MODIFICATION OF GRÜSS TYPE INEQUALITIES VIA CONFORMABLE FRACTIONAL INTEGRAL

By

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Candidate of **Master of Science in Mathematics** at National University of Modern Languages do here by declare that the thesis **Modification of Grüss Type Inequalities via Conformable Fractional Integral** submitted by me in fractional fulfillment of **MS Mathematics** degree, is my original work, and has not been submitted or published earlier. I also solemnly declare that it shall not, in future, be submitted by me for obtaining any other degree from this or any other university or institution. I also understand that if evidence of plagiarism is found in my thesis/dissertation at any stage, even after the award of a degree, the work may be cancelled and degree revoked.

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DEDICATION

I dedicate my thesis to my parents and teachers for their endless support and encouragement throughout my pursuit for education. I hope this achievement will fulfill the dream they envisioned for me.

ABSTRACT

In this thesis, Grüss type integral inequalities were established for conformable fractional integral given by Katugampola [11]. Grüss type integral provides the estimation of a function to its integral mean. It is useful in error estimations of the quadrature rules in numerical analysis. Grüss type integral inequality to weighted Ostrowski-Grüss type inequality are modified for differentiable mapping in terms of the upper and lower bounds of the first derivative via Katugampola conformable fractional integral. The inequality is then applied to numerical integration. Afterward, the application to numerical integration of modified Grüss type inequality to weighted Ostrowski-Grüss inequality via conformable fractional integral for $\alpha-$fractional differentiable mapping is described.

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Sumbal Farooq

Contents

Chapter 1

Introduction and Preliminaries

1.1 Introduction

Gorenflo *et al.* [1] provided the idea of fractional calculus, it is a branch of mathematics concerned with the study and application of arbitrary order integrals and derivatives. Despite being uncommon, the word "fractional" is nonetheless used frequently. Fractional calculus is a topic that is both old and new. However, it may be considered an unknown area as well, because it has only been objecting to specialist conferences and treatises for a little more than twenty years. The use of fractional calculus in numerical analysis and other branches of science and engineering, probably including fractal phenomena, has generated a great deal of interest in calculus recent years.

With the development of differential and integral calculus, the differential and integral inequalities also grew very rapidly. These inequalities have great importance in mathematical sciences. These inequalities provide us lower and upper bounds on functions and their derivatives and have wide applications in special means and numerical integration.

Mathematics and allied subjects cover numerous aspects of convexity. Furthermore, Finance and biology both heavily rely on the concept of convexity. This represents a powerful principle that can be used in many ways. The technique for investigating a diverse range of unconnected topics in pure and applied. This includes looking at the topic from different angles, as well as exploring it. The development of the theory of inequalities is strongly related to the convexity concept. Differential equation is an important tool for understanding certain properties of it.

In the modelling of engineering and science issues, fractional differential and integral equations are becoming increasingly significant. It has been demonstrated that in many cases, these models produce better outcomes than equivalent models using integer derivatives. The theory of fractional differential equations and the calculus of fractional order derivatives have been thoroughly investigated. The existing result for fractional differential equations is established through a fixed point technique in the majority of the current literature. Qualitative features of fractional differential equations utilizing the Riemann-Liouville (R-L) and Caputo derivatives for differential and integral inequalities. Scientists develop a comparative conclusion for the R-L type of integral inequalities by utilising the conventional Lipschitz condition on the nonlinear component. In this case, the comparison theorem and the explicit declaration come in handy. One of the most important use for fractional integral inequalities is determining solutions to numerical problems. This can be done in many different fields, such as quadrature, transform theory and probability. Additionally, these inequalities provide upper and lower bounds on answers to equations which proceed them. In mathematics, this is often very valuable information when trying to solve a problem or find an optimal solution. A fractional differential equation with variable coefficients can be used to demonstrate the existence of the fractional differential equation. It is critical to remember this when working on equations and issues involving these sorts of equations, since it will make solving them much easier. A quasi-linearization approach is used to solve nonlinear fractional differential equations. This is a novel approach for solving these equations, and it is superior.

The purpose of this chapter is to look into generalizations of integral inequalities for n-times differentiable mappings. Explicit limits for interior point rules are established using the contemporary theory of inequality and a generic Peano kernel. Where n times differentiable functions are taken into account. The acquired integral equalities are then utilised to create inequalities for n-times differentiable mappings on the three norms $||.||_{\infty}, ||.||_{\rho}, ||.||_{1}$. Specifically, explicit limits for perturbed trapezoid, midpoint, Simpson's, Newton-Cotes, and left and right rectangle rules are explored. The inequalities are also applied to various composite quadrature rules, and the analysis allows for the determination of the partition necessary to ensure that the accuracy of the result is within a specified error tolerance.

Integration using weight functions is employed in a wide range of mathematical issues, including approximation theory and spectral analysis, statistical analysis, and distribution theory. Grüss proposed an integral inequality and Ostrowski discovered an intriguing integral inequality related to differentiable mappings, which has important implications in numerical integration, probability and optimization theory, stochastic, statistics, information, and integral operator theory. Many scholars have focused their emphasis in recent years on the study and generalizations of the two inequalities.

Recently, a novel inequality was built utilizing the weighted Peano kernel, which is more generic than earlier inequalities discovered and discussed. The weighted Peano kernel method not only broadened the results but also provided several more intriguing inequalities as special examples. By using the weighted Grüss inequality for bounded differentiable mappings, they created another version of the Ostrowski-Grüss type inequality, which generalizes the earlier inequalities discovered. It is also possible to derive perturbed midpoint and trapezoid inequalities. This inequality is expanded to accommodate for numerical integration applications.

Many mathematicians have great contributions in this direction in the form of books, like, writing of the book by Mitrinović *et al.* [2] and the classical book by Hardy Littlewood and Polya [3].

Ostrowski introduced the classical integral inequality as follows:

Theorem 1 *: Let* $f : [s_1, s_2] \to \mathbb{R}$, *with* $s_1 < s_2$ *be differentiable mapping on* (s_1, s_2) *whose derivative* $f' : (s_1, s_2) \to \mathbb{R}$ *is bounded on* (s_1, s_2),

$$|f'(t)| \leq M < \infty, \quad for\ all\ \ k \in [s_1, s_2].$$

Then, the inequality is given by

$$\left| f(k) - \frac{1}{s_2 - s_1} \int_{s_1}^{s_2} f(t)\,dt \right| \leq \left[\frac{1}{4} + \frac{\left(k - \frac{s_1+s_2}{2}\right)^2}{(s_2 - s_1)^2} (s_2 - s_1)\ M \right],$$

for all $k \in [s_1, s_2]$, *where* M *is some constant. The constant* $\frac{1}{4}$ *is the best possible. The constant* $\frac{1}{4}$ *is sharp in the sense that it can not be replaced by a smaller one.*

Many researchers introduced various kinds of Ostrowski type integral inequalities to achieve desired results [2, 4, 5]. Mathematical inequalities play a significant role in the study of mathematics and many related subjects, and their applications are diverse. In the case of fractional partial differential equations, fractional integral inequalities are useful in determining the uniqueness of solutions. They also give upper and lower boundaries for fractional boundary value problem solutions. These recommendations have led various researchers in the field of integral inequalities to inquire into certain extensions by involving fractional calculus operators.

In 1935, Grüss [6] introduced an integral inequality which estimates the difference between the integral of the product of two functions and product of their integrals and is given in the form of the following theorem:

Theorem 2 *: Let* f *and* g $:[s_1, s_2] \to \mathbb{R}$ *be two integrable functions such that* $\phi \leq f(k) \leq \psi$ *and* $\gamma \leq g(k) \leq \Gamma$ *for all* $k \in [s_1, s_2]$ *and* ϕ, ψ, γ *and* Γ *are constants. We,* *then, have the following inequality:*

$$\left| \frac{1}{s_2 - s_1} \int_{s_1}^{s_2} f(k) g(k)\, dk - \frac{1}{(s_2 - s_1)^2} \int_{s_1}^{s_2} f(k)\, dk \int_{s_1}^{s_2} g(k)\, dk \right| \leq \frac{1}{4} (\psi - \phi)(\Gamma - \gamma). \qquad (1.1)$$

The constant $\frac{1}{4}$ is sharp.

In the meantime, a number of mathematician introduced the concept of non-integer order of the derivatives and the integrals known as fractional derivatives and the fractional integrals [1, 7, 3, 8].

The Grüss type inequality is useful in a variety of situations. Difference equations, integral arithmetic mean, and h-integral arithmetic mean are examples. On the other hand, we investigate the Grüss type inequality in spaces with intern product, and as a result, several applications of the Mellin transform of sequences and polynomials in Hilbert spaces are investigated. In this regard, there are several notable inequalities that use integer order integrals, including Jensen's inequality, Holder's inequality, Minkowski's inequality [10], and reverse Minkowiski's inequality. The space of the p-integrable functions, $L_p(s_1,\ s_2)$, is particularly important for studying such inequalities, as well as functions, integrals, and norms. However, this employ the space of Lebesgue mensurable functions, which accepts the space $L_p(s_1,\ s_2)$. The advent of fractional calculus allows for a variety of modifications, conclusions, and key ideas in sciences, and other fields. As a result, various fractional integrals, such as Riemann-Liouville, Katugampola, Hadamard, Erd'elyi-Kober, Liouville, and Weyl types, might be defined. Other fractional integrals can be discovered. As a result, various inequalities incorporating such formulations have been established throughout the years using fractional integrals, such as the reverse Minkowski, Hermite-Hadamard inequalities, Ostrowski type inequalities, and Fejer type inequalities. It is also highlighted that there are extensions in the literature that use fractional integrals of Riemann-Liouville, Hadamard, and the q-fractional integral.

The suitable fractional derivative (or integral) depends on the system under consideration, and as a result, there are several works of literature devoted to various fractional operators. Katugampola [11] has introduced new fractional operators that generalize both the Riemann-Liouville and Hadamard fractional operators. Although the Katugampola fractional integral operator is an Erd'elyi-Kober type operator, the author contended that Hadamard equivalence operators cannot be obtained from Erd'elyi-Kober type operators. In this sense, Almeida, Malinowska, and Odzijewicz created the Caputo-Katugampola derivative, which generalizes the idea of Caputo and Caputo-Hadamard fractional derivatives. The new operator turns out to be the left inverse of the Katugampola fractional integral and retains some of the essential features of the Caputo and Caputo Hadamard fractional derivatives. This derivative generalizes the Caputo and Caputo Hadamard fractional derivatives.

In recent years, a limit based definition of conformable derivative and conformable fractional integral was presented by Katugampola [11] in order to overcome some of difficulties pointed out by [12].

1.2 Preliminaries

In this section, some definitions of fractional derivatives, fractional integrals [12, 13, 14], Riemann Integral and Katugampola derivative of fractional order [8] are covered. Some basic definitions and concepts presented here will be used throughout this dissertation.

Riemann Integral:

Let f be a function defined on an interval $I = [s_1,\ s_2]$. Suppose that there is a number R such that for each $\varepsilon > 0$, there is a $\delta > 0$ such that if $\dot{P} = \{([k_{i-1}, k_i], t_i) : i = 1, 2, 3, ..., n\}$ is any tagged partition of $[s_1, s_2]$, where $k_i - k_{i-1} < \delta$ for $i = 1, 2, 3, ..., n$, then

$$\left| \sum_{i=1}^{n} f(t_i)(k_i - k_{i-1}) - R \right| \leq \varepsilon.$$

Then, we write

$$R = \int_{s_1}^{s_2} f(k)dk,$$

and say that R is a Riemann integral of f over the interval $[s_1, s_2]$.

1.2.1 Fractional Derivatives

Liouville left-sided derivative :

$$D_{0^+}^{\alpha}[f(k)] = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dk^n} \int_0^k (k-t)^{-\alpha+n-1} f(t)dt, \quad k > 0.$$

Liouville right-sided derivative :

$$D^{\alpha}_{0^-}[f(k)] = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dk^n} \int_k^{\infty} (k-t)^{-\alpha+n-1} f(t)dt, \quad k < \infty.$$

Riemann-Liouville left-sided derivative:

If $f(k) \in C\ ([s_1, s_2])$ and $s_1 < k < s_2$ then the left sided Riemann-Lioville derivative is

$${}^{RL}D^{\alpha}_{s_1^+}[f\ (k)] = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dk^n} \int_{s_1}^{k} (k-t)^{n-\alpha-1} f(t)dt, \quad k \geq s_1.$$

Riemann-Liouville right-sided derivative:

If $f(k) \in C\ ([s_1, s_2])$ and $s_1 < k < s_2$ then the right sided Riemann-Lioville derivative is

$${}^{RL}D^{\alpha}_{s_2-}[f\ \ (k)] = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dk^n} \int_{k}^{s_2} (k-t)^{n-\alpha-1} f(t)\ dt, \quad k \leq s_2.$$

Caputo left-sided derivative:

The Caputo left-sided derivative of function $f(k)$ is defined as

$${}_*D^{\alpha}_{s_1^+}[f(k)] = \frac{1}{\Gamma(n-\alpha)} \int_{s_1}^{k} (k-t)^{n-\alpha-1} \frac{d^n}{dk^n}[f(t)]dt, \quad k \geq s_1.$$

Caputo right-sided derivative:

The Caputo right-sided derivative of function $f(k)$ is defined as

$${}_*D^{\alpha}_{s_2-}[f(k)] = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_{k}^{s_2} (k-t)^{n-\alpha-1} \frac{d^n}{dk^n}[f(t)]dt, \quad k \leq s_2.$$

Local fractional Yang derivative:

$$D_{-}^{\alpha}[f(k)]|_{k=k_0} = \lim_{k \to k_0} \frac{\Delta^{\alpha}[f(k) - f(k_0)]}{(k - k_0)}.$$

1.2.2 Fractional Integrals

Riemann-Liouville left-sided integral:

Let $f \in L_1 \, [s_1, \, s_2]$, the left-sided Riemann-Liouville $J_{s_1+}^{\alpha} f(k)$ of order $\alpha > 0$ with $s_1 \geq 0$ are defined by

$$J_{s_1+}^{\alpha} f(k) = \frac{1}{\Gamma(\alpha)} \int_{s_1}^{k} (k - t)^{\alpha-1} f(t) dt, \quad k \geq s_1,$$

where $\Gamma(\alpha)$ is the Gamma function.

Riemann-Liouville right-sided integral:

Let $f \in L_1 \, [s_1, s_2]$, the right-sided Riemann-Liouville $J_{s_2-}^{\alpha} f(k)$ of order $\alpha > 0$ with $s_1 \geq 0$ are defined by

$$J_{s_2-}^{\alpha} f(k) = \frac{1}{\Gamma(\alpha)} \int_{k}^{s_2} (t - k)^{\alpha-1} f \ (t) dt, \quad k \leq s_2,$$

where $\Gamma(\alpha)$ is the Gamma function

$$J_{s_1^+}^{0} f(k) = J_{s_2^-}^{0} f(k) = f(k).$$

Conformable Fractional Integral:

Let $\alpha \in (n,\ n{+}1)$ and $\delta = \alpha{-}n$. Then, the left and right-sided conformable integrals of order $\alpha > 0$ are given by:

$$J_{s_1}^{\alpha} f(k) = \frac{1}{n!} \int_{s_1}^{k} (k-t)^n (t-s_1)^{\delta-1} f(t)\ dt,$$

and

$$J_{s_1}^{s_2} f(k) = \frac{1}{n!} \int_{k}^{s_2} (t-k)^n (s_2-t)^{\delta-1} f(t) dt,$$

respectively. Note that for $\alpha = n{+}1$, and $\delta = 1$, $n = 0, 1, 2, ...$

The conformable fractional integral becomes RL-integrals, that is

$$\begin{aligned} J_{\alpha}^{s_1} f(k) &= J_{s_1^+}^{\alpha} f(k), \\ \text{and} \quad J_{\alpha}^{s_2} f(k) &= J_{s_2^-}^{\alpha} f(k). \end{aligned}$$

Local fractional Yang integral:

It is defined as follows:

$$J_{s_2}^{\alpha} f(k) = \frac{1}{\Gamma(1+\alpha)} \int_{s_1}^{s_2} f\ (t)(dt)^{\alpha}.$$

Hermite-Hadamard's Inequality:

Let f: $I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on an interval I of real numbers and $s_1,\ s_2 \in I$ with $s_1 < s_2$. The inequality

$$f(\frac{s_1+s_2}{2}) \leq \int_{s_1}^{s_2} f(k) dk \leq \frac{f(s_1)+f(s_2)}{2},$$

is well known in the literature as Hermite-Hadamard's inequality.

Katugampola Fractional Integral:

Let $[s_1, s_2] \subset R$ be a finite interval. Then the Katugampola fractional integral of order $\alpha > 0$ are defined by:

$$^{\rho}I^{\alpha}_{s_1+}f(k) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{s_1}^{k}\frac{t^{\rho-1}f(t)}{(k^{\rho}-t^{\rho})^{1-\alpha}}dt,$$

where $s_1 < k < s_2$, $\rho > 0$, Γ is Gamma function and the integral exist.

Katugampola Derivative of Fractional order and Conformable Fractional Integral:

The Katugampola derivative of fractional order $\alpha \in (0, 1]$ and for $s \in [0, \infty)$ is defined as:

$$\begin{aligned} D^{\alpha}(f(s)) &= \lim_{\varepsilon\to 0}\frac{f(se^{\varepsilon s^{-\alpha}}) - f(s)}{\varepsilon}, \qquad (1.2) \\ D^{\alpha}(f(0)) &= \lim_{s\to 0}D^{\alpha}(f(s)), \end{aligned}$$

provided the limit exists. If f is fully differentiable at s, then

$$D^{\alpha}(f(s)) = s^{1-\alpha}\frac{df(s)}{ds}. \qquad (1.3)$$

A function f is α-differentiable at a point $s \geq 0$, if the limit in (1.2) exists and is finite. Consequently, the results are given in the form of the following theorem:

Theorem 3 *: Let $\alpha \in (0,1]$ and f, g be $\alpha-$differentiable at a point $s > 0$, then*

(i) $D^{\alpha}(s_1 f + s_2 g) = s_1 D^{\alpha}(f) + s_2 D^{\alpha}(g)$, *for all* $s_{11}, s_{12} \in \mathbb{R}$,

(ii) $D^{\alpha}(s_1) = 0$, *for all constant functions* $f(s) = s_1$,

(iii) $D^{\alpha}(fg) = f D^{\alpha}(g) + g D^{\alpha}(f)$,

(iv) $D^{\alpha}(\frac{f}{g}) = \frac{g D^{\alpha}(f) - f D^{\alpha}(g)}{g^2}$,

(v) $D^{\alpha}(s^n) = n s^{n-\alpha}$, *for all* $n \in \mathbb{R}$,

(vi) $D^{\alpha}(fog)(s) = f^{'}(g(s)) D^{\alpha}(g(s))$, *for f is differentiable at* $g(s)$.

Conformable Fractional Integral : Let $\alpha \in (0,1]$ and $0 \leq s_1 < s_2$. A function $f\ : [s_1,\ s_2] \to \mathbb{R}$ is $\alpha-$fractional integral on $[s_1,\ s_2]$, if the integral

$$\int_{s_1}^{s_2} f(t) d_{\alpha} t = \int_{s_1}^{s_2} f(t) t^{\alpha-1} dt, \tag{1.4}$$

exists and is finite. Further,

$$J_{\alpha}^{s_1}(f(t)) = J_1^{\alpha}(t^{\alpha-1} f) = \int_{s_1}^{t} \frac{f(s)}{s^{1-\alpha}} ds,$$

where the integral is the Riemann improper integral with $\alpha \in (0,1]$.

The following results are also obvious:

Lemma 4 *: Let the conformable differential operator D^{α} be given as in theorem (3), where $\alpha \in (0,1]$ and $s \geq 0$ and suppose that f and g are $\alpha-$differentiable. Then*

(i) $D^{\alpha}(\ln s) = s^{-\alpha}$, *for* $s > 0$,

(ii) $D^{\alpha}\left[\int_{s_1}^{s} f(s,t)\right] d_{\alpha} t = f(s,s) + \int_{s_1}^{s} D^{\alpha}[f(s,t)] d_{\alpha} t$,

(iii) $\int_{s_1}^{s_2} f(s) D^{\alpha}(g(s)) d_{\alpha} s = fg|_{s_1}^{s_2} - \int_{s_1}^{s_2} g(s) D^{\alpha}(f(s)) d_{\alpha} s$.

p-Convex Function:

Let $I \subset (0, \infty)$ be a real interval and $p \in R\backslash\{0\}$. A function $f\,:\,I\rightarrow R$ is said to be $p-$convex, if

$$f\left([ts_1^p + (1-t)s_2^p]^{1/p}\right) \leq tf(s_1) + (1-t)f(s_2), \tag{1.5}$$

for all $s_1, s_2 \in I\ and\ t \in [0,\,1]$.

Chapter 2

Literature Survey and Motivations

2.1 Existing Literature

Kirmaci [15] provided some inequalities for differentiable convex mappings, including Hermite-integral Hadamard's inequality for convex functions. There are also some applications to specific methods of real numbers mentioned. Then, several midway formula error estimates are derived.

Alomari *et al.* [16] addressed various extensions of the Ostrowski integral inequality for limited variation maps. It also considered distinct Lipschitzian, monotonic, completely continuous, and n-times differentiable mapping equations with error estimates. In certain circumstances, specific techniques or numerical quadrature rules are used to do these computations. Their main goal is to prove several Ostrowski's type inequalities for the class of convex (concave) functions.

Theorem 5 *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[s_1, s_2]$, where s_1,$s_2 \in I$ with $s_1 < s_2$. If $|f'|$ is convex function on $[s_1, s_2]$, then the following inequality holds:*

$$\left| f(k) - \frac{1}{s_2 - s_1}\int_{s_1}^{s_2} f(u)du \right|$$
$$\leq \left[\frac{1}{6} + \frac{1}{3}\left(\frac{(k-s_1)}{(s_2-s_1)}\right)^3\right]|f'(s_2)| + \left[\frac{1}{6} + \frac{1}{3}\left(\frac{(s_2-s_1)}{(s_2-s_1)}\right)^3\right]|f'(s_1)|, \quad (2.1)$$

for all $k \in [s_1, s_2]$.

For functions with convex derivative absolute values, an Ostrowski like inequality may be deduced as follows:

Corollary 6 *In Theorem (5), additionally, if $|f'(k)| \leq M$, for $M > 0$, then the inequality*

$$\left| f(k) - \frac{1}{s_2 - s_1}\int_{s_1}^{s_2} f(u)du \right| \leq M(s_2 - s_1)\left[\frac{1}{3} + \frac{(s_2-k)^3 + (k-s_1)^3}{3(s_2-s_1)^3}\right], \quad (2.2)$$

holds. The constant $\frac{1}{3}$ is the best possible in terms of being replaceable by smaller one.

Corollary 7 *In Theorem (5), choose $k = \frac{s_1+s_2}{2}$, then:*

$$\left| f(\frac{s_1+s_2}{2}) - \frac{1}{s_2 - s_1}\int_{s_1}^{s_2} f(u)du \right| \leq \frac{5(s_2-s_1)}{24}[|f'(s_2)| + |f'(s_1)|]. \quad (2.3)$$

Liu [17] proved some super-multiplicative or super-additive Ostrowski type inequalities for h-convex functions are obtained using Riemann-Liouville fractional integrals. They present fresh estimates for various sorts of Ostrowski inequalities for fractional integrals.

Edmundo *et al.* [10] presented the concepts of fractional order derivatives and integrals used in Mathematics, Physics and different courses of Engineering. In contrast to what happens in fractional calculus, the derivative in classical calculus has a significant geometric interpretation. It is related with the idea of tangent. This disparity might be viewed as an issue for fractional calculus poor success up until 1900. In 1826, Abel solved an integral equation related to the tautochrone issue, which is regarded as the first use of fractional calculus. This is referred to as the first Liouville definition. The second definition proposed by Liouville is stated in terms of an integral and is known as the Liouville version for the integration of noninteger order. Riemann produced the most important article after a series of studies by Liouville, ten years after his passing. Furthermore, both the Liouville and Riemann formulations contain the so-called complementary function, which is a problem to be solved. The Caputo version is compared to the Riemann-Liouville formulation because of its importance. Caputo's concept inverts the order of integral and derivative operators with the noninteger order derivative of the Riemann-Liouville equation. The distinction between these two formulations is summarized as:

In Caputo, first compute the derivative of integer order, then compute the integral of noninteger order. In the Riemann-Liouville equation, first compute the integral of noninteger order, then compute the derivative of integer order. It is vital to note that the Caputo derivative may be used to confront situations when the beginning conditions are done in the function and in the relevant integer order derivatives.

Following the inaugural congress at the University of New Haven in 1974, fractional calculus has grown and various applications in many fields of scientific knowledge have arisen. As a result, multiple ways to solving derivative issues have been offered, and distinct definitions of the fractional derivative are accessible in the literature. In addition, it provides systematic formulations of existing fractional derivatives and integrals.

Qayyum *et al.* [18] generate inequalities of the Ostrowski type inequality. In terms of the second derivative's norms or limits, their work developed limits for departures from combinations of integrals across end intervals that span all feasible interval lengths. Furthermore, perturbed outcomes were discovered that may be explained statistically. It is also found that limitations on a specific quadrature rule for differentiable functions, which extend the two times differentiable functions. There is also discussion of some fresh disturbed outcomes. Their expanded inequalities have applications in approximation theory, probability theory, and numerical analysis. It demonstrate how the inequalities derived for the cumulative distribution function may be used. Using extended fractional integral operators, the following theorem is the proof of several novel inequalities related to the Ostrowski inequality.

Theorem 8 *Let* $f:[s_1,s_2]\to\mathbb{R}$ *be a two times differentiable function,*

$$\begin{aligned}\tau(k;\alpha,\beta) \ :&= \frac{1}{2(\alpha+\beta)}[\alpha(k-s_1)-\beta(s_2-k)]f^{'}(k)-f(k)\\ &+\frac{1}{\alpha+\beta}[\alpha M\ (f\ ;\ s_1,k)+\beta M\ (f\ ;\ k,s_2)],\end{aligned} \tag{2.4}$$

where $M\,(f;\ s_1,\ k)$ *is the integral mean defined in [19]. Then,*

$$|\tau(k;\alpha,\beta)| \leq \begin{cases} [\alpha(k-s_1)-\beta(s_2-k)]\frac{\|f''\|_\infty}{6(\alpha+\beta)}, \\ f'' \in L_\infty[s_1,s_2], \\ \frac{1}{(2m+1)^{1/m}} * [\alpha^m(k-s_1)^{m+1}-\beta^m(s_2-k)^{m+1}]^{1/s_2}\frac{\|f''\|_s}{2(\alpha+\beta)}, \\ f'' \in L_s[s_1,s_2], \\ s>1, \frac{1}{s}-\frac{1}{m}=1, \\ (\alpha(k-s_1)-\beta(s_2-k)+|\alpha(k-s_1)-\beta(s_2-k)|), \\ *\frac{\|f''\|_1}{4(\alpha+\beta)}, \quad f'' \in L_1[s_1,s_2], \end{cases}$$

for all $k \in [s_1, s_2]$.

Many scholars have recently focused on the theory of convexity. As a result, employing fresh and original ideas, the traditional notions of convex sets and convex functions have been expanded and developed in numerous areas. It is worth noting here that, in addition to classical convex functions, the class of p-convex functions also contains harmonically convex functions. Many academics have been drawn to the link between the theory of convex functions and the theory of inequalities. Hermite-Hadamard inequality, named after Hermite and Hadamard, is one of the most thoroughly studied inequalities for convex functions. This inequality is both required and sufficient for a function to be convex. For a recent study on Hermite-Hadamard type inequalities and look at the p-convex function class for two new integral identities for differentiable functions are derived. It will establish our main results, which are

Hermite-Hadamard type inequalities for differentiable p-convex functions, using these results. To solve our integrals, we employ hypergeometric functions. It is hoped that the concepts and methodologies presented in this study will spark additional research in this field.

Noor *et al.* [20] proved some Hermite-Hadamard type inequalities for differentiable p-convex function in the form of the following theorem:

Theorem 9 *Let $f : I = [s_1, s_2] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° (the interior of I) with $s_1 < s_2$, such that $f' \in L\,[s_1, s_2]$. If $|f'|$ is $p-$convex function, then we have:*

$$\left|\frac{f(s_1)+f(s_2)}{2} - \frac{p}{s_2^p - s_1^p}\int_{s_1}^{s_2}\frac{f(k)}{k^{1-p}}dk\right| \leq s_2^{1-p}.\frac{s_2^p - s_1^p}{2p}\{\mathcal{K}_1|f'(s_1)| + \mathcal{K}_2|f'(s_2)|\},$$

where

$$\begin{aligned}\mathcal{K}_1 &= \frac{2}{3}.{}_2F_1\left(1-\frac{1}{p},3;4;1-\frac{s_1^p}{s_2^p}\right) - \frac{1}{2}.{}_2F_1\left(1-\frac{1}{p},2;3;1-\frac{s_1^p}{s_2^p}\right) \\ &\quad + \frac{1}{12}.{}_2F_1\left(1-\frac{1}{p},2;4;\frac{1}{2}\left(1-\frac{s_1^p}{s_2^p}\right)\right), \end{aligned} \tag{2.5}$$

and

$$\begin{aligned}\mathcal{K}_2 &= \frac{1}{3}.{}_2F_1\left(1-\frac{1}{p},2;4;1-\frac{s_1^p}{s_2^p}\right) - \frac{1}{2}.{}_2F_1\left(1-\frac{1}{p},1;3;1-\frac{s_1^p}{s_2^p}\right) \\ &\quad + \frac{1}{2}.{}_2F_1\left(1-\frac{1}{p},1;3;\frac{1}{2}\left(1-\frac{s_1^p}{s_2^p}\right)\right) \\ &\quad - \frac{1}{12}.{}_2F_1\left(1-\frac{1}{p},2;4;\frac{1}{2}\left(1-\frac{s_1^p}{s_2^p}\right)\right). \end{aligned} \tag{2.6}$$

Theorem 10 *Let $f : I = [s_1, s_2] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° (the interior of I) with $s_1 < s_2$, such that $f' \in L[s_1, s_2]$. If $|f'|^q$ is $p-$convex function where $q \geq 1$, then, we have:*

$$\left| \frac{f(s_1) + f(s_2)}{2} - \frac{p}{s_2^p - s_1^p} \int_{s_1}^{s_2} \frac{f(k)}{k^{1-p}} dk \right| \leq s_2^{1-p} . \frac{s_2^p - s_1^p}{2p} \{\mathcal{K}_1 |f'(s_1)|^q + \mathcal{K}_2 |f'(s_2)|^q\}^{1/q},$$

where $\mathcal{K}_1, \mathcal{K}_1$ are given by (2.5) and (2.6) and

$$\begin{aligned} H &= {}_2F_1\left(1 - \frac{1}{p}, 2; 3; 1 - \frac{s_1^p}{s_2^p}\right) - 2\left(1 - \frac{1}{p}, 1; 2; 1 - \frac{s_1^p}{s_2^p}\right) \\ &\quad + {}_2F_1\left(1 - \frac{1}{p}, 1; 3; \frac{1}{2}\left(1 - \frac{s_1^p}{s_2^p}\right)\right). \end{aligned}$$

Convexity is currently receiving a lot of attention from many researchers, which has led to the extension and generalization of classical concepts such as convex sets and convex functions using innovative ideas. It ís worth mentioning here that this new research has had a number of important consequences for our understanding of mathematics. Besides the classic convex functions, there is also a class of p-convex functions which was introduced and studied recently. Many researchers have recently become interested in the interrelationship between convex function theory and inequality theory. One of the most extensively studied inequalities involving convex functions is Hermite-Hadamard inequality, which was developed by French mathematicians Pierre Hermite and Henri Hadamard.

Yıldız *et al.* [21] established numerous important inequalities for particular differentiable mappings using the Riemann-Liouville fractional integrals and are related to the well-known Ostrowski type integral inequality. Numerous authors have recently considered a number of generalizations of the Ostrowski integral inequality for mappings with bounded variation as well as Lipschitzian, monotonic, absolutely continuous, and n-times differentiable mappings with error estimates for certain special means and other numerical quadrature rules.

Theorem 11 *Let* $f : [s_1, s_2] \to \mathbb{R}$, *be a differentiable mapping on* (s_1, s_2) *with* $s_1 < s_2$ *such that* $f' \in L\,[s_1,\ s_2]$. *If* $|f'|$ *is convex on* $[s_1, s_2]$ *and* $k \in [s_1, s_2]$, *then the following inequality for fractional integrals with* $\alpha > 0$ *holds:*

$$\left|\left[\frac{(k - s_1)^{\alpha} + (s_2 - k)^{\alpha}}{(s_2 - s_1)^{\alpha+1}}\right] f(k) - \frac{\Gamma(\alpha+1)}{(s_2 - s_1)^{\alpha+1}}\left[J^{\alpha}_{k^+} f(s_2) + J^{\alpha}_{k^-} f(s_1)\right]\right|$$
$$\leq \frac{1}{\alpha+2}\left\{\begin{array}{c} \left(\frac{(s_2-k)^{\alpha+2}}{(s_2-k)^{\alpha+2}} + \frac{(k-s_1)^{\alpha+1}}{(s_2-s_1)^{\alpha+1}}\left[\frac{1}{\alpha+1} + \frac{(s_2-k)}{(s_2-s_1)}\right]\right)|f'(s_1)| \\ + \left(\frac{(k-s_1)^{\alpha+2}}{(s_2-k)^{\alpha+2}} + \frac{(s_2-k)^{\alpha+1}}{(s_2-s_1)^{\alpha+1}}\left[\frac{1}{\alpha+1} + \frac{(k-s_1)}{(s_2-s_1)}\right]\right)|f'(s_2)| \end{array}\right\},$$

where Γ *is a Euler Gamma function.*

Mehmet *et al.* [14] constructed Hermite- Hadamard- Fejér type inequalities for p-convex functions. He also constructed an integral identity and different Hermite-Hadamard-Fejér type integral inequalities for p- convex functions. Several Hermite-Hadamard and Hermite-Hadamard-Fejér inequalities for convex, harmonically convex, and p- convex functions were established. Because Hermite-Hadamard type inequalities and fractional integrals are widely used, many scholars expand their investigations to Hermite-Hadamard type inequalities employing fractional integrals rather than integer integrals. More and more Hermite-Hadamard inequalities involving fractional integrals have recently been established for various classes of functions. Some of the conclusions for p-convex functions are presented here.

Theorem 12 *Let $f : I \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a harmonically convex function and s_1, s_2 $\in I$ with $s_1 < s_2$. If $f \in L\,[s_1, s_2]$ and w: $[s_1, s_2] \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ is a non-negative, integrable and harmonically symmetric with respect to $\frac{2s_1s_2}{s_1 + s_2}$, then*

$$f\left(\frac{2s_1s_2}{s_1+s_2}\right)\int_{s_1}^{s_2}\frac{w(k)}{k^2}dk \leq \int_{s_1}^{s_2}\frac{f(k)w(k)}{k^2}dk \leq \frac{f(s_1)+f(s_2)}{2}\int_{s_1}^{s_2}\frac{w(k)}{k^2}dk.$$

Theorem 13 *Let $f : I \subset \mathbb{R}(0, \infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L\,[s_1, s_2]$, where $s_1, s_2 \in I^{\circ}$ and $s_1 < s_2$. If $|f'|$ is $p-$convex function on $[s_1, s_2]$ for $p \in \mathbb{R}\backslash\{0\}$, w: $[s_1, s_2] \to \mathbb{R}$ is continuous, then the following inequality holds:*

$$\begin{aligned}&\left|\int_{s_1}^{s_2}\frac{f(k)w(k)}{k^{1-p}}dk - f\left(\left[\frac{s_1^p+{s_2}^p}{2}\right]^{1/p}\right)\int_{s_1}^{s_2}\frac{w(k)}{k^{1-p}}dk\right|\\ \leq\ &\left(\frac{{s_2}^p+s_1^p}{p}\right)^2 ||w||_\infty[C_1(p)|f'(s_1)|+C_2(p)|f'(s_2)|],\end{aligned}$$

where

$$\begin{aligned}C_1(p) &= \left[\int_0^{\frac{1}{2}}\frac{t^2}{[ts_1^p+(1-t){s_2}^p]^{1-\frac{1}{p}}}dt+\int_{\frac{1}{2}}^{1}\frac{t-t^2}{[ts_1^p+(1-t){s_2}^p]^{1-\frac{1}{p}}}dt\right],\\ C_2(p) &= \left[\int_0^{\frac{1}{2}}\frac{t-t^2}{[ts_1^p+(1-t){s_2}^p]^{1-\frac{1}{p}}}dt+\int_{\frac{1}{2}}^{1}\frac{(1-t)^2}{[ts_1^p+(1-t){s_2}^p]^{1-\frac{1}{p}}}dt\right].\end{aligned}$$

Remark 14 *In theorem (13), Mehmet Kunt et al. [14] observed the followings:*

(1) For $p = 1$ and $w(k) = 1$, one has [[15], Theorem 2.2],

(2) If choose $w(k) = 1$, one has [[20], Theorem 3.3].

Gauhar *et al.* [22] proposed several new Grüss type inequalities which are generalizations of basic Grüss type inequality (1.1) for conformable fractional integrals in their study in the form of following theorem:

Theorem 15 *Let f be an integrable function on $[0, \infty)$. Assume that there exist two integrable functions ϕ_1, ϕ_2 on $[0, \infty)$ such that*

$$\phi_1(k) \leq f(k) \leq \phi_2(k), k \in [0, \infty). \tag{2.7}$$

Proof. For k, α, $\beta > 0$, we have:

$$^{\beta}\Im^{\mu}\phi_1(k)^{\alpha}\Im^{\mu}f(k) +^{\alpha} \Im^{\mu}\phi_2(k)^{\beta}\Im^{\mu}f(k) \geq^{\alpha} \Im^{\mu}\phi_2(k)^{\beta}\Im^{\mu}\phi_1(k) +^{\alpha} \Im^{\mu}f(k)^{\beta}\Im^{\mu}f(k). \tag{2.8}$$

From (2.7), for all τ, $\rho \geq 0$, it follows that

$$[\phi_2(\tau) - f(\tau)][f(\rho) - \phi_1(\rho)] \geq 0.$$

Therefore, we have

$$\phi_2(\tau)f(\rho) + \phi_1(\rho)f(\tau) \geq \phi_1(\rho)\phi_2(\tau) + f(\tau)f(\rho). \tag{2.9}$$

Multiplying both sides of (2.9) by $\frac{1}{\Gamma(\alpha)}(\frac{k^{\mu} - \tau^{\mu}}{\mu})^{\alpha-1}\tau^{\mu-1}$,

$$\begin{aligned} & f(\rho)\frac{1}{\Gamma(\alpha)}(\frac{k^{\mu} - \tau^{\mu}}{\mu})^{\alpha-1}\tau^{\mu-1}\phi_2(\tau) + \phi_1(\rho)\frac{1}{\Gamma(\alpha)}(\frac{k^{\mu} - \tau^{\mu}}{\mu})^{\alpha-1}\tau^{\mu-1}f(\tau) \\ \geq \; & \frac{1}{\Gamma(\alpha)}(\frac{k^{\mu} - \tau^{\mu}}{\mu})^{\alpha-1}\tau^{\mu-1}\phi_1(\rho)\phi_2(\tau) \\ & +f(\tau)f(\rho)\frac{1}{\Gamma(\alpha)}(\frac{k^{\mu} - \tau^{\mu}}{\mu})^{\alpha-1}\tau^{\mu-1}. \end{aligned} \tag{2.10}$$

Integrating (2.10) over $\tau \in (0,\, k)$

$$f(\rho)\int\limits_0^k \frac{1}{\Gamma(\alpha)}(\frac{k^\mu - \tau^\mu}{\mu})^{\alpha-1}\tau^{\mu-1}\phi_2(\tau)d\tau + \phi_1(\rho)\int\limits_0^k \frac{1}{\Gamma(\alpha)}(\frac{k^\mu - \tau^\mu}{\mu})^{\alpha-1}\tau^{\mu-1}f(\tau)d\tau$$

$$\geq \phi_1(\rho)\int\limits_0^k \frac{1}{\Gamma(\alpha)}(\frac{k^\mu - \tau^\mu}{\mu})^{\alpha-1}\tau^{\mu-1}\phi_2(\tau)d\tau + f(\rho)\int\limits_0^k \frac{1}{\Gamma(\alpha)}(\frac{k^\mu - \tau^\mu}{\mu})^{\alpha-1}\tau^{\mu-1}f(\tau)d\tau.$$

We know that,

$${}^\alpha\Im^\mu f(k) = \frac{1}{\Gamma(\alpha)}\int\limits_0^k (\frac{k^\mu - \tau^\mu}{\mu})^{\alpha-1}\frac{f(\tau)}{\tau^{1-\mu}}d\tau, \tag{2.11}$$

and

$$\begin{aligned} & f(\rho)\int\limits_0^k \frac{1}{\Gamma(\alpha)}(\frac{k^\mu - \tau^\mu}{\mu})^{\alpha-1}\tau^{\mu-1}\phi_2(\tau)d\tau \\ = \;& f(\rho)\frac{1}{\Gamma(\alpha)}\int\limits_0^k (\frac{k^\mu - \tau^\mu}{\mu})^{\alpha-1}\frac{\phi_2(\tau)}{\tau^{1-\mu}}d\tau, \\ = \;& f(\rho)\frac{1}{\Gamma(\alpha)}\int\limits_0^k (\frac{k^\mu - \tau^\mu}{\mu})^{\alpha-1}\frac{f(\tau)}{\tau^{1-\mu}}d\tau, \\ = \;& f(\rho)^\alpha\Im^\mu f(k) = f(\rho)^\alpha\Im^\mu\phi_2(k). \end{aligned}$$

The integration leads to

$$f(\rho)^\alpha\Im^\mu\phi_2(k) + \phi_1(\rho)^\alpha\Im^\mu f(k) \geq \phi_1(\rho)^\alpha\Im^\mu\phi_2(k) + f(\rho)^\alpha\Im^\mu f(k). \tag{2.12}$$

Multiplying both sides of (2.12) by $\frac{1}{\Gamma(\beta)}(\frac{k^\mu - \tau^\mu}{\mu})^{\beta-1}\rho^{\mu-1}$ gives,

$$\begin{aligned} & {}^\alpha\Im^\mu\phi_2(k)\frac{1}{\Gamma(\beta)}(\frac{k^\mu - \tau^\mu}{\mu})^{\beta-1}\rho^{\mu-1}f(\rho) \\ & +{}^\alpha\Im^\mu f(k)\frac{1}{\Gamma(\beta)}(\frac{k^\mu - \tau^\mu}{\mu})^{\beta-1}\rho^{\mu-1}\phi_1(\rho) \\ \geq \;& {}^\alpha\Im^\mu\phi_2(k)\frac{1}{\Gamma(\beta)}(\frac{k^\mu - \tau^\mu}{\mu})^{\beta-1}\rho^{\mu-1}\phi_1(\rho) \\ & +{}^\alpha\Im^\mu f(k)\frac{1}{\Gamma(\beta)}(\frac{k^\mu - \tau^\mu}{\mu})^{\beta-1}\rho^{\mu-1}f(\rho). \end{aligned} \tag{2.13}$$

Integrating (2.13) over $\rho \in (0,k)$

$$
\begin{aligned}
&{}^{\alpha}\Im^{\mu}\phi_2(k)\frac{1}{\Gamma(\beta)}\int_0^k(\frac{k^{\mu}-\tau^{\mu}}{\mu})^{\beta-1}\rho^{\mu-1}f(\rho)d\rho \\
&+{}^{\alpha}\Im^{\mu}f(k)\frac{1}{\Gamma(\beta)}\int_0^k(\frac{k^{\mu}-\tau^{\mu}}{\mu})^{\beta-1}\rho^{\mu-1}\phi_1(\rho)d\rho \\
\geq\ &{}^{\alpha}\Im^{\mu}\phi_2(k)\frac{1}{\Gamma(\beta)}\int_0^k(\frac{k^{\mu}-\tau^{\mu}}{\mu})^{\beta-1}\rho^{\mu-1}\phi_1(\rho)d\rho \\
&+{}^{\alpha}\Im^{\mu}f(k)\frac{1}{\Gamma(\beta)}\int_0^k(\frac{k^{\mu}-\tau^{\mu}}{\mu})^{\beta-1}\rho^{\mu-1}f(\rho)d\rho. \qquad (2.14)
\end{aligned}
$$

Then by integrating, we get

$$
{}^{\beta}\Im^{\mu}\phi_1(k){}^{\alpha}\Im^{\mu}f(k)+{}^{\alpha}\Im^{\mu}\phi_2(k){}^{\beta}\Im^{\mu}f(k)\geq{}^{\alpha}\Im^{\mu}\phi_2(k){}^{\beta}\Im^{\mu}\phi_1(k)+{}^{\alpha}\Im^{\mu}f(k){}^{\beta}\Im^{\mu}f(k).
$$

■

While studying the error bounds of different numerical methods, Ostrowski inequality plays a vital role. A quadrature rule is a mathematical formula that helps to calculate the position of points on a plane. This was motivated by the need to find refinements, generalizations, extensions, and applications for this tool. Researchers continue to search for new ways to improve or perfect these rules in order to better serve their purposes. Fractional calculus is a field of mathematics that studies non-integral order integral and differential operators. Many Mathematicians, notably Liouville, Riemann, weyl, and Fourier, have made significant contributions to this topic. Abel, Lacroix, Leibniz, and Grunwald expanded on the theory underlying fractional calculus. Riemann developed the first definition for an operator regulating integrals over nonlinear spaces in terms of a fractional derivative.

Hurye Kadakal [12] used a better technique than power-mean inequality and an identity for differentiable functions, he obtained inequalities for functions whose absolute value derivatives at specific powers are convex. It is demonstrated numerically that enhanced power-mean integral inequality provides a superior method than power-mean inequality. There are also some applications to special means of real numbers and some error estimates for the midpoint formula are described by this author.

Theorem 16 *Let $f : I^{\circ} \subset R \to R$ be a differentiable mapping on I°, $s_1, s_2 \in I$ with $s_1 < s_2$ and let $q \geq 1$. If the mapping $|f'|^q$ is convex on the interval $[s_1, s_2]$, then the following inequality holds:*

$$\begin{aligned} &\left| \frac{1}{s_2 - s_1} \int_{s_1}^{s_2} f(k)dk - f\left(\frac{s_1 + s_2}{2}\right) \right| \\ \leq \; &\left| \frac{s_2 - s_1}{4} \left(\frac{1}{q+2}\right)^{1/q} \left[\left(\left| f'\left(\frac{s_1 + s_2}{2}\right) \right|^q + \frac{|f'(s_2)|^q}{q+1} \right)^{1/q} \right. \right. \\ &\left. \left. + \left(\left| f'\left(\frac{s_1 + s_2}{2}\right) \right|^q + \frac{|f'(s_1)|^q}{q+1} \right)^{1/q} \right] \right|. \end{aligned} \tag{2.15}$$

Iqbal *et al.* [23] stated the fact that the Grüss type inequality connection between the integral of the product of two functions and the product of their integrals. The continuous and discrete cases of Grüss-type variants play a considerable role in examining the qualitative conduct of differential and integral equations. Their main purpose is to show some new and modified versions of the Grüss inequality by using a generalized k-fractional derivative. Such new versions of the inequalities are supposed to be vital and the exploration has continued to develop investigations for such kinds of variants. The Grüss inequality is one of the most fascinating inequalities amongst the field of inequalities. Some Ostrowski's type inequalities for the Riemann-Stieltjes integral of continuous complex valued integrands defined on the complex unit circle and various subclasses of integrators of bounded variation are given. Natural applications for functions of unitary operators in Hilbert spaces are provided as well. For the generalized fractional derivative, also used Young's inequality to find new forms of inequalities. Such conclusions for this novel and generalized fractional derivative are extremely useful and valuable in the domains of differential equations and fractional differential calculus, both of which have a strong connections to real-world situations. These findings may stimulate additional research in a variety of fields of pure and applied sciences. Mathematical inequalities play a significant role in the study of mathematics and many related subjects, and their applications are diverse. In the case of fractional partial differential equations, fractional integral inequalities are useful in determining the uniqueness of solutions. They also give upper and lower boundaries for fractional boundary value problem solutions. These recommendations have led various researchers in the field of integral inequalities to inquire into certain extensions by involving fractional calculus operatorsAnd also proved several new integral inequalities for the k-Hilfer fractional derivative operator, which is a fractional calculus operator.

Theorem 17 *Let $k>0$ and $(D_{s_1+,k}^{\xi+\eta(n-\xi)}\Omega)$ be a positive function on $[0,\infty)$, and let $({}^kD_{s_1+}^{\xi,\eta}f)$ denote the Hilfer k-Hilfer fractional derivative of order $\xi, 0<\xi<1$, and type $0<\eta\leq 1$. Suppose that there exist $(D_{s_1+,k}^{\xi+\eta(n-\xi)}\Psi_1)$, $(D_{s_1+,k}^{\xi+\eta(n-\xi)}\Psi_2)$ such that*

$$(D_{s_1+,k}^{\xi+\eta(n-\xi)}\Psi_1)(\xi)\leq(D_{s_1+,k}^{\xi+\eta(n-\xi)}\Omega)(\xi)\leq(D_{s_1+,k}^{\xi+\eta(n-\xi)}\Psi_2)(\xi), \tag{2.16}$$

for all $\xi\in[0,\infty)$. Then,

$$\begin{aligned}&({}^kD_{s_1+}^{\xi,\eta}\Psi_1)(\xi)({}^kD_{s_1+}^{\xi,\eta}\Omega)(\xi)+({}^kD_{s_1+}^{\xi,\eta}\Psi_2)(\xi)({}^kD_{s_1+}^{\xi,\eta}\Omega)(\xi)\\ \geq\ &({}^kD_{s_1+}^{\xi,\eta}\Psi_1)(\xi)({}^kD_{s_1+}^{\xi,\eta}\Psi_2)(\xi)+({}^kD_{s_1+}^{\xi,\eta}\Omega)(\xi)({}^kD_{s_1+}^{\xi,\eta}\Omega)(\xi).\end{aligned} \tag{2.17}$$

Proof. Using (2.16) for all $\gamma\geq 0,\delta\geq 0$, we have

$$\begin{aligned}&\Big[(D_{s_1+,k}^{\xi+\eta(n-\xi)}\Psi_2)(\gamma)-(D_{s_1+,k}^{\xi+\eta(n-\xi)}\Omega)(\gamma)\Big]\\ &\times\Big[(D_{s_1+,k}^{\xi+\eta(n-\xi)}\Omega)(\delta)-(D_{s_1+,k}^{\xi+\eta(n-\xi)}\Psi_1)(\delta)\Big]\geq 0,\end{aligned}$$

and then

$$\begin{aligned}&(D_{s_1+,k}^{\xi+\eta(n-\xi)}\Psi_2)(\gamma)(D_{s_1+,k}^{\xi+\eta(n-\xi)}\Omega)(\delta)+(D_{s_1+,k}^{\xi+\eta(n-\xi)}\Psi_1)(\delta)(D_{s_1+,k}^{\xi+\eta(n-\xi)}\Omega)(\gamma)\\ \geq\ &(D_{s_1+,k}^{\xi+\eta(n-\xi)}\Psi_1)(\delta)(D_{s_1+,k}^{\xi+\eta(n-\xi)}\Psi_2)(\gamma)+(\delta)(D_{s_1+,k}^{\xi+\eta(n-\xi)}\Omega)(\gamma)(D_{s_1+,k}^{\xi+\eta(n-\xi)}\Omega)\end{aligned} \tag{2.18}$$

If we multiply by $\frac{(\xi-\gamma)^{\frac{\eta(n-\xi)}{k}-1}}{k\Gamma_k(\eta(n-\xi))}$ on both sides of (2.18) and integrate the resulting identity for the variable γ over the interval (s_1, ξ), we get

$$\begin{aligned}
&(D_{s_1+,k}^{\xi+\eta(n-\xi)}\Omega)(\delta)\frac{1}{k\Gamma_k(\eta(n-\xi))}\int_{s_1}^{\xi}(\xi-\gamma)^{\frac{\eta(n-\xi)}{k}-1}(D_{s_1+,k}^{\xi+\eta(n-\xi)}\Psi_2)(\gamma)d\gamma \\
&+(D_{s_1+,k}^{\xi+\eta(n-\xi)}\Psi_1)(\delta)\frac{1}{k\Gamma_k(\eta(n-\xi))}\int_{s_1}^{\xi}(\xi-\gamma)^{\frac{\eta(n-\xi)}{k}-1}(D_{s_1+,k}^{\xi+\eta(n-\xi)}\Omega)(\gamma)d\gamma \\
\geq\ &(D_{s_1+,k}^{\xi+\eta(n-\xi)}\Psi_1)(\delta)\frac{1}{k\Gamma_k(\eta(n-\xi))}\int_{s_1}^{\xi}(\xi-\gamma)^{\frac{\eta(n-\xi)}{k}-1}(D_{s_1+,k}^{\xi+\eta(n-\xi)}\Psi_2)(\gamma)d\gamma \\
&+(D_{s_1+,k}^{\xi+\eta(n-\xi)}\Omega)(\delta)\frac{1}{k\Gamma_k(\eta(n-\xi))}\int_{s_1}^{\xi}(\xi-\gamma)^{\frac{\eta(n-\xi)}{k}-1}(D_{s_1+,k}^{\xi+\eta(n-\xi)}\Omega)(\gamma)d\gamma,
\end{aligned}$$

which can be written as follows:

$$\begin{aligned}
&(D_{s_1+,k}^{\xi+\eta(n-\xi)}\Omega)(\delta)({}^kD_{s_1+}^{\xi,\eta}\Psi_2)(\xi)+(D_{s_1+,k}^{\xi+\eta(n-\xi)}\Psi_1)(\delta)({}^kD_{s_1+}^{\xi,\eta}\Omega)(\xi) \\
\geq\ &(D_{s_1+,k}^{\xi+\eta(n-\xi)}\Psi_1)(\delta)({}^kD_{s_1+}^{\xi,\eta}\Psi_2)(\xi)+(D_{s_1+,k}^{\xi+\eta(n-\xi)}\Omega)(\delta)({}^kD_{s_1+}^{\xi,\eta}\Omega)(\xi). \qquad (2.19)
\end{aligned}$$

Now, multiplying by $\frac{(\xi-\gamma)^{\frac{\eta(n-\xi)}{k}-1}}{k\Gamma_k(\eta(n-\xi))}$ on both sides of (2.19) and integrate the resulting identity for the variable δ over the interval (s_1, ξ), we get

$$\begin{aligned}
&({}^kD_{s_1+}^{\xi,\eta}\Psi_1)(\xi)({}^kD_{s_1+}^{\xi,\eta}\Omega)(\xi)+({}^kD_{s_1+}^{\xi,\eta}\Psi_2)(\xi)({}^kD_{s_1+}^{\xi,\eta}\Omega)(\xi) \\
\geq\ &({}^kD_{s_1+}^{\xi,\eta}\Psi_1)(\xi)({}^kD_{s_1+}^{\xi,\eta}\Psi_2)(\xi)+({}^kD_{s_1+}^{\xi,\eta}\Omega)(\xi)({}^kD_{s_1+}^{\xi,\eta}\Omega)(\xi).
\end{aligned}$$

This completes the proof. ■

2.2 Conceptual Framework

The main objective of the present study is to establish weighted Ostrowski-Grüss type inequality for differentiable mapping in terms of the upper and lower bounds of the first derivative via Katugampola conformable fractional integral.

- In chapter-3, some integral inequalities via conformable fractional integrals are existing in literature are presented.

- In chapter-4, modification of the Grüss type inequality to weighted Ostrowski-Grüss type inequalities via conformable fractional integral are established.

- In chapter-5, application to numerical integration is given.

- In chapter-6, some conclusions and future research study in directions of further generalized inequalities are discussed. These generalized inequalities can be helpful in mathematical analysis.

Chapter 3

Modification of Grüss Type Inequalities to Weighted Ostrowski-Grüss Inequality

We shall discuss here the subjected modification which was introduced by Ahmad F. *et al.* [26].

3.1 Weight Function and Moments

Let the weight function w: $[s_1, s_2] \to \mathbb{R}$ be non-negative such that:

$$\int_{s_1}^{s_2} w(s)ds < \infty. \tag{3.1}$$

The domain of w may be finite or infinite and may vanish at the boundary points.

We donated the moments by m, M and the symbolic notation μ as follows:

$$\begin{aligned} m(s_1, s_2) &= \int_{s_1}^{s_2} w(s)ds, \\ M(s_1, s_2) &= \int_{s_1}^{s_2} sw(s)ds, \\ \text{and } \mu(s_1, s_2) &= \frac{M(s_1, s_2)}{m(s_1, s_2)}. \end{aligned} \tag{3.2}$$

3.2 Main Results

Dragomir modified Grüss type inequality to weighted Grüss inequality as follows [29]:

Theorem 18 *: Let f, g : $[s_1, s_2] \to \mathbb{R}$ be integrable functions such that $\phi \leq f(k) \leq \psi$ and $\gamma \leq g(k) \leq \Gamma$, for all $k \in [s_1, s_2]$ and ϕ, ψ, γ and Γ are constants.*

Then, we have the following inequality:

$$\begin{aligned} &\left| \frac{1}{m(s_1, s_2)} \int_{s_1}^{s_2} f(k)g(k)w(k)dk - \frac{1}{m(s_1, s_2)} \int_{s_1}^{s_2} f(k)w(k)dk \frac{1}{m(s_1, s_2)} \int_{s_1}^{s_2} g(k)w(k)dk \right| \\ \leq\ & \frac{1}{4}(\psi - \phi)(\Gamma - \gamma). \end{aligned} \tag{3.3}$$

The constant $\frac{1}{4}$ is sharp.

Dragomir and Rassias modified Grüss inequality to Ostrowski-Grüss type inequality for differentiable function in terms of the upper and lower bounds of first order derivative in the form of the theorem [17].

Theorem 19 *: Let $f : [s_1, s_2] \to \mathbb{R}$ be a continuous on $[s_1, s_2]$ and differentiable on (s_1, s_2), where first derivative satisfies the condition:*

$$\gamma \leq f'(k) \leq \Gamma, \quad \forall k \in (s_1, s_2),$$

then

$$\left| f(k) - \left(k - \frac{s_1 + s_2}{2}\right) \frac{f(s_2) - f(s_1)}{s_2 - s_1} - \frac{1}{s_2 - s_1} \int_{s_1}^{s_2} f(t)\,dt \right|$$
$$\leq \frac{1}{4}(s_2 - s_1)(\Gamma - \gamma), \ \forall\ k \in (s_1, s_2). \tag{3.4}$$

Ahmad F. *et al.* [26] pointed out new estimation of (3.4) giving much better results than estimation. The new estimation is given in the form of the following theorem:

Theorem 20 *: Let $f : [s_1, s_2] \to \mathbb{R}$ be a continuous on $[s_1, s_2]$ and differentiable on (s_1, s_2), with*

$$\phi \leq f'(k) \leq \psi, \quad k \in (s_1, s_2).$$

Then

$$\left| f(k) - (k - \mu(s_1, s_2))f'(k) - \frac{1}{m(s_1, s_2)} \int_{s_1}^{s_2} f(t)w(t)\,dt \right|$$
$$\leq \frac{1}{8}(\psi - \phi)(m(s_1, s_2)) + \left| \int_{s_1}^{s_2} w(t) sgn(t - k)\,dt \right|, \quad \forall\ k \in [s_1, s_2]. \tag{3.5}$$

Proof. Define the mapping $p(.,\,.) : [s_1, s_2]^2 \to \mathbb{R}$ by

$$p\,(k,\ t) = \begin{cases} \int_{s_1}^{t} w(s)\,ds, & \text{if } t \in [s_1, k], \\ \\ \int_{s_2}^{t} w(s)\,ds, & \text{if } t \in [k, s_2]. \end{cases} \tag{3.6}$$

Dragomir and Rassias proved the following identity [25] using integration by parts:

$$\int_{s_1}^{s_2} p(k,t)f^{'}(t)\,dt = m(s_1, s_2)f(k) - \int_{s_1}^{s_2} f(t)w(t)\,dt. \tag{3.7}$$

The proof of (3.7) is as follows:

$$\begin{aligned} \int_{s_1}^{s_2} p(k,t)f^{'}(t)\,dt &= \int_{s_1}^{k} \left(\int_{s_1}^{t} w(s)\,ds\right) f^{'}(t)\,dt + \int_{k}^{s_2} \left(\int_{s_2}^{t} w(s)\,ds\right) f^{'}(t)\,dt, \\ &= \int_{s_1}^{k} m(s_1,t)f^{'}(t)\,dt + \int_{k}^{s_2} m(s_2,t)f^{'}(t)\,dt, \\ &= m(s_1,t)f(t)|_{s_1}^{k} - \int_{s_1}^{k} f(t)w(t)\,dt + m(s_2,t)f(t)|_{k}^{s_2} \\ &\quad - \int_{k}^{s_2} f(t)w(t)\,dt, \\ &= m(s_1,k)f(k) - m(s_1,s_1)f(s_1) \\ &\quad - \int_{s_1}^{s_2} f(t)w(t)\,dt + m(s_2,s_2)f(s_2) - m(s_2,k)f(k). \end{aligned}$$

$$\begin{aligned} \int_{s_1}^{s_2} p(k,t)f^{'}(t)\,dt &= m(s_1,k)f(k) - m(s_2,k)f(k) - \int_{s_1}^{s_2} f(t)w(t)\,dt, \\ &= (m(s_1,k) + m(k,s_2))f(k) - \int_{s_1}^{s_2} f(t)w(t)\,dt, \\ \int_{s_1}^{s_2} p(k,t)f^{'}(t)\,dt &= m(s_1,s_2)f(k) - \int_{s_1}^{s_2} f(t)w(t)\,dt. \end{aligned}$$

The equation (2.4) is maintained by the following finding:

$$\int_{s_1}^{s_2} p(k,t)\,dt = m(s_1,s_2)(r - M(s_1,s_2)). \tag{3.8}$$

From (3.6), we have:

$$\begin{aligned}
\int_{s_1}^{s_2} p(k,t)\,dt &= \int_{s_1}^{k}\left(\int_{s_1}^{t} w(s)ds\right)dt + \int_{k}^{s_2}\left(\int_{s_2}^{t} w(s)ds\right)dt,\\
&= \left(\int_{s_1}^{t} w(s)ds\right)t|_{s_1}^{k} - \int_{s_1}^{k} tw(t)dt + \left(\int_{s_2}^{t} w(s)ds\right)t|_{k}^{s_2} - \int_{k}^{s_2} tw(t)dt,\\
&= k\int_{s_1}^{k} w(s)ds - \int_{s_1}^{k} tw(t)dt - k\int_{s_2}^{k} w(s)ds - \int_{k}^{s_2} tw(t)dt,\\
&= km(s_1,k) - M(s_1,k) - km(s_2,k) - M(k,s_2),\\
&= k(m(s_1,k)+m(k,s_2)) - (M(s_1,k)+M(k,s_2)),\\
&= km(s_1,s_2) - M(s_1,s_2),\\
\int_{s_1}^{s_2} p(k,t)\,dt &= m(s_1,s_2)\left(k - \frac{M(s_1,s_2)}{m(s_1,s_2)}\right),\\
\int_{s_1}^{s_2} p(k,t)\,dt &= m(s_1,s_2)\mu(s_1,s_2).
\end{aligned}$$

which is the required result. ■

They further proved the following estimation:

$$0 \le p(k,t) \le \max\begin{cases} \int_{k}^{s_2} w(s)ds, & \text{if } t\in[s_1,\frac{s_1+s_2}{2}],\\ \int_{s_1}^{k} w(s)ds, & \text{if } t\in(\frac{s_1+s_2}{2},s_2], \end{cases}$$

$$\begin{aligned}
&= \max\left(\int_{k}^{s_2} w(s)ds, \int_{s_1}^{k} w(s)ds\right),\\
&= \frac{1}{2}\left(\int_{s_1}^{s_2} w(t)dt + \left|\int_{s_1}^{s_2} sgn(t-k)w(t)dt\right|\right),\\
&= \frac{1}{2}\left(m(s_1,s_2) + \left|\int_{s_1}^{s_2} sgn(t-k)w(t)dt\right|\right).
\end{aligned}$$

Since, $w(k)$ is a non-negative function, therefore $g(k) = \frac{p(k,\ t)}{w(k)}$, we have:

$$\begin{aligned} 0 &\leq p\,(k,t) \\ &\leq \frac{1}{2}\left(m(s_1,s_2) + \left|\int_{s_1}^{s_2} sgn(t-k)w(t)\,dt\right|\right). \end{aligned} \tag{3.9}$$

Apply weighted Grüss type inequality (3.3) for mappings $f(.) = f^{'}(.)$, $g(.) = \frac{p(k,\ .)}{w(k)}$ and using (3.9), we get:

$$\begin{aligned} &\left|\frac{1}{m(s_1,s_2)}\int_{s_1}^{s_2} p(k,t)f^{'}(t)\,dt - \frac{1}{m(s_1,s_2)}\int_{s_1}^{s_2} p(k,t)\,dt\frac{1}{m(s_1,s_2)}\int_{s_1}^{s_2} f^{'}(t)w(t)\,dt\right| \\ \leq\; &\frac{1}{8}(\psi-\phi)\left[m(s_1,s_2) + \left|\int_{s_1}^{s_2} sgn(t-k)w(t)\,dt\right|\right]. \end{aligned}$$

Using mean value theorem for integrals, it further implies:

$$\begin{aligned} &\left|\frac{1}{m(s_1,s_2)}\int_{s_1}^{s_2} p(k,t)f^{'}(t)\,dt - \frac{1}{m(s_1,s_2)}\int_{s_1}^{s_2} p(k,t)\,dt\frac{f^{'}(k)}{m(s_1,s_2)}(m(s_1,s_2))\right| \\ \leq\; &\frac{1}{8}(\psi-\phi)\left[m(s_1,s_2) + \left|\int_{s_1}^{s_2} sgn(t-k)w(t)\,dt\right|\right], \\ &\left|\frac{1}{m(s_1,s_2)}\int_{s_1}^{s_2} p(k,t)f^{'}(t)\,dt - \frac{f^{'}(k)}{m(s_1,s_2)}\int_{s_1}^{s_2} p(k,t)\,dt\right| \\ \leq\; &\frac{1}{8}(\psi-\phi)\left[m(s_1,s_2) + \left|\int_{s_1}^{s_2} sgn(t-k)w(t)\,dt\right|\right]. \end{aligned} \tag{3.10}$$

Using (3.7)and (3.8) in (3.10), we get

$$\begin{aligned} &\left|\frac{1}{m(s_1,s_2)}(m(s_1,s_2)f(k) - \int_{s_1}^{s_2} f(t)w(t)\,dt - \frac{f^{'}(k)}{m(s_1,s_2)}(m(s_1,s_2)(k-\mu(s_1,s_2)))\right| \\ \leq\; &\frac{1}{8}(\psi-\phi)\left[m(s_1,s_2) + \left|\int_{s_1}^{s_2} sgn(t-k)w(t)\,dt\right|\right], \\ &\left|f(k) - \frac{1}{m(s_1,s_2)}\int_{s_1}^{s_2} f(t)\,dt - (k-\mu(s_1,s_2))f^{'}(k)\right| \\ \leq\; &\frac{1}{8}(\psi-\phi)\left[m(s_1,s_2) + \left|\int_{s_1}^{s_2} sgn(t-k)w(t)\,dt\right|\right], \end{aligned}$$

which is the required inequality.

Remark 21 *: If we put $w(t) = 1$ in (3.5), we get the following inequality:*

$$\left|f(k) - \left(k - \frac{s_1 + s_2}{2}\right) f^{'}(k) - \frac{1}{s_2 - s_1} \int_{s_1}^{s_2} f(t)dt\right| \leq \frac{1}{4}(\psi - \phi)\left(\frac{s_2 - s_1}{2} + |k - \frac{s_1 + s_2}{2}|\right).$$

Corollary 22 *: Under the assumption of theorem (20) and putting $k = \frac{s_1+s_2}{2}$ in (3.5), we have the mid-point like inequality:*

$$\begin{aligned} & \left|f\left(\frac{s_1 + s_2}{2}\right) - \left(\frac{s_1 + s_2}{2} - \mu(s_1, s_2)\right) f\left(\frac{s_1 + s_2}{2}\right) - \frac{1}{m(s_1, s_2)} \int_{s_1}^{s_2} f(t)w(t)dt\right| \\ \leq & \frac{1}{8}(\psi - \phi)\left(m(s_1, s_2) + \left|\int_{s_1}^{s_2} sgn(t - \frac{s_1 + s_2}{2})w(t)dt\right|\right). \end{aligned} \tag{3.11}$$

Corollary 23 *: Under the assumptions of theorem (20), we have the trapezoidal like inequality. The inequality is concluded from (3.3) with $k = s_1$ and $k = s_2$ adding the results and using the triangular inequality and then dividing by 2 :*

$$\begin{aligned} & \left|\frac{f(s_1) + f(s_2)}{2} - \frac{1}{2}\left(s_1 f^{'}(s_1) + s_2 f^{'}(s_2) - \mu(s_1, s_2)(f^{'}(s_2) + f^{'}(s_1)\right)\right. \\ & \left. - \frac{1}{m(s_1, s_2)} \int_{s_1}^{s_2} f(t)\mathrm{w}(t)dt\right| \\ \leq & \frac{1}{16}(\psi - \phi)\left(2m(s_1, s_2) + \left|\int_{s_1}^{s_2} w(t)sgn(t - s_2)dt\right| + \left|\int_{s_1}^{s_2} w(t)sgn(t - s_1)dt\right|\right), \\ = & \frac{1}{16}(\psi - \phi)\ 4\ m\ (s_1, s_2), \\ = & \frac{1}{4}(\psi - \phi)\ m\ (s_1, s_2). \end{aligned} \tag{3.12}$$

3.3 Applications

Let $J_n : s_1 = k_0 < k_1 < k_2 < k_3 < ... < k_{n-1} < k_n = s_2$ be the division of the interval $[s_1, s_2]$, $\beta_i \in [k_i, k_{i+1}]$, $i = 1, ..., n - 1$.

We have the following quadrature formula in the form of the following theorem:

Theorem 24 *: Let f: $[s_1, s_2] \to \mathbb{R}$ be continuous on $[s_1, s_2]$ and differentiable on (s_1, s_2) and $f' : [s_1, s_2] \to R$ satisfy the condition $\phi \leq f'(k) \leq \psi$, for all $k \in (s_1, s_2)$. Then, we have the following perturbed Riemann's type quadrature formula:*

$$\int_{s_1}^{s_2} f(s)w(s)ds = B(f, f', \beta, J_n) + R(f,\ f',\ \beta,\ J_n), \tag{3.13}$$

where

$$B\left(f, f', \beta, J_n\right) = \sum_{i=0}^{n-1} m(k_i,\ k_{i+1})f(\beta_i) - \sum_{i=0}^{n-1} m(k_i,\ k_{i+1}) - \mu(k_i,\ k_{i+1})f'(\beta_i),$$

and the remainder term gives the estimation:

$$\begin{aligned} & \left|R\left(f,\ f',\ \beta_i,\ J_n\right)\right| \\ \leq\ & \frac{1}{8}(\psi - \phi) \sum_{i=0}^{n-1} m(k_i,\ k_{i+1}) \left(m(k_i,\ k_{i+1}) + \left| \int_{k_i}^{k_{i+1}} sgn(s - \beta_i)w(s)ds \right| \right), \\ \forall \beta_i\ \in\ & [k_i,\ k_{i+1}],\ \text{where } h_i = k_{i+1} - k_i,\ \text{for } i = 1, 2, ..., n-1. \end{aligned} \tag{3.14}$$

Proof. Applying theorem (20) on the interval $[k_i,\ k_{i+1}]$, $\beta_i \in [k_i,\ k_{i+1}]$, where $h_i = k_{i+1} - k_i$, for $i = 1, 2, ..., n-1$, we get:

$$\begin{aligned} & \left| \int_{k_i}^{k_{i+1}} f(s)w(s)ds - m(k_i,\ k_{i+1})f(\beta_i) + m(k_i,\ k_{i+1})(\beta_i - \mu(k_i,\ k_{i+1}))f'(\beta_i) \right| \\ \leq\ & \frac{1}{8}(\psi - \phi)m(k_i,\ k_{i+1}) \left(m(k_i,\ k_{i+1}) + \left| \int_{k_i}^{k_{i+1}} sgn(s - \beta_i)w(s)ds \right| \right). \end{aligned}$$

Summing over i from *0* to *n-1* and using the generalized triangular inequality, we get the required estimation (3.14). ■

Corollary 25 *Under the assumption of theorem (24), by choosing* $\beta_i = \frac{k_i + k_{i+1}}{2}$*, we obtain mid-point quadrature formula:*

$$\int_{k_i}^{k_{i+1}} f(s)w(s)ds = B_M\left(f, f', \beta, J_n\right) + R_M\left(f, f', \beta, J_n\right),$$

where

$$\begin{aligned} B_M\left(f, f', \beta, J_n\right) &= \sum_{i=0}^{n-1} m(k_i,\ k_{i+1}) f\left(\frac{k_i + k_{i+1}}{2}\right) - \\ &\quad \sum_{i=0}^{n-1} m(k_i,\ k_{i+1}) \left(\frac{k_i + k_{i+1}}{2} - \mu(k_i,\ k_{i+1})\right) * f'\left(\frac{k_i,\ k_{i+1}}{2}\right), \end{aligned}$$

and the remainder term satisfies the estimation:

$$\begin{aligned} \left|R_M\left(f, f', \beta, J_n\right)\right| &\leq \frac{1}{8}(\psi - \phi) \sum_{i=0}^{n-1} m(k_i,\ k_{i+1}) * \\ &\quad \left(m(k_i,\ k_{i+1}) + \left|\int_{k_i}^{k_{i+1}} w(s) sgn(s - \frac{k_i,\ k_{i+1}}{2})\right|\right). \end{aligned}$$

Chapter 4

Modified Weighted Ostrowski-Grüss Inequality via Conformable Fractional Integral

Here, modification of the Grüss type inequality to Ostrowski-Grüss type inequality via conformable fractional integral is established. We shall generalize the results of Chapter 3 via Katugamopola Conformable Fractional Integral [8].

.

Let us denote the moments m, M and the symbolic notation μ via conformable fractional integral as follows:

$$\begin{aligned} m(s_1, s_2) &= \int_{s_1}^{s_2} w(s)\ d_\alpha s, \\ M(s_1, s_2) &= \int_{s_1}^{s_2} sw(s)\ d_\alpha s, \\ \text{and } \mu(s_1, s_2) &= \frac{M(s_1, s_2)}{m(s_1, s_2)}, \end{aligned} \tag{4.1}$$

where the weight function w: $[s_1, s_2] \rightarrow \mathbb{R}$ is non-negative such that

$$\int_{s_1}^{s_2} w(s) d_\alpha s < \infty. \tag{4.2}$$

4.1 Main Theorem

Let us now establish Grüss type inequality to weighted Ostrowski-Grüss inequality via Katugampola conformable fractional integral for differentiable mapping in the form of the following theorem:

Theorem 26 *Let $f : [s_1, s_2] \rightarrow \mathbb{R}$ be continuous on $[s_1, s_2]$ and α-fractional differentiable for $\alpha \in (0,\ 1]$, where $\phi \leq D^\alpha f(k) \leq \psi$, for all $k \in (s_1, s_2)$. Then, we have the following weighted Ostrowski-Grüss type inequality via Katugampola conformable fractional integral for differentiable mapping:*

$$\begin{aligned} &\left| f(k) - \frac{1}{m(s_1, s_2)} \int_{s_1}^{s_2} f(t) w(t) d_\alpha t - (k - \mu(s_1, s_2)) D^\alpha (f(k)) \right| \\ \leq\ & \frac{1}{8}(\psi - \phi) \left| m\ (s_1, s_2) + \int_{s_1}^{s_2} sgn\ (t-k) w(t) d_\alpha t \right|. \end{aligned} \tag{4.3}$$

Lemma 27 *Let us prove the following identity:*

$$\int_{s_1}^{s_2} p(k,t)\,d_\alpha t = m(s_1,s_2)(k-\mu(s_1,s_2)), \tag{4.4}$$

where the mapping $p(.,\,.)\ :[s_1\ ,s_2]^2\to\mathbb{R}$ *is defined by:*

$$p\,(k,t)=\begin{cases}\int_{s_1}^{t} w(s)\,d_\alpha s, & \text{if } t\in[s_1,k],\\ \int_{s_2}^{t} w(s)\,d_\alpha s, & \text{if } t\in(k,s_2].\end{cases}$$

Proof. Integration by parts gives the following:

$$\begin{aligned}
\int_{s_1}^{s_2} p(k,t)\,d_\alpha t &= \int_{s_1}^{k}(\int_{s_1}^{t} w(s)\,d_\alpha s)\,d_\alpha t + \int_{k}^{s_2}(\int_{s_2}^{t} w(s)\,d_\alpha s)\,d_\alpha t,\\
&= (\int_{s_1}^{t} w(s)\,d_\alpha s)t|_{s_1}^{k} - \int_{s_1}^{k} tw(t)\,d_\alpha t +\\
&\quad (\int_{s_2}^{t} w(s)\,d_\alpha s)t|_{k}^{s_2} - \int_{k}^{s_2} tw(t)\,d_\alpha t,\\
&= \int_{s_1}^{k} w(s)\,d_\alpha s - \int_{s_1}^{s_2} tw(t)\,d_\alpha t - k\int_{s_2}^{k} w(s)\,d_\alpha s,\\
&= km\,(s_1,k) - M\,(s_1,s_2) + k\int_{k}^{s_2} w(s)\,d_\alpha s,\\
&= k(m\,(s_1,k) + M(k,s_2)) - M(s_1,s_2),\\
&= km(s_1,s_2) - M(s_1,s_2),\\
&= m(s_1,s_2)\left(k - \frac{M(s_1,s_2)}{m(s_1,s_2)}\right),\\
\int_{s_1}^{s_2} p(k,t)\,d_\alpha t &= m\,(s_1,s_2)(k-\mu(s_1,s_2)).
\end{aligned} \tag{4.5}$$

■

Lemma 28 *: Let* $w : [s_1, s_2] \to \mathbb{R}$ *be a non-negative weight function and* $f : [s_1, s_2] \to \mathbb{R}$ *be* α*-fractional differentiable for* $\alpha \in (0,\ 1]$ *in the sense of Katugampola. Then we have the following identity:*

$$\int_{s_1}^{s_2} p(k,t)\, D_\alpha(f(t))\, d_\alpha t \ = m\,(s_1, s_2) f(k) - \int_{s_1}^{s_2} w(t) f(t)\, d_\alpha t, \tag{4.6}$$

where

$$p\,(k,t) = \begin{cases} m(s_1, t), & t \in [s_1,\ k], \\ m(s_2, t), & t \in (k,\ s_2]. \end{cases}$$

Proof. Using integration by parts, we have:

$$\begin{aligned} \int_{s_1}^{s_2} p(k,t) D_\alpha(f(t))\, d_\alpha t &= \int_{s_1}^{k} m(s_1, t) D_\alpha(f(t))\, d_\alpha t + \int_{k}^{s_2} m(s_2, t) D_\alpha(f(t))\, d_\alpha t, \\ &= m(s_1, t) f(t)|_{s_1}^{k} - \int_{s_1}^{k} f(t) w(t) s_{2\alpha} t + m(s_2, t)\, f(t)|_{k}^{s_2} \\ &\quad - \int_{k}^{s_2} f(t) w(t)\, d_\alpha t, \\ &= m(s_1, k)\, f(k) - m(s_2, k) f(k) - \int_{s_1}^{s_2} f(t) w(t)\, d_\alpha t, \\ &= (m(s_1, k) + m(k, s_2)\,) f(k) - \int_{s_1}^{s_2} f(t) w(t)\, d_\alpha t, \\ &= m(s_1, s_2) f(k) - \int_{s_1}^{s_2} f(t) w(t)\, d_\alpha t, \end{aligned}$$

which is the required identity. ■

4.2 Kernel Estimation

Let us prove that the kernel satisfies the estimation:

$$0 \leq p\,(k,\ t) \leq \max \begin{cases} \int_k^{s_2} w\,(s)\,d_\alpha s, & \text{if} \quad t \in [s_1,\, \frac{s_1+s_2}{2}], \\ \int_{s_1}^{k} w\,(s)\,d_\alpha s, & \text{if} \quad t \in (\frac{s_1+s_2}{2},\, s_2]. \end{cases}$$

$$\begin{aligned} 0 \quad &\leq \quad p\,(k,t) \leq \max\left(\int_{s_1}^{k} w(s)\,d_\alpha s,\, \int_{k}^{s_2} w(s)\,d_\alpha s\right) \\ &= \quad \frac{1}{2}\left(\int_{s_1}^{s_2} w(s)\,d_\alpha s + \left|\int_{s_1}^{s_2} sgn(s-k)w(s)\,d_\alpha s\right|\right), \\ &= \quad \frac{1}{2}\left(m\,(s_1,\,s_2) + \left|\int_{s_1}^{s_2} sgn(s-k)\mathrm{W}(s)\,d_\alpha s\right|\right). \end{aligned}$$

We further note that $w(k)$ is a non-negative function. This implies

$$0 \leq \frac{p\,(k,t)}{w(k)} \leq \frac{1}{2}\left[m(s_1,s_2) + \left|\int_{s_1}^{s_2} sgn(s-k)\mathrm{W}(s)\,d_\alpha s\right|\right]. \tag{4.7}$$

Proof. (Theorem 26)

Applying weighted Grüss inequality (3.3) for mapping $f(.) = D^\alpha(f(.))g(.) = \frac{p(k,\,.)}{w(k)}$

and using (4.7), we have:

$$\begin{aligned} &\left|\frac{1}{m(s_1,s_2)}\int_{s_1}^{s_2} p\,(k,t)\,D^\alpha(f(t))\,d_\alpha t \right. \\ &\left. -\frac{1}{m(s_1,s_2)}\int_{s_1}^{s_2} p\,(k,t)\,d_\alpha t * \frac{1}{m(s_1,s_2)}\int_{s_1}^{s_2} D^\alpha(f(t))w(t)\,d_\alpha t\right| \\ \leq \quad &\frac{1}{8}(\psi-\phi)\left[m(s_1,s_2) + \left|\int_{s_1}^{s_2} sgn(t-k)\mathrm{W}(t)\,d_\alpha t\right|\right]. \end{aligned}$$

Using mean value theorem for integrals, it further implies

$$\begin{aligned} &\left|\frac{1}{m(s_1,s_2)}\int_{s_1}^{s_2} p\,(k,t)\,D^\alpha(f(t))\,d_\alpha t - \frac{D^\alpha(f(t))}{m(s_1,s_2)}\int_{s_1}^{s_2} p\,(k,t)\,d_\alpha t\right| \\ \leq \quad &\frac{1}{8}(\psi-\phi)\left[m(s_1,s_2) + \left|\int_{s_1}^{s_2} sgn(t-k)\mathrm{W}(t)\,d_\alpha t\right|\right]. \end{aligned} \tag{4.8}$$

Using (4.7) and (4.6) in (4.8), we get the required inequality (4.3). ■

Remark 29 *: If we put $w(t) = 1$, $\alpha = 1$ in (4.3), we get the following inequality*

$$\left| f(k) - \left(k - \frac{s_1 + s_2}{2}\right) f'(k) - \frac{1}{s_2 - s_1} \int_{s_1}^{s_2} f(t)dt \right| \leq \frac{1}{4}(\psi - \phi) \left(\frac{s_2 - s_1}{2} + \left| \left(k - \frac{s_1 + s_2}{2}\right) \right| \right),$$

which is remark (21) in [26].

Corollary 30 *: Under the assumption of theorem (26), putting $k = \frac{s_1 + s_2}{2}$ and $\alpha = 1$, we get the mid-point inequality of [26].*

$$\left| f(\frac{s_1 + s_2}{2}) - \left(\frac{s_1 + s_2}{2}\right) - \mu(s_1, s_2) f' \left(\frac{s_1 + s_2}{2}\right) - \frac{1}{m\ (s_1, s_2)} \int_{s_1}^{s_2} f(t)w(t)dt \right| \leq \frac{1}{8}(\psi - \phi) \left[m\ (s_1, s_2) + \left| \int_{s_1}^{s_2} sgn \left(t - \frac{s_1 + s_2}{2}\right) \mathrm{w}(t)dt \right| \right].$$

Corollary 31 *: Under the assumption of theorem (26), with $k = s_1$ and $k = s_2$, adding the results and using triangular inequality and then dividing by 2 and gives us:*

$$\begin{aligned} & \left| \frac{f(s_1) + f(s_2)}{2} - \frac{1}{2} [s_1\ D^\alpha f(s_1) + s_2 D^\alpha f(s_2) - \mu(s_1, s_2) \left(D^\alpha f(s_2) + D^\alpha f(s_1)\right)] \right. \\ & \left. - \frac{1}{m\ (s_1, s_2)} \int_{s_1}^{s_2} f(t)w(t)d_\alpha t \right| \\ \leq\ & \frac{1}{16}(\psi - \phi) \left[2\ m\ (s_1, s_2) + \left| \int_{s_1}^{s_2} \mathrm{w}(t) sgn(t - s_2) d_\alpha t \right| + \left| \int_{s_1}^{s_2} \mathrm{w}(t) sgn(t - s_1) d_\alpha t \right| \right], \\ =\ & \frac{1}{16}(\psi - \phi) 4\ m\ (s_1, s_2), \\ =\ & \frac{1}{4}(\psi - \phi)\ m\ (s_1, s_2). \end{aligned}$$

Corollary 32 *: From corollary (31), for* $\alpha = 1$*, we get corollary (23) of [26].*

Chapter 5

Application in Numerical Integration

Fractional integral inequalities have been shown to be one of the most significant and potent tools for advancement in many fields of pure and applied mathematics. Determining the uniqueness of fractional boundary value issue solutions in numerical quadrature, transform theory, probability, and statistics is the most significant use of these inequalities. Additionally, they offer upper and lower bounds for the solutions to the previous equations.

The Katugampola fractional derivative was recently introduced by Katugampola. It is a limit-based fractional derivative that keeps many of the fundamental characteristics of ordinary derivatives, including the product, quotient, and chain rules. Fractional derivatives are often handled using an integral form and are hence non-local in nature. The current study begins with a fundamental property of the Katugampola fractional derivative $D^{\alpha}[y] = t^{1-\alpha}\frac{dy}{dx}$, and the corresponding differential operator $D^{\alpha} = t^{1-\alpha}D^{1}$. These operators, as well as their inverses, commutators, anti-commutators, and a number of key differential equations, are investigated.

The growth of differential and integral inequalities was accelerated by the invention of differential and integral calculus. In the mathematical sciences, these imperfections are quite significant. The lower and upper bounds on functions and their derivatives are provided by these inequalities, which also have several applications in special means and numerical integration. There are many different branches covered by mathematics and associated fields. Convexity is a key idea in both biology and finance. This illustrates a potent idea that can be applied in numerous contexts. The method is pure and applicable, and it can be used to study a wide variety of unrelated subjects. This entails both exploring the subject and approaching it from several perspectives. The convexity idea has a significant impact on the growth of inequality theory.

In the modeling of engineering and scientific issues, fractional differential and integral equations are becoming increasingly significant. It has been demonstrated that these models routinely outperform equivalent models that employ integer derivatives. A thorough investigation has been conducted in the theory of fractional differential equations and the calculus of fractional order derivatives. The majority of the present literature establishes the solution to fractional differential equations using a fixed point approach. The qualitative properties of the Riemann-Liouville (R-L) and Caputo derivatives are studied using differential and integral inequalities in fractional differential equations. We reach a comparable conclusion for the R-L type of integral inequalities using the common Lipschitz condition on the nonlinear component.

In this case, the comparison theorem and the explicit declaration are useful. One of the most important uses of fractional integral inequalities is the finding of numerical problem solutions. This is conceivable in a variety of fields, including probability, transform theory, and quadrature. These inequalities also provide upper and lower bounds on the solutions to the equations that follow. This knowledge is usually quite useful in mathematics when attempting to solve a problem or select the best answer. A fractional differential equation with variable coefficients can be used to show the existence of the fractional differential equation. It is critical to keep this in mind when dealing with equations and issues involving these types of equations since doing so will make the problems much simpler to solve. A quasi-linearization approach is used to solve nonlinear fractional differential equations. The Katugampola fractional derivative was defined using the Katugampola fractional integral, and it expands the capability of using real number powers or complex number powers of the integral and differential operators, just like any other fractional differential operator. These operators combine the fractional derivatives of Riemann-Liouville and Hadamard into a single form.

In Mathematics, Physics and different courses of Engineering, the concept of fractional order derivatives and integrals are used by Edmundo Capelas de Oliveira *et al.* [10]. In fractional calculus, the derivative in classical calculus has a significant geometric interpretation, it is concern with the idea of tangent. This disparity might be shown as an issue for fractional calculus poor success up until ninteen centuary. First use of fractional calculus is regarded as integral equation that was solved by Abel. This is known as the first Liouville definition. Liouville's second definition, given in terms of an integral, is currently known as the Liouville version for the integration of noninteger order. After a series of research by Liouville, Riemann wrote the most important piece 10 years after his death. It is also worth noting that both the Liouville and Riemann formulations contain the so-called complementary function, which is a problem to be solved. We will compare the Caputo version to the Riemann-Liouville formulation because of its importance. With the noninteger order derivative of the Riemann-Liouville equation, Caputo's concept inverts the order of integral and derivative operators. The contrast between these two formulations may be summed up as follows: In Caputo, first compute the integer order derivative, subsequently the noninteger order integral. Compute the integral of noninteger order first, then the derivative of integer order in the Riemann-Liouville equation. It is important to note that the Caputo derivative may be utilized to deal with scenarios where the initial conditions are done in the function and the necessary integer order derivatives. Fractional calculus has expanded since its inception at the University of New Haven in 1974, with numerous applications in many sectors of scientific knowledge. As a result, several solutions to derivative problems have been proposed, and numerous definitions of the fractional derivative are available in the literature. This study provides systematic formulations of current fractional derivatives and integrals.

A mathematical technique known as the quadrature rule aids in calculating the locations of points on a plane. Finding improvements, generalizations, expansions, and uses for this tool served as the driving force behind this. In order for these rules to better fulfil their intended functions, researchers are constantly looking for new methods to enhance or perfect them. Mathematicians who specialize in fractional calculus study non-integral order integral and differential operators. Numerous mathematicians have made substantial contributions to this field, including Liouville, Riemann, Weyl, and Fourier. The theory underlying fractional calculus was developed by Abel, Lacroix, Leibniz, and Grunwald. In terms of a fractional derivative, Riemann created the first formulation of an operator governing integrals over nonlinear spaces.

In this chapter, the applications to numerical integration of Grüss type inequality to weighted Ostrowski-Grüss inequality via conformable fractional integral for $\alpha-$fractional differentiable mapping is discussed.

Let J_n ; $s_1 = k_0 < k_1 < k_2 < k_3 < ... < k_{n-1} < k_n = s_2$ be the division of the interval $[s_1 , s_2]$, $\beta_i \in [k_i, k_{i+1}]$, $i = 1, ..., n-1$.

We have the following quadrature formula in the form of the following theorem:

Theorem 33 *: Let $f : [s_1, s_2] \to \mathbb{R}$ be continuous on $[s_1, s_2]$ and α-fractional differentiable function on (s_1, s_2) and $D^\alpha f(k) : (s_1, s_2) \to \mathbb{R}$ satisfy the condition $\phi \leq$*

$D^{\alpha}f(k) \leq \psi$, for all $k \in [s_1, s_2]$. Then, we have the following perturbed Riemann's type quadrature formula:

$$\int_{s_1}^{s_2} f(s)w(s)d_{\alpha}s = B(f, D^{\alpha}(f), \beta, J_n) + R(f, D^{\alpha}(f), \beta, J_n), \tag{5.1}$$

where

$$\begin{aligned} & B(f, D^{\alpha}(f), \beta, J_n) \\ = & \sum_{i=0}^{n-1} m(k_i, k_{i+1}) f(\beta_i) \\ & - \sum_{i=0}^{n-1} m(k_i, k_{i+1}) - \mu(k_i, k_{i+1}) D^{\alpha}(f(\beta_i)), \end{aligned}$$

and the remainder term gives the estimation:

$$\begin{aligned} |R(f, D^{\alpha}(f), \beta, J_n)| \leq & \frac{1}{8}(\psi - \phi) \sum_{i=0}^{n-1} m(k_i, k_{i+1}) * \\ & \left(m(k_i, k_{i+1}) + \left| \int_{k_i}^{k_{i+1}} sgn(s - \beta_i) w(s) d_{\alpha}s \right| \right), \end{aligned} \tag{5.2}$$

for all $\beta_i \in [k_i,\ k_{i+1}]$, where $h_i = k_{i+1} - k_i$, for $i = 1, 2, ..., n-1$.

Proof. Applying theorem (26), on the interval $[k_i,\ k_{i+1}], \beta_i \in [k_i,\ k_{i+1}]$, where $h_i = k_{i+1} - k_i$, for $i = 1, 2, ..., n-1$, we get

$$\begin{aligned} & \left| \int_{k_i}^{k_{i+1}} f(s)w(s)d_{\alpha}s - m\ (k_i, k_{i+1}) f(\beta_i) \right. \\ & \left. + m(k_i, k_{i+1})(\beta_i - \mu(k_i, k_{i+1})) D^{\alpha}(f(\beta_i)) \right| \\ \leq & \frac{1}{8}(\psi - \phi) m(k_i, k_{i+1}) \left(m(k_i, k_{i+1}) + \left| \int_{k_i}^{k_{i+1}} sgn(s - \beta_i) w(s) d_{\alpha}s \right| \right). \end{aligned}$$

Summing over i from 0 to $n-1$ and using the generalized triangular inequality, we get the required estimation (5.2). ■

Corollary 34 *: Under the assumption of theorem (33), by choosing* $\beta_i = \frac{k_i + k_{i+1}}{2}$, $\alpha = 1$, *we obtain mid-point quadrature of [26]:*

$$\int_{k_i}^{k_{i+1}} f(s)w(s)ds = B_M\left(f, f'(f), \beta, J_n\right) + R_M\left(f, f'(f), \beta, J_n\right),$$

where

$$\begin{aligned} & B_M\left(f, f'(f), \beta, J_n\right) \\ = & \sum_{i=0}^{n-1} m(k_i, k_{i+1}) f(\frac{k_i + k_{i+1}}{2}) \\ & - \sum_{i=0}^{n-1} m(k_i, k_{i+1})(\frac{k_i + k_{i+1}}{2} - \mu(k_i, k_{i+1}) \\ & * f'(f(\frac{k_i + k_{i+1}}{2})), \end{aligned}$$

and the remainder term satisfies the estimation:

$$\begin{aligned} & \left| R_M\left(f, f'(f), \beta, J_n\right) \right| \\ \leq & \frac{1}{8}(\psi - \phi) \sum_{i=0}^{n-1} m(k_i, k_{i+1}) \\ & \times \left(m(k_i, \ k_{i+1}) + \left| \int_{k_i}^{k_{i+1}} w(s) sgn(s - \frac{k_i + k_{i+1}}{2}) \right| \right). \end{aligned}$$

The notion of convexity has received fresh attention from several academics. As a result, the classical concepts of convex sets and convex functions have been expanded upon and modified in a variety of ways. It is important to note that the class of p-convex functions also includes harmonically convex functions in addition to classical convex functions. The connection between the theory of convex functions and the idea of inequalities has attracted the attention of many academics. One of the inequalities for convex functions that has received the most attention is the Hermite-Hadamard inequality, named after Hermite and Hadamard. This Hermite-Hadamard inequality must be satisfied by a function in order to be convex.

Chapter 6

Conclusions

In this research, first of all established some integral inequalities for conformable fractional integral given by Katugampola. Later modified Grüss type integral inequality to weighted Ostrowski-Grüss type inequality for differentiable mapping in terms of the upper and lower bounds of the first derivative via Katugampola conformable fractional integral has been derived. The inequality is then applied to numerical integration. Then, the applications to numerical integration of Grüss type inequality to weighted Ostrowski-Grüss inequality via conformable fractional integral for $\alpha-$fractional differentiable mapping are described. We can demonstrate a modification of Grüss type inequalities to weighted Ostrowski-Grüss inequality after reading and seeing so many research publications. We changed the weighted Ostrowski-Grüss inequality conformable fractional integral after changing the Grüss type inequality.

6.1 Concluding Remarks

In this thesis, we defined weight function and moments via Katugampola α-fractional differentiable mappings, α-fractional integrals. Also established two new identities via Katugampola conformable fractional integral. Then modified weighted Ostrowski-Grüss type inequality is generalized by using Katugampola fractional derivative and integral for α-fractional derivative. At the end, application to numerical integration is provided.

6.2 Recommendations

For several additional fractional integrals, we may develop a new modified weighted Ostrowski-Grüss inequality. Within this work, we can extend these inequalities to additional convex functions for differential mappings. Error estimations of Grüss-type inequalities can be looked into. New applications of numerical integration and special means methods can also be explored.

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