infty ,$

$$\le \{r_1 + (2 - 2r_1 - 2r_2 - b_1) + r_2 + 2b_k + \textstyle\sum_{s=1}^{k-1} d_s \}\|\boldsymbol{E^k}\|_\infty , \tag{4.37}$$

$$\le \{-r_1 - r_2 - b_1 + 2(1 + b_k) + \textstyle\sum_{s=1}^{k-1} d_s \}\|\boldsymbol{E^k}\|_\infty, \tag{4.38}$$

$$= \|\boldsymbol{E^k}\|_\infty, \tag{4.39}$$

also, $$\|\boldsymbol{E^{k+1}}\|_\infty \le \|\boldsymbol{E^0}\|_\infty. \tag{4.40}$$

As a result, the following theorem is proved.

**Theorem 2:** The Crank-Nicolson difference approximation defined by (4.22) and (4.23) is unconditionally stable.

## 4.3 Convergence Analysis

Suppose that $u(x_i, y_j, t_k), i = 0,1, \dots \dots . . l; j = 0,1, \dots \dots . m; k = 0,1, \dots \dots . n$ , be the precise resolution of the PDE for fractions at the mesh point. $(\boldsymbol{x_i}, \boldsymbol{t_k})$. Define $e_{i,j}^k = u(x_i , y_j, t_k) - u_{i,j}^k , i = 0,1, \dots \dots . . l; j = 0,1, \dots \dots . m; k = 0,1, \dots \dots . n$ , and $e^k = ( e_1^k, e_2^k, \dots . e_{m-1}^k)^T$, using $e^0 = 0$, were

$$e_i^k = \begin{bmatrix} e_{i,1}^k \\ e_{i,2}^k \\ . \\ . \\ . \\ e_{i,m-2}^k \\ e_{i,m-1}^k \end{bmatrix}, \quad i = 1,2 \dots \dots . . l - 1 ,$$

Substitution into (4.22) and (4.23) indications to

$$-r_1\big(e_{i+1,j}^1 + e_{i-1,j}^1\big) + (1 + 2r_1 + 2r_2)\, e_{i,j}^1 - r_2\big(e_{i,j+1}^1 + e_{i,j-1}^1\big) + e_{i,j}^{-1} = R_{i,j}^1, \tag{4.41}$$

$$-r_1(e_{i-1,j}^{k+1} + e_{i+1,j}^{k+1}) + (1 + 2r_1 + 2r_2)\, e_{i,j}^{k+1} - r_2\big(e_{i,j+1}^{k+1} + e_{i,j-1}^{k+1}\big) + e_{i,j}^{-1}b_n = r_1(e_{i-1,j}^k + e_{i+1,j}^k) + (2 - 2r_1 - 2r_2 - b_1)e_{i,j}^k + r_2(e_{i,j+1}^k + e_{i,j-1}^k) + \sum_{k=1}^{n-1} e_{i,j}^{n-k} d_k + R_{i,j}^{n+1}, \tag{4.42}$$

where, $$R_i^{n+1} = u(x_i , y_j, t_{n+1}) - 2u(x_i, y_j\, t_n) + u(x_i, y_j, t_{n-1}) + \sum_{k=1}^{n-1} b_k \{ u(x_i, y_j, t_{n-k+1}) - 2u(x_i, y_j, t_{n-k}) + u(x_i, y_j, t_{n-k-1})\} - \tag{4.43}$$

$r_1[u(x_{i+1}, y_j, t_{n+1}) - 2u(x_i, y_j, t_{n+1}) + u(x_{i-1}, y_j, t_{n+1}) + u(x_{i+1}, y_j, t_n) -$

$2u(x_i, y_j, t_n) + u(x_{i-1}, y_j, t_n) - r_2[u(x_i, y_{j+1}, t_{n+1}) - 2u(x_i, y_j, t_{n+1}) + u(x_i, y_{j-1}, t_{n+1}) + u(x_i, y_{j+1}, t_n) - 2u(x_i, y_j, t_n) + u(x_i, y_{j-1}, t_n)],$

$$R_i^{n+1} = \sum_{k=0}^{n-1} b_k \{ u(x_i, y_j, t_{n-k+1}) - 2u(x_i, y_j, t_{n-k}) + u(x_i, y_j, t_{n-k-1})\}$$
$$- r_1[u(x_{i+1}, y_j, t_{n+1}) - 2u(x_i, y_j, t_{n+1}) + u(x_{i-1}, y_j, t_{n+1}) + u(x_{i+1}, y_j, t_n) - 2u(x_i, y_j, t_n) + u(x_{i-1}, y_j, t_n) - r_2[u(x_i, y_{j+1}, t_{n+1}) - 2u(x_i, y_j, t_{n+1}) + u(x_i, y_{j-1}, t_{n+1}) + u(x_i, y_{j+1}, t_n) - 2u(x_i, y_j, t_n) + u(x_i, y_{j-1}, t_n)], \quad (4.44)$$

from the above equations (26), (27), (28), we have

$$D_t^\alpha u(x_i, y_j, t_{k+1}) + O(\tau) = \frac{\Delta t^{-\alpha}}{\Gamma(3-\alpha)}\left[\sum_{k=0}^{n}(u_{i,j}^{n-k+1} - 2u_{i,j}^{n-k} + u_{i,j}^{n-k-1})b_k\right], \quad (4.45)$$

$$\frac{\partial^2}{\partial x^2} u(x_i, y_j, t_{k+1}) + O(\Delta x^2) = \frac{1}{2}\left[\frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{\Delta x^2} + \frac{u_{i+1,j}^{n+1} - 2u_{i,j}^{n+1} + u_{i-1,j}^{n+1}}{\Delta x^2}\right], \quad (4.46)$$

$$\frac{\partial^2}{\partial y^2} u(x_i, y_j, t_{k+1}) + O(\Delta y^2) = \frac{1}{2}\left[\frac{u_{i,j-1}^n - 2u_{i,j}^n + u_{i,j+1}^n}{\Delta y^2} + \frac{u_{i,j-1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j+1}^{n+1}}{\Delta y^2}\right]. \quad (4.47)$$

Hence,

$$R_{i,j}^{k+1} = O(\tau^{1+\alpha} + \tau^\alpha(\Delta x)^2 + \tau^\alpha(\Delta y)^2). \quad (4.48)$$

$$\left|R_{i,j}^{k+1}\right| \le C\,(\tau^{1+\alpha} + \tau^\alpha(\Delta x)^2 + \tau^\alpha(\Delta y)^2), \quad (4.49)$$

$i = 0,1, \dots\dots.. l-1;\ j = 0,1, \dots\dots. m-1;\ k = 0,1, \dots\dots. n,$

where C is constant. Therefore, we achieve,

**Theorem 3:** $\|e^k\|_\infty \le C\, b_{k-1}^{-1}\,(\tau^{1+\alpha} + \tau^\alpha(\Delta x)^2 + \tau^\alpha(\Delta y)^2), k = 1,2, \dots\dots n,$

$where\ \|e^k\|_\infty = \|e^k\|_\infty = \max_{1\le i\le l-1; 1\le j\le m-1}\left|e_{i,j}^k\right|$ . And C is a constant term.

**PROOF:**

Using mathematical induction scheme, For K=1.

Let $\|e^k\|_\infty = \left|e_{p,q}^1\right| = \max_{1\le i\le l-1; 1\le j\le m-1}\left|e_{i,j}^1\right|$ ,

consider equation from (4.26)

$$\left|e_{i,j}^1\right| \le -r_1(\left|e_{p+1,q}^1\right| + \left|e_{p-1,q}^1\right|) + (1 + 2r_1 + 2r_2)\left|e_{p,q}^1\right| - r_2(\left|e_{p,q+1}^1\right| + \left|e_{p,q-1}^1\right|) + \left|e_{p,q}^{-1}\right|, \quad (4.50)$$

$$\le \left|-r_1(e_{p+1,q}^1 + e_{p-1,q}^1) + (1 + 2r_1 + 2r_2)e_{p,q}^1 - r_2(e_{p,q+1}^1 + e_{p,q-1}^1) + e_{p,q}^{-1}\right| = R_{p,q}^1, \quad (4.51)$$

$$\le C\, b_0^{-1}(\tau^{1+\alpha} + \tau^\alpha(\Delta x)^2 + \tau^\alpha(\Delta y)^2). \quad (4.52)$$

Suppose that $\|e^s\|_\infty \le C\, b_{s-1}^{-1}(\tau^{1+\alpha} + \tau^\alpha(\Delta x)^2 + \tau^\alpha(\Delta y)^2); s = 0,1,2, \dots., k-1.$

And $\left|e_{p,q}^{k+1}\right| = \max_{1\le i\le l-1;1\le j\le m-1}\left|e_{i,j}^{k+1}\right|$. Note that $b_s^{-1} \le b_k^{-1}, s = 0,1,\dots,k.$ we have

$$\left|e_{p,q}^{k+1}\right| \le -r_1\left(\left|e_{p+1,q}^{k+1}\right| + \left|e_{p-1,q}^{k+1}\right|\right) + (1+2r_1+2r_2)\left|e_{p,q}^{k+1}\right| - r_2\left(\left|e_{p,q+1}^{k+1}\right| + \left|e_{p,q-1}^{k+1}\right|\right) + b_n\left|e_{p,q}^{-1}\right|, \tag{4.53}$$

$$\le \left|-r_1\left(e_{p+1,q}^{k+1} + e_{p-1,q}^{k+1}\right) + (1+2r_1+2r_2)e_{p,q}^{k+1} - r_2\left(e_{p,q+1}^{k+1} + e_{p,q-1}^{k+1}\right) + b_n e_{p,q}^{-1}\right|, \tag{4.54}$$

$$= \left|r_1\left(e_{p+1,q}^{k} + e_{p-1,q}^{k}\right) + (2-2r_1-2r_2-b_1)e_{p,q}^{k} - r_2\left(e_{p,q+1}^{k} + e_{p,q-1}^{k}\right) + \sum_{k=1}^{n-1} e_{p,q}^{n-k} d_k + R_{p,q}^{n+1}\right|, \tag{4.55}$$

$$\le \left|r_1\left(e_{p+1,q}^{k} + e_{p-1,q}^{k}\right) + (2-2r_1-2r_2-b_1)e_{p,q}^{k} - r_2\left(e_{p,q+1}^{k} + e_{p,q-1}^{k}\right) + \sum_{k=1}^{n-1} e_{p,q}^{n-k} d_k\right| + \left|R_{p,q}^{n+1}\right|, \tag{4.56}$$

$$\le r_1\left(\left|e_{p+1,q}^{k}\right| + \left|e_{p-1,q}^{k}\right|\right) + (2-2r_1-2r_2-b_1)\left|e_{p,q}^{k}\right| - r_2\left(\left|e_{p,q+1}^{k}\right| + \left|e_{p,q-1}^{k}\right|\right) + \sum_{k=1}^{n-1} d_k\left|e_{p,q}^{n-k}\right| + C\left(\tau^{1+\alpha} + \tau^{\alpha}(\Delta x)^2 + \tau^{\alpha}(\Delta y)^2\right), \tag{4.57}$$

$$\le C_1(\|e^k\|_\infty + \|e^k\|_\infty) + C_2\|e^k\|_\infty + C_3(\|e^k\|_\infty + \|e^k\|_\infty) + \sum_{k=1}^{n-1} d_k\|e^{n-k}\|_\infty + C\left(\tau^{1+\alpha} + \tau^{\alpha}(\Delta x)^2 + \tau^{\alpha}(\Delta y)^2\right), \tag{4.58}$$

$$\le \left[C_1 + C_2 + C_3 + \textstyle\sum_{k=1}^{n-1} d_k) + b_k\right] b_k^{-1}\, C(\tau^{1+\alpha} + \tau^{\alpha}(\Delta x)^2 + \tau^{\alpha}(\Delta y)^2), \tag{4.59}$$

$$= b_k^{-1}\, C(\tau^{1+\alpha} + \tau^{\alpha}(\Delta x)^2 + \tau^{\alpha}(\Delta y)^2). \tag{4.60}$$

Because,

$$\lim_{k\to\infty}\frac{b_k^{-1}}{k^{\alpha}} = \lim_{k\to\infty}\frac{k^{-\alpha}}{(k+1)^{1-\alpha}-k^{1-\alpha}}, \tag{4.61}$$

$$= \lim_{k\to\infty}\frac{k^{-1}}{(1+\frac{1}{k})^{1-\alpha}-1}, \tag{4.62}$$

$$= \lim_{k\to\infty}\frac{k^{-1}}{(1-\alpha)k^{-1}} = \frac{1}{1-\alpha}. \tag{4.63}$$

Hence there is a constant C,

$$\|e^k\|_\infty \le Ck^{\alpha}\left(\tau^{1+\alpha} + (\tau^{1+\alpha} + \tau^{\alpha}(\Delta x)^2 + \tau^{\alpha}(\Delta y)^2)\right), \tag{4.64}$$

if $K\tau \le T$, is finite, then we get the following theorem.

**Theorem 4:** Let $u_{i,j}^{k}$ be the estimated value of $u(x_i, y_j, t_k)$, calculated by use of the difference scheme (8) and (9). Then there is a positive constant term C, such that

$$|u_{i,j}^{k} - u(x_i, y_j, t_k)| \leq C(\tau + (\Delta x)^2 + (\Delta y)^2),$$

$$i = 1,2, \ldots \ldots . . l-1;\ j = 1,2 \ldots \ldots . m-1;\ k = 1,2 \ldots \ldots . n.$$

## 4.4 Algorithm: 2

(Input data)

1: $a_1, b_1, a_2, b_2, \mathrm{T}, M_x, M_y, N, \theta \in N$ , $U(x,y,t)$ $1 < \alpha \leq 2$ (Evaluation of function)

2: $f := fracdiff(U(x,y,t),t,\alpha) - \frac{\partial^2}{\partial x^2}U(x,y,t) - \frac{\partial^2}{\partial y^2}U(x,y,t);$

3: for i from 0 by 1 while i ≤ $M_x$ do; (Step size in x direction)
   $x[i] := a1 + \Delta xi;$ end do:

4: for j from 0 by 1 while j ≤ $M_y$ do; (Step size in y direction)
   $y[j] := a2 + \Delta yj;$ end do:

5: for n from 0 by 1 while n ≤ N do; (Step size in time direction)
   $t[n] := \Delta tn;$ end do:

6: for i from 0 by 1 while i ≤ $M_x$ do;
   for j from 0 by 1 while j ≤ $M_y$ do; (Evaluation of initial conditions)
   $u[i,j,0] := eval(U,[x = i\Delta x, y = j\Delta y, t = 0]);$
   $u[i,j,-1] := u[i,j,1] + 2\Delta t.eval\left(\frac{d}{dt}U(x,y,t),[x = i\Delta x, y = j\Delta y, t = 0]\right);$
   end do:

6: for n from 1 by 1 while n ≤ N do;
   for j from 0 by 1 while j ≤ $M_y$ do; (Evaluation of boundary conditions)
   $u[0,j,n] := eval(U(x,y,t),[x = a1, y = j\Delta y, t = n\Delta t]);$
   $u[M_x,j,n] := eval(U(x,y,t),[x = b1, y = j\Delta y, t = n\Delta t]);$
   do;
   for i from 1 by 1 while i ≤ $M_x - 1$ do;
   $u[i,0,n] := eval(U(x,y,t),[x = i\Delta x, y = a2, t = n\Delta t]);$
   $u[i,M_y,n] := eval(U(x,y,t),[x = i\Delta x, y = b2, t = n\Delta t]);$
   do; do:

7: for n from 0 by 1 while n ≤ N-1 do;
   for i from 1 by 1 while i ≤$M_x - 1$ do;
   for j from 1 by 1 while j≤$M_y - 1$ do;
   R [i, j, n] = (Evaluation of equation 4.8)
   end do; end do;

8: Sol[n+1]: =fsolve ({seq (seq (R [i, j, n], i=1... $M_x - 1$), j=1…$M_y - 1$)});
   assign(op(sol[n+1]); end do:

# CHAPTER 5

# PERFORMANCE EVALUATION

## Numerical Implementation

Consider the following wave equation

$${}_0^C D_t^{\alpha} u(x,t) = \frac{\partial^2 u}{\partial x^2} + t\sin(x), \qquad 0 < x < 1, \qquad 0 < t \le 1, \qquad 1 < \alpha \le 2,$$

With the following initial conditions and boundary conditions,

$$u(x,0) = f_i = 0, \quad u_t(x,0) = g_i = \sin(x), \quad 0 \le x \le 1,$$
$$u(0,t) = 0, \quad u(1,t) = t\sin(1), \quad t > 0.$$

for $M = 4, \theta = 1/2$ and $N = 5$,

**Solution:** In this problem $a = 0, b = 1,\ \Delta x = 0.25\ ,\ \Delta t = 0.2, T = 1,\ \phi(x,t) = t\sin(x)$, and $\alpha = 1.5$ implies that

$$M = \frac{b-a}{\Delta x} = 4, N = \frac{T}{\Delta t} = 5,$$

and $x_i = a + i\Delta x = i/4$ , for $i = 0,1,2,3,4$ is given as:

$$x_0 = 0, x_1 = 0.25, x_2 = 0.50, x_3 = 0.75, x_4 = 1,$$

Similarly, $t_n = n\Delta t = n/5$,for $n = 0,1,2,3,4,5$ is given as:

$$t_0 = 0, t_1 = 0.2,\ t_2 = 0.4, t_3 = 0.6,\ t_4 = 0.8, t_5 = 1.$$

First initial conditions, $u(x_i, 0) = 0, \Longrightarrow$ for $i = 0,1,2,3$ is given as:

$$u(x_0,0) = 0, u(x_1,0) = 0, u(x_2,0) = 0, u(x_3,0) = 0, u(x_4,0) = 0,$$

respectively. The second initial conditions, $u(x_i, 1) = u(x_i, -1) + 2\Delta t g_i$, for $i = 0,1,2,3$ is given as:

$$u(x_0,1) = u(x_0,-1) + 2(0.2)\sin(0) \Longrightarrow u(x_0,1) = u(x_0,-1)$$
$$u(x_1,1) = u(x_1,-1) + 2(0.2)\sin(0.25) \Longrightarrow u(x_1,-1) = u(x_1,1) - 0.0989$$
$$u(x_2,1) = u(x_2,-1) + 2(0.2)\sin(0.50) \Longrightarrow u(x_2,-1) = u(x_2,1) - 0.1918$$
$$u(x_3,1) = u(x_3,-1) + 2(0.2)\sin(0.75) \Longrightarrow u(x_3,-1) = u(x_3,1) - 0.2726,$$

respectively. First boundary conditions, $u(0, t_n) = 0$, for $n = 1,2,3,4,5$ is given as:

$$u(0,t_1) = 0, u(0,t_2) = 0, u(0,t_3) = 0, u(0,t_4) = 0, u(0,t_5) = 0,$$

Respectively. The second boundary conditions, $u(1,t_n) = t_n\sin(1)$ for $n = 1,2,3,4,5$ is given as:

$$u(1,t_0) = 0\times\sin(1) = 0, u(1,t_1) = (0.2)\sin(1) = 0.1683, u(1,t_2) = (0.2)\sin(1) = 0.3366$$

$$u(1,t_3) = (0.2)\sin(1) = 0.5049, u(1,t_4) = (0.2)\sin(1) = 0.6732, u(1,t_5) = (0.2)\sin(1) = 0.8415$$

Since the source term $f(x,t) = t\sin(x)$. The tridiagonal matrices of order $3\times 3$ are given as:

$$A = \begin{bmatrix} 1+2\mathrm{r} & -\mathrm{r} & 0 \\ -\mathrm{r} & 1+2\mathrm{r} & -\mathrm{r} \\ 0 & -\mathrm{r} & 1+2\mathrm{r} \end{bmatrix} = \begin{bmatrix} 3.5365295 & -1.2682647 & 0 \\ -1.2682647 & 3.5365295 & -1.2682647 \\ 0 & -1.2682647 & 3.5365295 \end{bmatrix},$$

$$A^{-1} = \begin{bmatrix} 0.2408534 & 0.0730436 & 0.0204206 \\ 0.0730436 & 0.2612741 & 0.0730436 \\ 0.0204206 & 0.0730436 & 0.2408534 \end{bmatrix}$$

and

$$B = \begin{bmatrix} -2\mathrm{r} & \mathrm{r} & 0 \\ \mathrm{r} & -2\mathrm{r} & \mathrm{r} \\ 0 & \mathrm{r} & -2\mathrm{r} \end{bmatrix} = \begin{bmatrix} -2.5365294 & 1.2682647 & 0 \\ 1.2682647 & -2.5365294 & 1.2682647 \\ 0 & 1.2682647 & -2.5365294 \end{bmatrix},$$

Therefore, the Crank-Nicolson finite difference formula (3.12) for $n = 0$, is given as:

$$\begin{bmatrix} 3.5365295 & -1.2682647 & 0 \\ -1.2682647 & 3.5365295 & -1.2682647 \\ 0 & -1.2682647 & 3.5365295 \end{bmatrix}\begin{pmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \end{pmatrix}$$

$$= \begin{bmatrix} -2.5365294 & 1.2682647 & 0 \\ 1.2682647 & -2.5365294 & 1.2682647 \\ 0 & 1.2682647 & -2.5365294 \end{bmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} u_1^1 - 0.0989616 \\ u_2^1 - 0.1917702 \\ u_3^1 - 0.2726555 \end{pmatrix}$$

$$+\frac{1}{2}\left\{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0.0039222 \\ 0.0076005 \\ 0.0108063 \end{pmatrix}\right\} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0.10672 \end{pmatrix}, \Rightarrow \begin{pmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \end{pmatrix} = \begin{pmatrix} 0.04945 \\ 0.09588 \\ 0.13622 \end{pmatrix}$$

$$= \vec{\mathbf{U}}^{\mathbf{1}}.$$

The Crank-Nicolson finite difference formula (3.13) for $n = 1$, is given as:

$$\begin{bmatrix} 3.5365295 & -1.2682647 & 0 \\ -1.2682647 & 3.5365295 & -1.2682647 \\ 0 & -1.2682647 & 3.5365295 \end{bmatrix}\begin{pmatrix} u_1^2 \\ u_2^2 \\ u_3^2 \end{pmatrix}$$

$$= \begin{bmatrix} -2.5365294 & 1.2682647 & 0 \\ 1.2682647 & -2.5365294 & 1.2682647 \\ 0 & 1.2682647 & -2.5365294 \end{bmatrix}\begin{bmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \end{bmatrix} - b_1\begin{bmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \end{bmatrix} - (1-2b_1)\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-b_1\begin{pmatrix} u_1^1 - 0.0989616 \\ u_2^1 - 0.1917702 \\ u_3^1 - 0.2726555 \end{pmatrix} + \frac{1}{2}\left(\begin{pmatrix} 0.0041220 \\ 0.0079878 \\ 0.0113569 \end{pmatrix} + \begin{pmatrix} 0.0082440 \\ 0.0159755 \\ 0.0227137 \end{pmatrix}\right) + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\Rightarrow \begin{pmatrix} u_1^2 \\ u_2^2 \\ u_3^2 \end{pmatrix} = \begin{pmatrix} 0.098993 \\ 0.19183 \\ 0.27271 \end{pmatrix} = \vec{\mathbf{U}}^{\mathbf{2}}.$$

The Crank-Nicolson finite difference formula (3.14) for $n = 2$, is given as:

$$\begin{bmatrix} 3.5365295 & -1.2682647 & 0 \\ -1.2682647 & 3.5365295 & -1.2682647 \\ 0 & -1.2682647 & 3.5365295 \end{bmatrix} \begin{pmatrix} u_1^3 \\ u_2^3 \\ u_3^3 \end{pmatrix}$$

$$= \begin{bmatrix} -2.5365294 & 1.2682647 & 0 \\ 1.2682647 & -2.5365294 & 1.2682647 \\ 0 & 1.2682647 & -2.5365294 \end{bmatrix} \begin{pmatrix} 0.098993 \\ 0.19183 \\ 0.27271 \end{pmatrix} - b_1 \begin{pmatrix} 0.098993 \\ 0.19183 \\ 0.27271 \end{pmatrix}$$

$$-(1 - 2b_1 + b_2) \begin{pmatrix} 0.04945 \\ 0.09588 \\ 0.13622 \end{pmatrix} - b_1 \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} - (b_1 - 2b_2) \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} - b_2 \begin{pmatrix} -0.0494766 \\ -0.0958772 \\ -0.1363155 \end{pmatrix}$$

$$+\frac{1}{2}\left( \begin{pmatrix} 0.0078443 \\ 0.0152010 \\ 0.0216125 \end{pmatrix} + \begin{pmatrix} 0.0117665 \\ 0.0228014 \\ 0.0324187 \end{pmatrix} \right) + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} u_1^3 \\ u_2^3 \\ u_3^3 \end{pmatrix} = \begin{pmatrix} 0.14853 \\ 0.28778 \\ 0.40910 \end{pmatrix} = \vec{\mathbf{U}}^{\mathbf{3}}$$

The Crank-Nicolson finite difference formula (3.15) for $n = 3$, is given as:

$$\begin{bmatrix} 3.5365295 & -1.2682647 & 0 \\ -1.2682647 & 3.5365295 & -1.2682647 \\ 0 & -1.2682647 & 3.5365295 \end{bmatrix} \begin{pmatrix} u_1^4 \\ u_2^4 \\ u_3^4 \end{pmatrix}$$

$$= \begin{bmatrix} -2.5365294 & 1.2682647 & 0 \\ 1.2682647 & -2.5365294 & 1.2682647 \\ 0 & 1.2682647 & -2.5365294 \end{bmatrix} \begin{pmatrix} 0.14853 \\ 0.28778 \\ 0.40910 \end{pmatrix} - b_1 \begin{pmatrix} 0.14853 \\ 0.28778 \\ 0.40910 \end{pmatrix}$$

$$-(1 - 2b_1 + b_2) \begin{pmatrix} 0.098993 \\ 0.19183 \\ 0.27271 \end{pmatrix} - (b_3 - 2b_2 + b_1) \begin{pmatrix} 0.04945 \\ 0.09588 \\ 0.13622 \end{pmatrix} - (b_2 - 2b_3) \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-b_3 \begin{pmatrix} -0.0494766 \\ -0.0958772 \\ -0.1363155 \end{pmatrix} + \frac{1}{2}\left( \begin{pmatrix} 0.0117665 \\ 0.0228014 \\ 0.0324187 \end{pmatrix} + \begin{pmatrix} 0.0156887 \\ 0.0304019 \\ 0.0432249 \end{pmatrix} \right) + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \Rightarrow \begin{pmatrix} u_1^4 \\ u_2^4 \\ u_3^4 \end{pmatrix}$$

$$= \begin{pmatrix} 0.19806 \\ 0.38376 \\ 0.54551 \end{pmatrix} = \vec{\mathbf{U}}^4.$$

The Crank-Nicolson finite difference formula (3.15) for $n = 4$, is given as:

$$\begin{bmatrix} 3.5365295 & -1.2682647 & 0 \\ -1.2682647 & 3.5365295 & -1.2682647 \\ 0 & -1.2682647 & 3.5365295 \end{bmatrix} \begin{pmatrix} \boldsymbol{u_1^5} \\ \boldsymbol{u_2^5} \\ \boldsymbol{u_3^5} \end{pmatrix}$$

$$= \begin{bmatrix} -2.5365294 & 1.2682647 & 0 \\ 1.2682647 & -2.5365294 & 1.2682647 \\ 0 & 1.2682647 & -2.5365294 \end{bmatrix} \begin{pmatrix} 0.19806 \\ 0.38376 \\ 0.54551 \end{pmatrix} - b_1 \begin{pmatrix} 0.19806 \\ 0.38376 \\ 0.54551 \end{pmatrix}$$

$$-(1-2b_1+b_2)\begin{pmatrix}0.14853\\0.28778\\0.40910\end{pmatrix}-\sum_{k=1}^{2}(b_{k+2}-2b_{k+1}+b_k)\vec{\mathbf{U}}^{3-k}-(b_3-2b_4)\begin{pmatrix}0\\0\\0\end{pmatrix}$$
$$-b_4\begin{pmatrix}u_1^1-0.0989616\\u_2^1-0.1917702\\u_3^1-0.2726555\end{pmatrix}$$
$$+\frac{1}{2}\left(\begin{pmatrix}0.0156887\\0.0304019\\0.0432249\end{pmatrix}+\begin{pmatrix}0.0196109\\0.0380024\\0.0540312\end{pmatrix}\right)+\begin{pmatrix}0\\0\\0\end{pmatrix},\Rightarrow\begin{pmatrix}u_1^5\\u_2^5\\u_3^5\end{pmatrix}=\begin{pmatrix}0.24760\\0.47973\\0.68188\end{pmatrix}=\vec{\mathbf{U}}^5.$$

## 5.1Representation of graphs

To assess our proposed time-fractional hyperbolic partial differential equation introduced in Chapters 3 and 4, this chapter will show and discuss simulation results by using Tec plot software. Essentially, there are two parts to this section. Results analysis, which is an assessment of the effectiveness of TFWEs using various parameters and a comparison with the exact answer, are shown below. To evaluate the effectiveness of the suggested algorithm, numerous challenges have been resolved. In this chapter, the same results as well as some fresh results have been attained.

## 5.2 Comparison with Different Parameters in 1D

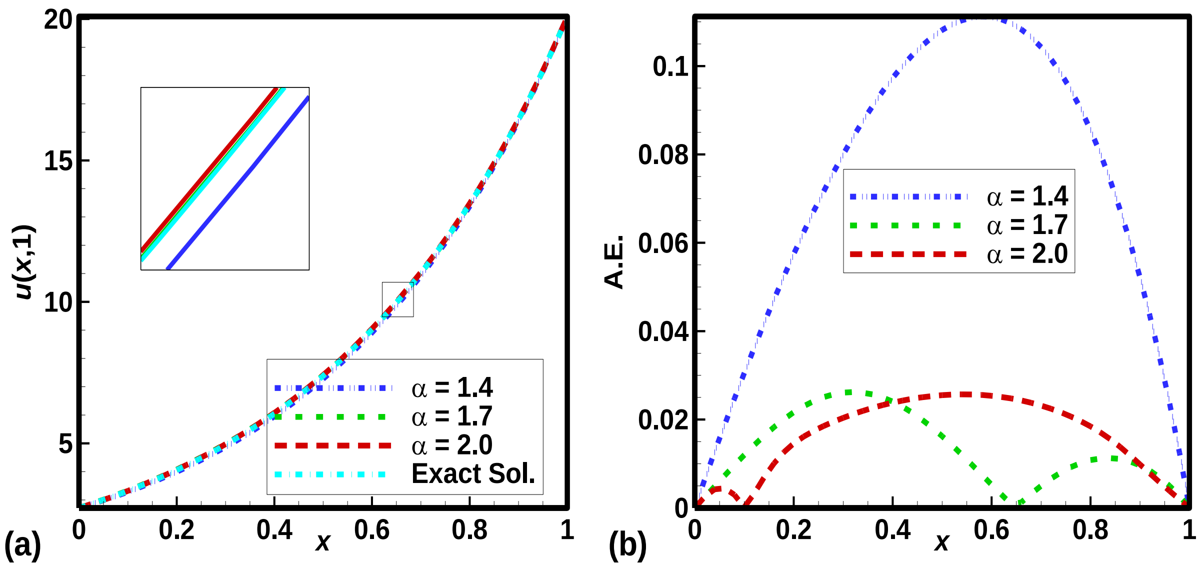


**Figure 5.2.1**: For $u(x,t)=e^{2x+t}$

In terms of the first problem, a precise and numerical solution is found for various values of α. Figure 5.2.1 Two-dimensional representation of **(a)** approximate solutions and exact solution $u(x,t) = e^{2x+t}$, and **(b)** absolute error against $\alpha$ when $M = 50$ and $N = 100$. Some intriguing findings can be seen in the graph 5.2.1 the approximate solution approaches the exact solution using the Crank-Nicolson difference scheme, and the error decreases. From this figure, it can be deduced that M=50 and N=100 both achieve the same level of accuracy.

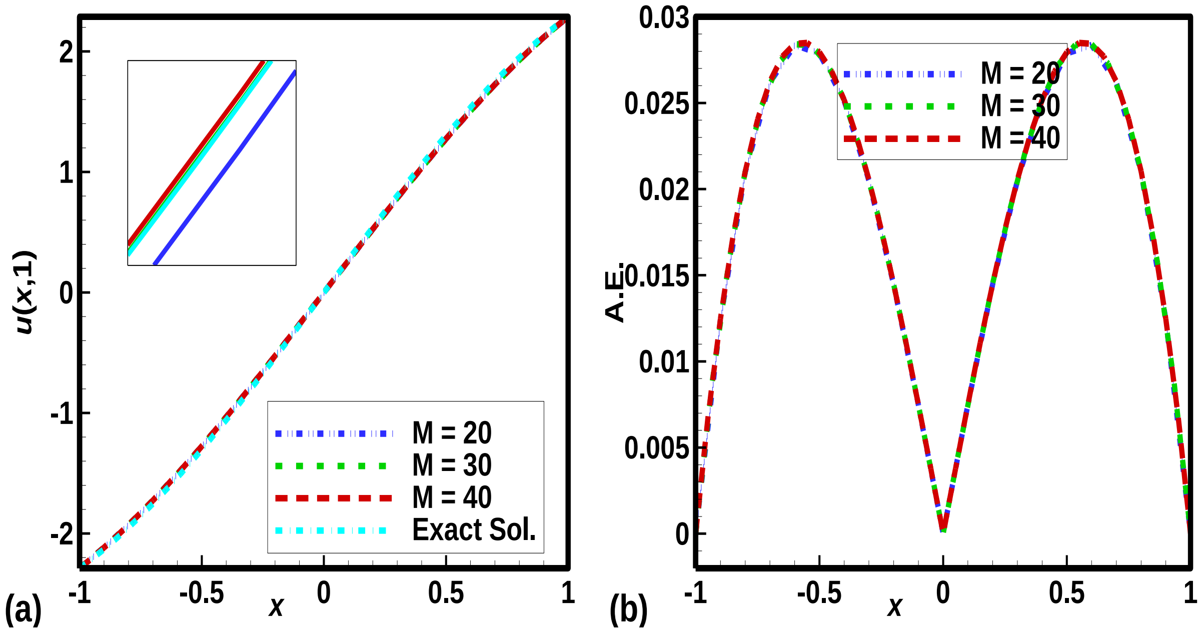


**Figure 5.2.2**: For $u(x,t) = e^{t}\sin(x)$

In figure 5.2.2 Two-dimensional representation of **(a)** approximate solutions and exact solution $u(x,t) = e^{t}\sin(x)$, and **(b)** absolute error against $M$ when $\alpha = 1.2$ and $N = 100$. The closed contact is validated with the accurate solution of example $U = e^{t}\sin(x)$, which also provides Two-dimensional Crank-Nicolson technique solution graphs when various value of M and N=100.

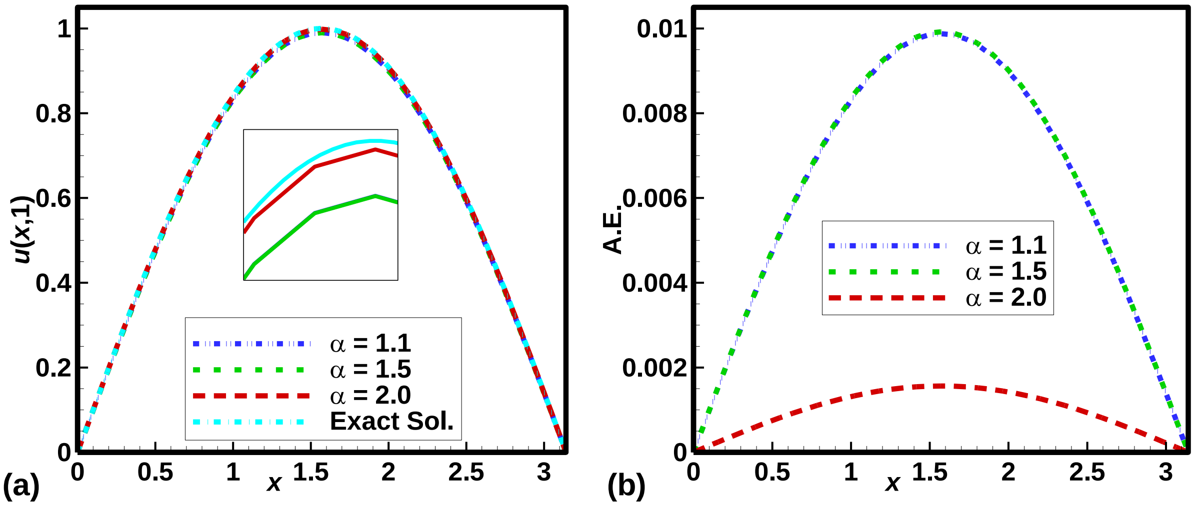


**Figure 5.2.3**: For $u(x,t) = t^2 \sin(x)$

In Figure 5.2.3 Two-dimensional representation of **(a)** approximate solutions and exact solution $u(x,t) = t^2 \sin(x)$, and **(b)** absolute error against various value of $\alpha$ when $M = 40$ and $N = 100$ that is the evidence that the suggested method is highly accurate and reliable.

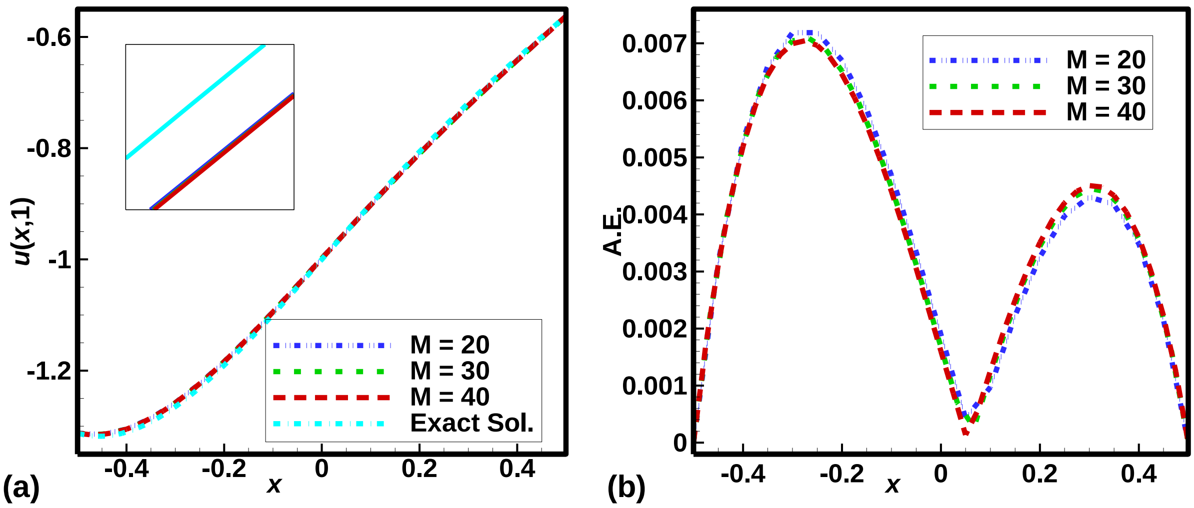


**Figure 5.2.4**: For $U(x,t) = x^4t^3 - x^3 - t^2 + xt$

In figure 5.2.4 Two-dimensional representation of **(a)** approximate solutions and exact solution $u(x,t) = x^4t^3 - x^3 - t^2 + xt$, and **(b)** absolute error against $M$ when $\alpha = 1.4$ and $N = 100$. The behaviour of the particular and numerical solution of the fractional order hyperbolic partial differential equation $U(x,t) = x^4t^3 - x^3 - t^2 + xt$ for α=1.4 is depicted in figure 5.2.4. The estimated solution gets closer and closer to the exact solution, resulting in a diminishing absolute error.

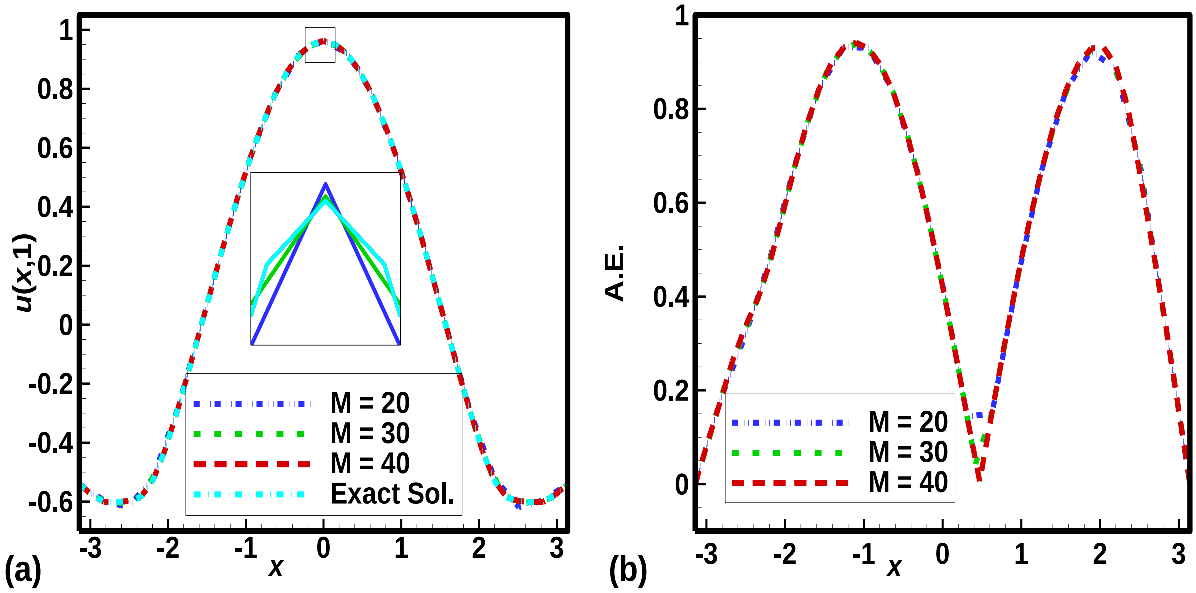


**Figure 5.2.5**: For $U(x,t) = \cos(x-t)$

Figure 5.2.5 Two-dimensional representation of **(a)** approximate solutions and exact solution $u(x,t) = \cos(x-t)$, agree pretty well with the numerical solutions for various values of M, and **(b)** absolute error against $M$ when $\alpha = 2$ and $N = 100$. the approximate and exact solutions for the Two-dimensional plot are displayed in Figure 5.2.5 to demonstrate the technique's correctness. It is clear that the answers are strikingly similar. Clearly, the suggested approach is quite precise and effective. Extremely high levels of agreement among the solutions are visible in the graph. Our method appears to give noticeably greater accuracy.

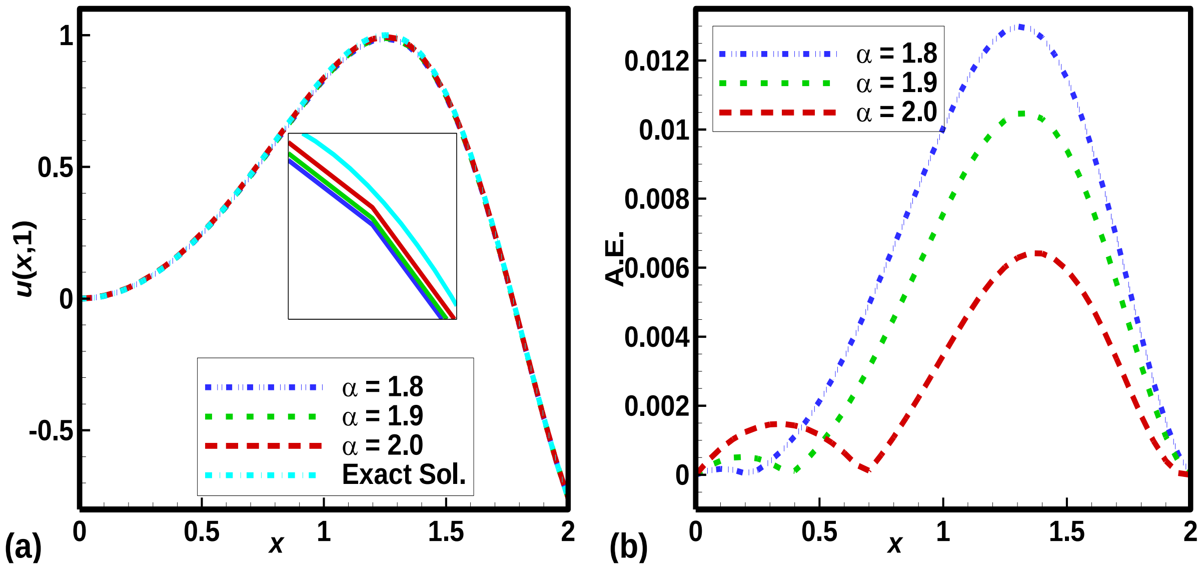


**Figure 5.2.6:** For $U(x,t) = t^3 \sin(x^2)$

The behaviour of the precise and numerical solution of the TFWEs for different values of α is shown in figure 5.2.6 are very similar to the precise answers, which also Two-dimensional representation of **(a)** approximate solutions and exact solution $u(x,t) = t^3 \sin(x^2)$, and **(b)** absolute error against $\alpha$ when $M = 40$ and $N = 100$. It is clear that the suggested approach is entirely correct and effective,

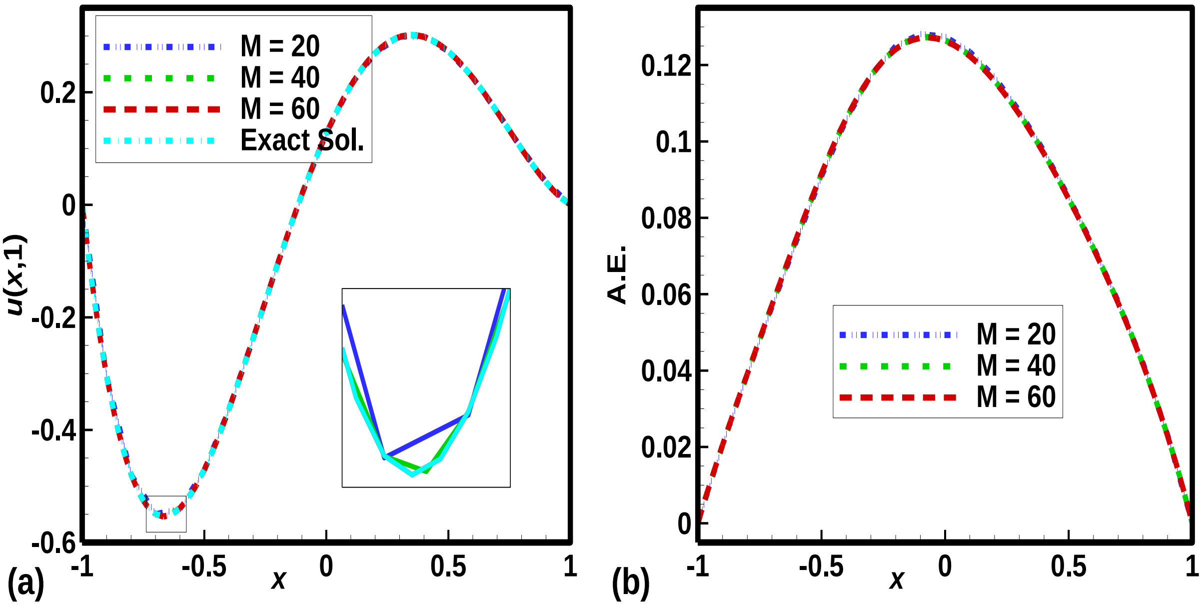


**Figure 5.2.7:** For $U(x,t) = x^4t^3 - x^3t^2 - x^2t + x$

In Figure 5.2.7 Two-dimensional representation of **(a)** approximate solutions and exact solution $u(x,t) = x^4t^3 - x^3t^2 - -x^2t + x$, and **(b)** absolute error against $M$ when $\alpha = 1.7$ and $N = 100$. The solutions in fractional order at α =1.7 serve to validate the proposed method for addressing issues involving fractional initial and boundary values. The findings go into great detail and are directly related to accurate solutions to the problems.

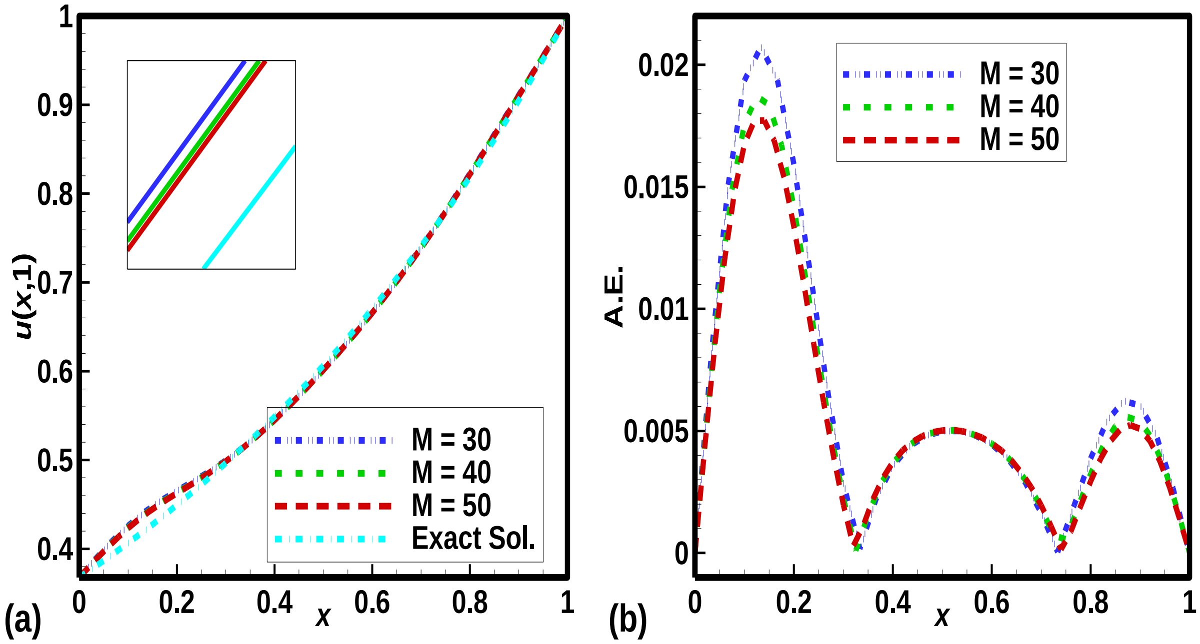


**Figure 5.2.8**: For $U(x,t) = e^{(x-t)}$

In Figure 5.2.8 Two-dimensional representation of **(a)** approximate solutions and exact solution $u(x,t) = \mathrm{e}^{x-t}$, and **(b)** absolute error against $M$ when $\alpha = 2$ and $N = 25$. The estimated solution gets closer and closer to the actual solution, resulting in a diminishing absolute error. Clearly, the suggested strategy is very accurate and efficient.

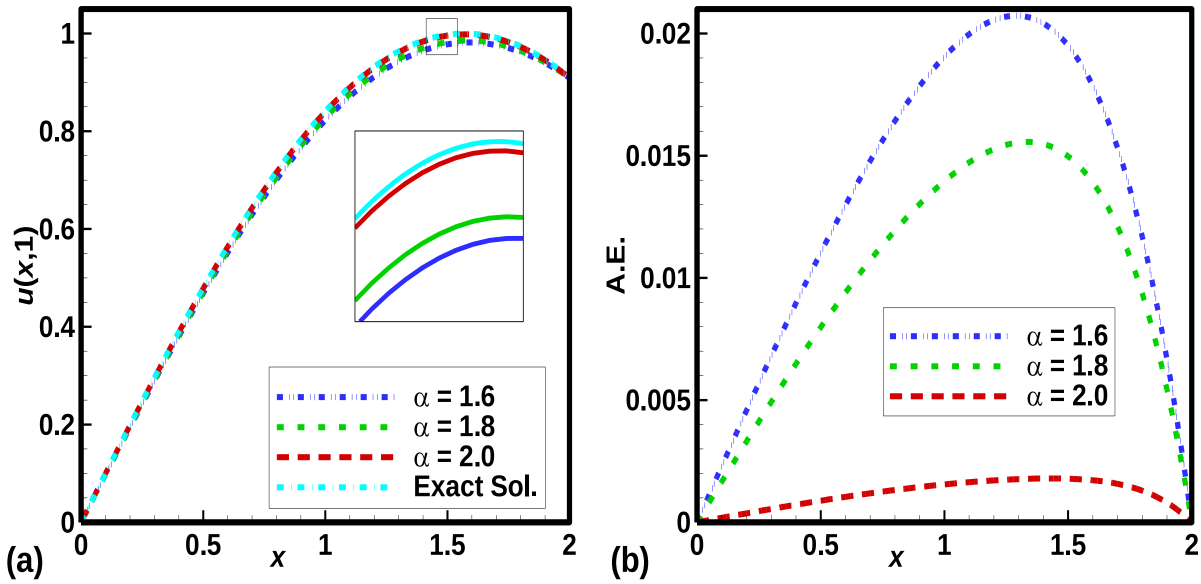


**Figure 5.2.9:** For $U(x,t) = t^5 \sin(\mathrm{x})$

Figure 5.2.9 Two-dimensional representation of **(a)** approximate solutions and exact solution $u(x,t) = t^5 \sin(x)$, and **(b)** absolute error against $\alpha$ when $M = 100$ and $N = 100$. The estimated solution approaches the exact solution more closely, resulting in a decreasing error. When M and N are both equal to 100, we can observe that the Crank-Nicolson technique produces approximate results.

## 5.3 Conclusion of 1 D

In this section, various problems have been solved by means of the proposed one-dimensional fractional order hyperbolic PDE using the Crank-Nicolson technique has been used to resolve a number of issues in one dimensional. It is observed that the obtained results are well-matched to the existing results. Additionally, it should be emphasized that the Crank-Nicolson approach effectively couples fractional order hyperbolic partial differential equations, particularly for highly nonlinear systems. The innovative nonlinear physical problem in the complicated geometry may now be solved using this effective strategy.

## 5.4 Comparison with Different Parameters in 2D

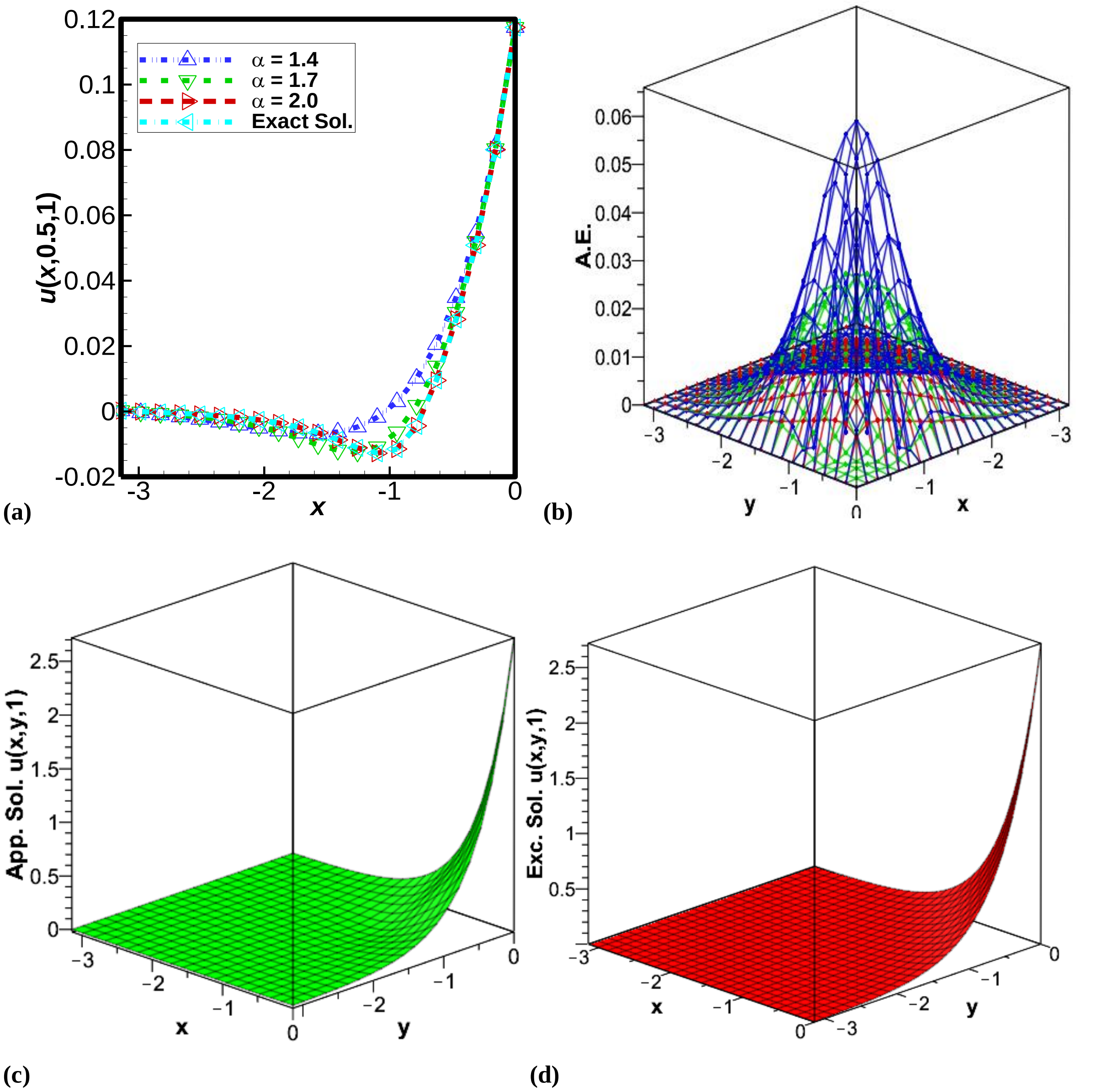


**Figure 5.4.1**: For $\mathbf{U(x,y,t) = e^{(2x+2y+t)}}$

Figure 5.4.1 Graphical illustration of **(a)** 2D behaviour of the numerical solution against $\alpha$ **(b)** 3D variation in the absolute error behaviour against $\alpha$ **(c)** exact solution $u(x,y,t) = e^{2x+2y+t}$ and **(d)** approximate solution (when $\alpha = 2$) in three-dimensions when $M = 20$ and $N = 20$. The precise and CNM solutions are seen to be in close proximity to one another. The graphs have demonstrated that the exact and produced results are extremely close and have validated the applicability of the current methodology.

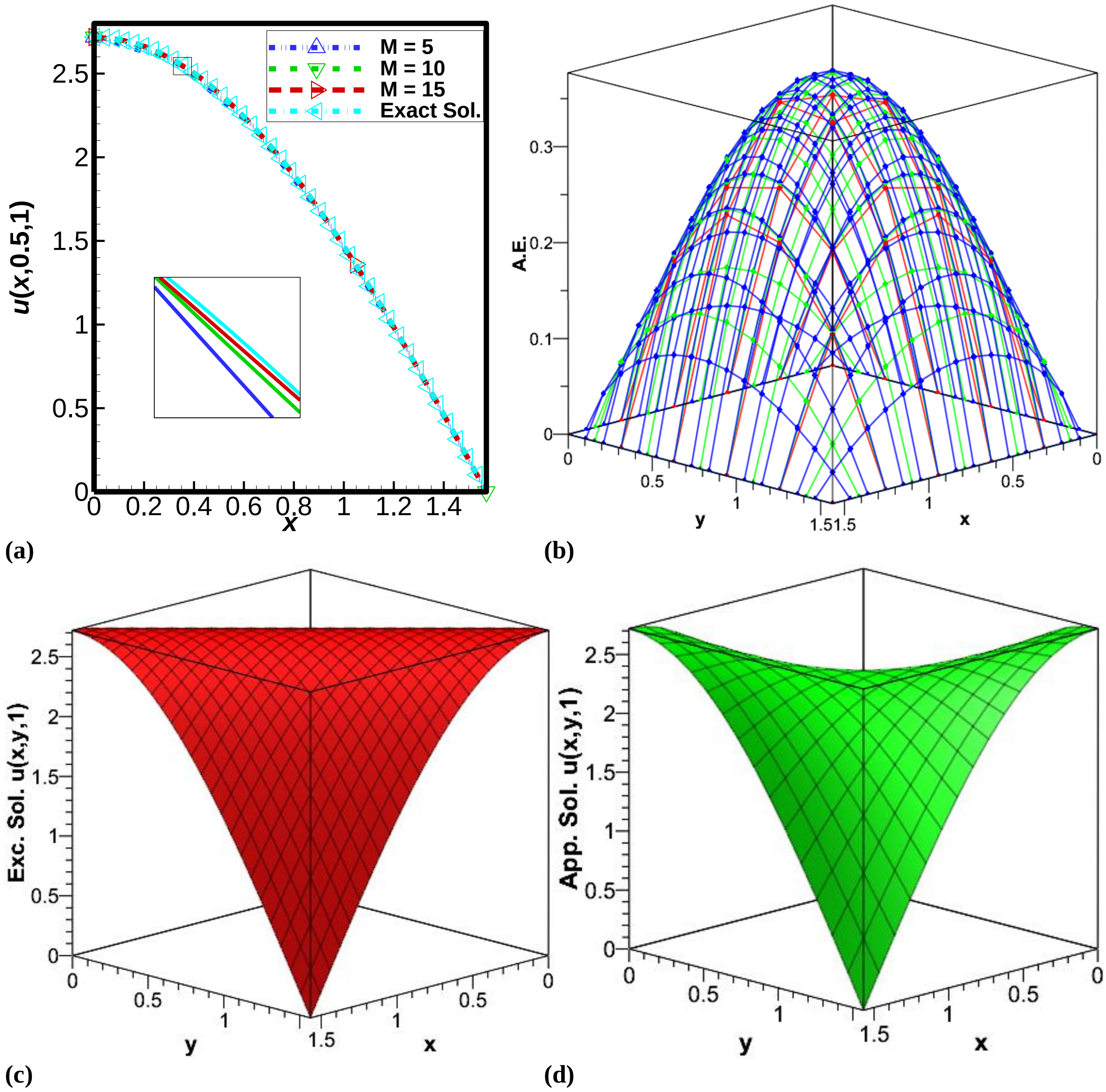


**Figure 5.4.2:** For $\mathbf{U}(\mathbf{x}, \mathbf{y}, \mathbf{t}) = \mathbf{e}^{\mathbf{t}}\mathbf{sin}(\mathbf{x}+\mathbf{y})$

Figure 5.4.2 Graphical illustration of **(a)** 2D behaviour of the numerical solution against $M = M_x = M_y$ **(b)** 3D variation in the absolute error behaviour against $M$ **(c)** exact solution $u(x, y, t) = \mathrm{e}^{t}\sin(x + y)$ and **(d)** approximate solution (when $M = 15$) in three-dimensions when $\alpha = 1.2$ and $N = 20$. Figure 5.4.2 also gives two-dimensional Crank-Nicolson technique solution graphs and validates the closed contact with the precise solution of Example $U = e^{t}sin(x + y)$. As a result, example have exact solution $U = e^{t}.sin(x + y)$ has been accurately resolved using the suggested way.

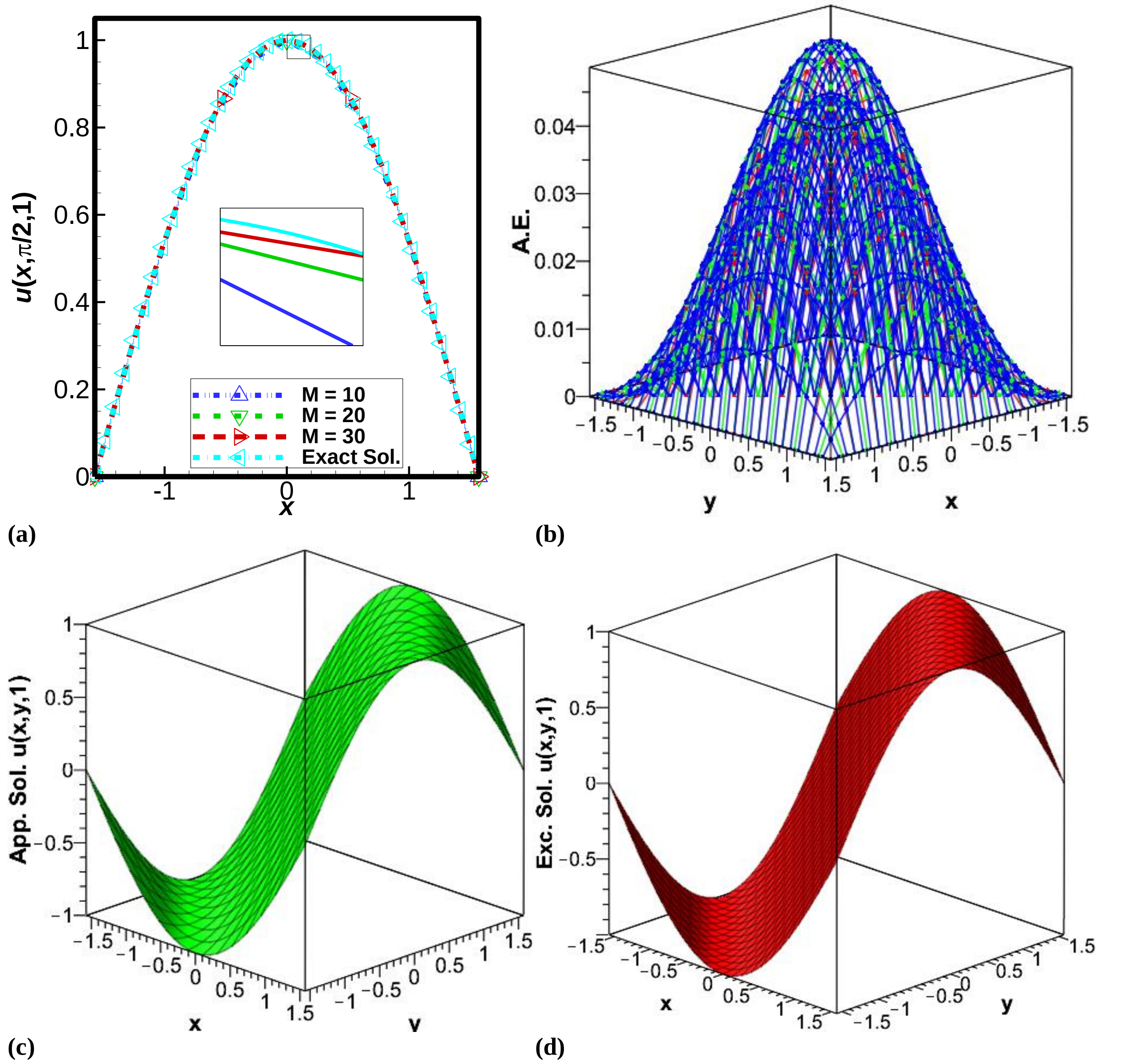


**Figure 5.4.3**: For $\boldsymbol{U(x,y,t) = t^2 sin(x+y)}$

Figure 5.4.3 Graphical illustration of **(a)** 2D behaviour of the numerical solution against $M = M_x = M_y$ **(b)** 3D variation in the absolute error behaviour against $M$ **(c)** exact solution $u(x,y,t) = t^2 \sin(x+y)$ and **(d)** approximate solution (when $M = 15$) in three-dimensions when $\alpha = 1.3$ and $N = 20$. The proposed method to tackle problems involving fractional initial and boundary values is validated by the solutions in fractional order at α=1.3. The findings are in-depth and directly tied to the precise answers to the issues.

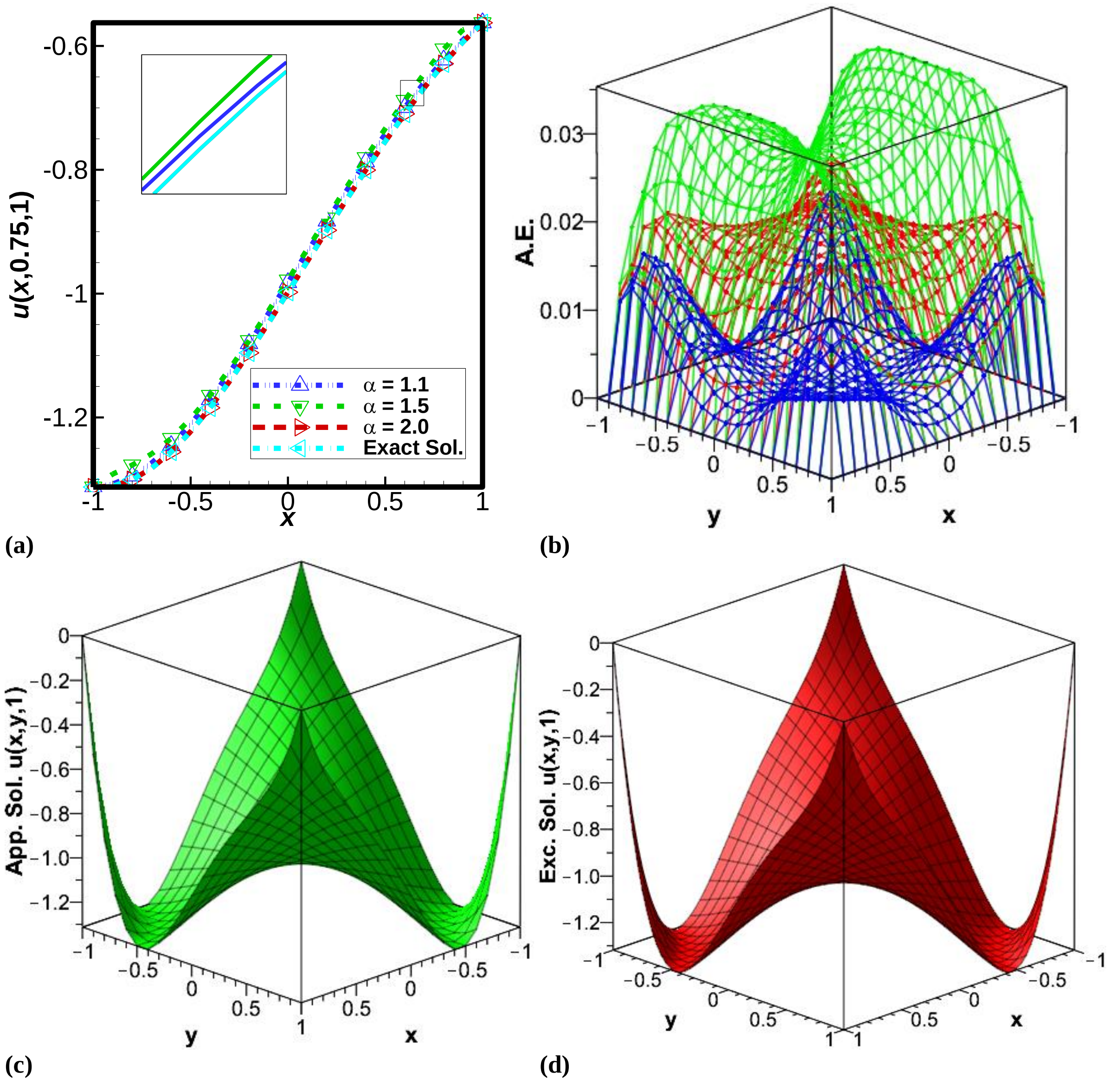


**Figure 5.4.4:** For $\mathbf{U(x, y, t) = x^4y^4t^3 - x^3y^3 - t^2 + xyt}$

Figure 5.4.4 Graphical illustration of **(a)** 2D behaviour of the numerical solution against $\alpha$ **(b)** 3D variation in the absolute error behaviour against $\alpha$ **(c)** exact solution $u(x, y, t) = x^4y^4t^3 - x^3y^3 - t^2 + xyt$ and **(d)** approximate solution (when $\alpha = 2$) in three-dimensions when $M = 30$ and $N = 20$. To show the accuracy of the technique, the approximate and exact solutions for the two-dimensional plot are shown in Figure 5.4.4 by setting values of the parameter α=1.1, α=1.5, and α=2. The solutions are obviously remarkably similar. Clearly, the suggested strategy is quite accurate and successful. The graph displays an incredibly high level of agreement between the solutions. Our technique seems to deliver significantly better accuracy.

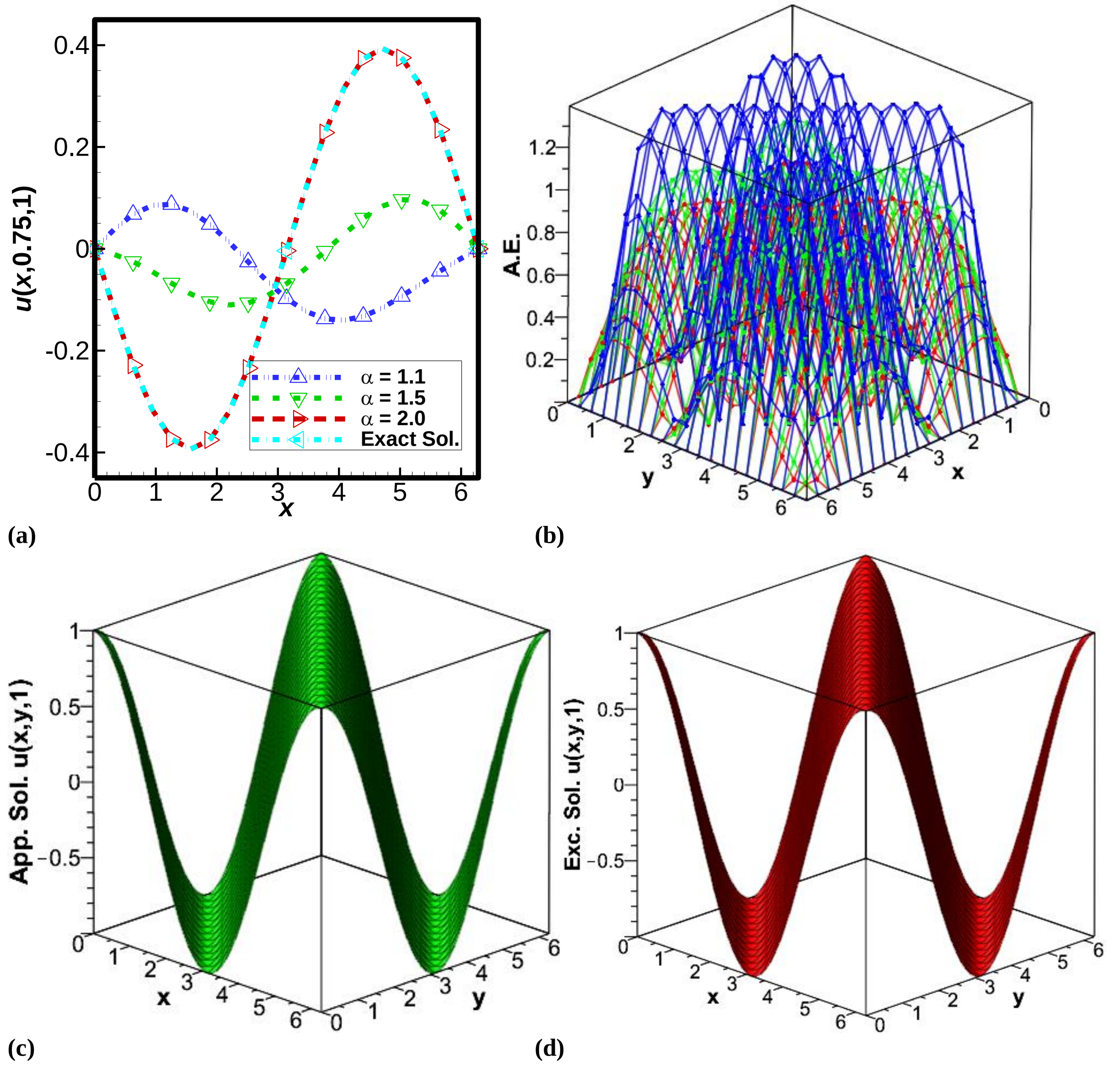


**Figure 5.4.5:** For $\mathbf{U(x,y,t) = tcos(x+y)}$

Figure 5.4.5 Graphical illustration of **(a)** 2D behaviour of the numerical solution against $\alpha$ **(b)** 3D variation in the absolute error behaviour against $\alpha$ **(c)** exact solution $u(x,y,t) = t\cos(x+y)$ and **(d)** approximate solution (when $\alpha = 2$) in three-dimensions when $M = 20$ and $N = 100$. In figure 5.4.5 plot the absolute error profile at various time scales to demonstrate the precision of the scheme. By fixing values of the parameter $\alpha = 1.1$, $\alpha = 1.5$, and $\alpha = 2$, the approximate and exact solutions for the 2D plot are displayed in Figure 5.4.5. It is clear that the solutions are remarkably similar. The proposed approach is obviously very precise and effective.

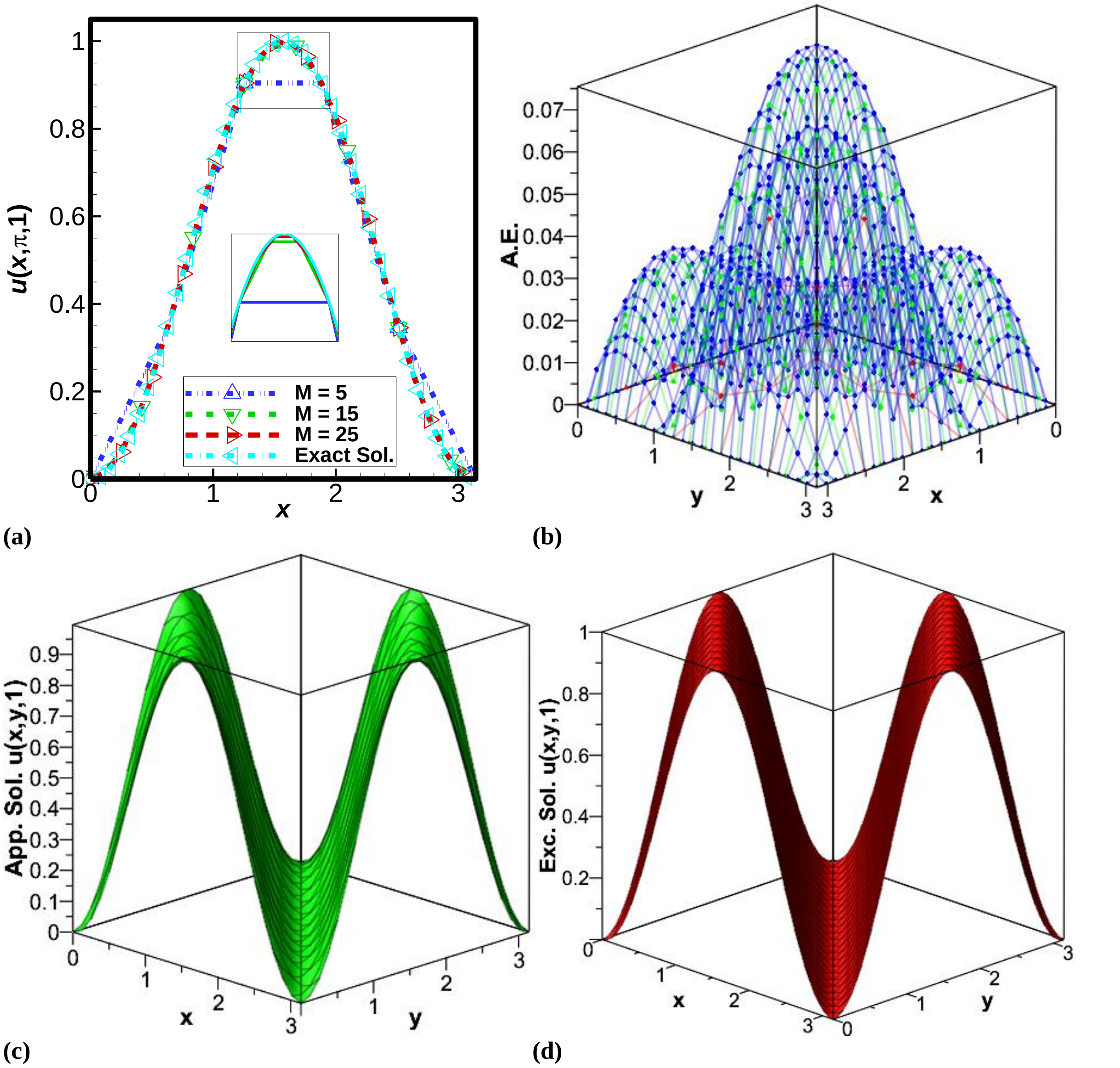


**Figure 5.4.6:** For $\boldsymbol{u(x,y,t) = t^3 \sin^2(x+y)}$

Figure 5.4.6 Graphical illustration of **(a)** 2D behaviour of the numerical solution against $M = M_x = M_y$ **(b)** 3D variation in the absolute error behaviour against $M$ **(c)** exact solution $u(x,y,t) = t^3 \sin^2(x+y)$ and **(d)** approximate solution (when $M = 25$) in three-dimensions when $\alpha = 1.6$ and $N = 20$. The proposed method is obviously highly precise and effective, and it has been observed that various values of α, the numerical solutions are closely related to the exact answers.

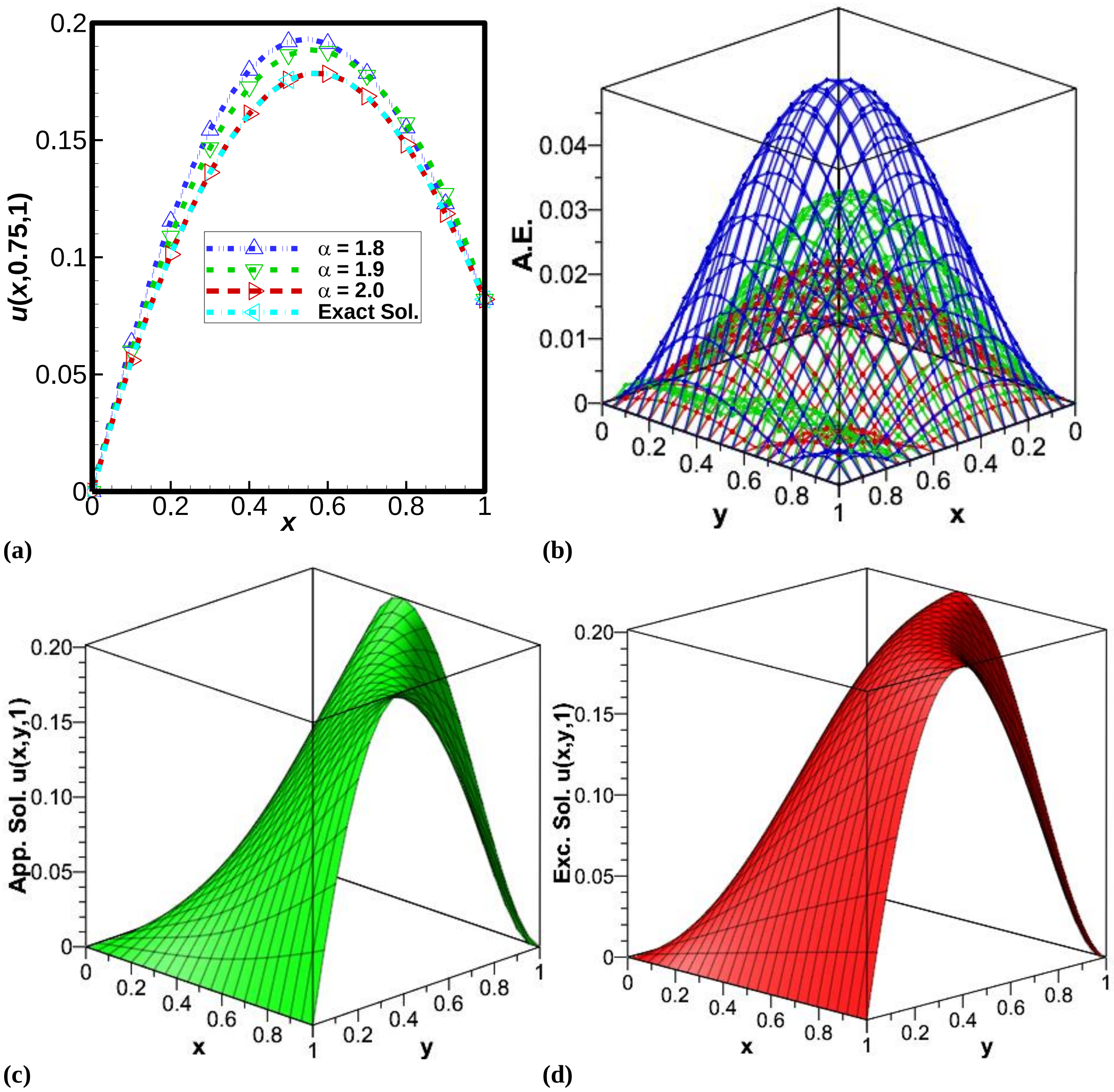


**Figure 5.4.7:** For $\mathbf{U(x, y, t) = x^4y^4t^3 - x^3y^3t^2 - x^2y^2t + xy}$

Figure 5.4.7 Graphical illustration of **(a)** 2D behaviour of the numerical solution against $\alpha$ **(b)** 3D variation in the absolute error behaviour against $\alpha$ **(c)** exact solution $u(x, y, t) = x^4y^4t^3 - x^3y^3t^2 - x^2y^2t + xy$ and **(d)** approximate solution (when $\alpha = 2$) in three-dimensions when $M = 20$ and $N = 20$. The two-dimensional graph 5.4.7 shows some surprising results: as the order of approximation increases when using the Crank-Nicolson scheme, the estimated solution approaches the particular solution, and the error rapidly declines.

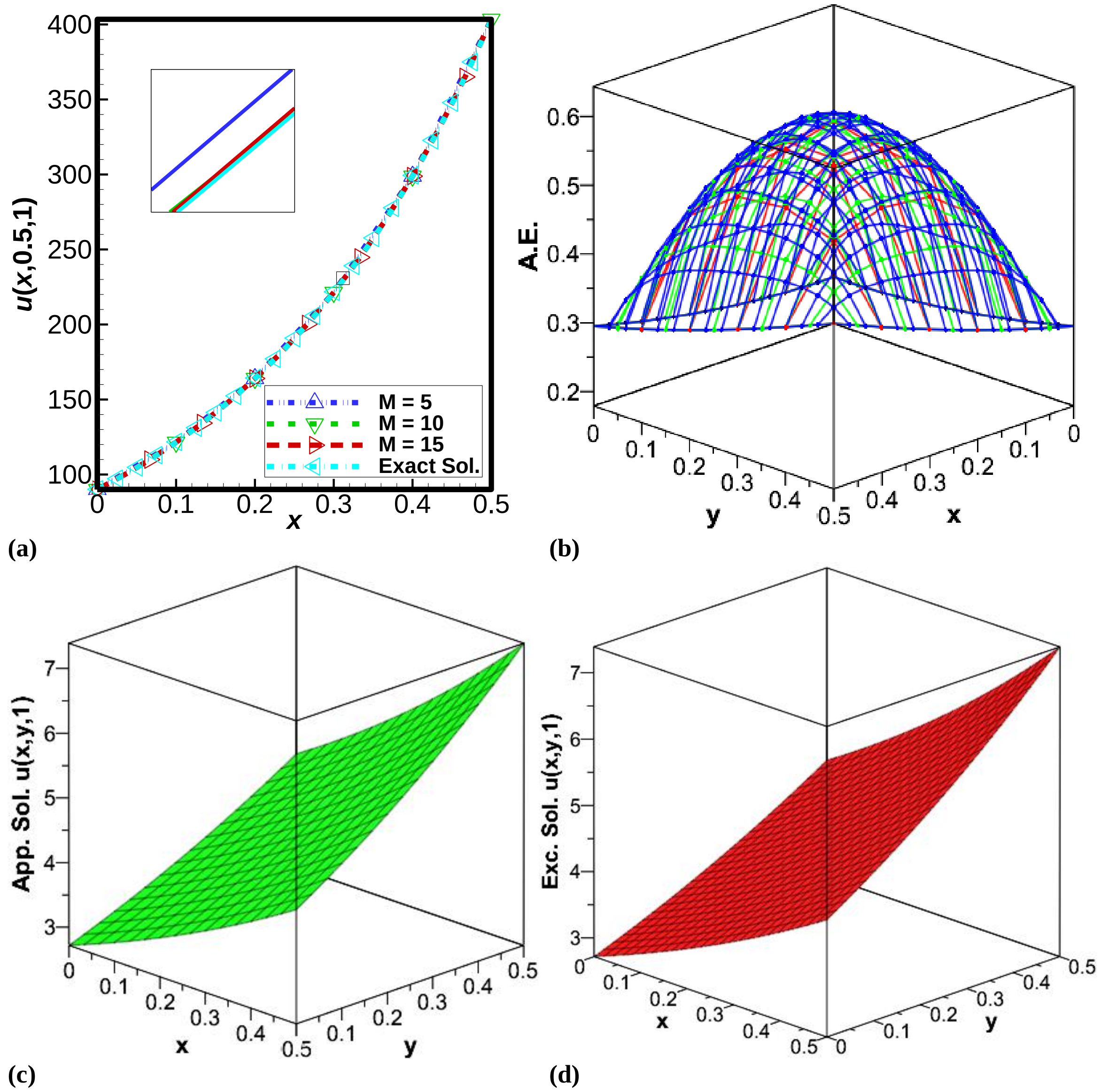


**Figure 5.4.8**: For $\boldsymbol{U(x,y,t)=e^{3(x+y+t)}}$

Figure 5.4.8 Graphical illustration of **(a)** 2D behaviour of the numerical solution against $M$ **(b)** 3D variation in the absolute error behaviour against $M$ **(c)** exact solution $u(x,y,t)=\mathrm{e}^{3(x+y+t)}$ and **(d)** approximate solution (when $M=15$) in three-dimensions when $\alpha=1.8$ and $N=20$. The behaviour of the numerical solution to the fractional wave equation and its precise solution is nearly identical. The estimated solution is thought to get closer to the exact solution, as well as the error minimized.

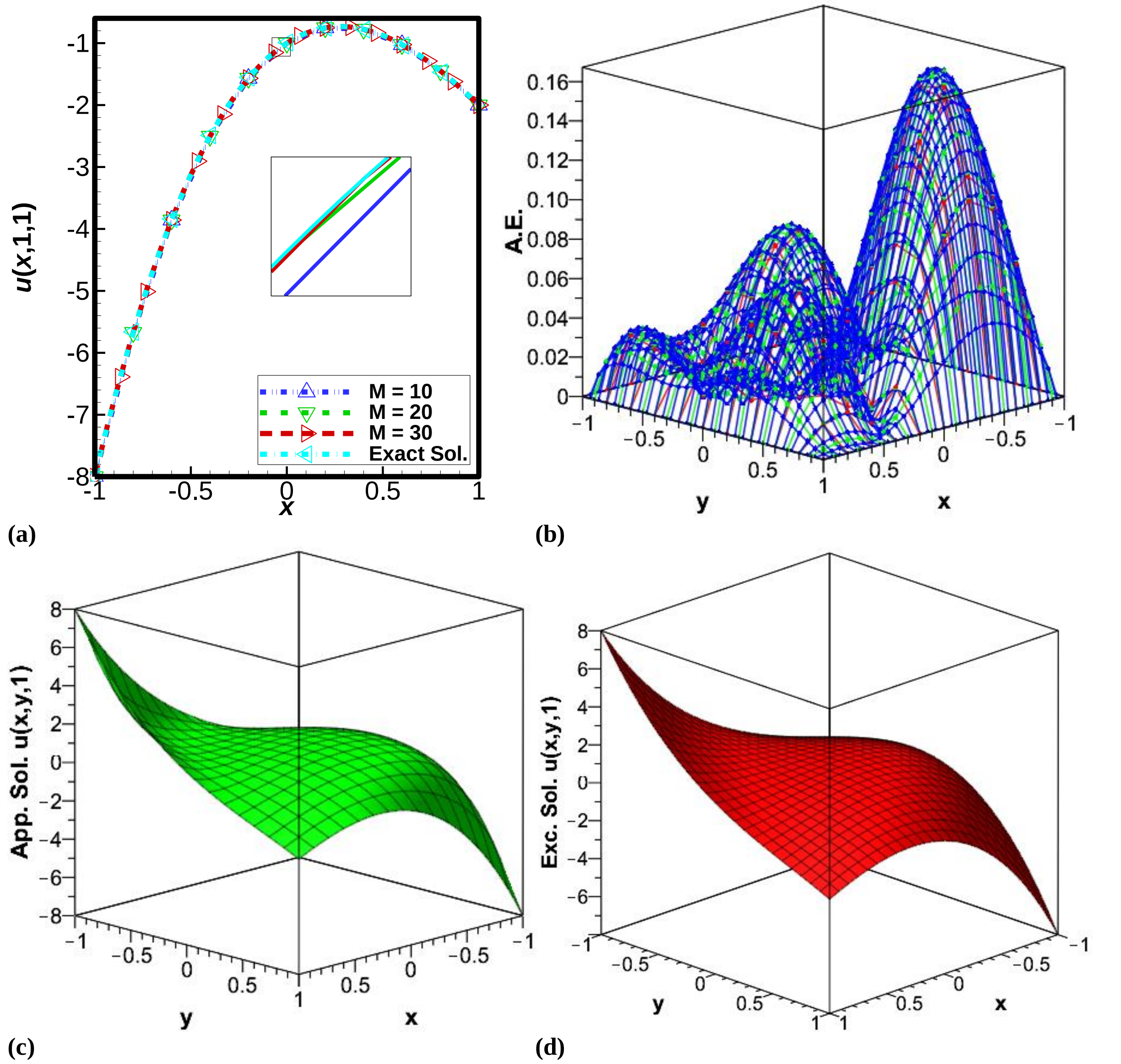


**Figure 5.4.9:** For $\mathbf{U(x,y,t) = (x-y)^3 t^5 + t^3 - (x+y)^2 + xyt}$

Figure 5.4.9 Graphical illustration of **(a)** 2D behaviour of the numerical solution against $M$ **(b)** 3D variation in the absolute error behaviour against $M$ **(c)** exact solution $u(x,y,t) = (x-y)^3 t^5 + t^3 - (x+y)^2 + xyt$ and **(d)** approximate solution (when $M = 30$) in three-dimensions when $\alpha = 1.9$ and $N = 20$. To show the accuracy of the technique, the approximate and exact solutions for the two-dimensional plots are shown in Figure 5.4.9 by setting values of the parameter α=1.9. The solutions are obviously remarkably similar. Clearly, the suggested strategy is quite accurate and successful.

## 5.5 Conclusion of 2D

In two dimensional the fractional wave equations are solved using a class of numerical techniques that are presented in this chapter. The finite difference approach based on the Crank-Nicolson formula underlies this class of procedures. The analysis of the numerical outcomes of the fractional finite difference scheme is given special consideration. The suggested problem's exact and numerical solutions are contrasted, and the resultant stability condition is quantitatively verified. This comparison allows us to draw the conclusion that the numerical solutions and exact solutions have a very good agreement. The Tec plot programming environment was used for all computations in this thesis work.

# CHAPTER 6

# CONCLUSION AND FUTURE WORK

## 6.1 Summary

This thesis extends and reviews P. Zhuang and F. Liu's finite difference modification for time fraction hyperbolic partial differential equations. First, we consider the fractional wave equations in one and two dimensions over a finite region. Differential equations have been used to create a model for the issue in physics and other domains. Stability is proved, and the convergence in a bounded domain is examined for the one-dimensional and two-dimensional Crank-Nicolson differential approximations. Numerical findings are presented using graphs. In order to investigate the exact solution of the problems, an effective connection between the fractional order wave equation and the Crank-Nicolson method is presented in this study. This method is based on discretizing the Caputo sense using a finite difference formulation. The stability of the system has been extensively studied through the use of techniques such as Von Neumann stability analysis. It is shown that the scheme is perfectly stable. The system is also given a convergence analysis. The main finding was as follows:

It has been determined and confirmed that the suggested approach is effective, well-suited, and accurate for dealing with fractional order wave models. By changing the value of $\alpha$(alpha), the graphs of the One dimensional and two-dimensional equations are approximately the same as the numerical solution. Some innovative results demonstrate that the proposed strategy can produce more accurate results with less computing effort and expense. Exact and numerical solutions are provided for t=0.1. There is excellent agreement between the numerical solution and one-dimensional numerical methods. The conformable fractional derivative brings great convenience to the study of fractional differential equations due to its unique properties. By taking a comparison with different parameters the equation gives an exact solution. By using an error analysis against the parameter $\alpha$, convergence of the provided strategy has been demonstrated. The Crank-Nicolson technique can be applied to solve FDEs. It is important to state that the suggested solution reduces computing time and work at a concrete level.

The examination story in the field of numerical techniques is not yet complete. The recommended approach can be expanded to investigate the numerical wave solution of several physical mechanisms that need to be investigated using suggested techniques, such as: Higher-dimensional fractional wave models, generalized fractional wave models, Physical problem arising in physical nature.

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