INVESTIGATION OF COEFFICIENT INEQUALITIES FOR CERTAIN NEW SUBCLASSES OF ANALYTIC FUNCTIONS

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Candidate of Master of Science in Mathematics at the National University of Modern Languages do hereby declare that the thesis Investigation of Coefficient Inequalities For Certain New Subclasses of Analytic Functions submitted by me in partial fulfillment of MS MATHS degree, is my original work and has not been submitted or published earlier. I also solemnly declare that it shall not, in the future, be submitted by me for obtaining any other degree from this or any other university or institution. I also understand that if evidence of plagiarism is found in my thesis/dissertation at any stage, even after the award of a degree, the work may be canceled and the degree revoked.

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ABSTRACT

Title: Investigation of coefficient inequalities for certain new subclasses of analytic functions

The aim of this research is to define and discuss some new subclasses of starlike and convex functions in an open unit disk. The q-theory and q-differential operators will be used to present the q-version of already existing results on starlike and convex functions. The classes of analytic functions with respect to the symmetric point will be explained with the help of q-derivative certain new classes of q-starlike and q-convex functions with respect to symmetric points subordinated with exponential functions will be introduced, and these classes will further be modified by using exponential function with subordination technique. Coefficient inequalities for the functions belonging to the new classes will be investigated. We will determine the possible upper bound of the 3rd Hankel determinant for the q-starlike and q-convex functions. The relevant connections of our new classes and results to known ones will be also pointed out.

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LIST OF SYMBOLS

E	Open unit disc
$\not\subset$	Set of complex numbers
A	The class of normalized analytic functions
S	Class of univalent functions
C	Class of convex univalent functions
S^*	Class of starlike univalent functions
$k(z)$	Koebe function
P	Class of analytic functions with positive real parts
Υ	Family of Schwarz functions
$\prec$	Subordination symbol
C_s	Class of convex functions with respect to symmetrical points
S_s^*	Class of starlike functions with respect to symmetrical points
$S_{s,q}^*$	Class of q-starlike functions with respect to symmetrical points
$C_{s,q}$	Class of q-starlike functions with respect to symmetrical points

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DEDICATION

This thesis work is dedicated to my parents, family, and my teachers throughout my education career who have not only loved me unconditionally but whose good examples have taught me to work hard for the things that I aspire to achieve.

CHAPTER 1

INTRODUCTION

1.1 Overview

This research approaches the interesting area of mathematics called Geometric Function Theory which covers the study of geometric properties of an analytic function. This is one of the most fascinating branches of mathematics and it was established at the start of the 20th century. Though similar ideas also appear in real analysis, in complex analysis the geometry of functions has had a greater impact. Geometric Function Theory has multiple applications in different areas of science and mathematics. Due to the application of fundamental hypergeometric series to a variety of topics, including combinatorics and quantum theory, the field has rapidly grown.

1.2 Riemann Mapping Theorem

To replace a reasonably complicated arbitrary domain, a result by Bernard Riemann was given known as the Riemann Mapping Theorem in 1851. This allows us to consider open unit disk $E = \{z : |z| < 1\}$ as a domain. As a base of Geometric Function Theory, this theorem plays a vital role. The base of modern function theory is laid by the prominent contribution of Cauchy, Riemann, and Weierstrass which was established in the 19th century.

1.3 The subclass of analytic and univalent functions

After that in 1907, univalent functions were studied by Koebe [1, 2] in which both univalent and analytic functions were considered in open unit disk E. He discovered functions in open unit disk E that are both analytic & univalent. the univalent analytic and normalized functions are collected in one class S, For more detail see [1, 2]. The geometry of the image domain indeed plays a very important role in the detailed study of analytic functions because these are classified into different classes and further into sub-classes entirely based upon the shapes of their image domains and other geometrical properties. Therefore, introducing and studying new geometrical structures as image domains and defining their associated analytic functions has always been a matter of discussion amongst researchers. The class S of normalized univalent function is a very significant class in Geometric Function Theory. The main sub-classes of class S are class S^* of starlike, the class C of convex univalent function class K of close-to-convex univalent function, and the class C^* of quasi-convex univalent function. In 1915, Alexander [3] linked two classes of convex & starlike functions by leading a relation called the Alexander relation. Any convex function can be taken to be lower semi-continuous with some possible redefinitions on the boundary.

1.4 Coefficient bounds

Geometry Function Theory deals with the geometry of complex-valued functions, the problem of discovering coefficient bounds plays a vital role and these functions are divided into various subfamilies of set A of normalized analytic functions because of its diverse geometrical analysis of image domains where A has functions $f(z)$ that are analytic in the open-unit disk E. The known coefficient assumption for the function was defined by de-Branges in 1985 and used by Bieberbach in 1916 which is the greatest emergent outcome. Numerous mathematicians have done remarkable work to disprove or prove this assumption and as a consequence they presented several sub-families of the class S of univalent function with deference to the geometrical approach of their images area between 1916-1985. For more detail, see [1, 2].

1.5 q-calculus

q-calculus is a methodology equal to the use course of calculus but which is centered on the solution of deriving q-analogous results with-out the use of limits. Jackson [4] deserves credit for the systematic introduction of q-calculus and Jackson [5] first introduced and provided definitions for q-derivative & q-integrals. In the 1740s, Euler established the theory of partitions commonly known as additive analytic number theory which was the genesis of q-analysis. Euler's works were not gathered and published until the early 1800s, under the name of the legendary Jacobi, despite the fact that he had always written in Latin. In 1829, Jacobi proposed his elliptic function which is theoretically equal to q-analysis as well as his triple, produce identity (also known as the Gauß Jacobi triple by-product identity). C. F. Gauß (1777–1855) was a significant contributor to the development of q-calculus. He is credited with the invention of the hypergeometrical series and associated contiguity relations in 1812.

A crucial topic of research in the realm of conventional mathematical-analysis is quantum or q-calculus. Its focus is on a useful gener-alization of integration and differentiation procedure from a theoretical stand point. A vast area of mathematics study with ancient roots and a renewed emphasis in the present day is quantum calculus. Significantly, the origins of quantum calculus may be traced all the way back to Bernoulli and Euler's function. This is an important aspect of the field. Nonetheless, because of its numerous uses, it has recently caught the attention of modern mathematicians. It is more challenge than other mathematics disciplines since it involves intricates calculations and computations. The main objective of this thesis/research will be to study and by using q-derivative investigate the geometric characteristics of analytic functions. Ismail *et al* [6] have introduced q-calculus in the field of Geometer Functions Theory for the very first time. Produced one of the very first contributions to the use of q-calculus in Geometric Function Theory. Who created the class of starlike function generalized versions. He did it by using the difference operator and at the same time, he made a suitable change in the domain of the functions. He gave his brand-new class the name class of q-starlike function when he first introduced it. Properties of q-starlike & q-convex functions investigated by various researchers, which have been introduced by [7, 8].

In 1907, Koebe [9] introduced the theory of univalent-functions which is a classical problem of complex analysis. He discovered functions in open unit disk E that are both analytic & univalent. The univalent analytic and normalized functions are collected in one class S. The group of functions like these that are analytic & univalent in the open-unit disk E, and satisfy the normalization conditions was then known as the class S. The class S of univalent function is the primary focus of investigation in the majority of the Geometric Function Theory. In 1921, Nevanlinna initiated the concept of a starlike function in open unit disk E, see [10]. The classes of convex & starlike functions were defined and their geometrical behavior was analyzed, see [1].

1.6 Hankel determinant

The Hankel matrix which is a square-toed matrix and was named after Hermann Hankel is distinguished by the fact that each ascending skew diagonal from right to left is constant. Determinants of such matrices are of great importance. In 1976, Noonan & Thomas [11] investigated the Hankel determinant of certain analytic functions.

Perhaps the most exciting part of Complex Function Theory is the interaction between geometry and analysis. These connections between geometric behavior and analytic structure are at the heart of theory regarding univalent functions. If $f(z_1) \neq f(z_2)$ if $z_1 \neq z_2$, then functions are considered theory regarding univalent functions accepting identical value twice. The current survey will concentrate on the subclasses of S functions $f(z) = z + a_2z^2 + a_3z^3 \ldots$ Univalent and analytic in the open-unit disk $|z| < 1$. This is the class that involves all univalent function that has been normalization by the condition $f(0) = 0$ & $f'(0) - 1 = 0$ With an emphasis on recent findings and open issues, we will focus on coefficient difficulties for the class S and any classes that are connected to it. Most of the methods we shall communicate have wide scopes and are not restricted to a coefficient problem, introduced by Duren [12] in 1977. The Hankel determinant of the univalent functions was introduced by Pommerenke [13] in 1967. Babalola [14] was the first person to examine the upper bound of $H_3(1)$ for subclasses of S. The estimate for the 3rd Hankel-determinant of the Taylor coefficient of the function $f(z)$ in the open unit disk falls under particular categories of analytic functions. Restrictions on the 3rd Hankel determinant for specific categories of analytic functions were introduced by Prajapat *et al* [15, 16].

This is quite natural to discuss the behavior for sub-classes of normalized univalent function in the unit-disk by the Fekete-Szego problem Kanas and Darwish obtained to see, [17] the convex and starlike univalent functions of complex order, Hankel determinant for subclass S^* and C of a starlike and convex function is investigated by Janteng *et al* [18]. Lecko *et al.* [19] investigate the third kind's sharp-bound of the Hankel determinant for starlike function with order $\frac{1}{2}$.

1.7 The starlike functions with respect to symmetric point

Analyzing symmetrical points w.r.t subclasses of analytic classes is another crucial component of these classes. In 1959, Sakaguchi [20] introduced the class S_s^* of univalent function starlike w.r.t symmetrical points. Zaprawa [21] studied the solution to coefficient inequalities for starlike functions w.r.t symmetric points. A number of coefficient problems have been resolved using the above-described innovative method.

1.8 Preface

The class of starlike functions with regard to symmetrical points connected to the exponential functions is the subject of our study. 3rd Hankel determinant for starlike and convex functions w.r.t symmetric points, see [22]. The chapter-wise description is given as under.

In Chapter 2, we will study the basic concept. Preliminary results will be used to derive the main results and we have discussed the basic classes like the class of univalent function. the class of caratheodory functions and the subclass of univalent functions and their related results. The coefficient inequality, Fekete Szeoga problems, and Hankel determinant and result have been discussed in this part of the thesis and we have also discussed the class of starlike function w.r.t the symmetrical points and subordination with an exponential function.

In Chapter 3, Focuses primarily on the foundational ideas of Geometric Functions Theory, which will be essential for understanding later chapters. It begins with the idea of analytic functions, moves on to the concept of normalized univalent function in open-unit disk E, and then discusses the fundamental subclasses of univalent function. Additionally, the relationship between the associated subclasses of the Caratheodory function class P will be examined. It will be emphasized how important methods like subordination are utilized to explore specific aspects of analytic functions. There is a thorough introduction to q-calculus as well as current classes of

q-functions. This chapter does not include any new results and all of the concepts and results discussed in this chapter are well-known and properly referred to throughout its entirety.

In Chapter 4, two generalized classes have been introduced and explored, the class $S_s^*(e^z)$ of starlike functions and the other class $C_s(e^z)$ of convex function by using subordination and a certain exponential function. We have looked into some intriguing aspects and a number of inclusion outcomes for these functions. By applying certain values to the various parameters, it is possible to extract a number of previously derived results as special instances based on our main results.

In Chapter 5, We will introduces new classes using q-calculus. We have cases defined and studied the new analytic classes $S_s^*,_q(e^z)$ of q-starlike functions and $C_s,_q(e^z)$ convex functions w.r.t symmetric points. We have investigated several interesting properties of these classes like coefficient inequalities, Fekete Szeago problem etc. We have found that many of the previously deriver results can be followed as particular cases by using our findings with certain Hankel determinants.

In Chapter 6, we concluded our research work.

CHAPTER 2

LITERATURE REVIEW

The concept of Geometric Functions Theory was initiated on the basis of analytic function. The analytic function was first defined by Duren [23] in 1983. He introduced the class A of analyte function that are nor-malized by conditions that is $f(0) = 0$ & $f'(0) = 1$ where $f(\{z) = \{z + \sum_{n=2}^{\infty} a_n\, z^n, \{z \in c\}, |z| < 1$. Robertson initiated the concepts of the theory of univalent function. we will talk about different kinds of analytic functions, see [24]In 1936. In 1964, Macgregor [25] introduced the class of univalent functions. In 1975, Silverman [26] studied univalent functions with negative coefficients that are starlike of order α and convex of order α coefficient distortion covering and coefficient inequalities are found. As is customary we will refer to C as the set of complex numbers. Let us also indicate by the letter S the subclasses of function in A that are univalent in U. In 2000, Altintas [27] investigated the author's demonstration of a number of inclusion connections connected to the (η, δ)neighbor-hoods of various sub-classes of starlike and convex functions of complex order using the well-known idea of a neighborhood of analytic function. In 2001, Frasin *et al* [28] introduced to look into some aspects of this class, we take into consideration the category of analytic functions $B(\alpha)$. In the open unit disk, we establish some intriguing requirements for the class of strongly starlike and strongly convex objects.

Pommerenke [29, 30] defined the Hankel determinant $H_q(n)$, where q and n are positive integers, for the functions as S, as shown in the following:

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} \tag{2.1}$$

For fixed positives integer q & n the growths of $H_q(n)$ as $n \to \propto$ have been determined by Noor [31], with bounded boundarys. Ehrenborg [32] investigated the Hankel determinant for exponential polynomial. For various values of q and n, the Hankel determinant of different orders is found. For instance when $q = 2$ and $n = 1$ the determinant

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix}, = |a_1a_3 - a_2^2|, (a_1 = 1) \tag{2.2}$$

This determinant is a special case of figuring out the highest value of a function $|a_3 - \mu a_2^2|$ in S when μ is real or complex. This is known as the Fekete Szego problem. An other researchers like Deniz *et al* [33], Lee *et al* [34],Cho and Owa [35], Keogh & Merkes [36], Ma [37], Magesh and Balaji [38], Murugusundaramoorthy *et al* [39], Reddy *et al* [40], Ravichandran *et al*[41], Tang *et al* [42]. have investigated the Fekete-Szego inequalitys for a variety of univalent analytic function subclasses. Now it can be determined that for $q = 2$ and $n = 2$.

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}, = |a_2a_4 - a_3^2| \tag{2.3}$$

In 1968, Hayman initialed the concept of a second Hankel determinant in univalent function, see [43]. In 1976, Noon-an and Thomas [44] introduce we calculate the second Hankel-determinants growth rate for an essentially mean p-valent function. A number of scholars have looked into what the highest possible value of $H_2(2)$ could be. Janteng *et al* [45, 46], defined the class the letter S the class of univalent function in the open unit disk E, are analytic, have been normalized, and have only one possible value. The class of S^* starlike and C convex function are the significant subclasses of S. Bansal [47], Lee *et al* [48], Liu *et al* [49], Orhan *et al* [50], Laxmi and Sharma [51], Shrigan [52], and Huey *et al* [53]. In 2018, Zaprawa [54] Provide a different approach to estimating the upper-bound of the Hankel determinant for $q = 2$ and $n = 3$ as $H_2(3)$ for the different subclasses of S. present a direction in assay the upper-bound of the Hankel determinant, This determinant we have

$$H_2(3) = \begin{vmatrix} a_3 & a_4 \\ a_4 & a_5 \end{vmatrix}, = |a_3a_4 - a_4^2| \tag{2.4}$$

Bansal *et al* [55], Lecko *et al* [56], Kowalczyk *et al* [57], Obradovic *et al* ([58], [59]),Kumar *et al* [60], Wang *et al* [61]. Investigated the formula for the Hankel determinant $H_3(1)$, also known as the 3rd order Hankel determinant can be found as following:

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}, \tag{2.5}$$

$$= a_3(a_2a_4 - {a_3}^2) - a_4(a_1a_4 - a_2a_3) + a_5(a_3 - {a_2}^2), a_1 = 1 \tag{2.6}$$

Takahashi & Nunokawa [62] acquire a specific relationship between the two classes of functions that we define, which are denoted by the notations $S^*(a,p)$ and $C(a,p)$, respectively. Kumar *et al* [63] investigated We look at two Ma–Minda–type sub-classes of starlike & convex functions that are related with the normalized analytic function This function transfers an open unit disk E onto a Nephroid–shaped bounded domains that is located in the right–half of the complex planes. We look into the properties of quasi-Hadamard and convolution products for these classes of functions. Singh *et al* [64] introduced we define several analytic function classes, and their subclasses, and derive sharp upper bounds on the functional $|a_3 - \zeta a_2^2|$ for the analytic function $f(z)$ belonging to these classes and subclasses. Tang *et al* [42] analyzed for a specific normalized analytic function defined on the open unit disk and the authors find Fekete-Szego inequality. Resides in a space that is both symmetric and starlike in relation to the real axis. Wang *et al* [65] introduced coefficient inequality for starlike functions. All of these bounds are sharp, including the bounds of the 1st and 3rd initial coefficients the bounds of inequality of the Fekete-Szego types and estimates of the 2^{nd} and 3rd Hankel determinants for the subclass of starlike functions.

In 1984, Mocanu [66] introduced w.r.t symmetric points on the starlike function. Aghalary *et al* [67] studied the problem of stability for the class of function that are uniformly starlike w.r.t symmetrical points and we offer the lower bounds of their stable radius. In 2015, Krishna *et al* [68] introduced a sharp-upper bounds for the 2nd Hankel functional associated with the k^{th} root transform $[f(z^k)]^{\frac{1}{k}}$ a starlike & convex functions w.r.t symmetric points constructed on the open unit disk in the complex plane using Toeplitz determinants is said to belong to the classes of normalized analytic functions $f(z)$ when it is shown to belong to this class. In 2020, Singh and Kaur [69] analyzed by making use of subordination, we were able to establish the Fekete Szego functional as well as the sharp upper bounds for the functions that belongs to a particular subclass of starlike function that is created by symmetric points. Senguttuvan *et al* [70] developed a thorough subclass of analytic functions with regard to symmetrical points, which are denoted by the notation (j,k). In addition, they have broadened the scope of the study by incorporating quantum calculus. In 2006, Shanmugam *et al* [71] introduced sharp upper bounds of $|a_3 - \mu a_2^2|$ are established for functions $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$ that belong to specific subclasses of starlike & convex functions w.r.t symmetrical points. In 2012, Singh *et al* [72] integral representation are developed, and precise coefficient estimations are found. In addition to this, the Fekete-Szego problem has been solved, and the second Hankel determinant has been taken into consideration for these classes. In 2014, Aouf *et al* [73] investigate the upper-bound of the form $|a_3 - \mu a_2^2|$ for the function $f(z)$ that belongs to a certain subclass of starlike function they w.r.t k-symmetrical points of complex order. These bounds are sharp. In addition to this, we provide applications of our findings to a variety of functions, each of which is defined by convolution with normalized analytic functions.

In 2021, Panigrahi *et al* [74] investigated this is the case when the function belongs to particular subclasses of starlike and convex functions w.r.t symmetrical points. In addition Fekete Szego inequalities for the function denoted by $\frac{z}{f(z)}$, as well as the inverse function, denoted by f, are explored for the classes that have been previously specified, and particular examples are highlighted for discussion. In 2022, Karthikeyan *et al* [75] introduced a new sub-class of multivalent functions w.r.t symmetrical points featuring highers order derivatives is presented here, as well as studied in detail. They have established the class subordinate to a conical region that is affected by Janowski function in order to unify and extend a variety of wellknown result. In doing so, they hoped to broaden their applicability. The most important findings of their study are the outcomes of inclusion, the subordination property, and the coefficient inequality of the designated class. In 2022, Mohamad *et al* [76] studied the functions W.r.t symmetric conjugate points in an open unit disk by defining new sub-classes $S^*_{s,c}(\alpha, \beta, A, B)$ and derived some of its fundamental characteristics. For functions in this new subclass it was examined how to estimate the Taylor-Maclaurin coefficients the Hankel determinant the Fekete-Szego inequality and the distortion and growth constraints. In 2016, Vamshee Krishna *et al* [77] studied this study uses Toeplitz determinants to give the best upper bound on the $H_3(1)$ Hankel determinant for starlike & convex functions w.r.t symmetric points. In 2017, Patil and Khairnar [78] introduced is to use the Toeplitz determinant to get the best upper-bound for the $H_3(1)$ Hankel-determinant for a starlike function w.r.t symmetric point. In 2022, Khan *et al* [79] introduced the class of starlike function w.r.t symmetrical points subordinated with sine functions. In addition to this, they research us to study the lower bounds & upper bound for the coefficients of the 3rd

Hankel-determinant for this stated class. In addition to this, the Zalcman functional $(a_3^2 - a5)$ is also determined.

In 2019, Naz *et al* [80] introduced Mocanu & Miller defined the class of admissible. Function (ω, q) such that the differential, subordination implies $p(\hat{z}) \prec q(\hat{z})$, where p is an analytic function in D with $p(0) - 1 = 0$. They analyzed how this class works when $q(z) = e^z$. As an application, they found several conditions that make it clear that normalized analytic function f belongs to the group of starlike function that are subordinated to the exponential function. In 2022, Lecko *et al* [81] introduced and explored a new category consisting of regularly occurring functions in the unit disc. In order to accomplish this, they make use of a modified variant of the intriguing analytic formula that Robertson presented (but did not fully use) for starlike functions with respect to a boundary point. This is done by subordinating the functions to an exponential function. In 2022, Singh [82] introduced the upper bound of a number of coefficient functional for a particular subclasses of analytic functions connected to the exponential functions in the open-unit disk. In 2000, Ehrenborg [83] studied a determinant of exponential polynomials called the Hankel determinant. In 2019, Zaprawa [84] introduced S_e^* and K_e, two classes of univalent functions. Both classes maintain their symmetry or invariance when subjected to rotations and a few issues relating to the coefficients of these types of functions. are investigated for that classes an estimation of the Hankel-determinants $H_{2,1}, H_{2,2}$, & $H_{3,1}$. In addition to this, nearly all of these inequalities are sharp. The primary concept that is presented in his study is based on the concept of representing the functionals that are being addressed in terms of their dependence on the fixed second coefficient of the functions in specific classes. In 2019, Shi [85] Studied a look at particular sub-families S_e^* & C_e of univalent function associate with exponential function that are symmetrical along the real axis in the scope of the open-unit disk E. Finding the boundaries of the Hankel determinant for order three is our objective for these classes. In addition the estimated of the 3rd Hankel determinant that is presented in this study for the family S_e^* improves the constraints that were examined not too long ago. In addition, research into 2-fold symmetrical functions and 3-fold symmetrical function has been conducted using the same constraints. In 2022, Shi *et al* [86] introduced the inverse function connected to a class of bounded turning function subordinated to the exponential function were the focus of our discussion of specific coefficient-related issues. We determined the upper and lower bounds for a few starting coefficients the Fekete Szegö types inequality and the estimation of the 2nd and 3rd Hankel determinant. These boundaries have all been shown to be precise. In 2022, Shi *et al* [87] analyzed the coefficients of the functions, and related function can be made using constraints on the logarithmic coefficient of analytic functions. This finding has led to a lot of interest in the study of logarithmic-related issues of a particular subclasses of univalental functions in recent years. A sub-class of starlike function S_e^* associated with the exponential mappings was taken into consideration in the current analysis.

Another important aspect of subclasses of starlike and convex with respectable to symmetrical points is related to an exponential function. Cho *et al* [88] defined the class S_α^* of the starlike function of order α. Of starlike functions associated with exponential functions. In addition to this, sharp radii problems have also been investigated in their article. In 2015, Mendiratta [89] extended the work of Zhang *et al* [90] by determining the Fekete Szegö problems for a class S_s^* and the 3rd Hankel deter-minant and upper-bound of the deter-minant $H_3(1)$ are also obtained. The inclusion relation coefficient estimates growth less and distortion result subordination theorem and different radii constant for function belonging to the class S_e^* are derived using the structural formula. In 2020, Ganesh *et al* [91] investigated the class $S_s^*(e^z)$ of starlike function with respect to symmetrical function. Various intention results related to coefficient inequalities and Hankel determinant for function belonging to the class $S_s^*(e^z)$ are derived subordination has been used to define the corresponding class $C_s(e^z)$ of convex function w.r.t symmetrical point associated with exponential functions.

It took a very long time for additional development in this area to take place, but it turned out to be a successful comeback when see, [92, 93] published their work on complexs operator along with their separate q-generalization. These are referred to as q-Picard singular integral operators and q-Gauss-Weierstrass singular integral operators respectively. By utilizing fundamental q-hypergeometric functions, in 2011, Srivastava [94] built a solid framework for q-calculus applications in Geometric Function Theory. Aral and Gupta [95, 96, 97] produced another series of contributions by defining the q-askakov Durrmeyer operator by using q-eta functions. These definitions were published in three separate years. They also used their q-extensions to come up with a number of geometric results. In 2012, Purohit [98] introduced was the first person to introduce and analyze a class in the open-unit disk for the multivalently analytic functions. Additionally, he was the first person to employ a specific operator of q-derivative in a work. He made a significant contribution by providing q-extensions for several findings in analytic function theory. In a way analogous to this a number of q-calculus operators such as integral and derivative in fractional form have been utilized to define and investigate a variety of different sub-classes of analytic functions. In 2013, Aldweby and Darus [99, 100] introduced the q-perators by utilizing the idea of convolution of analytic functions that are normalized, which was inspired by the research done in q-calculus. In addition, they talked about the geometrical structure of the defined operators in the classes of analytic functions which involve the q-version of hypergeometric function in compact disk. Selvakumaran *et al.* [101, 102] define the ideas of a fractionals form of q-calculus to explain the study being done in Geometric Function Theory about q-calculus. Using the unit disk as the domain, he defined and researched the q-integral operators for analytic functions. During the course of their additional research on these operators, the researchers noticed that the convexity of defined operators in classes of analytic functions that had actually been defined by a linear multiplier part q-differential integral operators was present. The development of a generalized class of starlike functions was one of the most recent contributions. In 2015, Agrawal and Sahoo [103] provided this information, which was of the alpha order.

Very recently, a large number of researchers in the field of Geometric Functions Theory such as Noor *et al.* ([104],[105],[106],[107],[108], [109],[110],) Ramachandran *et al.* [111] and Mahmood and Sokó [112], have used q-calculus to contribute to the creation of the results.

The work of the above researcher motivated us to define a new class $S^*_{s,q}(e^z)$ of q-starlike function w.r.t symmetrical points are subordinate with an exponential operator, and the class $C_{s,q}(e^z)$ of q-convex function w.r.t symmetric points subordinate with exponential Hankel determinant for our function belonging to our new classes defined and Fekete Szegö problems of functions belonging to class $S^*_{s,q}(e^z)$ and $C_{s,q}(e^z)$ will be determined we will show that our needy defined classes are the advancement of that have been above-mentioned classes defined by various researcher. It will also be shown that our new results are a refinement of results already derived in articles [90], and [91]. analytic approaches subordination techniques and concepts of q-calculus will be order to derive our main results.

CHAPTER 3

PRELIMINARY CONCEPTS

In this chapter, we provide a succinct overview of some fundamental ideas from Geometric Function Theory that will be relevant in our subsequent chapters. The class S of nor-malized univalent function and several of its sub-classes, which are determined by both geometric and analytic criteria, are introduced in this chapter.

3.1 Analytic Functions

Geometric Function Theory heavily relies on analytic functions as

Definition 3.1.1 [2] *In mathematics, A function that can be locally written as a convergent power series of complex numbers is referred to be an analytic function. This means that the function can be approximated by a polynomial function with increasing degrees of accuracy as we take more terms of the series. Analytic functions are important in complex analysis, a branch of mathematics that studies complex numbers and their functions.*

An important property of analytic functions is that they preserve many algebraic and geometric properties under composition, such as differentiability, continuity, and conformality (preserving angles and shapes). This makes analytic functions a powerful tool for solving problems in various fields, such as physics, engineering, and economics.

Theorem 3.1.2 ***Riemann Mapping Theorem*** [1] *Let D be a simply connected domain with at least 2nd boundary's points. Then there exist unique analytic functions that map onto an open unit disk in E.*

Instead of dealing with an arbitrary domain D, in Geometric Functions Theory we deal with an open unit disk E. The Riemann Mapping Theorem is responsible for the choice of an open unit. Riemann's fundamental mapping theorem, which he developed about 1850, provided a strong foundation for the study of Geometric Function Theory.

Definition 3.1.3 ***Class*** *A* [2] *The class A includes functions f that are analytic in an open unit disc E and have Taylor series of the type*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \ , z \in E. \tag{3.1}$$

3.2 Univalent Functions

In 1907, Koebe [9] presented the fundamental concept that is used in the investigation of univalent functions. Univalent functions are a class of functions that are defined on the open unit disc E of the complex plane and distinguished by the fact that they accept a value in E just once and map E onto a schlicht domain. Another name for this class of functions is schlicht functions (a German word used to describe a zone without branch points and without self-overlapping boundaries.).

Definition 3.2.1 *In complex analysis a function $f(z)$ is said to be univalent (one-to-one) in a region D of the complex planes if for any 2 distinct point z_1 and z_2 in D, $f(z_1) \neq f(z_2)$. In other words, the function does not map two different points in D to the same point in the range.*

A function $f(z)$ is called univalent in the entire complex plane if it is univalent in the entire plane. Univalent functions have several important properties, including the fact that they are conformal maps. That is, they preserve angles locally and therefore preserve the shapes of small regions. They are crucial to the study of complex analysis and Geometric Function Theory. Examples of univalent functions include polynomial functions, exponential functions, and certain types of trigonometric functions such as the sine and cosine functions.

3.3 The Class S of Univalent Functions

Let there be a function $f(z)$ that is univalent and has the form (3.1). When this occurs, we refer to the function as a normalized univalent function and the class of functions that share this characteristic is represented by the letter S and defined as follows.

Definition 3.3.1 *Class S is stated to be formed by the functions univalent and belonging to class A. In other words, S ={f ∈ A, and f is univalent in E}.*

The Koebe function is considered to be the most significant univalent function,

$$K(z) = \frac{1}{4}\left(\frac{1+z}{1-z}\right)^2 - \frac{1}{4} = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} nz^n \ , z \in E. \tag{3.2}$$

is a well-known example of a function that belongs to class S. With the exception of the negative real axis from $\frac{-1}{4}$ to,∞ the Koebe-function translates E onto the entire complex plane. In [2, 1] it is discussed how $K(z)$ maintains its invariance qualities when subjected to simple transformations such rotation, omitted-value transformation, disk automorphism, conjugation, and dilation.

Caratheordory functions are included in the set of functions that make up the class P. These functions all have a real portion that is greater than zero. This class serves as the foundation for the definition of a large number of subclasses of univalent functions. In relation to the class P, we go through fundamental ideas, the application of which will be necessary for our work.

3.4 Class P of Caratheodory Functions

It has been noted that there are functions with image domains limited to the open half-planes whenever there is numerous complex valued function whose image domains span the entire complex planes. Such function fall under the class P, see [2].

Definition 3.4.1 *If a function p that is analytic in E has the form then we say that it belongs to the class P.*

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \ . \tag{3.3}$$

condition

$$Re(p(z)) > 0.$$

Each and every $p \in P$ is considered to be a function in E with a positive real component.

Theorem 3.4.2 *The theorem of Caratheodory [113] If $p \in P$ and $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, so*

$$|c_n| \leq 2.$$

This strong inequality satisfies the requirements of the Mobius function as well.

$$L_o(z) = \frac{1+z}{1-z} = 1 + \sum_{n=1}^{\infty} z^n 2.$$

3.5 Subordination

Subordination is a mathematical technique that was first introduced in 1909. Furthermore,Littlewood [114, 115] and Rogosinski [116, 117] studied its properties.Subordination is a powerful technique in Geometric Function Theory that allows one to relate two functions defined on different domains. In particular, given two functions $f(z)$ and $g(z)$, defined on the unit disk $E = z : |z| < 1$, We state this g is subordinate to f if there exists a functions $w(z)$ analytic and univalent in D such that $g(z) = f(w(z))$, and $w(0) = 0, w'(0) > 0$. The Schwarz function, which is defined as follows, is a determinant of the concept of subordination.

Definition 3.5.1 *Assume that $g(E) = D$ and let $g \in s$. and $f(0) = g(0)$ if f is analytic in E and $f(E) \subset D$ then f is subordinate to g, written as $f \prec g$.*

3.6 Certain Subclasses of the class S

Several fundamental sub-classes of convex, star-like, close-to-convex & quasi-convex functions are included in this section. In addition, a review will be made of the relationship that these classes have with the caratheodory functions as well as some of the aspects that are already known about these classes.

3.6.1 Starlike and Convex Functions

The class of all starlike functions, which is represented by the symbol S^*, defines it as,

Definition 3.6.1 *[1],[2] Let $f \in E$ be defined by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n, z \in E$ Then we say that f is starlike with respect to the origin if f is univalent and the image $f(E)$ is a star-shaped domain with respect to the origin.*

The following is an analytic description of starlike function: [10]

$$S^* = \{f \in A : Re\frac{zf'(z)}{f(z)} > 0, z \in E\}. \tag{3.4}$$

One well-known example of this class is the Koebe function, which can be represented by the notation (3.2).

Definition 3.6.2 *[1],[2] A domain D is said to be convex if the line connecting any two points in the domain totally resides in the domain. A function f is referred to as a convex function if it maps E onto a convex domain D. The class of all convex functions, C, is used to represent it.*

study 1913

$$C = \{f \in A : Re(\frac{(zf'(z))'}{f'(z)}) > 0, z \in E\}. \tag{3.5}$$

The beautiful relation among these classes was introduced by Alexander [3] and is given by

$$f \in C \iff zf' \in S^*. \tag{3.6}$$

Definition 3.6.3 *(**Symmetric Points**) In Geometry Function Theory, symmetric points refer to points that are symmetric with respect to a given curve or surface, such as a circle or a sphere. Specifically, if we have a function $f(z)$ defined on a region of the complex plane, and a point z_0 in the region, the symmetric point of z_0 w.r.t the curve or surface defined by $|f(z)| = |f(z_0)|$ is a point z_1 such that: $|f(z_1)| = |f(z_0)|$, and the line connecting z_0 and z_1 is perpendicular to the curve or surface defined by $|f(z)| = |f(z0)|$.*

In other words, the symmetric point of z_0 w.r.t the curve or surface defined by $|f(z)| = |f(z_0)|$ is a point z_1 such that the line connecting z_0 and z_1 is perpendicular to the curve or surface, and the distances from z_0 and z_1 to the curve or surface are equal.

The concept of symmetric points is used in Geometry Function Theory to study the behavior of complex functions and their singularities. It is a powerful tool for analyzing the geometry of complex functions and their properties and is widely used in the study of complex analysis and related fields.

Definition 3.6.4 ***(Exponential Function)*** *If an exponential function is symmetric about a point, then the point of symmetry must lie on the vertical asymptote of the function. This is because the exponential function grows or decays very rapidly as x moves away from the point of symmetry.*

To be more precise, let's consider a kind of exponential function: $f(x) = a^x$.

Definition 3.6.5 ***(Subclass of a Starlike Function)*** *In, 2015 Mendiratta et al [89] defined the class of starlike function* $S_e^* = S^*(e^x)$ *we have*

$$\frac{zf'(z)}{f(z)} \prec e^x, z \in E.$$

Definition 3.6.6 ***(Convex Function in Symmetric Point)*** *In,1977 Da and Singh [118] introduced the class of convex functions define we have*

$$Re\{\frac{2\{zf'(z)\}'}{\{f(z) - f((-z)\}'}\} > 0, \forall z \in E.$$

Definition 3.6.7 ***(Starlike Functions with respect to Symmetric Point)*** *In, 2020 Ganesh et al. [91] introduced the class of starlike functions defined we have*

$$\frac{2[zf'(z)]}{f(z) - f(-z)} \prec e^z, z \in E.$$

3.7 Quantum Calculus or q-calculus

Quantum calculus has to do with the q-analogues of mathematic facts which can be written as $q \to 1^-$. Euler (1707–1783) was the first person to study q-calculus and Jackson did the same thing before the 20th century. Researchers have recently paid a lot of attention to this area because of how it can be used in Maths and Physics. Purohit [119] came up with a more general version of q-formula Taylor's by using ideas from fractional q-calculus. Jackson found the q-derivative and the q-integral in a systematical way [4, 6].

Researchers used the q-calculus preliminary results to study sub-classes of univalent functions in the field of Geometric Function Theory. Ismail [6] used q-calculus to define and research the generalized starlike functions. Recently, the definition of the q-close-to-convex functions and a number of intriguing findings were made; for more information, see [120]. Raghavendar and

Swaminathan explored a few of these functions' fundamental characteristics [121]. There are also q-analogues of integral transforms, and many of the important results from classical analysis have been extended to the q-analogues. q-operators are defined and studied with the help of the convolution of normalized analytic functions See ([122],[123], [124]). Calculus based on the fractional q-difference was first described by Al-Salam [125] and Agarwal [126]. Later on, Selvakumaran *et al*. [127], Ramachandran *et al*. [128] defined various classes of analytic functions and the convexity features of those functions using the fractional q-operator. The progression of q-theory can be traced back to the following references: ([129],[130],[131],[132],[133])

3.7.1 q-Derivative

The first person introduced the q-derivative by Jackson [4] In 1908.

Definition 3.7.1 *Let $f \in A$ Then q-derivative of f is*

$$D_q f(z) = \frac{f(qz) - f(z)}{qz - z}, z \neq 0 \text{ and } D_q f(0) = f'(0), 0 < q < 1. \tag{3.7}$$

we have $q \to 1^-, [n]_q \to n$, and $D_q f(z) \to f'(z)$ as $f'(z)$ is an ordinary derivative. from (3.7), (3.1) we have

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}, z \in E. \tag{3.8}$$

where

$$[n]_q = \{\frac{1 - q^n}{1 - q}. \tag{3.9}$$

[6] The following is an explanation of the properties of the q-derivative.

1. consider $g_1(z) = z^n$ to be a function. we have q-derivative

$$D_q g_1(z) = [n]_q z^{n-1}, as [n]_q, we have (3.9) \tag{3.10}$$

2. consider $f, g \in B \subset C$ such that q-derivative of f and $g(z)$ exist for all $z \in B$. Then For any constants a_1 and a_2, we have

$$D_q a_1 f(z) + a_2 g z) = a_1 D_q f(z) + a_2 D_q g(z) \tag{3.11}$$

$$D_q(f(z)g(z)) = g(z) D_q f(z) + f(qz) D_q g(z) \tag{3.12}$$

$$D_q(\frac{f(z)}{g(z)}) = \frac{g(z) D_q f(z) - f(qz) D_q g(z)}{g(z) g(z)(qz)}, g(z) g(z)(qz) \neq 0. \tag{3.13}$$

Ademgullari *et al* [134], In 2016. shown that for $f \in A$. we have

$$D_q(log f(z)) = \frac{D_q f(z)}{f(z)}, z \in E. \tag{3.14}$$

3.7.2 q-Starlike Function

Introduced the class q-starlike function by Ismail *et al* [6].

Definition 3.7.2 *consider* $f \in A$ *we have* $f \in s_q^*$ *as*

$$|\frac{z}{f(z)}(D_q f(z) - \frac{1}{1-q}| \leq \frac{1}{1-q}, z \in E, 0 < q < 1. \tag{3.15}$$

when $q \to 1^-$, *we get the well-known class* S^*

3.7.3 q-Convex Function

According to Srivastava and Owa's definition of class C_q in 1989 [135], which includes q-analogues of convex functions,

Definition 3.7.3 *cosider* $f \in A$, *we have* $f \in C_q$ *if*

$$|\frac{zD^2 f(z)}{D_q f(z)} - \frac{1}{1-q}| \leq \frac{1}{1-q}, \in E, 0 < q < 1 \tag{3.16}$$

when $q \to 1^-$, *we get the well-known class* C.

3.8 Preliminary Lemmas

Lemma 3.8.1 *[136] If* $p \in P, then |p_n| \leq 2, \forall n \in N$.

Lemma 3.8.2 *[137] If* $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + p_4 z^4 + \cdots$. *If* $Re(p(z)) > 0$ *in* E, *then for some* x, z *with* $|x| \leq 1, |z| \leq 1$, *we have*

$2p_2 = p_1^2 + x(4 - p_1^2)$, *for some* $x, |x| \leq 1$

$4P_3 = p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z$.

Lemma 3.8.3 *[120] If* $p \in P$, *then* $|p_2 - vp_1^2| \leq max|\{1, |2v - 1|\}$ *for any* $v \in C$.

CHAPTER 4

ANALYTIC FUNCTIONS WITH RESPECT TO SYMMETRIC POINTS

In this chapter, we will review the class of starlike functions w.r.t symmetric points related to exponential function where subordination technique will be used by in [91], where [91] he has investigated the same intertiary results related to the function belonging to the class of starlike function w.r.t symmetric point.

4.1 The Classes of starlike and Convex Function with Respect to Symmetric Points

The concept of univalent functions with respect to symmetrical points was first introduced by Sakaguchi [20] in 1959, who defined and studied the class S_s^* of starlike functions w.r.t symmetric points and showed that these functions are convex, and hence univalent. In 1975, Das and Singh [118] analyzed the corresponding class of convex function w.r.t symmetric points represented by C_s.

Definition 4.1.1 *[91] A function $f \in A$, and f is said to be in the class $S_s^*(e^z)$ if and only if*

$$\frac{2[zf'(z)]}{f(z)-f(-z)} \prec e^z, z \in E. \tag{4.1}$$

Definition 4.1.2 *[91]A function $f \in A$ $f \in C_s(e^z)$ if and only if*

$$\frac{2[zf'(z)]'}{(f(z)-f(-z))'} \prec e^z, z \in E. \tag{4.2}$$

4.2 Main Results

Theorem 4.2.1 *If $f \in S_s^*(e^z)$ then $|a_2| \leq \frac{1}{2}$, $|a_3| \leq \frac{1}{2}$, $|a_4| \leq \frac{19}{48}, |a_5| \leq \frac{13}{24}$.*

Proof: As $f \in S_s^*(e^z)$ as

$$\frac{2[zf'(z)]}{f(z)-f(-z)} = e^{w(z)}. \tag{4.3}$$

Consider, we have

$$f(z) = z + a_2z^2 + a_3z^3 + a_4z^4 + a_5z^5 + ..., \tag{4.4}$$

$$f'(z) = 1 + 2a_2z + 3a_3z^2 + 4a_4z^3 + 5a_5z^4 + \cdots. \tag{4.5}$$

Using (4.4) and (4.5), we consider

$$\begin{aligned}
\frac{2[zf'(z)]}{f(z)-f(-z)} &= \frac{2\left[z[1+2a_2z+3a_3z^2+4a_4z^3+5a_5z^4+....\right]}{(z+a_2z^2+a_3z^3+a_4z^4+a_5z^5+....)-(-z+a_2z^2-a_3z^3+a_4z^4-a_5z^5+....)} \\
&= \frac{2z\left[1+2a_2z+3a_3z^2+4a_4z^3+5a_5z^4+....\right]}{(z+a_2z^2+a_3z^3+a_4z^4+a_5z^5+....+z-a_2z^2-a_3z^3+a_4z^4+a_5z^5+....)} \\
&= \frac{2z[1+2a_2z+3a_3z^2+4a_4z^3+5a_5z^4+....]}{2z[1+a_3z^2+a_5z^4]} \\
&= \frac{1+2a_2z+3a_3z^2+4a_4z^3+5a_5z^4+\cdots}{1+a_3z^2+a_5z^4} \\
&= 1+2a_2z+2a_3z^2+(4a_4z^4-2a_3a_2)z^3+(4a_5z^5-2a_3^2)z^4+\cdots,
\end{aligned} \tag{4.6}$$

we have

$$p(z) = \frac{1+w(z)}{1-w(z)}.$$

Equivalent,

$$w(z) = \frac{p(z)-1}{p(z)+1}. \tag{4.7}$$

As we know that

$$p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + p_4z^4 + \cdots. \tag{4.8}$$

From (4.8) in (4.7) we have

$$w(z)=\frac{1+p_1z+p_2z^2+p_3z^3+p_4z^4+...-1}{1+p_1z+p_2z^2+p_3z^3+p_4z^4+...+1},$$

we have

$$=\frac{p_1z}{2}+\left(\frac{p_2}{2}-\frac{p_1^2}{4}\right)z^2+\left(\frac{p_3}{2}-\frac{p_1p_2}{2}+\frac{p_1^3}{8}\right)z^3+\left(\frac{p_4}{2}-\frac{p_1p_4}{2}-\frac{p_2^2}{4}+\frac{3p_1^2p_2}{8}-\frac{p_1^4}{16}\right)z^4+\cdots. \tag{4.9}$$

Since

$$e^{w(z)}=1+w(z)+\frac{(w(z))^2}{2!}+\frac{(w(z))^3}{3!}+\frac{(w(z))^4}{4!}+\cdots. \tag{4.10}$$

From (4.10), we get

$$e^w(z)=1+\frac{p_1z}{2}+\left(\frac{p_2}{2}-\frac{p_1^2}{4}\right)z^2+\left(\frac{p_3}{2}-\frac{p_1p_2}{2}+\frac{p_1^3}{8}\right)z^3+\left(\frac{p_4}{2}-\frac{p_1p_4}{2}-\frac{p_2^2}{4}+\frac{3p_1^2p_2}{8}-\frac{p_1^4}{16}\right)z^4+\cdots$$

$$+\frac{1}{2}(\frac{p_1z}{2}+\left(\frac{p_2}{2}-\frac{p_1^2}{4}\right)z^2+\left(\frac{p_3}{2}-\frac{p_1p_2}{2}+\frac{p_1^3}{8}\right)z^3+\left(\frac{p_4}{2}-\frac{p_1p_4}{2}-\frac{p_2^2}{4}+\frac{3p_1^2p_2}{8}-\frac{p_1^4}{16}\right)z^4+\cdots)^2$$
$$+\frac{1}{6}(\frac{p_1z}{2}+\left(\frac{p_2}{2}-\frac{p_1^2}{4}\right)z^2+\left(\frac{p_3}{2}-\frac{p_1p_2}{2}+\frac{p_1^3}{8}\right)z^3+\left(\frac{p_4}{2}-\frac{p_1p_4}{2}-\frac{p_2^2}{4}+\frac{3p_1^2p_2}{8}-\frac{p_1^4}{16}\right)z^4+\cdots)^3$$
$$+\frac{1}{24}(\frac{p_1z}{2}+\left(\frac{p_2}{2}-\frac{p_1^2}{4}\right)z^2+\left(\frac{p_3}{2}-\frac{p_1p_2}{2}+\frac{p_1^3}{8}\right)z^3+\left(\frac{p_4}{2}-\frac{p_1p_4}{2}-\frac{p_2^2}{4}+\frac{3p_1^2p_2}{8}-\frac{p_1^4}{16}\right)z^4+\cdots)^4$$
$$+\cdots.(4.11)$$

This gives us

$$e^w(z)=1+\frac{p_1z}{2}+\left(\frac{p_2}{2}-\frac{p_1^2}{4}+\frac{p_1^2}{8}\right)z^2+\left(\frac{p_3}{2}-\frac{p_1p_2}{2}+\frac{p_1^3}{8}-\frac{p_1^3}{8}+\frac{p_1p_2}{4}-\frac{p_1^3}{48}\right)z^3$$
$$+\left(\frac{p_4}{2}-\frac{p_1p_4}{2}-\frac{p_2^2}{4}+\frac{3p_1^2p_2}{8}-\frac{p_1^4}{16}+\frac{p_2^2}{8}+\frac{3p_1^4}{32}-\frac{p_1p_3}{4}+\frac{p_1^2p_2}{16}-\frac{p_1^4}{384}\right)z^4+\cdots,$$

Which implies that

$$e^w(z)=1+\frac{p_1z}{2}+\left(\frac{p_2}{2}+(\frac{-2p_1^2+p_1^2}{8})\right)z^2+\left(\frac{p_3}{2}+(\frac{-2p_1p_2+p_1p_2}{4})-\frac{p_1^3}{48}\right)z^3+$$

$$\left(\frac{p_4}{2}+\frac{-2p_1p_3+p_1p_3}{4}+\frac{-2p_2^2+p_2^2}{8}+\frac{6p_1^2p_2-4p_1^2p_2-p_1^2p_2}{16}+\frac{-24p_1^4+36p_1^4-12p_1^4+p_1^4}{384}\right)z^4+\cdots$$

$$=1+\frac{p_1z}{2}+\left(\frac{p_2}{2}-\frac{p_1^2}{8}\right)z^2+\left(\frac{p_3}{2}-\frac{p_1p_2}{4}+\frac{p_1^3}{48}\right)z^3+\left(\frac{p_4}{2}-\frac{p_1p_3}{4}-\frac{p_2^2}{8}+\frac{p_1^2p_2}{16}+\frac{p_1^4}{384}\right)z^4$$
$$+\cdots.(4.12)$$

From (4.6) and (4.12) we compare the coefficient which gives us

$$2a_2=\frac{p_1}{2},$$

This gives us

$$a_2 = \frac{p_1}{4}, \tag{4.13}$$

we have

$$2a_3 = \frac{p_2}{2} - \frac{p_1^2}{8}.$$

This implies that

$$a_3 = \frac{p_2}{4} - \frac{p_1^2}{16}. \tag{4.14}$$

Now, we consider

$$\begin{aligned}
4a_4 &= \frac{p_3}{2} - \frac{p_1p_2}{4} + \frac{p_1^3}{48} + 2a_3a_2 \\
4a_4 &= \frac{p_3}{2} - \frac{p_1p_2}{4} + \frac{p_1^3}{48} + 2(\frac{p_2}{4} - \frac{p_1^2}{16})\frac{p_1^3}{4} \\
&= \frac{p_3}{2} - \frac{p_1p_2}{4} + \frac{p_1^3}{48} + \frac{p_1p_2}{8} - \frac{p_1^3}{32}.
\end{aligned}$$

Thus we have

$$a_4 = \frac{p_3}{8} - \frac{p_1p_2}{32} - \frac{p_1^3}{384}. \tag{4.15}$$

Next, to find the coefficient value of a_5, we have

$$\begin{aligned}
4a_5 &= \frac{p_4}{2} - \frac{p_1p_3}{4} - \frac{p_2^2}{8} + \frac{p_2p_1^2}{16} + \frac{p_1^4}{384} + 2a_3^2 \\
4a_5 &= \frac{p_4}{2} - \frac{p_1p_3}{4} - \frac{p_2^2}{8} + \frac{p_2p_1^2}{16} + \frac{p_1^4}{384} + 2(\frac{p_2}{4} - \frac{p_1^2}{16})^2 \\
&= \frac{p_4}{2} - \frac{p_1p_3}{4} - \frac{p_2^2}{8} + \frac{p_2p_1^2}{16} + \frac{p_1^4}{384} + \frac{p_2^2}{8} + \frac{p_1^4}{128} - \frac{p_1^2p_2}{16},
\end{aligned}$$

We have

$$a_5 = \frac{p_4}{8} - \frac{p_1p_3}{16} + \frac{p_1^4}{384}. \tag{4.16}$$

which implies that

$$a_2 = \frac{p_1}{4}, a_3 = \frac{p_2}{4} - \frac{p_1^2}{16}, a_4 = \frac{p_3}{8} - \frac{p_1p_2}{32} - \frac{p_1^3}{384}, a_5 = \frac{p_4}{8} - \frac{p_1p_3}{16} + \frac{p_1^4}{384}. \tag{4.17}$$

Now, we use Lemma 3.8.1, in (4.13)

$$a_2 = \frac{p_1}{4},$$

So we get

$$|a_2| \leq \frac{1}{2}. \tag{4.18}$$

Apply the Lemma 3.8.3, in (4.14), we get

$$\begin{aligned}
|a_3| &\leq |\frac{p_2}{4} - \frac{p_1^2}{16}| \\
&\leq \frac{1}{4}|p_2 - \frac{p_1^2}{4}| \\
&\leq \frac{2}{4}max|\{1, |2(\frac{1}{4}) - 1|\} \\
&\leq \frac{1}{2}max[1, \frac{1}{2}].
\end{aligned}$$

Thus we have

$$|a_3| \leq \frac{1}{2}. \tag{4.19}$$

Now so we get again in (4.15) applying the Lemma 3.8.1 as

$$\begin{aligned}
a_4 &= \frac{p_3}{8} - \frac{p_1 p_2}{32} - \frac{p_1^3}{384} \\
|a_4| &= |\frac{p_3}{8} - \frac{p_1 p_2}{32} - \frac{p_1^3}{384}|.
\end{aligned}$$

We use

$$|a - b| \leq |a| + |b|,$$

which leads us

$$\begin{aligned}
&\leq \left| |\frac{p_3}{8} - \frac{p_1 p_2}{32}| + |\frac{p_1^3}{384}| \right| \\
&\leq \left| |\frac{p_3}{8}| + |\frac{p_1 p_2}{32}| + |\frac{p_1^3}{384}| \right| \\
&= |\frac{2}{8} + \frac{4}{32} + \frac{8}{384}|.
\end{aligned}$$

So, we get

$$|a_4| \leq \frac{19}{48}. \tag{4.20}$$

We use Lemma 3.8.1 in (4.16)

$$|a_5| \leq \left| |\frac{p_4}{8} - \frac{p_1 p_3}{16}| + \frac{4p_1^4}{384}| \right|.$$

As we know

$$|a - b| \leq |a| + |b|,$$

so we get

$$\leq \left| |\frac{p_4}{8}| + |\frac{p_1 p_3}{16}| + |\frac{4p_1^4}{384}| \right|$$

$$=\frac{2}{8}+\frac{4}{16}+\frac{16}{384},$$

which implies that

$$|a_5| \leq \frac{13}{24}. \tag{4.21}$$

This completes the proof.

Theorem 4.2.2 *If $f \in S_s^*(e^z)$ then $|a_3 - a_2^2| \leq \frac{1}{2}$*

Proof: Mathematical technique as used in the proof of the Theorem 4.2.1 and using (4.17), we get

$$|a_3 - a_2^2| = |\frac{p_2}{4} - \frac{p_1^2}{16} - \frac{p_1^2}{16}|.$$

We have

$$\begin{aligned}|a_3 - a_2^2| &= |\frac{p_2}{4} - \frac{2p_1^2}{16}| \\ &= |\frac{p_2}{4} - \frac{p_1^2}{8}| \\ &= \frac{1}{4}|p_2 - \frac{p_1^2}{2}|.\end{aligned}$$

Now, we use Lemma 3.8.3, which gives us

$$\leq \frac{2}{4} max|\{1, |2(\frac{1}{2}) - 1|\}$$

$$\leq \frac{1}{2} max[1, 0]$$

$$|a_3 - a_2^2| \leq \frac{1}{2}, \tag{4.22}$$

This completes the proof.

Theorem 4.2.3 *If $f \in S_s^*(e^z)$ then $|a_2a_3 - a_4| \leq \frac{529+57\sqrt{118}}{3468}$.*

Proof: Proceeding as proof the Theorem 4.2.1 and using (4.17), we get

$$\begin{aligned}|a_2a_3 - a_4| &= |\frac{p_1}{4}(\frac{p_2}{4} - \frac{p_1^2}{16}) - (\frac{p_3}{8} - \frac{p_1p_2}{32} - \frac{p_1^3}{384}| \\ &= |(\frac{p_1p_2}{16} - \frac{p_1^3}{64} - \frac{p_3}{8} + \frac{p_1p_2}{32} + \frac{p_1^3}{384})|.\end{aligned}$$

Thus we have

$$|a_2a_3 - a_4| = |\frac{3p_1p_2}{32} - \frac{p_3}{8} - \frac{5p_1^3}{384}|. \tag{4.23}$$

Now we use Lemma 3.8.2, this gives us

$$
\begin{aligned}
|a_2a_3-a_4| &= \left|\frac{3p_1}{32}[\frac{p_1^2}{2}+\frac{x(4-p_1^2)}{2}]-\frac{1}{8}[\frac{p_1^3}{4}+\frac{p_1(4-p_1^2)x}{2}-\frac{p_1(4-p_1^2)x^2}{4}+\frac{(4-p_1^2)(1-|x|^2)z}{2}]-\frac{5p_1^3}{384}\right| \\
&= \left|\frac{3p_1^3}{64}+\frac{3x(4-p_1^2)p_1}{64}-\frac{p_1^3}{32}-\frac{p_1(4-p_1^2)x}{16}+\frac{p_1(4-p_1^2)x^2}{32}-\frac{(4-p_1^2)(1-|x|^2)z}{16}-\frac{5p_1^3}{384}\right| \\
&= \left|\frac{p_1(4-p_1^2)x^2}{32}+\frac{3x(4-p_1^2)p_1-4p_1(4-p_1^2)x}{64}-\frac{(4-p_1^2)(1-|x|^2)z}{16}-\frac{p_1^3}{32}+\frac{3p_1^3}{64}-\frac{5p_1^3}{384}\right| \\
&= \left|\frac{p_1(4-p_1^2)x^2}{32}-\frac{x(4-p_1^2)p_1}{64}-\frac{(4-p_1^2)(1-|x|^2)z}{16}+\frac{(-12+18-5)p_1^3}{384}\right|
\end{aligned}
$$

$$
|a_2a_3-a_4| = \left|\frac{p_1(4-p_1^2)x^2}{32}-\frac{x(4-p_1^2)p_1}{64}-\frac{(4-p_1^2)(1-|x|^2)z}{16}+\frac{p_1^3}{384}\right|. \tag{4.24}
$$

Denotes $|x| = t \in [0,1], p_1 = c \in [0,2]$.Then, by employing the equation of triangle inequality (4.24) gives us

$$
|a_2a_3-a_4| \leq \frac{c(4-c^2)t^2}{32}+\frac{t(4-c^2)c}{64}+\frac{(4-c^2)}{16}+\frac{c^3}{384}.
$$

Suppose that

$$
F(c,1) \equiv \frac{c(4-c^2)t^2}{32}+\frac{t(4-c^2)c}{64}+\frac{(4-c^2)}{16}+\frac{c^3}{384},
$$

we have

$$
\frac{\partial F}{\partial t} = \frac{c(4-c^2)}{64}+\frac{t(4-c^2)c}{16} \geq 0.
$$

The function $F(c,t)$ does not decrease for any value of t within [0,1]. This demonstrates that the maximum value of $F(c,t)$ is reached when $t = 1$.

$$
MaxF(c,t) = F(c,1) = \frac{c(4-c^2)t^2}{32}+\frac{t(4-c^2)c}{64}+\frac{(4-c^2)}{16}+\frac{c^3}{384}.
$$

Let us define

$$
M(c) = \frac{c(4-c^2)t^2}{32}+\frac{t(4-c^2)c}{64}+\frac{(4-c^2)}{16}+\frac{c^3}{384}. \tag{4.25}
$$

Which implies that

$$
M'(c) = -\frac{17c^2}{128}+\frac{3}{16}-\frac{c}{8}. \tag{4.26}
$$

$$
-\frac{17c^2}{128}-\frac{c}{8}+\frac{3}{16} = 0, \tag{4.27}
$$

Simplify

$$
c = -\frac{8}{17}+\frac{2\sqrt{118}}{17}.
$$

$$
M''(c) = -\frac{17c}{64}-\frac{1}{8}.
$$

$M'(c)$ vanishes at $c = r^* = -\frac{8}{17} + \frac{2\sqrt{118}}{17}$. The result of a computation calculation is that $M''(c) < 0$, which indicates that the function $M(c)$ is able to reach its highest value at $M(c)$. $r^* = -\frac{8+2\sqrt{118}}{17}$. we get

From (4.25) we have

$$M(r^*) = \frac{529+59\sqrt{118}}{3468}.$$

$$|a_2a_3 - a_4| \leq \frac{529+59\sqrt{118}}{3468}. \tag{4.28}$$

This completes the proof.

Theorem 4.2.4 *If $f \in S_s^*(e^z)$ then $|a_2a_4 - a_3^2| \leq \frac{3}{8}$*

Proof: From equation (4.17) of Theorem 4.2.1, we get

$$\begin{aligned}|a_2a_4 - a_3^2| &= |\frac{p_1}{4}(\frac{p_3}{8} - \frac{p_1p_2}{32} - \frac{p_1^3}{384}) - (\frac{p_2}{4} - \frac{p_1^2}{16})^2| \\ &= |\frac{p_3p_1}{32} - \frac{p_1^2p_2}{128} - \frac{p_1^4}{1536} - \frac{p_2^2}{16} - \frac{p_1^2p_2}{32} - \frac{p_1^4}{256}|.\end{aligned}$$

Use Lemma 3.8.2 we get,

$$|a_2a_4 - a_3^2| = \left|\frac{p_1}{32}\left[\frac{p_1^3}{4} + \frac{p_1(4-p_1^2)x}{2} - \frac{p_1(4-p_1^2)x^2}{4} + \frac{(4-p_1^2)(1-|x|^2)z}{2}\right] - \frac{p_1^2}{128}\left[\frac{p_1^2}{2} + \frac{x(4-p_1^2)}{2}\right]\right.$$

$$-\frac{p_1^4}{1536} - \frac{1}{16}\left(\frac{p_1^2}{2} + \frac{x(4-p_1^2)}{2}\right)^2 - \frac{p_1^2}{32}\left[\frac{p_1^2}{2} + \frac{x(4-p_1^2)}{2}\right] - \frac{p_1^4}{256}|$$

$$= \left|\frac{p_1^4}{128} + \frac{p_1^2(4-p_1^2)x}{64} - \frac{p_1^2(4-p_1^2)x^2}{128} + \frac{p_1(4-p_1^2)(1-|x|^2)z}{64} - \frac{p_1^4}{256} - \frac{p_1^2x(4-p_1^2)}{256} - \frac{p_1^4}{1536}\right.$$

$$-\frac{p_1^4}{64} - \frac{x(4-p_1^2)p_1^2}{32} - \frac{x^2(4-p_1^2)^2}{64} - \frac{2p_1^4}{64} + \frac{x(4-p_1^2)p_1^2}{64} - \frac{p_1^4}{256}|.$$

Simplify

$$|a_2a_4 - a_3^2| = \left|\frac{p_1(4-p_1^2)(1-|x|^2)z}{64} - \frac{p_1^2(4-p_1^2)x^2}{128} - \frac{p_1^2x(4-p_1^2)}{256} - \frac{x^2(4-p_1^2)p_1^2}{64} - \frac{p_1^4}{1536}\right|. \tag{4.29}$$

Denote $|x| = t \in [0,1], p_1 = c \in [0,2]$ by employing the equation of triangle inequality

$$|a_2a_4 - a_3^2| \leq \frac{(4-c^2)}{32} + \frac{c^2(4-c^2)t^2}{128} + \frac{c^2t(4-c^2)}{256} + \frac{t^2(4-c^2)^2}{64} + \frac{c^4}{1536}.$$

Let us consider

$$F(c,t) = \frac{(4-c^2)}{32} + \frac{c^2(4-c^2)t^2}{128} + \frac{c^2t(4-c^2)}{256} + \frac{t^2(4-c^2)^2}{64} + \frac{c^4}{1536}. \tag{4.30}$$

Thus we get

$$\frac{\partial F}{\partial t} = \frac{(4-c^2)c^2}{256} + \frac{(4-c^2)c^2 t}{64} + \frac{(4-c^2)^2 t}{32} \geq 0.$$

Which gives that $F(c,t)$ is increasing for any then t in [0,1]. this show that $F(c,t)$ has maximum value at $t = 1$.

$$MaxF(c,t) = F(c,t) = \frac{(4-c^2)}{32} + \frac{c^2(4-c^2)}{128} + \frac{c^2(4-c^2)}{256} + \frac{(4-c^2)^2}{64} + \frac{c^4}{1536}.$$

Let us define

$$M(c) = \frac{(4-c^2)}{32} + \frac{c^2(4-c^2)}{128} + \frac{c^2(4-c^2)}{256} + \frac{(4-c^2)^2}{64} + \frac{c^4}{1536}. \tag{4.31}$$

We get

$$M'(c) = \frac{-c^3}{48} - \frac{5(-c^2+4)c}{128} - \frac{c}{16}.$$

We have

$$M''(c) = \frac{7c^2}{128} - \frac{7}{32}.$$

If $c = 0$, $M'(c)$ disappears. A quick calculation reveals that the function $M(c)$ has its maximum values at $c = 0$, which indicates that $M''(c) < 0$.we get equation (4.31) $M(0) = \frac{16}{64} + \frac{4}{32}$ simplify

$$|a_2 a_4 - a_3^2| \leq M(0) = \frac{16+8}{64}.$$

As

$$|a_2 a_4 - a_3^2| \leq M(0) = \frac{3}{8}, \tag{4.32}$$

Hence we obtain this proof.

Theorem 4.2.5 *If $f \in S_s^*(e^z)$ then* $|H_3(1)| = 0.5893$

Proof: Consider

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}$$
$$= a_3 \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} - a_4 \begin{vmatrix} a_1 & a_3 \\ a_2 & a_4 \end{vmatrix} + a_5 \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix},$$

since $a_1 = 1$

$H_3(1) = a_3\,(a_2\,a_4 - a_3^2) - a_4\,(a_4 - a_2\,a_3) + a_5\,(a_3 - a_2{}^2)$, by applying triangle triangle inequality we get

$$|H_3\ (1)| \leq |a_3|\,|\,(a_2\,a_4 - a_3{}^2)\,| + |\,a_4\,|\,|\,(a_4 - a_2 a_3)\,| + |\,a_5\,|\,|\,(a_3 - a_2{}^2)|. \tag{4.33}$$

Now, substituting the equations (4.19), (4.20), (4.21), (4.22), (4.28), and (4.32) in (4.33) we get

$$|H_3(1)| \leq 0.5893.$$

This completes the proof.

Theorem 4.2.6 *If $f \in C_s(e^z)$ then $|a_2| \leq \frac{1}{4}$, $|a_3| \leq \frac{1}{6}$, $|a_4| \leq \frac{19}{192}$, $|a_5| \leq \frac{13}{120}$.*

Proof: Consider

$$\frac{2\,[\,z\,f'\,(\,z\,)\,]'}{(f\,(\,z)-f\,(\,-z\,)\,)'} \;=\; e^{w\,(z)}. \tag{4.34}$$

As we know

$$f(z) = z + a_2z^2 + a_3z^3 + a_4z^4 + a_5z^5 + ...,$$

$$f'(z) = 1 + 2a_2z + 3a_3z^2 + 4a_4z^3 + 5a_5z^4 + ...,$$

$$zf'(z) = z + 2a_2z^2 + 3a_3z^3 + 4a_4z^4 + 5a_5z^5 + ...,$$

Thus, we have

$$(zf'(z))' = 1 + 4a_2z + 9a_3z^2 + 16a_4z^3 + 25a_5z^4 + \cdots. \tag{4.35}$$

As

$$f(z) - f(-z) = [2z + 2a_3z^3 + 2a_5z^5 + ...]$$

$$(f(z) - f(-z))' = [2 + 6a_3z^2 + 10a_5z^4 + ...],$$

Which leads us

$$(f(z) - f(-z))' = 2[1 + 3a_3z^2 + 5a_5z^4 + ...]. \tag{4.36}$$

From (4.35) and (4.36) we get

$$\begin{aligned} \frac{2[zf'(z)]'}{(f(z)-f(-z))'} &= \frac{2[1+4a_2z+9a_3z^2+16a_4z^3+25a_5z^4+...]}{2[1+3a_3z^2+5a_5z^4+...]} \\ &= \frac{[1+4a_2z+9a_3z^2+16a_4z^3+25a_5z^4+...]}{[1+3a_3z^2+5a_5z^4+...]}. \end{aligned}$$

Implies that

$$\frac{2[zf'(z)]'}{(f(z)-f(-z))'} = 1 + 4a_2z + 6a_3z^2 + (16a_4 - 12a_3a_2)z^3 + (20a_5 - 18a_3^2)z^4 + \cdots, \tag{4.37}$$

we know that

$$p\,(\,z\,) \;=\; \frac{1\,+\;w\,(\,z\,)}{1\;-w\,(\,z\,)},$$

we can have

$$w(z)=\frac{p\,(\,z\,)\,-\,1}{p\,(\,z\,)\,+\,1}. \tag{4.38}$$

As we know that

$$p(z)=1+p_1z+p_2z^2+p_3z^3+p_4z^4+\cdots. \tag{4.39}$$

From (4.38) and (4.39) we have

$$w(z)=\frac{1+p_1z+p_2z^2+p_3z^3+p_4z^4+...-1}{1+p_1z+p_2z^2+p_3z^3+p_4z^4+...+1}.$$

$$=\frac{p_1z}{2}+\left(\frac{p_2}{2}-\frac{p_1^2}{4}\right)z^2+\left(\frac{p_3}{2}-\frac{p_1p_2}{2}+\frac{p_1^3}{8}\right)z^3+\left(\frac{p_4}{2}-\frac{p_1p_4}{2}-\frac{p_2^2}{4}+\frac{3p_1^2p_2}{8}-\frac{p_1^4}{16}\right)z^4+\cdots\quad. \tag{4.40}$$

Since we have

$$e^{w(z)}=1+w(z)+\frac{(w(z))^2}{2!}+\frac{(w(z))^3}{3!}\quad+\frac{(w(z))^4}{4!}+\cdots. \tag{4.41}$$

From (4.40) in (4.41) we get

$$e^w(z)=1+\frac{p_1z}{2}+\left(\frac{p_2}{2}-\frac{p_1^2}{4}\right)z^2+\left(\frac{p_3}{2}-\frac{p_1p_2}{2}+\frac{p_1^3}{8}\right)z^3+\left(\frac{p_4}{2}-\frac{p_1p_4}{2}-\frac{p_2^2}{4}+\frac{3p_1^2p_2}{8}-\frac{p_1^4}{16}\right)z^4+\cdots$$

$$+\frac{1}{2}(\frac{p_1z}{2}+\left(\frac{p_2}{2}-\frac{p_1^2}{4}\right)z^2+\left(\frac{p_3}{2}-\frac{p_1p_2}{2}+\frac{p_1^3}{8}\right)z^3+\left(\frac{p_4}{2}-\frac{p_1p_4}{2}-\frac{p_2^2}{4}+\frac{3p_1^2p_2}{8}-\frac{p_1^4}{16}\right)z^4+\cdots)^2$$
$$+\frac{1}{6}(\frac{p_1z}{2}+\left(\frac{p_2}{2}-\frac{p_1^2}{4}\right)z^2+\left(\frac{p_3}{2}-\frac{p_1p_2}{2}+\frac{p_1^3}{8}\right)z^3+\left(\frac{p_4}{2}-\frac{p_1p_4}{2}-\frac{p_2^2}{4}+\frac{3p_1^2p_2}{8}-\frac{p_1^4}{16}\right)z^4+\cdots)^3$$
$$+\frac{1}{24}(\frac{p_1z}{2}+\left(\frac{p_2}{2}-\frac{p_1^2}{4}\right)z^2+\left(\frac{p_3}{2}-\frac{p_1p_2}{2}+\frac{p_1^3}{8}\right)z^3+\left(\frac{p_4}{2}-\frac{p_1p_4}{2}-\frac{p_2^2}{4}+\frac{3p_1^2p_2}{8}-\frac{p_1^4}{16}\right)z^4+..)^4+\cdots \tag{4.42}$$

This gives us

$$e^w(z)=1+\frac{p_1z}{2}+\left(\frac{p_2}{2}-\frac{p_1^2}{4}+\frac{p_1^2}{8}\right)z^2+\left(\frac{p_3}{2}-\frac{p_1p_2}{2}+\frac{p_1^3}{8}-\frac{p_1^3}{8}+\frac{p_1p_2}{4}-\frac{p_1^3}{48}\right)z^3$$
$$+\left(\frac{p_4}{2}-\frac{p_1p_4}{2}-\frac{p_2^2}{4}+\frac{3p_1^2p_2}{8}-\frac{p_1^4}{16}+\frac{p_2^2}{8}+\frac{3p_1^4}{32}-\frac{p_1p_3}{4}+\frac{p_1^2p_2}{16}-\frac{p_1^4}{384}\right)z^4+\cdots, \tag{4.43}$$

which implies that

$$e^w(z)=1+\frac{p_1z}{2}+\left(\frac{p_2}{2}-\frac{p_1^2}{8}\right)z^2+\left(\frac{p_3}{2}-\frac{p_1p_2}{4}+\frac{p_1^3}{48}\right)z^3\quad+\left(\frac{p_4}{2}-\frac{p_1p_3}{4}-\frac{p_2^2}{8}+\frac{p_1^2p_2}{16}+\frac{p_1^4}{384}\right)z^4+\cdots. \tag{4.44}$$

From (4.37) and (4.44) we comparing coefficient which gives us

$$4a_2=\frac{p_1}{2}.$$

This gives us

$$a_2 = \frac{p_1}{8}. \tag{4.45}$$

We have

$$6a_3 = \frac{p_2}{2} - \frac{p_1^2}{8}.$$

This implies that

$$a_3 = \frac{p_2}{12} - \frac{p_1^2}{48}. \tag{4.46}$$

Now we consider

$$(16a_4 - 12a_3a_2) = \frac{p_3}{2} - \frac{p_1p_2}{4} + \frac{p_1^3}{48}.$$

We have

$$16a_4 = \frac{p_3}{2} - \frac{p_1p_2}{4} + \frac{p_1^3}{48} + 12(\frac{p_2}{12} - \frac{p_1^2}{48})\frac{p_1}{8}.$$

This implies that

$$a_4 = \frac{p_3}{32} - \frac{p_1p_2}{128} - \frac{p_1^3}{1536}. \tag{4.47}$$

As

$$(20a_5 - 18a_3^2) = \frac{p_4}{2} - \frac{p_1p_3}{4} - \frac{p_2^2}{8} + \frac{p_2p_1^2}{16} + \frac{p_1^4}{384}.$$

Next, to find the coefficient value of a_5, we have

$$20a_5 = \frac{p_4}{2} - \frac{p_1p_3}{4} - \frac{p_2^2}{8} + \frac{p_2p_1^2}{16} + \frac{p_1^4}{384} + 18a_3^2$$
$$20a_5 = \frac{p_4}{2} - \frac{p_1p_3}{4} - \frac{p_2^2}{8} + \frac{p_2p_1^2}{16} + \frac{p_1^4}{384} + 18(\frac{p_2}{12} - \frac{p_1^2}{48})^2$$
$$20a_5 = \frac{p_4}{2} - \frac{p_1p_3}{4} - \frac{p_2^2}{8} + \frac{p_2p_1^2}{16} + \frac{p_1^4}{384} + 18(\frac{p_2^2}{144} + \frac{p_1^4}{2304} - \frac{p_1^2p_2}{384})$$
$$a_5 = \frac{p_4}{40} - \frac{p_1p_3}{80} - \frac{p_2^2}{160} + \frac{p_2p_1^2}{320} + \frac{p_1^4}{7680} + \frac{p_2^2}{160} - \frac{p_1^2p_2}{320} + \frac{p_1^4}{2560}.$$
$$a_5 = \frac{p_4}{40} - \frac{p_1p_3}{80} - \frac{p_2^2}{160} + \frac{p_2p_1^2}{320} + \frac{p_1^4}{7680} + \frac{p_2^2}{160} - \frac{p_1^2p_2}{320} + \frac{p_1^4}{2560}.$$

This implies that

$$a_5 = \frac{p_4}{40} - \frac{p_1p_3}{80} + \frac{8p_1^4}{15360}. \tag{4.48}$$

As

$$a_2 = \frac{p_1}{8}, a_3 = \frac{p_2}{12} - \frac{p_1^2}{48}, a_4 = \frac{p_3}{32} - \frac{p_1p_2}{128} - \frac{p_1^3}{1536}, a_5 = \frac{p_4}{40} - \frac{p_1p_3}{80} + \frac{8p_1^4}{15360}. \tag{4.49}$$

From (4.45) we use Lemma 3.8.1, which gives us

$$|a_2| \leq \frac{1}{4}. \tag{4.50}$$

Use the Lemma 3.8.3 in (4.46).

$$\begin{aligned} a_3 &= \frac{p_2}{12} - \frac{p_1^2}{48} \\ |a_3| &\le \frac{2}{12} max|\{1, |2(\frac{1}{4}) - 1\}| \\ &\le \frac{1}{6} max[1, \frac{1}{2}]. \end{aligned}$$

Thus we have

$$|a_3| \le \frac{1}{6}. \tag{4.51}$$

Now so we get again, we apply the Lemma 3.8.1 in (4.47).

$$|a_4| = |\frac{p_3}{32} - \frac{p_1 p_2}{128} - \frac{p_1^3}{1536}|,$$

we use

$$|a - b| \le |a| + |b|,$$

which leads us

$$\begin{aligned} &\le \left| |\frac{p_3}{32} - \frac{p_1 p_2}{128}| + |\frac{p_1^3}{1536}| \right| \\ &\le \left| |\frac{p_3}{32}| + |\frac{p_1 p_2}{128}| + |\frac{p_1^3}{1536}| \right| \\ &= |\frac{2}{32} + \frac{4}{128} + \frac{8}{1536}|, \end{aligned}$$

so, we get

$$|a_4| \le \frac{19}{192}. \tag{4.52}$$

We use the Lemma 3.8.1 in (4.48)

$$\begin{aligned} |a_5| &= |\frac{p_4}{40} - \frac{p_1 p_3}{80} + \frac{p_1^4}{1920}| \\ &\le \left| |\frac{p_4}{40} - \frac{p_1 p_3}{80}| + \frac{p_1^4}{1920}| \right|. \end{aligned}$$

As we know

$$|a - b| \le |a| + |b|.$$

So we get

$$\begin{aligned} &\le \left| |\frac{p_4}{40}| + |\frac{p_1 p_3}{80}| + |\frac{p_1^4}{1920}| \right| \\ &= \frac{2}{40} + \frac{4}{80} + \frac{16}{1920} \end{aligned}$$

We have

$$|a_5| \leq \frac{13}{120}. \tag{4.53}$$

Hence, we obtain this proof

Theorem 4.2.7 *If $f \in C_s(e^z)$ then $|a_3 - a_2^2| \leq \frac{1}{6}$,*

Proof: Using the similar Mathematical techniques as used in the proof of the Theorem 4.2.6 and using (4.49), we get

$$|a_3 - a_2^2| = |\frac{p_2}{12} - \frac{p_1^2}{48} - \frac{p_1^2}{64}|,$$

we have

$$|a_3 - a_2^2| = |\frac{p_2}{12} - \frac{7p_1^2}{192}|.$$

Now, we use Lemma 3.8.3, which gives us

$$\leq \frac{2}{12} max|\{1, |2(\frac{7}{16}) - 1\}|$$

$$|a_3 - a_2^2| \leq \frac{1}{6}. \tag{4.54}$$

This completes the proof.

Theorem 4.2.8 *If $f \in C_s(e^z)$ then $|a_2a_3 - a_4| \leq \frac{829+85\sqrt{170}}{21168}$*

Proof: Using the similar Mathematical techniques as used in the proof of the Theorem 4.2.6 and using (4.49), we get

$$|a_2a_3 - a_4| = |\frac{p_1}{8}[\frac{p_2}{12} - \frac{p_1^2}{48}] - [\frac{p_3}{32} - \frac{p_1p_2}{128} - \frac{p_1^3}{1536}]|.$$

We have

$$= |\frac{7p_1p_2}{384} - \frac{p_3}{32} - \frac{3p_1^3}{1536}|. \tag{4.55}$$

Use the Lemma 3.8.2, we have

$$= |\frac{7p_1}{384}[\frac{p_1^2}{2} + \frac{x(4-p_1^2)}{2}] - \frac{1}{32}[\frac{p_1^3}{4} + \frac{p_1(4-p_1^2)x}{2} - \frac{p_1(4-p_1^2)x^2}{4} + \frac{(4-p_1^2)(1-|x|^2)z}{2}] - \frac{3p_1^3}{1536}|$$

$$|a_2a_3 - a_4| = |\frac{p_1(4-p_1^2)x^2}{128} - \frac{5p_1x(4-p_1^2}{768} - \frac{(4-p_1^2)(1-|x|^2)z}{64} - \frac{p_1^3}{1536}|. \tag{4.56}$$

Denotes $|x| = t \in [0,1]$, $p = c \in [0,2]$ then by using triangle inequality (4.56) gives us

$$|a_2a_3 - a_4| \leq \frac{(4-c^2)ct^2}{128} + \frac{5(4-c^2)ct}{768} + \frac{(4-c^2)}{64} + \frac{c^3}{1536}.$$

$$(F,1) \equiv \frac{(4-c^2)c}{128} + \frac{5(4-c^2)c}{768} + \frac{(4-c^2)}{64} + \frac{c^3}{1536}$$

Thus we get

$$\frac{\partial F}{\partial t} = \frac{(4-c^2)ct}{64} + \frac{5(4-C^2)c}{768}.$$

The function $F(c,t)$ is increasing for any t in $[0,1]$ this shows that $F(c,t)$ has max value at $t=1$.

$$MaxF(c,t) = F(c,1) = \frac{(4-C^2)c}{128} + \frac{5(4-c^2)c}{768} + \frac{4-c^2}{64} + \frac{c^3}{1536}.$$

Consider

$$M(c) = \frac{(4-C^2)c}{128} + \frac{5(4-c^2)c}{768} + \frac{4-c^2}{64} + \frac{c^3}{1536}. \tag{4.57}$$

$$M'(c) = \frac{-21c^2}{512} + \frac{11}{192} - \frac{c}{32}$$

$$c = \frac{-8+2\sqrt{170}}{21}. \tag{4.58}$$

From (4.58) in (4.57) we get,

$$|a_2a_3 - a_4| \leq \frac{829+85\sqrt{170}}{21168}. \tag{4.59}$$

This completes the proof.

Theorem 4.2.9 *If $f \in C_s(e^z)$ then $|a_2a_4 - a_3^2| \leq \frac{25}{576}$,*

Proof: From equation (4.49) of Theorem 4.2.6 we get

$$|a_2a_4 - a_3^2| = |\frac{p_1}{8}(\frac{p_3}{32} - \frac{p_1p_2}{128} - \frac{p_1^3}{1536}) - (\frac{p_2}{12} - \frac{p_1^2}{48})^2|$$

$$= |\frac{p_3p_1}{256} - \frac{p_1^2p_2}{1024} - \frac{p_1^4}{12288} - \frac{p_2^2}{144} + \frac{p_1^2p_2}{288} - \frac{p_1^4}{2304}|.$$

Use the Lemma 3.8.2 we get,

$$|a_2a_4 - a_3^2| = |\frac{p_1}{256}[\frac{p_1^3}{4} + \frac{p_1(4-p_1^2)x}{2} - \frac{p_1(4-p_1^2)x^2}{4} + \frac{(4-p_1^2)(1-|x|^2)z}{2}] - \frac{p_1^2}{1024}[\frac{p_1^2}{2} + \frac{x(4-p_1^2)}{2}]$$

$$-\frac{p_1^4}{12288} - \frac{1}{144}(\frac{p_1^2}{2} + \frac{x(4-p_1^2)}{2})^2 - \frac{p_1^2}{288}[\frac{p_1^2}{2} + \frac{x(4-p_1^2)}{2}] - \frac{p_1^4}{2304}|$$

$$= |\frac{p_1^4}{1024} + \frac{p_1^2(4-p_1^2)x}{512} - \frac{p_1^2(4-p_1^2)x^2}{1024} + \frac{p_1(4-p_1^2)(1-|x|^2)z}{512} - \frac{3p_1^4}{8192} - \frac{3p_1^2x(4-p_1^2)}{8192} - \frac{p_1^4}{12288}$$

$$-\frac{p_1^4}{576} - \frac{x(4-p_1^2)p_1^2}{288} - \frac{x^2(4-p_1^2)^2}{576} + \frac{p_1^4}{576} + \frac{x(4-p_1^2)p_1^2}{576} - \frac{p_1^4}{2304}|.$$

Simplify

$$|a_2a_4-a_3^2| = |\frac{p_1(4-p_1^2)(1-|x|^2)z}{512} - \frac{p_1^2(4-p_1^2)x^2}{1024} + \frac{25p_1^2x(4-p_1^2)}{73728} - \frac{x^2(4-p_1^2)p_1^2}{576} + \frac{17p_1^4}{36864}|. \tag{4.60}$$

Denote $|x| = t \in [0,1], p_1 = c \in [o,2]$ then using triangle inequality, we get

$$|a_2a_4-a_3^2| \leq \frac{(4-c^2)}{256} + \frac{c^2(4-c^2)t^2}{1024} + \frac{25c^2t(4-c^2)}{73728} + \frac{t^2(4-c^2)c^2}{576} + \frac{17c^4}{36864},$$

we have

$$F(c,t) = \frac{(4-c^2)}{256} + \frac{c^2(4-c^2)t^2}{1024} + \frac{25c^2t(4-c^2)}{73728} + \frac{t^2(4-c^2)c^2}{576} + \frac{17c^4}{36864}. \tag{4.61}$$

Which implies that

$$\frac{\partial F}{\partial t} = \frac{2(4-c^2)c^2t}{1024} + \frac{25(4-c5^2)c^2}{73728} + \frac{2(4-c^2)^2t}{576} \geq 0.$$

Which that $F(c,t)$ is rising for any value of t in the [0,1]. this demonstrates the maximum value of $F(c,t)$ at $t = 1$

$$MaxF(c,t) = F(c,t) = \frac{(4-c^2)}{256} + \frac{c^2(4-c^2)}{1024} + \frac{25c^2(4-c^2)}{73728} + \frac{(4-c^2)c^2}{576} + \frac{17c^4}{36864}.$$

Let us define

$$M(c) = \frac{(4-c^2)}{256} + \frac{c^2(4-c^2)}{1024} + \frac{25c^2(4-c^2)}{73728} + \frac{(4-c^2)c^2}{576} + \frac{17c^4}{36864}. \tag{4.62}$$

We get

$$M'(c) = \frac{-29c^3}{36864} + \frac{(-25c^2+100)c}{36864} - \frac{(-23c^2+4)c}{4608} - \frac{c}{128}.$$

We have

$$M''(c) = \frac{65c^2}{6144} - \frac{77}{3072}.$$

If $c = 0$, $M'(c)$ disappears. A quick calculation reveals that the function $M(c)$ has its maximum values at $c = 0$, which indicates that $M''(c) < 0$. Hence, we have (4.62) we get, $M(0) = \frac{4}{256} + \frac{16}{576}$

Simplify

$$|a_2a_4-a_3^2| \leq M(0) = \frac{36+64}{2304}.$$

As

$$|a_2a_4-a_3^2| \leq \frac{25}{576}. \tag{4.63}$$

This completes the proof.

Theorem 4.2.10 *If $f \in C_s(e^z)$ then $|H_3(1)| \leq 0.11678$*

Proof: Consider

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}$$

$$= a_3 \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} - a_4 \begin{vmatrix} a_1 & a_3 \\ a_2 & a_4 \end{vmatrix} + a_5 \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix}$$

Since $a_1 = 1$

$H_3(1) = a_3(a_2 a_4 - a_3{}^2) - a_4(a_4 - a_2 a_3) + a_5(a_3 - a_2{}^2)$, by applying triangle triangle inequality, we get

$$|H_3(1)| \leq |a_3||(a_2 a_4 - a_3{}^2)| + |a_4||(a_4 - a_2 a_3)| + |a_5||(a_3 - a_2{}^2)|. \quad (4.64)$$

Now, substituting the equations (4.51), (4.52), (4.53, (4.54), (4.59), (4.63) in (4.64), we get

$$|H_3(1)| \leq 0.11678 \quad (4.65)$$

Thus the theorem is proved.

CHAPTER 5

THE NEW CLASSES $S^{*}_{s},_{q}\,(e^{z})$ AND $C_{s},_{q}\,(e^{z})$ ASSOCIATED WITH EXPONENTIAL FUNCTION

In this chapter making use of a linear operator, we will introduce and study some new classes of analytic functions with respect to symmetrical points. These classes generalize some known classes of analytic and univalent functions. We will also discuss their interrelations, coefficient bounds, and some other results. Moreover, we will investigate some interesting proportions of functions that belong to our new classes.

5.1 The Classes of q-Starlike and q-Convex Function with Respect to Symmetric Points

The concept of univalent functions with respect to symmetric points.

Definition 5.1.1 *A function $f \in A$, and f is said to be in the class $S^{*}_{s,q}(e^{z})$ if and only if*

$$\frac{2[zD_q f(z)]}{f(z) - f(-z)} \prec e^{z}, z \in E. \tag{5.1}$$

Definition 5.1.2 *A function $f \in A$, is in the class $f \in C_{s,q}(e^{z})$ if and only if*

$$\frac{2[D_q(zD_q f(z))]}{D_q[f(z) - f(-z)]} \prec e^{z}, z \in E. \tag{5.2}$$

5.2 Main Results

For the following results, we consider $[1]_q = q_1,\ [2]_q = q_2,\ [3]_q = q_3,\ [4]_q = q_4,\ [5]_q = q_5, q \in (0,1)$ unless otherwise stated.

Theorem 5.2.1 *If $f \in S^*_{s,q}(e^z)$ then*

$$|a_2| \leq \frac{1}{q_2},$$

$$|a_3| \leq \frac{1}{q_3 - q_1},$$

$$|a_4| \leq |\frac{1}{q_4} + (\frac{q_2}{q_4 q_2 (q_3 - q_1)} - \frac{1}{q_4}) + (\frac{1}{6q_4} - \frac{q_2}{2q_2(q_3 - q_1)q_4})|,$$

$$|a_5| \leq |\frac{1}{(q_5 - q_1)} - \frac{1}{q_5 - q_1} + (\frac{-1}{2(q_5 - q_1)} + \frac{(4q_3 - q_1)}{(2(q_3 - q_1))^2(q_5 - q_1)}) + (\frac{1}{2(q_5 - q_1)} - \frac{1}{16(q_3 - q_1)(q_5 - q_1)}) + (\frac{1}{24(q_5 - q_1)} + \frac{16(q_3 - q_1)}{(8(q_3 - 8))^2(q_5 - q_1))})|.$$

Proof: Consider

$$\frac{2[zD_q f(z)]}{f(z) - f(-z)} = e^{w(z)}. \tag{5.3}$$

We get

$$\frac{2[zD_q f(z)]}{f(z) - f(-z)} = q_1 + a_2 q_2 z + (a_3 q_3 - q_1 a_3) z^2 + (a_4 q_4 - a_3 a_2 q_2) z^3 + (a_5 q_5 - q_1 a_5 - a_3^2 q_3 + q_1 a_3^2) z^4 + \cdots. \tag{5.4}$$

Let us define the function

$$p(z) = \frac{1 + w(z)}{1 - w(z)}.$$

Equivalently

$$w(z) = \frac{p(z) - 1}{p(z) + 1}. \tag{5.5}$$

That is

$$w(z) = \frac{p_1 z}{2} + \left(\frac{p_2}{2} - \frac{p_1^2}{4}\right) z^2 + \left(\frac{p_3}{2} - \frac{p_1 p_2}{2} + \frac{p_1^3}{8}\right) z^3 + \left(\frac{p_4}{2} - \frac{p_1 p_4}{2} - \frac{p_2^2}{4} + \frac{3p_1^2 p_2}{8} - \frac{p_1^4}{16}\right) z^4 + \cdots.$$

Since we have

$$e^{w(z)} = 1 + w(z) + \frac{(w(z))^2}{2!} + \frac{(w(z))^3}{3!} + \frac{(w(z))^4}{4!} + \cdots. \tag{5.6}$$

From (5.6), we get

$$e^{w(z)} = 1 + \frac{p_1 z}{2} + \left(\frac{p_2}{2} - \frac{p_1^2}{4}\right) z^2 + \left(\frac{p_3}{2} - \frac{p_1 p_2}{2} + \frac{p_1^3}{8}\right) z^3 + \left(\frac{p_4}{2} - \frac{p_1 p_4}{2} - \frac{p_2^2}{4} + \frac{3p_1^2 p_2}{8} - \frac{p_1^4}{16}\right) z^4 + \cdots$$

$$+ \frac{1}{2}(\frac{p_1 z}{2} + \left(\frac{p_2}{2} - \frac{p_1^2}{4}\right) z^2 + \left(\frac{p_3}{2} - \frac{p_1 p_2}{2} + \frac{p_1^3}{8}\right) z^3 + \left(\frac{p_4}{2} - \frac{p_1 p_4}{2} - \frac{p_2^2}{4} + \frac{3p_1^2 p_2}{8} - \frac{p_1^4}{16}\right) z^4 + \cdots)^2$$

$$+ \frac{1}{6}(\frac{p_1 z}{2} + \left(\frac{p_2}{2} - \frac{p_1^2}{4}\right) z^2 + \left(\frac{p_3}{2} - \frac{p_1 p_2}{2} + \frac{p_1^3}{8}\right) z^3 + \left(\frac{p_4}{2} - \frac{p_1 p_4}{2} - \frac{p_2^2}{4} + \frac{3p_1^2 p_2}{8} - \frac{p_1^4}{16}\right) z^4 + \cdots)^3$$

$$+\frac{1}{24}(\frac{p_1 z}{2}+\left(\frac{p_2}{2}-\frac{p_1^2}{4}\right)z^2+\left(\frac{p_3}{2}-\frac{p_1p_2}{2}+\frac{p_1^3}{8}\right)z^3+\left(\frac{p_4}{2}-\frac{p_1p_4}{2}-\frac{p_2^2}{4}+\frac{3p_1^2p_2}{8}-\frac{p_1^4}{16}\right)z^4+\cdots)^4+\cdots.$$

This gives us

$$e^{w(z)}=1+\frac{p_1 z}{2}+\left(\frac{p_2}{2}-\frac{p_1^2}{8}\right)z^2+\left(\frac{p_3}{2}-\frac{p_1p_2}{4}+\frac{p_1^3}{48}\right)z^3+\left(\frac{p_4}{2}-\frac{p_1p_3}{4}-\frac{p_2^2}{8}+\frac{p_1^2p_2}{16}+\frac{p_1^4}{384}\right)z^4+\cdots. \tag{5.7}$$

Now we compare the coefficient (5.4), (5.7) which gives us

$$a_2=\frac{p_1}{2q_2}. \tag{5.8}$$

We have

$$a_3=\frac{p_2}{2q_3-q_1}-\frac{p_1^2}{8q_3-q_1}. \tag{5.9}$$

Now, we consider

$$a_4=\frac{p_3}{2q_4}-\frac{p_1p_2}{4q_4}+\frac{p_1p_2q_2}{q_4(4q_2(q_3-q_1))}+\frac{p_1^3}{48q_4}-\frac{p_1^3q_2}{16q_2(q_3-q_1)q_4}, \tag{5.10}$$

and

$$a_5=\frac{p_4}{2(q_5-q_1)}-\frac{p_1p_3}{4(q_5-q_1)}-\frac{p_2^2}{8(q_5-q_1)}+\frac{p_1^2p_2}{16(q_5-q_1)}+\frac{p_1^4q}{384(q_5-q_1)}$$
$$+\left(\frac{p_2^2}{(2(q_3-q_1))^2}+\frac{p_1^4}{(8(q_3-q_1))^2}-\frac{2p_1^2p_2}{2(q_3-q_1)8(q_3-q_1)}\right)\frac{q_3-q_1}{q_5-q_1}. \tag{5.11}$$

By apply Lemma 3.8.3 and Lemma 3.8.1, we get

$$|a_2|\leq \frac{1}{q_2}, \tag{5.12}$$

$$|a_3|\leq \frac{1}{q_3-q_1}. \tag{5.13}$$

Now, so we get again apply the Lemma 3.8.1 as

$$|a_4|=\left|\frac{p_3}{2q_4}-\frac{p_1p_2}{4q_4}+\frac{p_1p_2q_2}{q_4(4q_2(q_3-q_1))}+\frac{p_1^3}{48q_4}-\frac{p_1^3q_2}{16q_2(q_3-q_1)q_4}\right|.$$

We use $|a-b|\leq|a|+|b|$ which leads us

$$|a_4|\leq\left|\frac{1}{q_4}+(\frac{q_2}{q_4q_2(q_3-q_1)}-\frac{1}{q_4})+(\frac{1}{6q_4}-\frac{q_2}{2q_2(q_3-q_1)q_4})\right|. \tag{5.14}$$

By Lemma 3.8.1, and triangular inequalities, we get

$$|a_5|\leq\left|\frac{1}{(q_5-q_1)}-\frac{1}{q_5-q_1}+(\frac{-1}{2(q_5-q_1)}+\frac{(4q_3-q_1)}{(2(q_3-q_1))^2(q_5-q_1)})\right.$$
$$\left.+(\frac{1}{2(q_5-q_1)}-\frac{1}{16(q_3-q_1)(q_5-q_1)})+(\frac{1}{24(q_5-q_1)}+\frac{16(q_3-q_1)}{(8(q_3-8))^2(q_5-q_1))})\right|. \tag{5.15}$$

Hence, this completes the proof

Taking $q\to 1^-$ in the above result we get the results that have been already proved in [91] as shown in the following corollary.

Corollary: If $f \in S_s^*(e^z)$ then $|a_2| \leq \frac{1}{2}, |a_3| \leq \frac{1}{2}, |a_4| \leq \frac{19}{48}, |a_4| \leq \frac{13}{24}$.

Theorem 5.2.2 *If $f \in S_s^*,_q(e^z)$ then $|a_3 - a_2^2| \leq \frac{1}{q_3 - q_1}$,*

Proof: Using the similar Mathematical techniques as used in the proof the Theorem 5.2.1, from (5.8), (5.9) we have

$$|a_3 - a_2^2| = \left| \frac{p_2}{2(q_3 - q_1)} - \frac{p_1^2}{8(q_3 - q_1)} - \frac{p_1^2}{(2q_2)^2} \right|$$
$$= \left| \frac{p_2}{2(q_3 - q_1)} - \frac{p_1^2(4q_2^2 - 8(q_3 - q_1))}{(2q_2)^2 8(q_3 - q_1)} \right|$$
$$= \frac{1}{2(q_3 - q_1)}(|p_2 - \frac{p_1^2 4q_2^2 - 8(q_3 - q_1)}{(2q_2)^2 4}|.$$

Now, we use Lemma 3.8.3 this gives us

$$|a_3 - a_2^2| \leq \frac{2}{2(q_3 - q_1)} max|\{1, |2\frac{(4q_2^2 - 8(q_3 - q_1))}{(2q_2)^2 4} - 1|\}$$
$$\leq \frac{1}{(q_3 - q_1)} max|\{1, \frac{(4q_2^2 - 8(q_3 - q_1)) - (2q_2)^2 2}{(2q_2)^2 2}|\}$$

$$|a_3 - a_2^2| \leq \frac{1}{q_3 - q_1}, \tag{5.16}$$

which is the required result. Taking $q \to 1^-$ in the above result we get the results that have been already proved in [91] as shown in the following corollary.

Corollary: If $f \in S_s^*(e^z)$ then $|a_3 - a_2^2| \leq \frac{1}{2}$.

Theorem 5.2.3 *If $f \in S_s^*,_q(e^z)$ then*

$$|a_2a_3 - a_4| \leq 1/3 \frac{1}{q_2 q_4 \left((q_1 - q_3 - 9) q_2 - 3 q_4\right)^2 (q_1 - q_3)} \Big(\Big(\Big(-4 q_1^2 + (8 q_3 + 2) q_1 - 4 q_3^2 - 2 q_3 - 18\Big)$$
$$q_2^2 + 2 q_4 (q_1 - q_3 - 12) q_2 - 6 q_4^2 \sqrt{(4 q_1^2 + (-8 q_3 - 2) q_1 + 4 q_3^2 + 2 q_3 + 18) q_2^2 - 2 q_4 (q_1 - q_3 - 12)}$$
$$\sqrt{q_2 + 6 q_4^2} - 5 q_2 \left(\left(q_1 - q_3 + \tfrac{63}{5}\right)(q_1 - q_3 - 3) q_2^2 + \tfrac{(12 q_1 - 12 q_3 - 90) q_4 q_2}{5} - 9/5 q_4^2\right)(q_1 - q_3).$$

Proof: Using the similar Mathematical techniques as used in the proof the Theorem 5.2.1 and using the value of coefficients, (5.8), (5.9), (5.10), we get

$$|a_2a_3 - a_4| = |\frac{p_1p_2}{4q_2(q_3 - q_1)} + \frac{p_1p_2}{4q_4} - \frac{p_1p_2q_2}{q_4q_2 4(q_3 - q_1)} - \frac{p_3}{2q_4} - \frac{p_1^3}{16q_2(q_3 - q_1)} - \frac{p_1^3}{48q_4}$$
$$+ \frac{p_1^3 q_2}{16q_2(q_3 - q_1)q_4}|.$$

Now applying the Lemma 3.8.2, we have

$$|a_2a_3 - a_4| = |\frac{p_1(4 - p_1^2)x^2}{8q_4} + \frac{p_1x(4 - p_1^2)}{8q_2(q_3 - q_1)} + \frac{p_1x(4 - p_1^2)}{8q_4} - \frac{p_1q_2x(4 - p_1^2)}{8q_4q_2(q_3 - q_1)} - \frac{p_1(4 - p_1^2)x}{4q_4}$$

$$-\frac{(4-p_1^2)(1-|x|^2)z}{4q_4}+\frac{p_1^3}{8q_2(q_3-q_1)}+\frac{p_1^3}{8q_4}-\frac{p_1^3q_2}{8q_4q_2(q_3-q_1)}-\frac{p_1^3}{8q_4}-\frac{p_1^3}{16q_2(q_3-q_1)}$$

$$-\frac{p_1^3}{48q_4}+\frac{p_1^3q_2}{16q_2(q_3-q_1)q_4}|. \tag{5.17}$$

Denote $|x|=t\in[0,1], p_1=c\in[0,2]$. Then using triangle inequality, we have

$$|a_2a_3-a_4|\leq\frac{(4-c^2)c}{8q_4}+(\frac{(4-c^2)c}{8q_2(q_3-q_1)}+\frac{(4-c^2)c}{8q_4(q_3-q_1)}-\frac{(4-c^2)c}{8q_4})+\frac{4-c^2}{4q_4}+(\frac{c^3}{16q_2(q_3-q_1)}$$

$$-\frac{c^3}{48q_4}-\frac{c^3}{16q_4(q_3-q_1)}).$$

which implies that

$$F(c,1)\equiv\frac{(4-c^2)ct^2}{8q_4}+(\frac{(4-c^2)ct}{8q_2(q_3-q_1)}+\frac{(4-c^2)ct}{8q_4(q_3-q_1)}-\frac{(4-c^2)ct}{8q_4})+\frac{4-c^2}{4q_4}+(\frac{c^3}{16q_2(q_3-q_1)}$$

$$-\frac{c^3}{48q_4}-\frac{c^3}{16q_4(q_3-q_1)}).$$

By computing the value of $\frac{\partial F}{\partial t}$, we note that the function $F(c,t)$ is non-decreasing for any t in $[0,1]$. This shows that $F(c,t)$ has a maximum value at $t=1$, so we get

$$M(c)=\frac{(4-c^2)c}{8q_4}+(\frac{(4-c^2)c}{8q_2(q_3-q_1)}+\frac{(4-c^2)c}{8q_4(q_3-q_1)}-\frac{(4-c^2)c}{8q_4})+\frac{4-c^2}{4q_4}$$

$$+(\frac{c^3}{16q_2(q_3-q_1)}-\frac{c^3}{48q_4}-\frac{c^3}{16q_4(q_3-q_1)}). \tag{5.18}$$

as

$$M'(c)=\frac{4-3c^2}{8q_2(q_3-q_1)}+\frac{4-3c^2}{8q_4(q_3-q_1)}-\frac{c}{2q_4}+\frac{3c^2}{16q_2(q_3-q_1)}-\frac{c^2}{16q_4}-\frac{3c^2}{16q_4(q_3-q_1)}.$$

$M'(c)$ vanishes at $c=r^*$, where

$$r^*=\frac{1}{q_1q_2-q_2q_3-9q_2-3q_4}(2(2q_2q_1-2q_2q_3+\sqrt{4q_1^2q_2^2-8q_1q_2^2q_3+4q_2^2q_3^2-2q_1q_2^2-}$$

$$\sqrt{2q_1q_2q_4+2q_2^2q_3+2q_2q_3q_4+18q_2^2+24q_2q_4+6q_4^2})).$$

This means that the function $M(c)$ can take the maximum value at r^*,

$$|a_2a_3-a_4|\leq 1/3\,\frac{1}{q_2q_4\left((q_1-q_3-9)\,q_2-3\,q_4\right)^2(q_1-q_3)}\left(\left(\left(-4\,q_1{}^2+(8\,q_3+2)\,q_1-4\,q_3{}^2-2\,q_3-18\right)\right.\right.$$

$$q_2^2+2\,q_4\,(q_1-q_3-12)\,q_2-6\,q_4{}^2\sqrt{(4\,q_1{}^2+(-8\,q_3-2)\,q_1+4\,q_3{}^2+2\,q_3+18)\,q_2{}^2-2\,q_4\,(q_1-q_3-12)}$$

$$\sqrt{q_2+6\,q_4{}^2}-5\,q_2\left(\left(q_1-q_3+\tfrac{63}{5}\right)(q_1-q_3-3)\,q_2{}^2+\tfrac{(12\,q_1-12\,q_3-90)q_4q_2}{5}-9/5\,q_4{}^2\right)(q_1-q_3)\tag{5.19}$$

which gives us the required result.

Taking $q \to 1^-$ in the above result we get the results that have been already proved in [91] as shown in the following corollary.

Corollary: If $f \in S_s^*(e^z)$ then $|a_2a_3 - a_4| \leq \frac{529+57\sqrt{118}}{3468}$.

Theorem 5.2.4 *If $f \in S_{s}^*{,}_q(e^z)$ then* $|a_2a_4 - a_3^2| \leq \frac{4}{(2(q_3-q_1))^2} + \frac{1}{q_2q_4}$

Proof: Using the similar Mathematical techniques as used in the proof the theorem 5.2.1, we begin the proof with, (5.8), (5.9), (5.10), we get

$$|a_2a_4 - a_3^2| = |\frac{p_1p_3}{4(q_2q_4)} + (\frac{p_1^2p_2q_2}{q_2q_4(8q_2(q_3-q_1))} - \frac{p_1^2p_2}{8q_4q_2}) + (\frac{p_1^4}{48q_4q_2} - \frac{p_1^4q_2}{16q_2(q_3-q_1q_4q_2})$$

$$-\frac{p_2^2}{(2q_3-q_1)^2} - \frac{p_1^4}{(8q_3-q_1)^2} + \frac{2p_2p_1^2}{(2(q_3-1)8(q_3-q_1)}|. \tag{5.20}$$

Use the Lemma 3.8.2, and by denoting $|x| = t \in [0,1], p_1 = c \in [0,2]$ with some simplification, we get

$$|a_2a_4 - a_3^2| \leq \frac{(4-c^2)}{4(q_2q_4)} + \frac{c^2(4-c^2)t^2}{16q_2q_4} + \frac{c^2(4-c^2)t}{8q_2q_4} - \frac{c^2t(4-c^2)}{16(q_2q_4)} - \frac{c^2t(4-c^2)}{8(q_2q_4)} + \frac{c^2t(4-c^2)}{16(q_2q_4)}$$

$$+\frac{c^2t(4-c^2)q_2}{16(q_2q_4q_2)(q_3-1)} - \frac{c^2t(4-c^2)}{2(2(q_3-1))^2} + \frac{c^2t(4-c^2)}{(2(q_3-1))(8(q_3-1)} + \frac{(4-c^2)^2t^2}{4(2(q_3-q_1))^2} + \frac{c^4}{16q_2q_4}$$

$$+\frac{q_2c^4}{16q_2q_4q_2(q_3-q_1)} - \frac{c^4}{16q_2q_4} + \frac{c^4}{48q_2q_4} - \frac{c^4q_2}{16q_2(q_3-q_11)q_4q_2} - \frac{c^4}{(8(q_3-q_1))^2}$$

$$+\frac{c^4}{2(q_3-q_1)(8(q_3-q_1))} - \frac{c^4}{4([2(q_3-q_1)])^2} + \frac{c^4}{16q_2q_4}. \tag{5.21}$$

Let us consider

$$F(c,t) = \frac{(4-c^2)}{4(q_2q_4)} + \frac{c^2(4-c^2)t^2}{16q_2q_4} + \frac{c^2(4-c^2)t}{8q_2q_4} - \frac{c^2t(4-c^2)}{16(q_2q_4)} - \frac{c^2t(4-c^2)}{8(q_2q_4)} + \frac{c^2t(4-c^2)}{16(q_2q_4)}$$
$$+\frac{c^2t(4-c^2)q_2}{16(q_2q_4q_2)(q_3-q_1)} - \frac{c^2t(4-c^2)}{2(2(q_3-q_1))^2} + \frac{c^2t(4-c^2)}{(2(q_3-q_1))(8(q_3-q_1)} + \frac{(4-c^2)^2t^2}{4(2(q_3-q_1))^2}$$
$$+\frac{c^4}{16q_2q_4} + \frac{q_2c^4}{16q_2q_4q_2(q_3-q_1)} - \frac{c^4}{16q_2q_4} + \frac{c^4}{48q_2q_4} - \frac{c^4q_2}{16q_2(q_3-q_1)q_4q_2}$$
$$-\frac{c^4}{(8(q_3-q_1))^2} + \frac{c^4}{2(q_3-1)(8(q_3-q_1))} - \frac{c^4}{4([2(q_3-q_1)])^2} + \frac{c^4}{16q_2q_4}.$$

Since $\frac{\partial F}{\partial t}$ shows that $F(c,t)$ is increasing for any then t in [0,1], and $F(c,t)$ has maximum value at $t = 1$, so we may write $M(c) = F(c,1)$ and

$$M'(c) = -\frac{c^3}{4q_2q_4} + \frac{(c^2+4)c}{4q_2q_4} + \frac{2q_2c^3}{16(q_2q_4q_2(q_3-q_1))} + \frac{2q_2(-c^2+4)c}{16([2]_2q_4q_4(q_3-q_1))}$$

$$+\frac{2(-c^2+4)c}{(2(q_3-q_1))^2}+\frac{2c^2}{2(q_3-q_1)8(q_3-q_1)}+\frac{2(-c^2+4)}{2(q_3-q_1)8(q_3-q_1)}-\frac{c}{2q_2q_4}$$

$$+\frac{4q_2c^3}{16q_2q_4q_2(q_3-q_1}-\frac{4c^3}{8(q_3-q_1)}. \tag{5.22}$$

As $M'(c)$ vanishes at $c=0$ and $M''(c)<0$ which means that the function $M(c)$ has maximum value at $c=0$. Hence we have $|a_2a_4-a_3^2|\leq M(0)$ which implies that

$$|a_2a_4-a_3^2|\leq\frac{4}{(2(q_3-q_1))^2}+\frac{1}{q_2q_4}, \tag{5.23}$$

which is the required result.

Taking $q\to 1^-$ in the above result we get the results that have been already proved in [91] as shown in the following corollary.

Corollary: If $f\in S_s^*(e^z)$ then $\leq |a_2a_4-a_3^2|\leq\frac{3}{8}$

Theorem 5.2.5 *If $f\in S^*_{s},_{q}(e^z)$ then*

$$H_3(1)\leq\frac{1}{q_3-q_1}\left(4\,(2q_3-2q_1)^{-2}+\frac{1}{q_2q_4}\right)+1/18\,\frac{1}{|q_4|\,(q_1-q_3)\,q_2q_4\,((q_1-q_3-9)\,q_2-3\,q_4)^2}$$

$$\left(6\left|\frac{q_3-q_1-1}{q_3-q_1}\right|+\left|\frac{q_3-q_1-3}{q_3-q_1}\right|+6\right)(((-4q_1{}^2+(8q_3+2)\,q_1-4q_3{}^2-2q_3-18)\,q_2{}^2+2q_4$$

$$(q_1-q_3-12)\,q_2-6q_4{}^2\sqrt{(4q_1{}^2+(-8q_3-2)\,q_1+4q_3{}^2+2q_3+18)\,q_2{}^2-2q_4\,(q_1-q_3-12)}$$

$$\sqrt{q_2+6q_4{}^2}-5\left((q_1-q_3-3)\left(q_1-q_3+\frac{63}{5}\right)q_2{}^2+\frac{(12q_1-12q_3-90)\,q_4q_2}{5}-9/5\,q_4{}^2\right)(q_1-q_3)\,q_2$$

$$+\frac{1}{q_3-q_1}\left(2\,(q_5-q_1)^{-1}+\frac{q_3-q_1}{(-q_1\,(q_5-q_1)+q_3\,(q_5-q_1))^2}-(-q_1\,(q_5-q_1)+q_3\,(q_5-q_1))^{-1}\right.$$

$$\left.+(24q_5-24q_1)^{-1}+\frac{q_3-q_1}{4\,(q_1-q_3)^2\,q_5-4q_1\,(q_3{}^2+q_1-2q_3)}\right).$$

Proof: Consider

$$H_3(1)=\begin{vmatrix}a_1&a_2&a_3\\a_2&a_3&a_4\\a_3&a_4&a_5\end{vmatrix}=a_3\begin{vmatrix}a_2&a_3\\a_3&a_4\end{vmatrix}-a_4\begin{vmatrix}a_1&a_3\\a_2&a_4\end{vmatrix}+a_5\begin{vmatrix}a_1&a_2\\a_2&a_3\end{vmatrix},\text{ Since } a_1=1$$

$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2)$, by applying triangle triangle inequality, we get

$$|H_3(1)| \leq |a_3||(a_2a_4 - a_3^2)| + |a_4||(a_4 - a_2a_3)| + |a_5||(a_3 - a_2^2)|. \tag{5.24}$$

Now, substituting the equations, in (5.24) we get the required result.

$$|H_3(1)| \leq \frac{1}{q_3 - q_1}\left(4\,(2q_3 - 2q_1)^{-2} + \frac{1}{q_2q_4}\right) + 1/18\,\frac{1}{|q_4|\,(q_1 - q_3)\,q_2q_4\,((q_1 - q_3 - 9)\,q_2 - 3\,q_4)^2}$$
$$\left(6\left|\frac{q_3 - q_1 - 1}{q_3 - q_1}\right| + \left|\frac{q_3 - q_1 - 3}{q_3 - q_1}\right| + 6\right)\left(\left(\left(-4\,{q_1}^2 + (8\,q_3 + 2)\,q_1 - 4\,{q_3}^2 - 2\,q_3 - 18\right)q_2{}^2 + 2\,q_4\right.\right.$$
$$(q_1 - q_3 - 12)\,q_2 - 6\,{q_4}^2\sqrt{(4\,{q_1}^2 + (-8\,q_3 - 2)\,q_1 + 4\,{q_3}^2 + 2\,q_3 + 18)\,{q_2}^2 - 2\,q_4\,(q_1 - q_3 - 12)}$$
$$\sqrt{q_2 + 6\,{q_4}^2} - 5\left((q_1 - q_3 - 3)\left(q_1 - q_3 + \frac{63}{5}\right){q_2}^2 + \frac{(12\,q_1 - 12\,q_3 - 90)\,q_4q_2}{5} - 9/5\,{q_4}^2\right)(q_1 - q_3)\,q_2$$
$$+\frac{1}{q_3 - q_1}\left(2\,(q_5 - q_1)^{-1} + \frac{q_3 - q_1}{(-q_1\,(q_5 - q_1) + q_3\,(q_5 - q_1))^2} - (-q_1\,(q_5 - q_1) + q_3\,(q_5 - q_1))^{-1}\right.$$
$$+ (24\,q_5 - 24\,q_1)^{-1} + \frac{q_3 - q_1}{4\,(q_1 - q_3)^2\,q_5 - 4\,q_1\,({q_3}^2 + q_1 - 2\,q_3)}. \tag{5.25}$$

which are the required results.

Taking $q \to 1^-$ in the above result we get the results that have been already proved in [91] as shown in the following corollary.

Corollary: If $f \in S_s^*(e^z)$ then $|H_3(1)| \leq 0.5893$.

Theorem 5.2.6 *If $f \in C_{s,q}(e^z)$ then*

$|a_2| \leq \frac{1}{q_2^2}$

$|a_3| \leq \frac{1}{q_3^2 - q_3}$

$|a_4| \leq |\frac{1}{q_4^2} + (\frac{-1}{q_4^2} + \frac{q_3}{(q_3^2 - q_3)q_4^2}) + (\frac{-q_3}{2(q_3^2 - q_3)q_4^2} + \frac{1}{6q_4^2})|$

$|a_5| \leq \frac{1}{(q_5^2 - q_5q_1)} - \frac{1}{(q_5^2 - q_5q_1)} + (\frac{4(q_3^3 - q_3^2q_1)}{(2(q_3^2 - q_3q_1))^2(q_5^2 - q_5q_1)} - \frac{1}{2(q_5^2 - q_5q_1)})$
$+ (\frac{1}{2(q_5^2 - q_5q_1)} - \frac{(q_3^3 - q_3^2q_1)}{(q_3^2 - q_3q_1)(q_3^2 - q_3q_1)(q_5^2 - q_5q_1)}) + (\frac{16(q_3^3 - q_3^2q_1)}{(8(q_3^2 - q_3q_1)^2(q_5^2 - q_5q_1)} + \frac{1}{24(q_5^2 - q_5q_1)})|.$

Proof: Consider

$$\frac{2D_q[zD_qf(z)]}{D_q(f(z) - f(-z))} = e^{w(z)}. \tag{5.26}$$

This gives us

$$\frac{2D_q[zD_qf(z)]}{D_q(f(z)-f(-z))}=\frac{2\left[q_1+q_2^2a_2z+q_3^2a_3z^2+q_4^2a_4z^3+q_5^2a_5z^4+....\right]}{2\left[q_1+q_3a_3z^2+q_5a_5z^4\right]+...]},$$

which implies that

$$=q_1+q_2^2a_2z+\left(q_3^2a_3-q_1q_3a_3\right)z^2+\left(q_4^2a_4-q_2^2q_3a_3a_2\right)z^3+\left(q_5^2-q_1q_5)a_5-(q_3^3-q_1q_3^2)a_3^2)\right)z^4+\cdots. \tag{5.27}$$

Let us define the function

$$p(z)=\frac{1+w(z)}{1-w(z)}.$$

That is

$$w(z)=\frac{p(z)-1}{p(z)+1}. \tag{5.28}$$

So, we have

$$w(z)=\frac{1+p_1z+p_2z^2+p_3z^3+p_4z^4+...-1}{1+p_1z+p_2z^2+p_3z^3+p_4z^4+...+1}.$$

That is

$$w(z)=\frac{p_1z}{2}+\left(\frac{p_2}{2}-\frac{p_1^2}{4}\right)z^2+\left(\frac{p_3}{2}-\frac{p_1p_2}{2}+\frac{p_1^3}{8}\right)z^3+\left(\frac{p_4}{2}-\frac{p_1p_4}{2}-\frac{p_2^2}{4}+\frac{3p_1^2p_2}{8}-\frac{p_1^4}{16}\right)z^4+\cdots. \tag{5.29}$$

Using the taylor series of $e^{w(z)}$, we may write the above equation as

$$e^w(z)=1+\frac{p_1z}{2}+\left(\frac{p_2}{2}-\frac{p_1^2}{8}\right)z^2+\left(\frac{p_3}{2}-\frac{p_1p_2}{4}+\frac{p_1^3}{48}\right)z^3\quad+\left(\frac{p_4}{2}-\frac{p_1p_3}{4}-\frac{p_2^2}{8}+\frac{p_1^2p_2}{16}+\frac{p_1^4}{384}\right)z^4 +\cdots. \tag{5.30}$$

Now compare the coefficient which gives us

$$a_2=\frac{p_1}{2q_2^2}, \tag{5.31}$$

$$a_3=\frac{p_2}{2(q_3^2-q_3q_1)}-\frac{p_1^2}{8(q_3^2-q_3q_1)}, \tag{5.32}$$

$$a_4=\frac{p_3}{2q_4^2}-\frac{p_1p_2}{4q_4^2}+\frac{p_2p_1q_3}{4(q_3^2-q_3q_1)q_4^2}-\frac{p_1^3q_3}{16(q_3^2-q_3q_1)q_4^2}+\frac{p_1^3}{48q_4^2}. \tag{5.33}$$

By some computation, we get the value of a_5 as

$$a_5=\frac{p_4}{2(q_5^2-q_5q_1)}-\frac{p_1p_3}{4(q_5^2-q_5q_1)}+\frac{p_2^2(q_3^3-q_3^2q_1)}{(2(q_3^2-q_3q_1))^2(q_5^2-q_5q_1)}-\frac{p_2^2}{8(q_5^2-q_5q_1)}$$
$$+\frac{p_1^4(q_3^3-q_3^2q_1)}{(8(q_3^2-q_3q_1))^2(q_5^2-q_5q_1)}+\frac{p_1^4}{384(q_5^2-q_5q_1)}+\frac{p_2p_1^2}{16(q_5^2-q_5q_1)}$$
$$-\frac{2p_1^2p_2(q_3^3-q_3^2q_1)}{(16(q_3^2-q_3)(q_3^2-q_3)(q_5^2-q_5q_1)}. \tag{5.34}$$

Now, we use Lemma 3.8.1 to get the coefficient bound of a_2 as

$$|a_2| \leq \frac{1}{q_2^2}, \tag{5.35}$$

and by Lemma 3.8.3, we get

$$|a_3| \leq \frac{1}{q_3^2 - q_3 q_1}. \tag{5.36}$$

Now again an application of Lemma 3.8.1 gives us

$$|a_4| \leq \left| \frac{1}{q_4^2} + (\frac{-1}{q_4^2} + \frac{q_3}{(q_3^2 - q_3)q_4^2}) + (\frac{-q_3}{2(q_3^2 - q_3)q_4^2} + \frac{1}{6q_4^2}) \right|, \tag{5.37}$$

and

$$|a_5| \leq \frac{1}{(q_5^2 - q_5 q_1)} - \frac{1}{(q_5^2 - q_5 q_1)} + (\frac{4(q_3^3 - q_3^2 q_1)}{(2(q_3^2 - q_3 q_1))^2 (q_5^2 - q_5 q_1)} - \frac{1}{2(q_5^2 - q_5 q_1)})$$

$$+(\frac{1}{2(q_5^2 - q_5 q_1)} - \frac{(q_3^3 - q_3^2 q_1)}{(q_3^2 - q_3 q_1)(q_3^2 - q_3 q_1)(q_5^2 - q_5 q_1)}) + (\frac{16(q_3^3 - q_3^2 q_1)}{(8(q_3^2 - q_3 q_1)^2 (q_5^2 - q_5 q_1)} +$$

$$\frac{1}{24(q_5^2 - q_5 q_1)})|. \tag{5.38}$$

Hence, we obtain this completes the proof

Taking $q \to 1^-$ in the above result we get the results that have been already proved in [91] as shown in the following corollary.

Corollary: If $f \in C_s(e^z)$ then $|a_2| \leq \frac{1}{4}$, $|a_3| \leq \frac{1}{6}$, $|a_4| \leq \frac{19}{192}$,$|a_5| \leq \frac{13}{120}$.

Theorem 5.2.7 *If $f \in C_{s,q}(e^z)$ then $|a_3 - a_2^2| \leq \frac{1}{q_3^2 - q_3 q_1}$*

Proof: Using the similar Mathematical techniques as used in the proof of the Theorem 5.2.6 and using (5.31), (5.32) we get

$$|a_3 - a_2^2| = |\frac{p_2}{2(q_3^2 - q_3 q_1)} - \frac{p_1^2}{8(q_3^2 - q_3 q_1)} - \frac{p_1^2}{4q_2^2 q_2^2}|$$

$$= |\frac{p_2}{2(q_3^2 - q_3 q_1)} - (\frac{(p_1^2 4q_2^2 q_2^2 + p_1^2 8(q_3^2 - q_3 q_1))}{8(q_3^2 - q_3 q_1) 4q_2^2 q_2^2})|.$$

Now, we use Lemma 3.8.3 this gives us

$$\leq \frac{1}{q_3^2 - q_3 q_1} max|\{1, |2(\frac{1}{q_3^2 - q_3 q_1}) - 1|\}$$

$$\leq \frac{1}{q_3^2 - q_3 q_1} max|\{1, |(\frac{2 - q_3^2 - q_3 q_1}{q_3^2 - q_3 q_1})|\}$$

$$|a_3 - a_2^2| \leq \frac{1}{q_3^2 - q_3 q_1}, \tag{5.39}$$

which is the required result

Taking $q \to 1^-$ in the above result we get the results that have been already proved in [91] as shown in the following corollary.

Corollary: If $f \in C_s(e^z)$ then $|a_3 - a_2^2| \leq \frac{1}{6}$

Theorem 5.2.8 *If $f \in C_{s,q}(e^z)$ then* $|a_2a_3 - a_4| \leq \frac{1}{726\left({q_2}^2{q_3}^2+\left(-{q_2}^2+3/11\,q_4\right)q_3-3/11\,{q_4}^2\right)^2{q_2}^2{q_4}^2(q_3-1)q_3}$

$$\left(12\sqrt{3}\left({q_2}^4{q_3}^4+\left(-2\,{q_2}^4+\tfrac{17\,{q_2}^2q_4}{6}\right){q_3}^3+\left({q_2}^4+\left(-\tfrac{17\,{q_4}^2}{6}-\tfrac{17\,q_4}{6}\right)q_2^2+1/2\,{q_4}^2\right){q_3}^2+\left(\tfrac{17\,{q_2}^2{q_4}^2}{6}\right.\right.\right.$$
$$-{q_4}^3q_3+1/2\,{q_4}^4\sqrt{\left(\left({q_3}^2-q_3\right){q_2}^2+1/3\,q_4\left(q_3-q_4\right)\right)\left(\left({q_3}^2-q_3\right){q_2}^2+3/8\,q_4\left(q_3-q_4\right)\right)}+1370\,{q_3}^6{q_2}^6+$$
$$\left(-4110\,{q_2}^6+1104\,{q_2}^4q_4\right){q_3}^5+\left(4110\,{q_2}^6+\left(-1104\,{q_4}^2-2208\,q_4\right){q_2}^4+297\,{q_2}^2{q_4}^2\right){q_3}^4+\left(-1370\,{q_2}^6\right.$$
$$+\left(2208\,{q_4}^2+1104\,q_4\right){q_2}^4+\left(-594\,{q_4}^3-297\,{q_4}^2\right){q_2}^2+27\,{q_4}^3{q_3}^3+\left(-1104\,{q_2}^4{q_4}^2+297\,{q_4}^3\left(q_4+2\right)\right.$$
$${q_2}^2-81\,{q_4}^4{q_3}^2+\left(-297\,{q_2}^2{q_4}^4+81\,{q_4}^5\right)q_3-27\,{q_4}^6$$

Proof: Using the coefficient bounds investigated in the previous result and similar Mathematical techniques as used in the proof of the Theorem 5.2.6, and using (5.31), (5.32 (5.33), we get

$$|a_2a_3 - a_4| = |\frac{p_1}{2(q_2)^2}[\frac{p_2}{2((q_3)^2 - q_3q_1)} - \frac{p_1^2}{8((q_3)^2 - q_3q_1)}] - [\frac{p_3}{2q_4^2} - \frac{p_1p_2}{4q_4^2}$$
$$+\frac{p_2p_1q_3}{4(q_3^2 - q_3q_1)q_4^2} - \frac{p_1^3q_3}{16(q_3^2 - q_3q_1)q_4^2} + \frac{p_1^3}{48q_4^2}]|.$$

We have

$$|a_2a_3 - a_4| = |\frac{p_1p_2}{4(q_2)^2((q_3)^2 - q_3q_1)} + \frac{p_1p_2}{4(q_4)^2} - \frac{p_1p_2q_3}{4(q_2)^2((q_3)^2 - q_3q_1)q_4} - \frac{p_3}{2(q_4)^2}$$
$$-\frac{p_1^3}{16(q_2)^2((q_3)^2 - q_3q_1)} + \frac{p_1^3q_3}{16(q_2)^2((q_3)^2 - q_3q_1)q_4} - \frac{p_1^3}{48(q_4)^2}|. \tag{5.40}$$

By the Lemma 3.8.2, we get

$$|a_2a_3 - a_4| = |\frac{p_1(4-p_1^2)x^2}{8(q_4)^2} + \frac{p_1(4-p_1^2)x}{8(q_2)^2((q_3)^2) - q_3)} + \frac{p_1(4-p_1^2)x}{8(q_4)^2} - \frac{p_1(4-p_1^2)xq_3}{8(q_2)^2((q_3)^2) - q_3)q_4}$$
$$-\frac{p_1(4-p_1^2)x}{4(q_4)^2} - \frac{(4-p_1^2)(1-|x|^2)z}{4(q_4)^2} + \frac{p_1^3}{8(q_2)^2((q_3)^2 - q_3)} + \frac{p_1^3}{8(q_4)2} - \frac{p_1^3q_3}{8(q_2)^2((q_3)^2 - q_3)q_4}$$
$$-\frac{p_1^3}{8(q_4)^2} - \frac{p_1^3}{16(q_2)^2((q_3)^2 - q_3)} + \frac{p_1^3q_3}{16(q_2)^2((q_3)^2 - q_3)q_4} - \frac{p_1^3}{48(q_4)^2}|.$$

Simplify

$$|a_2a_3 - a_4| = |\frac{p_1(4-p_1^2)x^2}{8(q_4)^2} + \frac{p_1(4-p_1^2)x}{8(q_2)^2((q_3)^2) - q_3)} - \frac{p_1(4-p_1^2)x}{8(q_4)^2} - \frac{p_1(4-p_1^2)xq_3}{8(q_2)^2((q_3)^2) - q_3)q_4}$$

$$-\frac{(4-p_1^2)(1-|x|^2)z}{4(q_4)^2}+\frac{p_1^3}{16(q_2)^2((q_3)^2-q_3)}-\frac{p_1^3}{48(q_4^)2}-\frac{p_1^3q_3}{16(q_2)^2((q_3)^2-q_3)q_4}|. \tag{5.41}$$

Denote $|x|=t\in[0,1]p=c\in[0,2]$. then using triangle inequality 4.2 gives us

$$|a_2a_3-a_4|\leq\frac{c(4-c^2)t^2}{8(q_4)^2}-\frac{c(4-c^2)t}{8(q_2)^2((q_3)^2)-q_3)}+\frac{c(4-c^2)t}{8(q_4)^2}+\frac{c(4-c^2)tq_3}{8(q_2)^2((q_3)^2)-q_3)q_4}$$

$$+\frac{(4-c^2)}{4(q_4)^2}-\frac{c^3}{16(q_2)^2((q_3)^2-q_3)}+\frac{c^3}{48q_4^2}+\frac{c^3q_3}{16(q_2)^2((q_3)^2-q_3)q_4}.$$

Consider

$$|a_2a_3-a_4|\equiv\frac{c(4-c^2)t^2}{8(q_4)^2}-\frac{c(4-c^2)t}{8(q_2)^2((q_3)^2)-q_3)}+\frac{c(4-c^2)t}{8(q_4)^2}+\frac{c(4-c^2)tq_3}{8(q_2)^2((q_3)^2)-q_3)q_4}$$

$$+\frac{(4-c^2)}{4(q_4)^2}-\frac{c^3}{16(q_2)^2((q_3)^2-q_3)}+\frac{c^3}{48(q_4)^2}+\frac{c^3q_3}{16(q_2)^2((q_3)^2-q_3)q_4}.$$

Thus we get

$$\frac{\partial F}{\partial t}=\frac{c(4-c^2)t}{4(q_4)^2}-\frac{c(4-c^2)}{8(q_2)^2((q_3)^2)-q_3)}+\frac{c(4-c^2)}{8(q_4)^2}+\frac{c(4-c^2)q_3}{8(q_2)^2((q_3)^2)-q_3)q_4}$$

$$MaxF(c,t)=F(c,1)=\frac{c(4-c^2)t^2}{8(q_4)^2}-\frac{c(4-c^2)t}{8(q_2)^2((q_3)^2)-q_3)}+\frac{c(4-c^2)t}{8(q_4)^2}+\frac{c(4-c^2)tq_3}{8(q_2)^2((q_3)^2)-q_3)q_4}$$

$$+\frac{(4-c^2)}{4(q_4)^2}-\frac{c^3}{16(q_2)^2((q_3)^2-q_3)}+\frac{c^3}{48q_42}+\frac{c^3q_3}{16(q_2)^2((q_3)^2-q_3)q_4}.$$

Let us define

$$M(c)=\frac{c(4-c^2)t^2}{8(q_4)^2}-\frac{c(4-c^2)t}{8(q_2)^2((q_3)^2)-q_3)}+\frac{c(4-c^2)t}{8(q_4)^2}+\frac{c(4-c^2)tq_3}{8(q_2)^2((q_3)^2)-q_3)q_4}$$

$$+\frac{(4-c^2)}{4(q_4)^2}-\frac{c^3}{16(q_2)^2((q_3)^2-q_3)}+\frac{c^3}{48q_4^2}+\frac{c^3q_3}{16(q_2)^2((q_3)^2-q_3)q_4}. \tag{5.42}$$

We have

$$M'(c)=\frac{-7c^2}{16(q_4)^2}+\frac{-c^2+4}{4(q_4)^2}+\frac{c^2}{16(q_2)^2((q_3)^2-q_3)}-\frac{-c^2+4}{8(q_2)^2((q_3)^2-q_3)}$$

$$-\frac{c^2q_3}{16(q_2)^2((q_3)^2-q_3)q_4}-\frac{(-c^2+4)q_3}{8(q_2)^2((q_3)^2-q_3)}-\frac{c}{2(q_4)^2}.$$

$M'(c)$ vanishes at $c=r^*$, where

$$c=-\frac{1}{11(q_2)^2(q_3)^2-11(q_2)^2q_3+3q_3q_4-3(q_4)^2}(2(2(q_2)^2(q_3)^2-2(q_2)^2q_3+$$

$$\sqrt{48q_2^4q_3^4-96q_2^4q_3^3+48q_2^4q_3^2+34q_2^2q_3^3q_4-34q_2^2q_3^2q_4^2-34q_2^2q_3^2q_4+34q_2^2q_3q_4^2+6q_3^2q_4^2-12q_3q_4^3+6q_4^4})),$$

which means that the function $M(c)$ can take maximum value at c in (5.40), we get

$$|a_2a_3-a_4|\leq\frac{1}{726\left(q_2{}^2q_3{}^2+\left(-q_2{}^2+3/11\,q_4\right)q_3-3/11\,q_4{}^2\right)^2q_2{}^2q_4{}^2\left(q_3-1\right)q_3}$$

$$\Big(12\sqrt{3}\Big(q_2{}^4q_3{}^4+\Big(-2\,q_2{}^4+\tfrac{17\,q_2{}^2q_4}{6}\Big)q_3{}^3+\Big(q_2{}^4+\Big(-\tfrac{17\,q_4{}^2}{6}-\tfrac{17\,q_4}{6}\Big)q_2^2+1/2\,q_4{}^2\Big)q_3{}^2+\Big(\tfrac{17\,q_2{}^2q_4{}^2}{6}\ \text{-}$$

$$q_4^3q_3+1/2\,q_4{}^4\sqrt{\left(\left(q_3{}^2-q_3\right)q_2{}^2+1/3\,q_4\left(q_3-q_4\right)\right)\left(\left(q_3{}^2-q_3\right)q_2{}^2+3/8\,q_4\left(q_3-q_4\right)\right)}+1370\,q_3{}^6q_2{}^6+$$

$$\left(-4110\,q_2{}^6+1104\,q_2{}^4q_4\right)q_3{}^5+\left(4110\,q_2{}^6+\left(-1104\,q_4{}^2-2208\,q_4\right)q_2{}^4+297\,q_2{}^2q_4{}^2\right)q_3{}^4+\left(-1370\,q_2{}^6\right.$$

$$+\left(2208\,q_4{}^2+1104\,q_4\right)[2]_q{}^4+\left(-594\,q_4{}^3-297\,q_4{}^2\right)[2]_q{}^2+27\,q_4{}^3q_3{}^3+\left(-1104\,q_2{}^4q_4{}^2+297\,q_4{}^3\left(q_4+2\right)\right.$$

$$q_2{}^2-81\,q_4{}^4q_3{}^2+\left(-297\,q_2{}^2q_4{}^4+81\,q_4{}^5\right)q_3-27\,q_4{}^6(5.43)$$

which is the required result.

Taking $q\to 1^-$ in the above result we get the results that have been already proved in [91] as shown in the following corollary.

Corollary: If $f\in C_s(e^z)$ then $|a_2a_3-a_4|\leq\frac{829+85\sqrt{170}}{21168}$.

Theorem 5.2.9 *If* $f\in C_{s,q}(e^z)$ *then*

$$|a_2a_4-a_3^2|\leq\frac{1}{q_2^2q_4^2}+\frac{16}{4(2(q_3^2-q_3))^2}$$

Proof: Using the similar Mathematical techniques as used in the proof of the Theorem 5.2.6 and using (5.31), (5.32), (5.33), we get

$$|a_2a_4-a_3^2|=|\frac{p_1}{2q_2^2}(\frac{p_3}{2q_4^2}-\frac{p_1p_2}{4q_4^2}+\frac{p_2p_1q_4q_3}{4q_2^2(q_3^2-q_3)q_4^2}-\frac{p_1^3q_4q_3}{16q_2^2(q_3^2-q_3)q_4^2})-(\frac{p_2}{2(q_3^2-q_3)}-\frac{p_1^2}{8(q_3^2-q_3)})^2|.$$

This gives us

$$=|\frac{p_1p_3}{4(q_2^2q_4^2)}+(\frac{p_1^2p_2q_4q_3}{(2q_2^2)4q_2^2(q_3^2-q_3)q_4^2}-\frac{p_1^2p_2}{8q_4^2q_2^2})+(\frac{p_1^4}{(8(q_3^2-q_3))^2}-\frac{p_1^4q_4q_3}{32q_2^4(q_3^2-q_3)q_4^2})$$

$$-\frac{p_2^2}{(2(q_3^2-q_3))^2}-\frac{p_1^4}{(8(q_3^2-q_3))^2}+\frac{2p_2p_1^2}{2(q_3^2-q_3)8(q_3^2-q_3)})|. \tag{5.44}$$

Use the Lemma 3.8.2 we get

$$|a_2a_4-a_3^2|=|\frac{p_1}{4(q_2^2q_4^2)}[\frac{p_1^3}{4}+\frac{p_1(4-p_1^2)x}{2}-\frac{p_1(4-p_1^2)x^2}{4}+\frac{(4-p_1^2)(1-|x|^2)z}{2}]$$

$$+\frac{p_1^2q_4q_3}{(8q_2^4)(q_3^2-q_3)q_4^2}[\frac{p_1^2}{2}+\frac{x(4-p_1^2)}{2}]-\frac{p_1^2}{8q_4^2q_2^2}[\frac{p_1^2}{2}+\frac{x(4-p_1^2)}{2}]+\frac{p_1^4}{(8(q_3^2-q_3))^2}$$

$$-\frac{p_1^4q_4q_3}{32q_2^4(q_3^2-q_3)q_4^2}-\frac{(\frac{p_1^2}{2}+\frac{x(4-p_1^2)}{2})^2}{(2(q_3^2-q_3))^2}-\frac{p_1^4}{(8(q_3^2-q_3))^2}+\frac{2[\frac{p_1^2}{2}+\frac{x(4-p_1^2)}{2}]p_1^2}{(2(q_3^2-q_3))(8(q_3^2-q_3))}|,$$

simplify

$$|a_2a_4-a_3^2|=|(\frac{p_1(4-p_1^2)(1-|x|^2)z}{8(q_2^2q_4^2)})-\frac{p_1^2(4-p_1^2)x^2}{16q_2^2q_4^2}+(\frac{p_1^2(4-p_1^2)x}{8q_2^2q_4^2}+\frac{p_1^2x(4-p_1^2)q_4q_3}{16(q_2^4(q_3^2-q_3)q_4^2)}$$

$$-\frac{p_1^2x(4-p_1^2)}{16(q_2^2q_4^2)}+\frac{p_1^2x(4-p_1^2)}{2(q_3^2-q_3)8(q_3^2-q_3)}-\frac{p_1^2(4-p_1^2)}{2(2(q_3^2-q_3)^2)})-\frac{x^2(4-p_1^2)^2}{4(2q_3^2-q_3)^2}+(\frac{q_4q_3p_1^4}{16(q_2^4(q_3^2-q_3)q_4^2q_2}$$

$$-\frac{p_1^4}{16q_2^2q_4^2}-\frac{p_1^4q_3q_4}{32q_2^4(q_3^2-q_3)q_4^2}-\frac{p_1^4}{4(2(q_3^2-q_3))^2}+\frac{p_1^4}{2(q_3^2-q_3)(8(q_3^2-q_3))}$$

$$-\frac{p_1^4}{4([2(q_3^2-q_3)])^2}+\frac{p_1^4}{16q_2^2q_4^2})|. \tag{5.45}$$

Denote $|x|=t\in[0,1], p_1=c\in[0,2]$ then using triangle inequality,we get

$$|a_2a_4-a_3^2|\leq\frac{(4-c^2)}{4(q_2^2q_4^2)})+\frac{c^2(4-c^2)t^2}{16q_2^2q_4^2}+(\frac{c^2(4-c^2)t}{8q_2^2q_4^2}+\frac{c^2t(4-p_1^2)q_4q_3}{16(q_2^4(q_3^2-q_3)q_4^2)}-\frac{c^2t(4-p_1^2)}{16(q_2^2q_4^2)}$$

$$+\frac{c^2t(4-c^2)}{2(q_3^2-q_3)8(q_3^2-q_3)}-\frac{c^2(4-c^2)}{2(2(q_3^2-q_3)^2)})+\frac{t^2(4-c^2)^2}{4(2q_3^2-q_3)^2}+(\frac{q_4q_3c^4}{16(q_2^4(q_3^2-q_3)q_4^2q_2}$$

$$-\frac{c^4}{16q_2^2q_4^2}-\frac{c^4q_3q_4}{32q_2^4(q_3^2-q_3)q_4^2}-\frac{c^4}{(8(q_3^2-q_3))^2}+\frac{c^4}{2(q_3^2-q_3)(8(q_3^2-q_3))}$$

$$-\frac{c^4}{4([2(q_3^2-q_3)])^2}+\frac{c^4}{16q_2^2q_4^2}). \tag{5.46}$$

Let us consider

$$F(c,t)=\frac{(4-c^2)}{4(q_2^2q_4^2)})+\frac{c^2(4-c^2)t^2}{16q_2^2q_4^2}+(\frac{c^2(4-c^2)t}{8q_2^2q_4^2}+\frac{c^2t(4-p_1^2)q_4q_3}{16(q_2^4(q_3^2-q_3)q_4^2)}-\frac{c^2t(4-p_1^2)}{16(q_2^2q_4^2)}$$

$$+\frac{c^2t(4-c^2)}{2(q_3^2-q_3)8(q_3^2-q_3)}-\frac{c^2(4-c^2)}{2(2(q_3^2-q_3)^2)})+\frac{t^2(4-c^2)^2}{4(2q_3^2-q_3)^2}+(\frac{q_4q_3c^4}{16(q_2^4(q_3^2-q_3)q_4^2q_2}$$

$$-\frac{c^4}{16q_2^2q_4^2}-\frac{c^4q_3q_4}{32q_2^4(q_3^2-q_3)q_4^2}-\frac{c^4}{(8(q_3^2-q_3))^2}+\frac{c^4}{2(q_3^2-q_3)(8(q_3^2-q_3))}$$

$$-\frac{c^4}{4([2(q_3^2-q_3)])^2}+\frac{c^4}{16q_2^2q_4^2}).$$

Thus we get

$$\frac{\partial F}{\partial t}=-\frac{c^2(4-c^2)t}{8q_2^2q_4^2}+\frac{c^2(4-c^2)}{8q_2^2q_4^2}+\frac{c^2(4-p_1^2)q_4q_3}{16(q_2^4(q_3^2-q_3)q_4^2)}-\frac{c^2(4-p_1^2)}{16(q_2^2q_4^2)}+\frac{c^2(4-c^2)}{2(q_3^2-q_3)8(q_3^2-q_3)}$$

$$+\frac{t(4-c^2)^2}{(2q_3^2-q_3)^2},$$

which gives that $F(c,t)$ is increasing for any then t in [0,1]. this show that $F(c,t)$ has maximum value at $t=1$.

$$MaxF(c,t)=F(c,1)=\frac{(4-c^2)}{4(q_2^2q_4^2)}+\frac{c^2(4-c^2)}{16q_2^2q_4^2}+(\frac{c^2(4-c^2)}{8q_2^2q_4^2}+\frac{c^2(4-p_1^2)q_4q_3}{16(q_2^4(q_3^2-q_3)q_4^2)}-\frac{c^2(4-p_1^2)}{16(q_2^2q_4^2)}$$

$$+\frac{c^2(4-c^2)}{2(q_3^2-q_3)8(q_3^2-q_3)}-\frac{c^2(4-c^2)}{2(2(q_3^2-q_3)^2)})+\frac{(4-c^2)^2}{4(2q_3^2-q_3)^2}+(\frac{q_4q_3c^4}{16(q_2^4(q_3^2-q_3)q_4^2q_2}$$

$$-\frac{c^4}{16q_2^2q_4^2}-\frac{c^4q_3q_4}{32q_2^4(q_3^2-q_3)q_4^2}-\frac{c^4}{(8(q_3^2-q_3))^2}+\frac{c^4}{2(q_3^2-q_3)(8(q_3^2-q_3))}$$

$$-\frac{c^4}{4([2(q_3^2-q_3)])^2}+\frac{c^4}{16q_2^2q_4^2}). \tag{5.47}$$

Let us define

$$M(c)=\frac{(4-c^2)}{4(q_2^2q_4^2)})+\frac{c^2(4-c^2)}{16q_2^2q_4^2}+(\frac{c^2(4-c^2)}{8q_2^2q_4^2}+\frac{c^2(4-p_1^2)q_4q_3}{16(q_2^4(q_3^2-q_3)q_4^2)}-\frac{c^2(4-p_1^2)}{16(q_2^2q_4^2)}$$

$$+\frac{c^2(4-c^2)}{2(q_3^2-q_3)8(q_3^2-q_3)}-\frac{c^2(4-c^2)}{2(2(q_3^2-q_3)^2)})+\frac{(4-c^2)^2}{4(2q_3^2-q_3)^2}+(\frac{q_4q_3c^4}{16(q_2^4(q_3^2-q_3)q_4^2q_2}$$

$$-\frac{c^4}{16q_2^2q_4^2}-\frac{c^4q_3q_4}{32q_2^4(q_3^2-q_3)q_4^2}-\frac{c^4}{(8(q_3^2-q_3))^2}+\frac{c^4}{2(q_3^2-q_3)(8(q_3^2-q_3))}$$

$$-\frac{c^4}{4([2(q_3^2-q_3)])^2}+\frac{c^4}{16q_2^2q_4^2}). \tag{5.48}$$

If $M'(c)$ vanishes at $c=0$ A simple computation yields that $M''(c)<0$ which means that the function $M(c)$ has maximum value at $c=0$. Hence we have

$$|a_2a_4-a_3^2|\leq M(0)=\frac{4}{4q_2^2q_4^2}+\frac{16}{4((2(q_3^2-q_3))^2},$$

which implies that

$$|a_2a_4-a_3^2|\leq\frac{1}{q_2^2q_4^2}+\frac{16}{4(2(q_3^2-q_3))^2}, \tag{5.49}$$

which is the required result.

Taking $q\to1^-$ in the above result we get the results that have been already proved in [91] as shown in the following corollary.

Corollary: If $f\in C_s(e^z)$ then $|a_2a_4-a_3^2|\leq\frac{25}{576}$.

Theorem 5.2.10 *If $f \in C_{s,q}(e^z)$ then*

$$|H_3(1)| \leq \frac{1}{8712\left(\left(q_3{}^2-q_3\right)q_2{}^2+3/11\,q_4\left(q_3-q_4\right)\right)^2 q_3{}^3 q_4{}^4 q_2{}^2\left(q_1-q_3\right)\left(q_3-1\right)^2}$$

$$(66\,q_3{}^2\sqrt{(3\,q_2{}^2q_3{}^2-3\,q_2{}^2q_3+q_3q_4-q_4{}^2)(8\,q_2{}^2q_3{}^2-8\,q_2{}^2q_3+3\,q_3q_4-3\,q_4{}^2)}\,(q_3-\tfrac{14}{11})$$
$$(q_3{}^2(q_3-1)^2q_2{}^4+\tfrac{17\,q_3q_4(q_3-1)(q_3-q_4)q_2{}^2}{6}+1/2\,q_4{}^2(q_3-q_4)^2)\,(q_1-q_3)\sqrt{2}+30140\,q_3{}^2$$
$$(-\tfrac{396\,q_4{}^4}{685}+q_3{}^3(q_3-\tfrac{14}{11})(q_1-q_3)(q_3-1))\,(q_3-1)^2q_2{}^6+24288\,(\tfrac{9\,q_4{}^5}{23}-\tfrac{9\,q_3q_4{}^4}{23}-q_3{}^3((q_3-\tfrac{14}{11})$$
$$q_1-\tfrac{13\,q_3{}^2}{46}-\tfrac{41\,q_3}{253}+\tfrac{33}{46}(q_3-1)\,q_4+q_3{}^4(q_3-\tfrac{14}{11})(q_1-q_3)(q_3-1)\,q_3q_4(q_3-1)\,q_2{}^4+6534\,(q_3-q_4)$$
$$q_4{}^2(\tfrac{24\,q_4{}^5}{121}-\tfrac{24\,q_3q_4{}^4}{121}-q_3{}^3((q_3-\tfrac{14}{11})\,q_1+\tfrac{5\,q_3{}^2}{11}-\tfrac{18\,q_3}{11}+\tfrac{16}{11})(q_3-1)\,q_4+q_3{}^4(q_3-\tfrac{14}{11})(q_1-q_3)(q_3-1))$$
$$q_2{}^2+594\,(q_3-q_4)^2(((-q_3+\tfrac{14}{11})\,q_1-\tfrac{13\,q_3{}^2}{11}+\tfrac{34\,q_3}{11}-\tfrac{24}{11})\,q_4+q_3(q_3-\tfrac{14}{11})(q_1-q_3))\,q_3{}^2q_4{}^3$$

Proof: *Consider*

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_3\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} - a_4\begin{vmatrix} a_1 & a_3 \\ a_2 & a_4 \end{vmatrix} + a_5\begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix}, \text{ Since } a_1 = 1$$

$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2)$, *by applying triangle inequality, we get*

$$|H_3(1)| \leq |a_3||(a_2a_4 - a_3^2)| + |a_4||(a_4 - a_2a_3)| + |a_5||(a_3 - a_2^2)|. \tag{5.50}$$

Now substituting all these values in (5.50) we get the desired inequality for

$$|H_3(1)| \leq \frac{1}{8712\left(\left(q_3{}^2-q_3\right)q_2{}^2+3/11\,q_4\left(q_3-q_4\right)\right)^2 q_3{}^3 q_4{}^4 q_2{}^2\left(q_1-q_3\right)\left(q_3-1\right)^2}$$

$$(66\,q_3{}^2\sqrt{(3\,q_2{}^2q_3{}^2-3\,q_2{}^2q_3+q_3q_4-q_4{}^2)(8\,q_2{}^2q_3{}^2-8\,q_2{}^2q_3+3\,q_3q_4-3\,q_4{}^2)}\,(q_3-\tfrac{14}{11})$$
$$(q_3{}^2(q_3-1)^2q_2{}^4+\tfrac{17\,q_3q_4(q_3-1)(q_3-q_4)q_2{}^2}{6}+1/2\,q_4{}^2(q_3-q_4)^2)\,(q_1-q_3)\sqrt{2}+30140\,q_3{}^2$$
$$(-\tfrac{396\,q_4{}^4}{685}+q_3{}^3(q_3-\tfrac{14}{11})(q_1-q_3)(q_3-1))\,(q_3-1)^2q_2{}^6+24288\,(\tfrac{9\,q_4{}^5}{23}-\tfrac{9\,q_3q_4{}^4}{23}-q_3{}^3((q_3-\tfrac{14}{11})$$
$$q_1-\tfrac{13\,q_3{}^2}{46}-\tfrac{41\,q_3}{253}+\tfrac{33}{46}(q_3-1)\,q_4+q_3{}^4(q_3-\tfrac{14}{11})(q_1-q_3)(q_3-1)\,q_3q_4(q_3-1)\,q_2{}^4+6534\,(q_3-q_4)$$
$$q_4{}^2(\tfrac{24\,q_4{}^5}{121}-\tfrac{24\,q_3q_4{}^4}{121}-q_3{}^3((q_3-\tfrac{14}{11})\,q_1+\tfrac{5\,q_3{}^2}{11}-\tfrac{18\,q_3}{11}+\tfrac{16}{11})(q_3-1)\,q_4+q_3{}^4(q_3-\tfrac{14}{11})(q_1-q_3)(q_3-1))$$
$$q_2{}^2+594\,(q_3-q_4)^2(((-q_3+\tfrac{14}{11})\,q_1-\tfrac{13\,q_3{}^2}{11}+\tfrac{34\,q_3}{11}-\tfrac{24}{11})\,q_4+q_3(q_3-\tfrac{14}{11})(q_1-q_3))\,q_3{}^2q_4{}^3. \tag{5.51}$$

Taking $q \to 1^-$ in the above result we get the results that have been already proved in [91] as shown in the following corollary.

Corollary: If $f \in C_s(e^z)$ then $|H_3(1) \leq 0.11678$.

CHAPTER 6

CONCLUSION

In this thesis, we have focussed on the coefficients of the functions that are analytic, univalent, and normalized in an open unit disk. Firstly, we summarized the basic definitions and results from Geometric Function Theory and those preliminary concepts have been further used to drive our new results, we have also explored the inventions in Quantum Calculus. The applications of the q-derivative operator in Geometric Functions Theory has been studied in detail, and further, the q-theory has been used to present certain novel classes of analytic function with respect to symmetric points.

Our work is centered on the classes of starlike functions and convex functions with respect to symmetric points and the q-extension of these classes has been investigated. We have summarized the existing work done see [91] on class $S_s^*(e^z)$ of starlike functions with respect to symmetric points and its corresponding class of $C_s(e^z)$ of convex functions with regard to symmetric points. The functions in these classes are subordinated to the exponential function. we have presented the q-version of the above-mentioned classes by defining the class $S_{s,q}^*(e^z)$ of q-starlike functions with respect to symmetric points associated with exponential function and the class $C_{s,q}(e^z)$ of q-convex functions with respect to symmetric points subordinated to exponential functions. These classes have been introduced with the utilization q-derivative operator. The subordination technique has been used to investigate these classes.

We have explored certain interesting properties of the functions belonging to our new classes including the coefficient bound and Fekete Szego problems. The third-order Hankel determinant determinant of functions in classes $S_{s,q}^*(e^z)$ and $C_{s,q}(e^z)$ have been determined. It has been noted that our new classes are the refined ones as compared to the existing classes and our new results are the advancement of the already derived theorems by various researchers in the field of Geometric Functions Theory. We have validated our newly derived results by taking the limit as $q \to 1^-$, this gave us the known results.

We hope that our research will make a remarkable contribution to the field of univalent functions theory and it is pleased to mention here that a part of this thesis has been published in a well-reputed journal given as under.

Zameer Abbas and Sadia Riaz, Coefficient inequalities for certain subclass of starlike functions with respect to symmetric points related with q-exponential function, Scientific Inquiry and Review, (2023), 7(4), 35-52.

6.1 Future work

The work presented in this thesis is all about the starlike function and convex functions with respect to symmetric points associated with the exponential function. We have two ideas for the extension of our derived results in the future. First, The results presented in our thesis can be investigated for the advanced classes of q-close-to-convex function and q-quasi convex function and the inclusion results can be derived. Secondly, Using the Subordination technique and the concepts of Quantum calculus, the new classes associated with the boundary points can be investigated and its comparison with the classes presented in our research can be shown.

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