DEVELOPMENT OF OSTROWSKI TYPE INEQUALITIES FOR FRACTONAL INTEGRAL

By

Fawad Ali

NATIONAL UNIVERSITY OF MODERN LANGUAGES ISLAMABAD

June, 2023

Development of Ostrowski type Inequalities for Fractional Integral

By

FAWAD ALI

A THESIS SUBMITED IN PARTIAL FULLFILMENT OF THE REQUIREMENT FOR THE DEGREE OF

MASTER OF SCIENCE

In **Mathematics**

To

FACULTY OF ENGINEERING & COMPUTER SCIENCE

NATIONAL UNIVERSITY OF MODERN LANGUAGES ISLAMABAD

NATIONAL UNIUVERSITY OF MODERN LANGUAGES FACULTY OF ENGINEERIING & COMPUTER SCIENCE

THESIS AND DEFENSE APPROVAL FORM

The undersigned certify that they have read the following thesis, examined the defense, are satisfied with overall exam performance, and recommend the thesis to the Faculty of Engineering and Computer Sciences for acceptance.

Thesis Title: Development of Ostrowski type Inequalities for Fractional Integral

Submitted By: Fawad Ali **Registration #:** 3/MS/Math/S20

Master of Science in Mathematics (MS Math)
Title of the Degree

Mathematics
Name of Discipline

Dr. Naila Rafiq
Name of Research Supervisor ______________________ Signature of Research Supervisor

Dr. Sadia Riaz
Name of HOD (MATH) ______________________ Signature of HOD (MATH)

Dr. Muhammad Noman Malik
Name of Dean (FE&CS) ______________________ Signature of Dean (FE&CS)

Date: 26 June, 2023

AUTHOR'S DECLARATION

I Fawad Ali

Son of Shahzad ALi

Registration #3/MS/MATH/S20

Discipline Mathematics

Candidate of **Master of Science in Mathematics** at National University of Modern Languages do here by declare that the thesis **Development of Ostrowski Type Inequalities For Fractional Integral** submitted by me in fractional fulfillment of **MS Mathematics** degree, is my original work, and has not been submitted or published earlier. I also solemnly declare that it shall not, in future, be submitted by me for obtaining any other degree from this or any other university or institution. I also understand that if evidence of plagiarism is found in my thesis/dissertation at any stage, even after the award of a degree, the work may be cancelled and degree revoked.

Signature of Candidate ………………..

Name of Candidate. Fawad Ali……

Date: 26 June, 2023

DEDICATION

I dedicate my thesis to my parents and teachers for their endless support and encouragement throughout my pursuit for education. I hope this achievement will fulfill the dream they envisioned for me.

ACKNOWLEDGMENT

All praises of Almighty Allah, the most benefient and the most merciful, who created this univese and gave us linkling to ascertain it. I am highly obliged to Almighty Allah for His benediction, direction and assistance in each and every pace of my life. He sanctifies us with the Holy prophet Muhammad (PBUH), who is a beacon of light for humanity and exhorts his followers to pursue knowledge from birth to tomb. All praises of Almighty Allah, the most beneficent and the most merciful, who created this universe and gave us inkling to ascertain it. I am highly obliged to Almighty Allah for His benediction,direction and assistance in each and every pace of my life.

I cannot wholly prompt gratefulness to my Supervisor Dr.NAILA RAFIQ underneath whose supervision and illuminating assistance, the research work presented in the thesis became conceivable. Her assistance and revitalizing will always endure source of stimulation for me.

I would also like to thank exclusively to Prof. Dr. NAZIR AHMAD MIR for his valuable directions.

I am highly grateful from the core of my heart to my father and late mother for backing and help, who always given me love, attention and joyfulness their unremit ting confidence in me directed me to where I stand today and who supported me in any way during accomplishment of my MS thesis.

Fawad Ali

ABSTRACT

In this thesis, first of all various types of convex functions and fractional integrals, their applications and various related identities and well-known inequalities are discussed. Then a new identity for differentiable, GA-convex function is established. Using this identity, Ostrowski type inequalities for fractional integral are developed. Then, two versions of Ostrowski type inequality for GA-convex differentiable and bounded function for Hadamard fractional integral are developed. Consequently, Ostrowski type inequalities for GA-convex nth differentiable bounded function for Hadamard fractional integral version-I and version-II are generalized. Accordingly, some applications to special means, such as arithmetic-, geometric-, logarithmic- and p-logarithmic means in subsequent sections are also provided. Further, Ostrowski type inequalities for first time differentiable and n-time differentiable GA-convex function via fractional integral are established using power mean inequality. At the end, some conclusions and recommendations for further research work are provided..

Contents

Chapter 1

Introduction and Preliminaries

1.1 Introduction

Mathematical inequalities of the Ostrowski type link the difference between a function and its nearest convex function to the function's second derivative. Named after the mathematician Alexander Ostrowski, who originally investigated them in the early 20th century, these inequalities. They now have several applications across many different domains and are a significant topic of research in functional analysis.

The basic form of an Ostrowski type inequality is as follows:

$$|f(x) - g(x)| \leq M \, |f''(x)|.$$

where $g(x)$ is the closest convex function to $f(x)$, M is a constant, and $f(x)$ is a function. The inequality asserts that the absolute value of the second derivative of the function $f(x)$, multiplied by a constant M, determines the boundary of the difference between the function $f(x)$ and its closest convex function, $g(x)$.

There are numerous modifications and generalisations of Ostrowski type inequalities in addition to their fundamental form. For instance, there are inequalities of the Ostrowski type that link a function's second derivative to the distance between the function and its closest concave function.Additionally, there are inequalities of the Ostrowski type that link higher order derivatives of a function to the variation between a function and its closest convex function. For instance, there are inequalities of the Ostrowski type that connect a function's fourth derivative to the distance between it and its nearest convex function.Ostrowski type inequalities have many applications in various fields such as approximation theory, numerical analysis, and optimization. In approximation theory.

Ostrowski type inequalities are used to bound the error between a function and its closest convex function. In numerical analysis, Ostrowski type inequalities are used to bound the error between a numerical approximation of a function and the true function. In optimization, Ostrowski type inequalities are used to bound the error between a function and its closest convex function, which can be used to improve the efficiency of optimization algorithms.In addition, Ostrowski type inequalities have applications in various branches of mathematics and physics, such as differential equations, partial differential equations, and quantum mechanics. In differential equations, Ostrowski type inequalities are used to bound the error between a solution of a differential equation and the true solution. In partial differential equations, Ostrowski type inequalities are used to bound the error between a numerical approximation of a solution of a partial differential equation and the true solution.

Since the writing of the book, Inequalities involving function and their integral and derivatives by D. S. Mitrinodic, et. al., in 1991 [1] and classical book by Hardy Little Wood and Polya [2], the subject of integral and differential inequalities grew very rapidly and significantly. In 1938, the classical integral inequality established by Ostrowski is given in the form of the following theorem:

Theorem 1 *:Let $f : [a_1, b_1] \to \mathbb{R}$, where $I \subset \mathbb{R}$ with $a_1 < b_1$ be a differentiable mapping on (a_1, b_1) whose derivative $f' : (a_1, b_1) \to \mathbb{R}$ is bounded on (a_1, b_1),*

$$i.e., \quad |f'(t)| \leq M < \infty, \quad \forall\, t \in [a_1, b_1].$$

Then,we have the following inequality:

$$\left| f(x) - \frac{1}{b_1 - a_1} \int\limits_{a_1}^{b_1} f(t)dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a_1+b_1}{2}\right)^2}{(b_1 - a_1)^2} \right] (b_1 - a_1)M, \tag{1.1}$$

for all $x \in [a_1, b_1]$, where M is some constant.The constant $\frac{1}{4}$ is the best possible The constant $\frac{1}{4}$ is sharp in the sense that it can not be replaced by a smaller one.

The inequality (1.1) gives an approximation for upper bound at point $x \in [a_1, b_1]$ of the integral average $\frac{1}{b_1-a_1} \int\limits_{a_1}^{b_1} f(t)dt$ by the value $f(x)$.

Recently, several generalization of the Ostrowski type integral inequalities for mapping of bounded variation and Lipschitzian monotonic, absolutely continuous and n-times differentiable mappings with error estimates for some special means and for some numerical quadrature rules are considered by many authors for fractional integral as well [3, 4].

1.2 Preliminaries

Here, we present some basic definitions and concepts which we will be used throughout this dissertation.

Convex Function

Convex function is a special type of mathematical function that has certain properties related to its shape. In functional analysis, the study of convex functions is an important area of research due to its connections to optimization, equilibrium problems, and other areas of mathematics and engineering. A function $f(x)$ is said to be convex [4] if for any two points x and y in the domain of the function and for any λ in the interval [0,1].

A function f :$[a_1, b_1] \in \mathbb{R} \to \mathbb{R}$ is said to be *convex*, if the following inequality holds:

$$\begin{aligned} f(\lambda x + (1-\lambda) y) &\leq \lambda f(x) + (1-\lambda) f(y), \\ \forall\, x,\ y &\in [a_1, b_1] \text{ and } \lambda \in [0,1]. \end{aligned}$$

According to this inequality, the function's value is always greater than or equal to the weighted average of the function's values at x and y, no matter where on the line segment between those two points it occurs. To put it another way, the function never deviates from the line joining x and y.

A convex function always has an upward-facing graph, which means that as one advances down the x-axis, the graph slopes higher. This characteristic of the function is known as its "convexity," and it has several significant implications for functional analysis.

One of the most important implications of convexity is that it allows for efficient optimization of the function. Because the function never dips below the line connecting any two points, there is always a global minimum of the function. Additionally, any local minimum of a convex function is also a global minimum, meaning that optimization algorithms can be stopped as soon as a local minimum is found.

Convex functions also have the useful virtue of being utilised to model equilibrium issues. Finding a system's equilibrium state in physics and engineering entails balancing the forces acting on the system. Because convex functions have the characteristic of being reduced when the forces operating on the system are in equilibrium, they can be utilised to represent these equilibrium issues.

Convex functions also have numerous other characteristics that make them significant in other mathematical disciplines, such as functional analysis. The sum of two convex functions is also convex because they are closed under addition. They are also closed under positive scaling, which means that for any positive scaling, if $f(x)$is convex, then $(a_1)(f(x))$ is also convex value of a.

In functional analysis, particularly in the study of Hilbert spaces, convex functions are very crucial. A unique class of vector space known as a Hilbert space is complete with regard to an inner product. A function $f(x)$ in a Hilbert space is referred to as convex if it is the pointwise limit of a series of affine functions. This characteristic, known as weak convexity, is a weaker version of convexity than the prior traditional definition.

In summary, convex functions are special type of mathematical functions that have certain properties related to its shape. They have many important implications in optimization, equilibrium problems, and other areas of mathematics and engineering. They play important role in functional janalysis, particularly in the study of Hilbert spaces and have many properties that make them important in various fields.

We say that f is concave, if $(-f)$ is convex.

Convex Function with application

Convex functions have several applications in functional analysis, a field of mathematics concerned with the study of function spaces [5]. Optimization, equilibrium issues, and the study of Hilbert spaces are some of the most prominent applications of convex functions in functional analysis.

Equilibrium issues are another major use of convex functions in functional analysis. Equilibrium issues in physics and engineering require determining the condition of a system in which the forces operating on it are balanced. Convex functions, which have the feature of being minimised when the forces operating on the system are balanced, may be utilised to describe these equilibrium situations. Convex functions are also useful in the study of Hilbert spaces.

Convex functions are frequently used in statistics and machine learning to describe a variety of phenomena, including data distributions and loss functions for model training. Convex optimization, which is the process of minimising a convex function subject to constraints, is a common use of convex functions in machine learning.

In summary, convex functions have a wide range of applications in functional analysis. They are particularly useful in optimization problems, equilibrium problems, and the study of Hilbert spaces. They are also widely used in machine learning and statistics. The property of a convex function, that is, it never dips below the line that connects x and y, implies that there is always a global minimum of the function, which makes it highly useful in optimization problems. Convex functions can also be used to model equilibrium problems and in the study of Hilbert spaces.

GA-Convex Function

A function $f \;\; [a_1, b_1] \subseteq (0, \infty) \;\to \mathbb{R}$ is called *GA-convex* on $[a_1,\ b_1]$ [6], if

$$f\left(x^{\lambda}\ y^{1-\lambda}\right) \;\leq\; \lambda\ f(x) + (1-\lambda)\ f\left(y\right),$$

$$\forall\ x,\ y \;\in\; [a_1, b_1] \text{ and } \lambda \in [0, 1]\,.$$

h- Convex Function

A function $f : [a_1, b_1] \in \mathbb{R} \rightarrow \mathbb{R}$ is said to be *h-convex* [5], if the following inequality holds:

$$\begin{aligned} f\left(h\left(\lambda\right)x + h\left(1-\lambda\right)y\right) &\leq h\left(\lambda\right)f\left(x\right) + h(1-\lambda)\ f\ \left(y\right), \\ \forall x, y &\in [a_1, b_1] \text{ and } \lambda \in [0,1]. \end{aligned}$$

s- Convex Function

A function $f : [a_1, b_1] \in \mathbb{R} \rightarrow \mathbb{R}$ is said to be s- convex [5], if the following inequality holds:

$$f\left(h\left(\lambda\right)x + h\left(1-\lambda\right)y\right) \leq h\left(\lambda\right)f\left(x\right) + h(1-\lambda)f\left(y\right),$$

where,

$$h\left(\lambda\right) = \lambda^s, \quad h\left(1-\lambda\right) = \left(1-\lambda\right)^s, \quad \forall \lambda \in [0,1], \quad x, y \in [a_1, b_1].$$

s-convex functions are also employed in financial mathematics. s-convex functions are used in mathematical finance to estimate the pricing of financial derivatives such as options and futures. The prices of these derivatives are frequently treated as partial differential equation solutions, which may be characterised using s-convex functions.

In summary, s-convex functions are a subclass of convex functions with certain unique qualities that make them valuable in functional analysis and other fields of mathematics and engineering. They find widespread use in optimization, partial differential equations, mathematical finance, optimum control, and other fields. The property of a s-convex function, that is, it dips below the line that connects x and y, implies that there is always a global minimum of the function, which is unique and it makes them particularly useful in optimization problems where the goal is to find the global minimum of a function and it is unique. s-convex functions can also be used to model certain types of partial differential equations, such as the Hamilton-Jacobi equations, which are used to describe the dynamics of a wide range of physical systems.

p-Convex Function

Suppose $A \subset (0, \infty) = \mathbb{R}_+$ and $p \in \mathbb{R} \ / \ \{0\}$Then, a function $f : \ A \to \mathbb{R}$ is called p-convex [5], if

$$f\left(\left[\lambda a_1^p + (1-\lambda)\, a_2^p \right]^{\frac{1}{p}} \right) \leq \lambda f\,(a_1) + (1-\lambda)\, f\,(a_2)$$

holds, $\forall \ a_1, a_2 \in A$ and $\lambda \in [0,1]$. If the inequality is in opposite direction, then f is called *p-concave.*

GA convex Function Integral Inequalities

GA convex function integral inequalities have many applications in various fields such as optimization, numerical analysis, and approximation theory [7]. In optimization, these inequalities can be used to bound the error between a function and its closest convex function, which can be used to improve the efficiency of optimization algorithms. In numerical analysis, these inequalities can be used to bound the error between a numerical approximation of a function and the true function. In approximation theory, these inequalities can be used to bound the error between a function and its closest convex function.

In addition, GA convex function integral inequalities have applications in other areas of mathematics and engineering, such as physics, control theory, and statistics. In physics, these inequalities can be used to bound the error between a physical system and its mathematical model. In control theory, these inequalities can be used to bound the error between a control system and its desired behavior. In statistics, these inequalities can be used to bound the error between a probability distribution and its approximating function.

Furthermore, these inequalities have different versions, such as GA convex function integral inequalities for higher-order derivatives, GA convex function integral inequalities for higher-dimensional functions, etc.

GG convex Function Integral Inequalities

GG (Gradient and Gauss-Newton) Convex Function Integral Inequalities are a type of mathematical inequity that connects a function's integral [8] and its nearest convex function to the function's gradient and Gauss-Newton matrix. These inequalities are obtained using the GG convex approximation approach. These inequalities are useful in functional analysis, optimization, and other branches of mathematics and engineering.

Where$f(x)$ is a function, $g(x)$ is the convex function closest to $f(x)$, $g(x)$ is the Gauss-Newton matrix (also known as the Gauss-Newton approximation of the Hessian), and a_1 and b_1 are the endpoints of an interval where the inequality holds. According to the inequality, the difference between the integral of the function $f(x)$ across the interval (a_1, b_1).

The Gauss-Newton matrix is a matrix that approximates the Hessian matrix (matrix of second-order partial derivatives) of a function. It is used in optimization, especially in non-linear least squares problems, where it is a common approximation to the Hessian. The Gauss-Newton matrix is defined as the Jacobian matrix of the function, multiplied by its transpose.

The concept of the closest convex function to a given function is important in functional analysis. A convex function is a special type of function that has certain properties related to its shape. Specifically, for any two points x and y in the domain of the function and for any λ in the interval $[0, 1]$, the following inequality holds:

$$\begin{aligned} f\left(\lambda x+(1-\lambda)\, y\right) &\leq \lambda f\left(x\right)+(1-\lambda) f\left(y\right), \\ \forall\ x,\ y &\in [a_1, b_1] \text{ and } \lambda \in [0,1]\,. \end{aligned}$$

This inequality asserts that the function's value at any point on the line segment between x and y is always larger than or equal to the weighted average of the function's values at x and y. In other words, the function never deviates from the line connecting x and y. The convex function that is closest to a given function $f(x)$ in some way is the convex function that is closest to $f(y)$. It might be the function that minimises the distance between $f(x)$ and the set of convex functions.

GG convex function integral inequalities have many applications in various fields such as optimization, numerical analysis, and approximation theory. In optimization, these inequalities can be used to bound the error between a function and its closest convex function, which can be used to improve the efficiency of optimization algorithms. In numerical analysis, these inequalities can be used to bound the error between a numerical approximation of a function and the true function. In approximation theory, these inequalities can be used to bound the error between a function and its closest convex function.

Fractional Integral

Fractional calculus is a branch of mathematics that deals with the study of derivatives and integrals of fractional order [9]. It is an extension of the traditional calculus of integer order, which deals with derivatives and integrals of integer order. Fractional calculus has applications in various fields such as physics, engineering, and finance. The fractional integral is one of the key ideas of fractional calculus. The definition of the fractional integral is the extension of the conventional integral to non-integer order. It is represented by the symbol $\int\limits_{a_1}^{\alpha}$,where α is the order of the integral and a is the lower limit of integration. There are several different ways to define the fractional integral, but one of the most common ways is to use the Riemann-Liouville fractional integral. The Riemann-Liouville fractional integral of a function $f(x)$ with respect to x is defined as:

$$\int_{a_1}^{\alpha} f\left(x\right) dx = \left(\frac{1}{\Gamma\left(n-\alpha\right)}\right) \int_{a_1}^{b_1} \left(x-a_1\right)^{\left(n-\alpha-1\right)} f\left(x\right) dx.$$

where Γ is the gamma function, n is a positive integer, and α is the order of the integral. The lower limit of integration a and the upper limit of integration b are arbitrary. The Riemann-Liouville fractional integral is a left-sided operator, meaning that it acts on the function to the left of the integral sign.

One of the properties of the fractional integral is that it is a non-local operator, meaning that it depends on the values of the function over an interval rather than at a single point. This is in contrast to the traditional integral, which is a local operator. This non-locality is a consequence of the fact that the fractional integral is defined in terms of a fractional power of the difference between the integration variable and the lower limit of integration.

The fractional derivative is another key idea in fractional calculus. The fractional derivative is the inverse operation of the fractional integral and is defined as a non-integer order extension of the classical derivative. It is represented by the symbol D, where denotes the order of the derivative. The fractional derivative, like the fractional integral, may be defined in a variety of ways, but one of the most frequent is to utilise the Riemann-Liouville fractional derivative. A function f(xRiemann-Liouville

)'s fractional derivative with regard to x is defined as:

$$D^{\alpha} f\left(x\right) = \left(\frac{d}{dx}\right)^{n} \left(\frac{\int\limits_{a_1}^{\alpha} f\left(t\right) dt}{\left(x - a_1\right)^{(n-\alpha)}}\right).$$

where n is a positive integer and α is the order of the derivative. The lower limit of integration a is arbitrary. The Riemann-Liouville fractional derivative is a left-sided operator, meaning that it acts on the function to the left of the derivative sign.

In addition to the Riemann-Liouville fractional integral and derivative, there are other ways to define fractional integrals and derivatives such as the Caputo fractional derivative and the Grunwald-Letnikov fractional derivative. Each of these methods have their own advantages and disadvantages, and the choice of which method to use depends on the specific application.

Riemann-liouville Integrals

Incorrect integrals of the Riemann-Liouville variety are employed in functional analysis, a discipline of mathematics concerned with the study of spaces of functions. These integrals were independently studied in the 19th century by the mathematicians Bernhard Riemann and Joseph Liouville [10], who are honoured by their names. They serve to define the Riemann-Liouville concept of a function being differentiable or integrable.

The basic form of a Riemann-Liouville integral is as follows:

$$\int a_1^{x^n} f(t)dt,$$

Where n is a positive integer and

$$-\infty < a_1 < x < \infty.$$

This integral is defined as the sum of the definite integrals of the function $f(t)$ over the intervals $(a_1, x), (a_1, x-1), (a_1, x-2), ..., (a_1, a_1+n)$ multiplied by the corresponding fractional power of $(x-t)$.

One of the main applications of Riemann-Liouville integrals is in the study of fractional calculus. Fractional calculus is the branch of mathematics that deals with the study of derivatives and integrals of non-integer order. Riemann-Liouville integrals provide a way to define derivatives and integrals of non-integer order, which are known as fractional derivatives and integrals.

Another important application of Riemann-Liouville integrals is in the study of differential equations. These integrals can be used to define the solutions of certain types of differential equations, such as fractional differential equations. These equations are used to model a wide range of physical systems, such as viscoelastic materials and anomalous diffusion processes.

Riemann-Liouville integrals also have applications in other areas of mathematics and engineering, such as signal processing and control theory. In signal processing, these integrals are used to define the concept of fractional order filtering, which is used to analyze signals that have non-integer power spectral density. In control theory, Riemann-Liouville integrals are used to define the solutions of fractional order control systems, which are used to control a wide range of physical systems.

In summary, Riemann-Liouville integrals are a type of improper integral that are used in functional analysis. They provide a way to define derivatives and integrals of non-integer order, which are known as fractional derivatives and integrals. These integrals have important applications in various fields such as fractional calculus, differential equations, signal processing, and control theory. Riemann-Liouville integrals are also used to define the solutions of fractional differential equations, fractional order filtering, and fractional order control systems. These integrals are used to model a wide range of physical systems and to analyze signals that have non-integer power spectral density.

Trapezoid Type Inequalities

Trapezoid type inequalities [11] are a class of mathematical inequalities that relate the difference between a function and its closest convex function to the integral of the function over an interval. These inequalities are named after the shape of the graph of a function and its closest convex function, which resembles a trapezoid. They have important applications in functional analysis, optimization, and other areas of mathematics and engineering.

The basic form of a trapezoid type inequality is as follows:

$$|f(x) - g(x)| \leq (b_1 - a_1)\ M\ \left|f'(x)\right|$$

where $f(x)$ is a function, $g(x)$ is the closest convex function to $f(x), M$ is a constant, and a and b are the endpoints of an interval over which the inequality holds. The inequality states that the difference between the function $f(x)$ and its closest convex function $g(x)$ is bounded by the integral of the absolute value of the first derivative of $f(x)$over the interval (a_1, b_1),multiplied by a constant M.

The concept of the closest convex function to a given function is important in functional analysis. A convex function is a special type of function that has certain properties related to its shape. Specifically, for any two points x and y in the domain of the function and for any λ in the interval$[0, 1]$the following inequality holds:

$$\begin{aligned} f(h(\lambda)x + h(1-\lambda)y) &\leq h(\lambda) f(x) + h(1-\lambda)\; f\;\; (y), \\ \forall x, y &\in [a_1, b_1] \text{ and } \lambda \in [0,1]. \end{aligned}$$

According to this inequality, any point on the line segment between x and y has a function value that is larger than or equal to the weighted average of the function values at x and y, no matter where it is located. The function never deviates from the line that joins x and y, in other words. The convex function that is most similar to $f(x)$ in some way is said to be the closest convex function to the given function $f(x)$. The function that minimises the distance between the set of convex functions and $f(y)$, for instance, could be the answer.Trapezoid type inequalities have many applications in various fields such as optimization, numerical analysis, and approximation theory. In optimization, trapezoid type inequalities are used to bound the error between a function and its closest convex function, which can be used to improve the efficiency of optimization algorithms. In numerical analysis, trapezoid type inequalities are used to bound the error between a numerical approximation of a function and the true function. In approximation theory, trapezoid type inequalities are used to bound the error between a function and its closest convex function.

In addition, trapezoid type inequalities have applications in other areas of mathematics and engineering, such as physics, control theory, and statistics. In physics, trapezoid type inequalities are used to bound the error between a physical system and its mathematical model. In control theory, trapezoid type inequalities are used to bound the error between a control system and its desired behavior. In statistics, trapezoid type inequalities are used to bound the error between a probability distribution and its approximating function.

Furthermore, trapezoid type inequalities have different versions such as, trapezoid type inequalities for second-order derivatives, trapezoid type inequalities for higher-order derivatives, trapezoid type inequalities for higher-dimensional functions, etc. These different versions of trapezoid type inequalities are used to bound the error between a function and its closest convex function in different scenarios and different dimensions

Left and Right-sided RL-Integral

Let $f \in L[a_1, b_1]$. The Left and Right sided Riemann-Lioville integral [12]

$$J_{a_1^+}^{\alpha} f\left(x\right) \quad \text{and} \quad J_{b_1^-}^{\alpha} f\left(x\right)$$

of order $\alpha \geq 0$ are defined by

$$J_{a_1+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{a_1}^{x} (x-\lambda)^{\alpha-1} f(\lambda) d\lambda, \quad x > a_1, \tag{1.2}$$

$$J_{b_1-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{b_1} (\lambda-x)^{\alpha-1} f(\lambda) d\lambda, \quad x < b_1, \tag{1.3}$$

respectively, where $\Gamma(\alpha)$ is the Gamma function and by convension

$$J_{a_1^+}^{0} f(x) = J_{b_1^-}^{0} f(x) = f(x).$$

Hermite–Hadamard–Mercer inequalities for GA-convex Functions

Hermite-Hadamard-Mercer inequalities are a class of mathematical inequalities that relate the integral of a function to the integral of its convex hull [11] over a given interval. These inequalities are named after the mathematicians Charles Hermite, Jacques Hadamard and John Mercer, who independently studied them in the 19th and 20th century. They have important applications in functional analysis, optimization, and other areas of mathematics and engineering, particularly in the context of GA-convex functions.

The basic form of a Hermite-Hadamard inequality for a GA-convex function is as follows:

A GA-convex function is a function that is both geodesically convex and Arithmetic convex. Geodesically convex functions are defined over a Riemannian manifold, and they are convex along any geodesic. Arithmetic convex functions are defined over a vector space, and they are convex with respect to the usual arithmetic operations.

The Hermite-Hadamard inequality is a special case of the more general Hermite-Hadamard-Mercer inequality, which relates the integral of a function to the integral of its convex hull over a given interval. The inequality states that for any function $f(x)$ that is GA-convex on the interval [a,b] and for any convex function $g(x)$ that bounds $f(x)$ on the interval $[a_1, b_1]$ the following inequality holds:

$$\int b_1^{a_1} f(x)\, dx \geq (b_1 - a_1) f\left(\frac{a_1 + b_1}{2}\right).$$

A GA-convex function is a function that is both geodesically convex and Arithmetic convex. Geodesically convex functions are defined over a Riemannian manifold, and they are convex along any geodesic. Arithmetic convex functions are defined over a vector space, and they are convex with respect to the usual arithmetic operations.

This inequality stipulates that the integral of the function over the interval must be larger than or equal to the product of the interval's length and the convex function's value at the interval's midpoint, which functions as the function's boundary.

The Hermite-Hadamard-Mercer inequalities have many important applications in various fields such as optimization, numerical analysis, and approximation theory. In optimization, these inequalities can be used to bound the error between a function and its convex hull, which can be used to improve the efficiency of optimization algorithms. In numerical analysis, these inequalities can be used to bound the error between a numerical approximation of a function and the true function. In approximation theory, these inequalities can be used to bound the error between a function and its closest convex function.

In functional analysis, the Hermite-Hadamard-Mercer inequalities are used to study spaces of GA-convex functions and to understand the properties of these functions. These inequalities provide bounds on the integral of a function and its convex hull, which can be used to understand the smoothness and other properties of the function. Additionally, these inequalities are used to study the properties of functions defined over Riemannian manifolds and vector spaces, which are important in the study of optimization and other areas of mathematics and engineering.

Hadamard Fractional Integral

Let $f \in L[a_1,b_1]$. Then, the left and right sided Hadamard integral [13] $J^{\alpha}_{a_1+}f\left(x\right)$ and $J^{\alpha}_{b_1-}f\left(x\right)$ of order $\alpha \geq 0$ are defined by

$$J^{\alpha}_{a_1+}f\left(x\right) = \frac{1}{\Gamma(\alpha)}\int_{a_1}^{x} \ln\left(\frac{x}{u}\right)^{\alpha-1}\frac{f(u)}{u}du, \tag{1.4}$$

$$J^{\alpha}_{b_1-}f\left(x\right) = \frac{1}{\Gamma(\alpha)}\int_{x}^{b_1} \ln\left(\frac{u}{x}\right)^{\alpha-1}\frac{f(u)}{u}du, \tag{1.5}$$

where

$$\Gamma(\alpha) = \int_{0}^{\infty} e^{-u}u^{\alpha-1}du = (\alpha-1)!,$$

by convension

$$J^{0}_{a_1+}f\left(x\right) = J^{0}_{b_1-}f\left(x\right) = f\left(x\right).$$

Cauchy Schwarz Inequality

If f and g are integrable on (a_1,b_1), then we have the inequality [13]:

$$\int_{a_1}^{b_1} |fg|\, d\alpha \leq \left(\int_{a_1}^{b_1} |f|^2\, d\alpha\right)^{\frac{1}{2}} \left(\int_{a_1}^{b_1} |g|^2\, d\alpha\right)^{\frac{1}{2}}. \tag{1.6}$$

Hölder Inequality

,Hölder's inequality is a fundamental inequality in functional analysis that relates the integral of the product of two functions to the integral of the power of their absolute values [13]. This inequality is named after the German mathematician Otto Hölder, who introduced it in his paper "Über einen Mittelwertsatz" (On a mean value theorem) in 1894. Hölder's inequality has many important applications in various areas of mathematics and engineering, including functional analysis, optimization, and numerical analysis.

where f and g are two functions defined on the interval [a,b], p and q are positive exponents such that $\frac{1}{p}+\frac{1}{q}= 1$, and the integral is taken over the interval [a,b]. The inequality states that the integral of the product of the two functions is bounded by the product of the integral of the power of their absolute values raised to the exponent of $\frac{1}{p}$ and $\frac{1}{q}$.

If f and g are integrable on (a_1,b_1), and let

$$p, q > 0, \quad p > 1, \quad \frac{1}{p} + \frac{1}{q} = 1,$$

then we have the inequality:

$$\int_{a_1}^{b_1} |fg|\, d\alpha \leq \left(\int_{a_1}^{b_1} |f|^p \, d\alpha\right)^{\frac{1}{p}} \left(\int_{a_1}^{b_1} |g|^q \, d\alpha\right)^{\frac{1}{q}}$$

One of the main applications of Hölder's inequality is in the study of spaces of functions, such as Sobolev spaces. These spaces are used to study the smoothness of functions and are important in the study of partial differential equations and other areas of mathematics and engineering. Hölder's inequality is used to define the norm of a function in a Sobolev space and to understand the properties of these spaces.

Hölder's inequality also has important applications in optimization, where it is used to bound the error between a function and its closest convex function. In numerical analysis, Hölder's inequality is used to bound the error between a numerical approximation of a function and the true function. In approximation theory, Hölder's inequality is used to bound the error between a function and its closest convex function.

In addition, Hölder's inequality has many applications in other areas of mathematics and engineering, such as probability and statistics, signal processing, and control theory. In probability and statistics, Hölder's inequality is used to bound the probability of the product of two random variables. In signal processing, Hölder's inequality is used to analyze signals with different types of singularities. In control theory, Hölder's inequality is used to analyze control systems with different types of dynamics.

Minkowski Inequality

If $f, g \in L_p$ on $[a,b]$ [13], then

$$\left| \int_{a_1}^{b_1} |f+g|^p \, d\alpha \right|^{\frac{1}{p}} \leq \left(\int_{a_1}^{b_1} |f|^p \, d\alpha \right)^{\frac{1}{p}} \left(\int_{a_1}^{b_1} |g|^p \, d\alpha \right)^{\frac{1}{p}}, \tag{1.7}$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Monotonic Function

Let $f : X \to [a_1,b_1]$ be a real valued function, then,f is said to be monotonic on $[a_1,b_1]$, [12] if f is either increasing or decreasing.

Bounded Functions

A function f is said to be bounded on $[a_1,b_1]$, [12], if there exists a number M, such that

$$|f(x)| \leq M \ , \quad \text{for all } \ x \in [a_1, b_1].$$

The Space $\mathbf{L}_p[a_1,b_1]$

The space $L_p = L_p[[a_1,b_1]$ consists of all p-Lebesgue integrable functions defined on $[a_1,b_1]$, [12], Thus $f \in L_{p,}$ if and only if:

$$\int_{a_1}^{b_1} |f(x)|^p \, dx < \infty.$$

L_p is a normed space under the normed defined by:

$$||f||_p = \left(\int_{a_1}^{b_1} |f(x)|^p \, dx \right)^{\frac{1}{p}}, \quad f \in L_p.$$

The Space $\mathbf{L}_1[a_1,b_1]$

The space $L_1 = L_1[a_1,b_1]$ consists of all the Lebesgue integrable functions [6] defined on $[a_1,b_1]$.Thus, $f \in L_1$ if and only if:

$$\int_{a_1}^{b_1} |f(x)|\, dx < \infty.$$

L_1 is a normed space under the norm defined by:

$$||f||_1 = \left(\int_{a_1}^{b_1} |f(x)|\, dx \right), \quad f \in L_1.$$

Power Mean Inequality

Let $p \geq 1$ [13]and

$$\frac{1}{p} + \frac{1}{q} = 1.$$

If f and g are real functions defined on $[a_1,b_1]$ and if $|f|$, $|g|^q$ are integral functions on $[a_1,b_1]$,then

$$\int_{a_1}^{b_1} |f(x)\, g(x)|\, dx \leq \left(\int_{a_1}^{b_1} |f(x)|\, dx \right)^{1-\frac{1}{q}} \left(\int_{a_1}^{b_1} |f(x)|\, |g|^q \right)^{\frac{1}{q}}. \tag{1.8}$$

Riemann Integral

If f is Riemann integrable on $[a_1,b_1]$,

$$i.e., \quad f \in R[a_1,b_1],$$

Then we have:

$$\left|\int_{a_1}^{b_1} f(x)\,dx\right| \leq \int_{a_1}^{b_1} |f(x)|\,dx.$$

If f is Riemann integrable on $[a_1, b_1]$ and $\widetilde{c}$ is such that $a_1 \leq \widetilde{c} \leq b_1$, then

$$\int_{a_1}^{b_1} f(x)\,dx = \int_{a_1}^{\widetilde{c}} f(x)\,dx + \int_{a_1}^{b_1} f(x)\,dx.$$

Chapter 2

Review of Literature and Conceptual Framework

In this chapter, we shall discuss different variants of Ostrowski type inequalities existing in the literature. We shall also provide here motivation of this research and a complete framework of this dissertation.

2.1 Existing Literature

- Alomari et. al., [4] proved an inequality for differentiable fuctions on its domain and absolute value of its first derivative is convex(concave) in the form of following theorem:

Theorem 2 *: Let* $f : I \subset (0, \infty) \to \mathbb{R}$ *be a differentiable mapping on* I^o *(the interior* *of I) such that* $f^{'} \in L[a_1, b_1]$ *then* $a_1, b_1 \in I$ *with* $a_1 < b_1$. *If* $\left|f^{'}\right|^{\frac{p}{p-1}}$ *is convex on* $[a_1, b_1]$, *then the following inequality holds:*

$$\left|f(x) - \frac{1}{b_1 - a_1}\int_{a_1}^{b_1} f(x)\,dx\right| \leq \frac{1}{(b_1-a_1)(p+1)^{\frac{1}{p}}}\left[(b_1-x)^2\left(\frac{|(f^{'}(x))^q + (f^{'}(b_1))^q|}{2}\right)^{\frac{1}{q}} + (x-a_1)^2\left(\frac{|(f^{'}(x))^q + (f^{'}(a_1))^q|}{2}\right)^{\frac{1}{q}}\right]^{\frac{1}{2}}, \quad (2.1)$$

$$\leq \frac{1}{(b_1-a_1)(p+1)^{\frac{1}{p}}},$$

for each $x \in [a_1, b_1]$, *when* $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 3 *Let* $f : I \subset (0, \infty) \to \mathbb{R}$ *be a differentiable mapping on* I^o *(the interior of I) such that* $f^{'} \in L[a_1, b_1]$ *then* $a_1, b_1 \in I$ *with* $a_1 < b_1$. *If* $\left|f^{'}\right|^{\frac{p}{p-1}}$ *is concave on* $[a_1, b_1]$, *then the following inequality holds:*

$$\left|f(x) - \frac{1}{b_1 - a_1}\int_{a_1}^{b_1} f(x)\,dx\right| \leq \frac{(b_1-a_1)}{(p+1)^{\frac{1}{p}}}\left[\left(\frac{b_1-x}{b_1-a_1}\right)^2\left|f^{'}\left(\frac{b_1+x}{2}\right)\right| + \left(\frac{x-a_1}{b_1-a_1}\right)^2\left|f^{'}\left(\frac{a_1+x}{2}\right)\right|\right] \quad (2.2)$$

for each $x \in [a_1, b_1]$, *where* $p > 1$.

Theorem 4 *Let* $f : I \subset (0, \infty) \to \mathbb{R}$ *be a differentiable mapping on* I^o *(the interior of I) such that* $f^{'} \in L[a_1, b_1]$ *then* $a_1, b_1 \in I$ *with* $a_1 < b_1$. *If* $\left|f^{'}\right|^q$ *is concave on* $[a_1, b_1]$, $q \geq 1$ *and* $|f'(x)| \leq M$, *then the following inequality holds:*

$$\left|f(x) - \frac{1}{b_1 - a_1}\int_{a_1}^{b_1} f(x)\,dx\right| \leq \frac{(b_1-a_1)}{2}\left[\left(\frac{b_1-x}{b_1-a_1}\right)^2\left|f^{'}\left(\frac{b_1+2x}{3}\right)\right| + \left(\frac{x-a_1}{b_1-a_1}\right)^2\left|f^{'}\left(\frac{a_1+2x}{3}\right)\right|\right] \quad (2.3)$$

In [12] Yildiz et. al., proved an inequality as follows:

Theorem 5 *Let $f: I \to \mathbb{R}$ be a differentiable mapping on I^o (the interior of I) such that $f' \in L[a_1, b_1]$ then $a_1, b_1 \in I$ with $a_1 < b_1$. If $|f'|$is convex on $[a_1, b_1]$ and $x \in [a_1, b_1]$, then the following inequality for fractional integral with $\alpha > 0$ holds:*

$$\begin{aligned} &\left| \left[\frac{(x-a_1)^\alpha + (b_1-x)^\alpha}{(b_1-a_1)\alpha + 1} \right] f(x) - \frac{\Gamma(\alpha+1)}{(b_1-x)^{\alpha+1}} \left[J^\alpha_{x+} f(b_1) + J^\alpha_{x-} f(a_1) \right] \right| \\ \leq\ & \frac{1}{\alpha+2} \left\{ \left(\frac{(b_1-x)^{\alpha+2}}{(b_1-a_1)^{\alpha+2}} + \frac{(x-a_1)^{\alpha+1}}{(b_1-a_1)^{\alpha+1}} \left[\frac{1}{\alpha+1} + \frac{b_1-x}{b_1-a_1} \right] \right) \left| f'(a_1) \right| \right. \\ &\left. + \left(\frac{(x-a_1)^{\alpha+2}}{(b_1-a_1)^{\alpha+2}} + \frac{(b_1-x)^{\alpha+1}}{(b_1-a_1)^{\alpha+1}} \left[\frac{1}{\alpha+1} + \frac{x-a_1}{b_1-a_1} \right] \right) \left| f'(b_1) \right| \right\}, \qquad (2.4) \end{aligned}$$

where Γ is Euler Gamma Function.

Theorem 6 *Let $f: [a_1, b_1] \to \mathbb{R}$,be a differentiable mapping on (a_1, b_1) with $a_1 < b_1$ such that $f' \in L[a_1, b_1]$. If $|f'|^q$ is convex on $[a_1, b_1]$, $q > 1$and $x \in [a_1, b_1]$, then the following inequalities for fractional integrals holds:*

$$\begin{aligned} &\left| \left[\frac{(x-a_1)^\alpha + (b_1-x)^\alpha}{(b_1-a_1)^{\alpha+1}} \right] f(x) - \frac{\Gamma(\alpha+1)}{(b_1-a_1)^{\alpha+1}} \left[J^\alpha_{x+} f(b_1) + J^\alpha_{x-} f(a_1) \right] \right| \\ \leq\ & \frac{1}{(b_1-a_1)^{\alpha+1} (\alpha p + 1)^{\frac{1}{p}}} \left[(b_1-x)^{\alpha+1} \left(\frac{|f'(x)|^q + |f'(b_1)|^q}{2} \right)^{\frac{1}{q}} \right. \\ &\left. - (x-a_1)^{\alpha+1} \left(\frac{|f'(x)|^q + |f'(a_1)|^q}{2} \right) \right], \qquad (2.5) \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1, \alpha > 0.$

- In 2012, Set [14] obtained some new inequalities of Ostrowski type for mappings whose derivatives are s-convex in the second sense via fractional integrals. These fractional Ostrowski type inequalities are given below in the form of following theorem:

Theorem 7 *Let $f:[a_1,b_1]\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on (a_1,b_1) with $a_1<b_1$ such that $f^{'}\in L[a_1,b_1]$. If $\left|f^{'}\right|^q$is s-concave in the second sense on $[a_1,b_1]$ for some fixed $s\in(0,1]$ and $p,q>1$, then the following inequality for fractional integral holds:*

$$\begin{aligned}&\left|\left[\frac{(x-a_1)^{\alpha}+(b_1-x)^{\alpha}}{(b_1-a_1)}\right]f(x)-\frac{\Gamma(\alpha+1)}{(b_1-a_1)}\left[J_{x-}^{\alpha}f(b_1)+J_{x+}^{\alpha}f(a_1)\right]\right|\\ \le\ &\frac{2^{(s-1)/q}}{(1+p\alpha)^{\frac{1}{p}}(b_1-a_1)}\left[(x-a_1)^{\alpha+1}\left|f^{'}\left(\frac{x+a_1}{2}\right)\right|+(b_1-x)^{\alpha+1}\left|f^{'}\left(\frac{b_1+x}{2}\right)\right|\right],\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $\alpha>0$ and Γ is Euler Gamma Function.

- In 2012, Liu [5] established some Ostrowski type inequalities involving RL-fractional integral for h-convex function. He also provided new estimates of Ostrowski type inequalities for fractional integrates. In this paper, some Ostrowski type inequalities via Riemann-Liouville fractional integrals for h-convex functions, which are super-multiplicative or super-additive,are given. These results not only generalize those of [14], but also provide new estimates on these types of Ostrowski inequalities for fractional integrals.

Theorem 8 *Let $h:J\subseteq\mathbb{R}\to\mathbb{R}\,([0,1]\subseteq J)$ be a non-negative and super-multiplicative function, $h(t)\geq t$ for $0\leq t\leq 1$, $f:[a_1,b_1]\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on (a_1,b_1) with $a_1<b_1$ such that $f^{'}\in L_1[a_1.b_1]$.*

If $\left|f'\right|$ *is h-convex on* $[a_1, b_1]$ *and* $\left|f'(x)\right| \leq M,\ x \in [a_1, b_1]$, *then the following inequalities for fractional integral with* $\alpha > 0$ *holds:*

$$\left|\left(\frac{(x-a_1)^{\alpha}+(b_1-x)^{\alpha}}{b_1-a_1}\right) f(x) - \frac{\Gamma(\alpha+1)}{b_1-a_1}\left[J_{x^-}^{\alpha} f(a_1) + J_{x^+}^{\alpha} f(b_1)\right]\right|$$

$$\leq \frac{M\left[(x-a_1)^{\alpha+1}+(b_1-x)^{\alpha+1}\right]}{b_1-a_1}\int_0^1 \left[t^{\alpha} h(t) + t^{\alpha} h(1-t)\right] dt \tag{2.6}$$

$$\leq \frac{M\left[(x-a_1)^{\alpha+1}+(b_1-x)^{\alpha+1}\right]}{b_1-a_1}\int_0^1 \left[h\left(t^{\alpha+1}\right) + h\left(t^{\alpha}(1-t)\right)\right] dt. \tag{2.7}$$

In Theorem 1, if we choose h(t) = t, then (2.6) and (2.7) reduce the inequality in [14].

Corollary 9 *Let* $h : J \subseteq \mathbb{R} \rightarrow \mathbb{R}\,([0,1] \subseteq J)$ *be a non-negative and super-additive function,and* $f : [a_1, b_1] \subset [0, \infty) \rightarrow \mathbb{R}$ *be a differentiable mapping on* (a_1, b_1) *with* $a_1 < b_1$ *such that* $f' \in L_1[a_1, b_1]$. *If* $\left|f'\right|^q$ *is h-convex function on* $[a_1, b_1]$, $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, *and* $\left|f'(x)\right| \leq M,\ x \in [a_1, b_1]$ *then the following inequalities for the fractional integral with* $\alpha > 0$ *holds:*

$$\left|\left[\frac{(x-a_1)^{\alpha}+(b_1-x)^{\alpha}}{b_1-a_1}\right] f(x) - \frac{\Gamma(\alpha+1)}{b_1-a_1}\left[J_{x^-}^{\alpha} f(a_1) + J_{x^+}^{\alpha} f(b_1)\right]\right|$$

$$\leq \frac{M\left[(x-a_1)^{\alpha+1}+(b_1-x)^{\alpha+1}\right]}{(1+p\alpha)^{\frac{1}{p}}(b_1-a_1)}\left(\int_0^1 \left[h(t) + h(1-t)\right]\right)^{\frac{1}{q}}$$

$$\leq \frac{M\left[(x-a_1)^{\alpha+1}+(b_1-x)^{\alpha+1}\right]}{(1+p\alpha)^{\frac{1}{p}}(b_1-a_1)} h^{\frac{1}{q}}(1). \tag{2.8}$$

Corollary 10 *If we choose $h(t) = t^s, s \in (0,1]$, in Theorem 8, then we have*

$$\left| \left[\frac{(x-a_1)^\alpha + (b_1-x)^\alpha}{b_1-a_1} \right] f(x) - \frac{\Gamma(\alpha+1)}{b_1-a_1} \left[J_{x^+}^\alpha f(b_1) + J_{x^-}^\alpha f(a_1) \right] \right|$$
$$\leq \frac{M}{(1+p\alpha)^{\frac{1}{p}}} \left(\frac{2}{s+1} \right)^{\frac{1}{q}} \frac{(x-a_1)^{\alpha+1} + (b_1-x)^{\alpha+1}}{b_1-a_1}, \tag{2.9}$$

due to the fact that

$$\int_0^1 [h(t) + h(1-t)]\, dt = \frac{2}{s\ +1}. \tag{2.10}$$

This is the inequality established in ([15], Theorem 8.)

- In 2014, Alginevic [16] gave Ostrowski type inequalities for fractional integrals for function whose fractional derivatives belong to Lp spaces in the form of following theorem:

Theorem 11 *Suppose that all the assumptions of Theorem 1 holds. Additionally, assume that (p,q) is a pair of conjugate exponents; that is 1$\leq p, q \leq \infty, (1/p) + (1/q) = 1$, and $f' \in L_p[a_1, b_1]$. Then the following inequalities holds:*

$$\left| f(x) - \frac{\Gamma(\alpha+1)}{(g(b_1) - g(a_1))^\alpha} J_{b_1;g}^\alpha f(b_1) \right| \leq ||P_4(x,t)||_q \, ||f'||_{p'} \tag{2.11}$$

where the q-norm is calculated with respect to variable t.

The constant $||P_4(x,t)||_q$ is sharp for 1< $p \leq \infty$ and the best possible for p=1, where

$$P_4(x,t) = \begin{cases} 1 - \left(\frac{g(b_1)-g(t)}{g(b_1)-g(a_1)} \right)^\alpha & a_1 \leq t \leq x, \\ -\left(\frac{g(b_1)-g(t)}{g(b_1)-g(a_1)} \right)^\alpha & x < t \leq b_1, \end{cases} \tag{2.12}$$

$${}_{a_1}J_{x;g}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_{a_1}^x (g(x) - g(t))^{\alpha-1} g'(t) f(t)\, dt$$

is the following integral of order a > 0 of a function f with respect to another function g.

- In 2017, Farid, G. [3] found a new version of Ostrowski type inequalities for RL-fractional integrals.

Fractional Ostrowski Type Inequality. Remaining within the assumption of Ostrowski inequality following more general inequality is observed.

Theorem 12 *Under the assumption of Theorem 1 we have:*

$$\begin{aligned} & \left| f(x)\left((b_1-a_1)^{\beta}+(x-a_1)^{\alpha}\right)-\left(\Gamma(\beta+1)\, I_{b_1^-}^{\beta} f(x)+\Gamma(\alpha+1)\, I_{a_1^+}^{\alpha} f(x)\right)\right| \\ \leq\ & M\left(\frac{\beta}{\beta+1}(b_1-a_1)^{\beta+1}+\frac{\alpha}{\alpha+1}(x-a_1)^{\alpha-1}\right), \quad x\in[a_1,b_1]. \end{aligned} \tag{2.13}$$

where $\alpha, \beta > 0$.

Theorem 13 *Let* $f : I \to \mathbb{R},$ *where* I *is an interval in* $\mathbb{R}$ *be a differentiable in* I^0*,the inyerior of* I *and* $a,\ b \in I^0$, $a_1 < b_1$. *If* $m < f'(t) \leq M$ *for all* $t \in [a_1, b_1]$ *,then we have*

$$\begin{aligned} & \left((x-a_1)^{\alpha}+(b_1-x)^{\beta}\right) f(x)-\left(\Gamma(\alpha+1)\, I_{a_1^+}^{\alpha} f(x)+\Gamma(\beta+1)\, I_{b_1^-}^{\beta} f(x)\right) \\ <\ & \frac{M\alpha}{\alpha+1}(x-a_1)^{\alpha-1}-\frac{m\beta}{\beta+1}(b_1-a_1)^{\beta+1}, \quad x\in[a_1,b_1]. \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} & \left((x-a_1)^{\alpha}+(b_1-x)^{\beta}\right) f(x)-\left(\Gamma(\alpha+1)\, I_{a_1^+}^{\alpha} f(x)+\Gamma(\beta+1)\, I_{b_1^-}^{\beta} f(x)\right) \\ \leq\ & M\left(\frac{\beta}{\beta+1}(b_1-a_1)^{\beta+1}+\frac{\alpha}{\alpha+1}(x-a_1)^{\alpha-1}\right), \quad x\in[a_1,b_1]. \end{aligned} \tag{2.15}$$

where $\alpha, \beta > 0$.

Theorem 14 *Under the assumption of Theorem 1 we have*

$$\begin{aligned}&\left|\left((b_1-x)^{\beta}+(x-a_1)^{\alpha}\right)-\left(\Gamma(\beta+1)\,I^{\beta}_{x^+}f(b_1)+\Gamma(\alpha+1)\,I^{\alpha}_{x^-}f(a_1)\right)\right|\\ \leq\ & M\left(\frac{\beta}{\beta+1}(b_1-x)^{\beta+1}+\frac{\alpha}{\alpha+1}(x-a_1)^{\alpha-1}\right),x\ \in[a_1,b_1] \qquad (2.16)\end{aligned}$$

where $\alpha,\beta>0$.

- In 2017, Dragomir [17], introduced several generalizations of Ostrowski type inequalities involving RL fractional integrals of bounded variations and of Hőlder continuous functions in the form of following theorem:

Theorem 15 *Assume that* $f:[a_1,b_1]\to\mathbb{C}$ *is r-H- HÖlder continuous functions on* $[a_1,b_1]$ *with* $r\in(0,1]$ *and* $H>0$ *and* g *is a strictly increasing on* (a_1,b_1), *having a continous derivative* $g^{'}$*on* (a_1,b_1). *Then,*

(i) For any $x\,\epsilon\,(a_1,b_1)$, *we have the inequalites:*

$$\begin{aligned}&\left|I^{\beta}_{a_1^+,g}f(x)+I^{\alpha}_{b_1^-,g}f(x)-\frac{1}{\Gamma(\alpha+1)}\left([g(x)-g(a_1)]^{\alpha}+[g(b_1)-g(x)]^{\alpha}\right)f(x)\right|\\ \leq\ & \frac{H}{\Gamma(\alpha)}\left[\int_{\alpha}^{x}\frac{g^{'}(t)(x-t)^{r}\,dt}{[g(x)-g(t)]^{1-\alpha}}+\int_{x}^{b_1}\frac{g^{'}(t)(t-x)^{r}\,dt}{[g(t)-g(x)]^{1-\alpha}}\right]\\ \leq\ & \frac{H}{\Gamma(\alpha+1)}\left|[g(x)-g(a_1)]^{\alpha}(x-a_1)^{r}+[g(b_1)-g(x)]^{\alpha}(b_1-x)^{r}\right|, \qquad (2.17)\end{aligned}$$

and

$$\begin{aligned}&\left|I^{\beta}_{x^-,\,g}\ f(a_1)+I^{\alpha}_{x^+,\,g}\ f(b_1)-\frac{1}{\Gamma(\alpha+1)}\left([g(x)-g(a_1)]^{\alpha}+[g(b_1)-g(x)]^{\alpha}\right)f(x)\right|\\ \leq\ & \frac{H}{\Gamma(\alpha)}\left[\int_{\alpha}^{x}\frac{g^{'}(t)(x-t)^{r}\,dt}{[g(t)-g(a_1)]^{1-\alpha}}+\int_{x}^{b_1}\frac{g^{'}(t)(t-x)^{r}\,dt}{[g(b_1)-g(t)]^{1-\alpha}}\right]\\ \leq\ & \frac{H}{\Gamma(\alpha+1)}\left|[g(x)-g(a_1)]^{\alpha}(x-a_1)^{r}+[g(b_1)-g(x)]^{\alpha}(b_1-x)^{r}\right| \qquad (2.18)\end{aligned}$$

(ii) We have the inequalities

$$\left|\frac{I^{\beta}_{b_1^-,\,g}\ f(a_1)+I^{\alpha}_{a_1^+,g}\ f(b_1)}{2}-\frac{1}{\Gamma(\alpha+1)}[g(b_1)-g(a_1)]^{\alpha}\frac{f(b_1)+f(a_1)}{2}\right|$$
$$\leq \frac{H}{2\Gamma(\alpha)}\left[\int_{\alpha}^{b_1}\frac{g'(t)(b_1-t)^r\,dt}{[g(b_1)-g(t)]^{1-\alpha}}+\int_{a_1}^{b_1}\frac{g'(t)(t-a_1)^r\,dt}{[g(t)-g(x)]^{1-\alpha}}\right]$$
$$\leq \frac{H}{\Gamma(\alpha+1)}(b_1-a_1)^r[g(b_1)-g(a_1)]^{\alpha}. \tag{2.19}$$

Corollary 16 *With the assumption of Theorem 15, we have:*

$$\left|I^{\alpha}_{a_1|,g}\ f(M_g(c,b_1))+I^{\alpha}_{b_1|,g}\ f(M_g(a_1,b_1))-\frac{[g(b_1)-g[a_1]^{\alpha}]}{2^{\alpha-1}\Gamma(\alpha+1)}f(M_g(a_1,b_1))\right|$$
$$\leq \frac{H}{\Gamma(\alpha)}\left[\int_{c}^{M_g(a_1,b_1)}\frac{g'(t)(M_g(a_1,b_1)-t)^r\,dt}{[g(M_g(a_1,b_1))-g(t)]^{1-\alpha}}+\int_{M_g(a_1,b_1)}^{b_1}\frac{g'(t)(t-M_g(a_1,b_1))^r\,dt}{[g(t)-g(M_g(a_1,b_1))]^{1-\alpha}},\right]$$
$$\leq \frac{H}{2^{\alpha}\Gamma(\alpha+1)}[g(b_1)-g(a_1)]^{\alpha}[(M_g(a_1,b_1)-a_1)^r+(b_1-M_g(a_1,b_1))^r],$$

and

$$\left|I^{\alpha}_{M_g(a_1,b_1)-g}\ f(a_1)+I^{\alpha}_{M_g(a_1,b_1)+g}\ f(b_1)-\frac{[g(b_1)-g[a_1]^{\alpha}]}{2^{\alpha-1}\Gamma(\alpha+1)}f(M_g(a_1,b_1))\right|$$
$$\leq \frac{H}{\Gamma(\alpha)}\left[\int_{c}^{M_g(a_1,b_1)}\frac{g'(t)(M_g(a_1,b_1)-t)^r\,dt}{[g(t)-g(a_1)]^{1-\alpha}}+\int_{M_g(a_1,b_1)}^{b_1}\frac{g'(t)(t-M_g(a_1,b_1))^r\,dt}{[g(b_1)-g(t)]^{1-\alpha}}\right]$$
$$\leq \frac{H}{2^{\alpha}\Gamma(\alpha+1)}[g(b_1)-g(a_1)]^{\alpha}[(M_g(a_1,b_1)-a_1)^r+(b_1-M_g(a_1,b_1))^r]. \tag{2.20}$$

It has also obserevd that the inequalities have applications for the geometric mean that is

$$for\ x\ =\ G(a_1,b_1)\ =\sqrt{a_1b_1},$$

of two numbers. Moreover, some particular cases for Hadamard fractional integrals are also discussed in [17].

- In 2018, Yildiz [18] and Set obtained some new Ostrowski type inequalities for generalized fractional integral operators as follows:

Theorem 17 *Let* $f : [a_1, b_1] \subseteq \mathbb{R}$ *be a differentiable on* (a_1, b_1) *and* $\left|f'(x)\right| \leq M,$ *for every* $x \in [a_1, b_1]$*. Then, the following generalized Ostrowski type fractional integral operators inequality holds:*

$$\left|\frac{\varphi(b_1 - a_1)}{b_1 - x} f(x) - {}_{a_1^+} I_{\varphi}(P_1(x, b_1) f(b_1)) - \frac{1}{(b_1 - a_1)}\ {}_{a+}I_{\varphi} f(b_1)\right| \leq \Im \tag{2.21}$$

where $P_1(x,t)$ *is the Peano kernal and*

$$\Im := \frac{M}{b_1 - a_1}\left[\int_{a_1}^{x}\left|\frac{\varphi(b_1 - t)}{b_1 - t}\right|(t - a_1)dt - \int_{x}^{b_1}\left|\frac{\varphi(b_1 - t)}{b_1 - t}\right|(b_1 - t)dt\right]. \tag{2.22}$$

We shall here, however generalize Ostrowski type inequalities for GA- convex functions involving fractional integrals. We shall further establish some new inequality which will be generalization of many Ostrowski type inequalities existing in the literature.

- In 2012, Set [14] obtained some new inequalities of Ostrowski type for mappings whose derivatives are s-convex in the second sense via fractional integrals. These fractional Ostrowski type inequalities are given below in the form of following theorem:

Theorem 18 *Let* f *:*$[a_1,\ b_1] \subset [0, \infty) \rightarrow \mathbb{R}$ *be a differentiable mapping on* (a_1, b_1) *with* $a_1 < b_1$ *such that* $f' \in L[a_1, b_1]$*. If* $\left|f'\right|^q$ *is s-concave in the seconed sense on* $[a_1, b_1]$

for some fixed $s \in (0,1]$ *and* $p, q > 1$, *then the following inequality for fractional integral holds:*

$$\begin{aligned}
&\left| \left[\frac{(x-a_1)^{\alpha} + (b_1 - x)^{\alpha}}{(b_1 - a_1)} \right] f(x) - \frac{\Gamma(\alpha+1)}{(b_1 - a_1)} \left[J_{x-}^{\alpha} f(b_1) + J_{x+}^{\alpha} f(a_1) \right] \right| \\
\leq\ & \frac{2^{(s-1)/q}}{(1+p\alpha)^{\frac{1}{p}} (b_1 - a_1)} \left[(x-a_1)^{\alpha+1} \left| f'\left(\frac{x+a_1}{2} \right) \right| \right. \\
& \left. + (b_1 - x)^{\alpha+1} \left| f'\left(\frac{b_1 + x}{2} \right) \right| \right]. \qquad (2.23)
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha > 0$ *and* Γ *is Euler Gamma Function.*

We shall here, however generalize Ostrowski type inequalities for GA- convex functions involving fractional integrals. We shall further establish some new inequality which will be generalization of many Ostrowski type inequalities existing in the literature.

2.2 General Concept

Some authors proved Ostrowski type inequality for the class of convex function via fractional integral [5, 18].

Motivated by the above results, a new identity for a differentiable, GA-convex function is established. With the aid of this identity, we create inequalities of the Ostrowski type for fractional integral. In order to account for this, we generalise Ostrowski type inequalities for GA-convex first and nth differentiable bounded functions for fractional integral. As a result, some applications are also provided in the sections that follow. A power mean inequality of the Ostrowski type is also established at the conclusion.

- In chapter-3, a new identity for Ostrowski type inequality for GA-convex function is established. Then, two versions of Ostrowski type inequality for GA-convex differentiable and bounded function for fractional integral are also developed.
- In chapter-4, these inequalities are generalized to n^{th} differentiable GA-convex bounded function and application to special means is also discussed.
- In chapter-5, Ostrowski type inequalities for first time differentiable and n-time differentiable GA-convex function fractional integral are established using power mean inequality. The inequalities of chapter-3 and chapter-4 are its special cases.
- At the end, some conclusions and recommendations for further research work are provided in chapter 6.

Chapter 3

Ostrowski Type Inequalities for Fractional Integral

In recent years, researchers have made numerous modifications in classical Ostrowski type integral inequality and applied it to reduce error bound in numerical integration.

Here, in this chapter new identity is constructed for Ostrowski type inequality for GA-convex function which helps us to develop two new versions of Ostrowski type inequality for GA-convex differentiable and bounded function for fractional integral.

3.1 A New Identity for Ostrowski Type Inequality for GA-Convex Function

To prove our main results, the following identity for Lebesgue integrable functions defined on $[a,b]$ is established in the form of a lemma as follows:

Lemma 19 *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable mapping on I° (I° is interior of I) such that $f(x) \in L[a_1,b_1]$, then*

$$f(x) = \frac{1}{(\ln b_1 - \ln a_1)} \left[J_{b_1^-} f(x) - J_{a_1^+} f(x) \right] \tag{3.1}$$

Proof. Let

$$p(\lambda) = \begin{cases} \lambda, & \lambda \in \left[0, \frac{\ln(b_1/x)}{\ln(b_1/a_1)}\right] \\ \lambda - 1, & \lambda \in \left(\frac{\ln(b_1/x)}{\ln(b_1/a_1)}, 1\right] \end{cases} \tag{3.2}$$

then, applying definition of function (3.2) in the integral we have:

$$\begin{aligned} I &= \int_0^1 p(\lambda)\, df\left(a_1^\lambda\ b_1^{1-\lambda}\right) \\ &= \int_1^{\ln(b_1/x)\ /\ \ln(b_1/a_1)} \lambda df\left(a_1^\lambda\ b_1^{1-\lambda}\right) + \int_{\ln(b_1/x)/\ln(b_1/a_1)}^1 (\lambda - 1)\, df\left(a_1^\lambda\ b_1^{1-\lambda}\right). \end{aligned} \tag{3.3}$$

On integrating (3.3) by parts we have:

$$\begin{aligned} I = \; & \lambda f\left(a_1^\lambda\ b_1^{1-\lambda}\right)\Big|_0^{\ln(b_1/x)/\ln(b_1/a_1)} - \int_0^{\ln(b_1/x)/\ln(b_1/a_1)} f\left(a_1^\lambda\ b_1^{1-\lambda}\right) d\lambda \\ & + (\lambda - 1)\, f\left(a_1^\lambda\ b_1^{1-\lambda}\right)\Big|_{\ln(b_1/x)/\ln(b_1/a_1)}^1 - \int_{\ln(b_1/x)/\ln(b_1/a_1)}^1 f\left(a_1^\lambda b_1^{1-\lambda}\right) d\lambda \end{aligned}$$

On applying limit of integrals, we have:

$$\begin{aligned} I &= \left[\ln\left(b_1/x\right)/\ln\left(b_1/a_1\right) - \ln\left(a_1/x\right)/\ln\left(b_1/a_1\right)\right] f\left(x\right) \\ &\quad - \int_0^{\ln(b_1/x)/\ln(b_1/a_1)} f\left(a_1^{\lambda} b_1^{1-\lambda}\right) d\lambda - \int_{\ln(b_1/x)/\ln(b_1/a_1)}^{1} f\left(a_1^{\lambda} b_1^{1-\lambda}\right) d\lambda \\ &= f\left(x\right) - \int_0^{\ln(b_1/x)/\ln(b_1/a_1)} f\left(a_1^{\lambda}\ b_1^{1-\lambda}\right) d\lambda - \int_{\ln(b_1/x)/\ln(b_1/a_1)}^{1} f\left(a_1^{\lambda}\ b_1^{1-\lambda}\right) d\lambda \quad (3.4) \end{aligned}$$

Changing variables, we have:

$$u = a_1 \quad {}^{\lambda}\ {b_1}^{1-\lambda}$$

$$\begin{aligned} \text{if } \lambda &= 0, \text{ then } u = b_1 \\ \text{and if } \lambda &= 1, \text{ then } u = a_1. \end{aligned}$$

Further,

$$\begin{aligned} \ln u &= \ln\left(a_1^{\lambda}\ b_1^{1-\lambda}\right) \\ &= \lambda \ln a_1 + (1-\lambda)\ln b_1, \end{aligned}$$

this implies

$$\begin{aligned} \ln u &= \lambda \ln\left(a_1/b_1\right) + \ln b_1 \\ \frac{1}{u} du &= \ln\left(a_1/b_1\right) d\lambda \\ d\lambda &= \frac{1}{u \ln\left(a_1/b_1\right)} du. \end{aligned}$$

Therefore, from (3.4) we have:which is the required identity. ■

3.2 Ostrowski Type Inequality for Fractional Integral (Version-I)

Consider, $f^{'}$ is GA-convex and bounded on interval $[a_1,b_1]$ for arbitrary differentiable mapping f with $a_1 < b_1$. Then, we have first version of Ostrowski type inequality for fractional integrals in the form of following theorem:.

Theorem 20 *Let $f : I \to \mathbb{R}$ be a differentiable mapping on I^o then a_1, $b_1 \in I$ with $a_1 < b_1$, such that $f^{'} \in L[a_1, b_1]$. If $f^{'}$ is GA-convex on $[a_1,b_1]$ and bounded, then:*

$$\begin{aligned} & \left| f(x) - \frac{1}{\ln(b_1/x)/\ln(b_1/a_1)} \left[J_{b_1^-} f(b_1) + J_{a_1^+} f(a_1) \right] \right| \\ \leq & \; b_1 \ln(a_1/b_1) \left[B(a_1/b_1, 2, 0) + C(a_1/b_1, 1, 1) \left| f^{'}(a_1) \right| \right. \\ & \left. + B(a_1/b_1, 1, 1) + C(a_1/b_1, 0, 2) \left| f^{'}(b_1) \right| \right], \end{aligned} \tag{3.5}$$

where,

$$\begin{aligned} B(u,m,n) &= \int_0^{\ln(b_1/x)/\ln(b_1/a_1)} u^{\lambda} \lambda^m (\lambda - 1)^n \, d\lambda, \\ C(u,m,n) &= \int_{\ln(b_1/x)/\ln(b_1/a_1)}^{1} u^{\lambda} \lambda^m (\lambda - 1)^n \, d\lambda \end{aligned}$$

and

$$\left| f^{'}(a_1) \right| \leq M_1, \quad \left| f^{'}(b_1) \right| \leq M_2.$$

Proof. Now,

$$
\begin{aligned}
I &= \int\limits_{0}^{\ln(b_1/x)/\ln(b_1/a_1)} \lambda df\left(a_1^{\lambda}\ b_1^{1-\lambda}\right) + \int\limits_{\ln(b_1/x)/\ln(b_1/a_1)}^{1} (\lambda-1)\, df\left(a_1^{\lambda}\ b_1^{1-\lambda}\right) \\
&= \int\limits_{0}^{\ln(b_1/x)/\ln(b_1/a_1)} \lambda f^{'}\left(a_1^{\lambda}\ b_1^{1-\lambda}\right) d\left(a_1^{\lambda}\ b_1^{1-\lambda}\right) \\
&\quad + \int\limits_{\ln(b_1/x)/\ln(b_1/a_1)}^{1} (\lambda-1)\, f^{'}\left(a_1^{\lambda}\ b_1^{1-\lambda}\right) d\left(a_1^{\lambda}\ b_1^{1-\lambda}\right) \\
&= \int\limits_{0}^{\ln(b_1/x)/\ln(b_1/a_1)} \lambda f^{'}\left(a_1^{\lambda}\ b_1^{1-\lambda}\right)\left(a_1^{\lambda}\ b_1^{1-\lambda}\right)(\ln a_1 - \ln b_1)\, d\lambda \\
&\quad + \int\limits_{\ln(b_1/x)/\ln(b_1/a_1)}^{1} (\lambda-1)\, f^{'}\left(a_1^{\lambda}\ b_1^{1-\lambda}\right)\left(a_1^{\lambda}\ b_1^{1-\lambda}\right)(\ln a_1 - \ln b_1)\, d\lambda \\
&= b_1 \ln(a_1/b_1) \int\limits_{0}^{\ln(b_1/x)/\ln(b_1/a_1)} (a_1/b_1)^{\lambda}\, \lambda f^{'}\left(a_1^{\lambda}\ b_1^{1-\lambda}\right) d\lambda \\
&\quad + b_1 \ln(a_1/b_1) \int\limits_{\ln(b_1/x)/\ln(b_1/a_1)}^{1} (a_1/b_1)^{\lambda}\, (\lambda-1)\, f^{'}\left(a_1^{\lambda}\ b_1^{1-\lambda}\right) d\lambda.
\end{aligned}
$$

This implies

$$\begin{aligned}
|I| &\leq b_1 \ln(a_1/b_1) \left(\int_0^{\ln(b_1/x)/\ln(b_1/a_1)} (a_1/b_1)^\lambda \lambda. \left[\lambda \left|f'(a_1)\right| + |1-\lambda| \left|f'(b_1)\right|\right] d\lambda \right. \\
&\quad \left. + \int_{\ln(b_1/x)/\ln(b_1/a_1)}^{1} (a_1/b_1)^\lambda |\lambda - 1| . \left[\lambda \left|f'(a_1)\right| + |1-\lambda| \left|f'(b_1)\right|\right] d\lambda \right) \\
&= b_1 \ln(a_1/b_1) \left[\int_0^{\ln(b_1/x)/\ln(b_1/a_1)} \left((a_1/b_1)^\lambda \lambda^2 |f'(a_1)| + (a_1/b_1)^\lambda \lambda |\lambda - 1| + |f'(b_1)|\right) d\lambda \right. \\
&\quad \left. + \int_{\ln(b_1/x)/\ln(b_1/a_1)}^{1} \left((a_1/b_1)^\lambda \lambda |\lambda - 1| |f'(a_1)| + (a_1/b_1)^\lambda (\lambda-1)^2 \left|f'(b_1)\right| d\lambda\right) \right], \\
&= b_1 \ln(a_1/b_1) \left[\left(\int_0^{\ln(b_1/x)/\ln(b_1/a_1)} (a_1/b_1)^\lambda \lambda^2 d\lambda + \int_{\ln(b_1/x)/\ln(b_1/a_1)}^{1} \left((a_1/b_1)^\lambda \lambda |\lambda - 1| d\lambda\right) \right) |f'(a_1)| \right. \\
&\quad \left. + \begin{pmatrix} \int_0^{\ln(b_1/x)/\ln(b_1/a_1)} ((a_1/b_1)^\lambda \lambda |\lambda - 1|) d\lambda \\ + \int_{\ln(b_1/x)/\ln(b_1/a_1)}^{1} \left((a_1/b_1)^\lambda |\lambda - 1|^2\right) d\lambda \end{pmatrix} |f'(b_1)| \right]. \qquad (3.6)
\end{aligned}$$

If

$$\begin{aligned}
B(u, m, n) &= \int_0^{\ln(b_1/x)/\ln(b_1/a_1)} u^\lambda \lambda^m (\lambda - 1)^n d\lambda, \\
C(u, m, n) &= \int_{\ln(b_1/x)/\ln(b_1/a_1)}^{1} u^\lambda \lambda^m (\lambda - 1)^n d\lambda,
\end{aligned}$$

then

$$\begin{aligned}
I &\leq b_1 \ln(a_1/b_1) \left[\{B(a_1/b_1, 2, 0) + C(a_1/b_1, 1, 1)\} \left|f'(a_1)\right| \right. \\
&\quad \left. + \{B(a_1/b_1, 1, 1) + C(a_1/b_1, 0, 2)\} \left|f'(b_1)\right| \right].
\end{aligned}$$

Combining (3.1) and (3.6) give us the required result (3.5) i.e.,

$$\begin{aligned} |I\ | \quad &\leq \quad b_1 \ln(a_1/b_1)\left([B(a_1/b_1,2,0)+C(a_1/b_1,1,1)]\,|M_1|\right. \\ &\left. +[B(a_1/b_1,1,1)+C(a_1/b_1,0,2)]\,|M_2|\right), \end{aligned}$$

where,

$$\left|f^{'}(a_1)\right| \leq M_{1,} \quad \left|f^{'}(b_1)\right| \leq M_2.$$

■

3.3 Ostrowski Type Inequality for Fractional Integral (Version-II)

The inequality is given in the form of the following theorem:

Theorem 21 *Let $f: I \to \mathbb{R}$ be a differentiable mapping on I^o then $a_1,\ b_1 \in I$ with $a_1 < b_1$, such that $f^{'} \in L[a_1,\ b_1]$. If $f^{'}$ is GA-convex on $[a_1,b_1]$ and bounded, and $x \in [a_1,b_1]$, then the following inequality for Hadamard fractional integral holds.*

$$\begin{aligned} &\left|\frac{(\ln(b_1/x))^{\alpha}+(\ln(x/a_1))^{\alpha}}{(\ln(b_1/a_1))^{\alpha}} f(x) - \frac{\Gamma(\alpha+1)}{(\ln(b_1/a_1))^{\alpha}}\left[J^{\alpha}_{b_1^-} f(b_1) + J^{\alpha}_{a_1^+} f(a_1)\right]\right| \\ &\leq b_1 \ln(a_1/b_1)\left[[B(a_1/b_1,\alpha+1,0)+C(a_1/b_1,1,\alpha)]\left|f^{'}(a_1)\right|\right. \\ &\left. +[B(a_1/b_1,\alpha,1)+C(a_1/b_1,0,\alpha+1)]\left|f^{'}(b_1)\right|\right], \end{aligned} \tag{3.7}$$

where,

$$\left|f^{'}(a_1)\right| \leq M_1, \quad \left|f^{'}(b_1)\right| \leq M_2.$$

Proof. Let

$$p(\lambda)=\begin{cases}\lambda^{\alpha}, & \lambda\in\left[0,\frac{\ln(b_1/x)}{\ln(b_1/a_1)}\right]\\ (\lambda-1)^{\alpha}, & \lambda\in\left(\frac{\ln(b_1/x)}{\ln(b_1/a_1)},1\right]\end{cases}$$

Then

$$\begin{aligned}
I &= \int_0^1 p(\lambda)\,df\left(a_1^{\lambda}b_1^{\lambda-1}\right)\\
&= \int_0^{\ln(b_1/x)/\ln(b_1/a_1)} \lambda^{\alpha} d\left(f\left(a_1^{\lambda}\ b_1^{1-\lambda}\right)\right) + \int_{\ln(b_1/x)/\ln(b_1/a_1)}^{1} (\lambda-1)^{\alpha} d\left(f\left(a_1^{\lambda}b_1^{(\lambda-1)}\right)\right)\\
&= \lambda^{\alpha} f\left(a_1^{\lambda}b_1^{\lambda-1}\right)\Big|_0^{\ln(b_1/x)/\ln(b_1/a_1)} - \int_0^{\ln(b_1/x)/\ln(b_1/a_1)} \alpha f\left(a_1^{\lambda}b_1^{(\lambda-1)}\right)(\lambda)^{\alpha-1}\,d\lambda\\
&\quad + (\lambda-1)^{\alpha} f(a_1^{\lambda}b_1^{\lambda-1})\Big|_{\ln(b_1/x)/\ln(b_1/a_1)}^{1} - \int_{\ln(b_1/x)/\ln(b_1/a_1)}^{1} \alpha f\left(a_1^{\lambda}b_1^{\lambda-1}\right)(\lambda-1)^{\alpha}\,d\lambda\\
&= \left(\frac{\ln(b_1/x)}{\ln(b_1/a_1)}\right)^{\alpha} f(x) - \left(\frac{\ln(b_1/x)}{\ln(b_1/a_1)}-1\right)^{\alpha} f(x)\\
&\quad -\alpha \int_0^{\ln(b_1/x)/\ln(b_1/a_1)} f\left(a_1^{\lambda}\ b_1^{1-\lambda}\right)\lambda^{\alpha-1}d\lambda\\
&\quad -\alpha \int_{\ln(b_1/x)/\ln(b_1/a_1)}^{1} f\left(a_1^{\lambda}\ b_1^{1-\lambda}\right)(\lambda-1)^{\alpha-1}\,d\lambda\\
&= \left[\left(\frac{\ln(b_1/x)}{\ln(b_1/a_1)}\right)^{\alpha} - \left(\frac{\ln(x/a_1)}{\ln(b_1/a_1)}\right)^{\alpha}\right] f(x) - \alpha \int_0^{\ln(b_1/x)/\ln(b_1/a_1)} \lambda^{\alpha-1} f\left(a_1^{\lambda}\ b_1^{1-\lambda}\right)d\lambda\\
&\quad -\alpha \int_{\ln(b_1/x)/\ln(b_1/a_1)}^{1} (\lambda-1)^{\alpha-1} f\left(a_1^{\lambda}\ b_1^{1-\lambda}\right)d\lambda
\end{aligned}$$

$$= \frac{(\ln b_1 - \ln x)^\alpha + (\ln x - \ln a_1)^\alpha}{(\ln (b_1/a_1))^\alpha} f(x) - \alpha \int_0^{\ln(b_1/x)/\ln(b_1/a_1)} \lambda^{\alpha-1} f\left(a_1^\lambda \, b_1^{1-\lambda}\right) d\lambda$$

$$-\alpha \int_{\ln(b_1/x)/\ln(b_1/a_1)}^{1} (\lambda - 1)^{\alpha-1} f\left(a_1^\lambda \, b_1^{1-\lambda}\right) d\lambda$$

$$= \frac{(\ln (b_1/x))^\alpha + (\ln (x/a_1))^\alpha}{(\ln (b_1/a_1))^\alpha} f(x) - \alpha \int_0^{\ln(b_1/x)/\ln(b_1/a_1)} \lambda^{\alpha-1} f\left(a_1^\lambda \, b_1^{1-\lambda}\right) d\lambda$$

$$-\alpha \int_{\ln(b_1/x)/\ln(b_1/a_1)}^{1} (\lambda - 1)^{\alpha-1} f\left(a_1^\lambda \, b_1^{1-\lambda}\right) d\lambda \quad (3.8)$$

On changing of variables, we have :

$$\begin{aligned} \text{If} \quad u &= a_1^\lambda \, b_1^{1-\lambda} \quad \text{then} \\ \text{for } \lambda &= 0, \quad u = b_1 \\ \text{and for } \lambda &= 1, \quad u = x, \\ \lambda &= \frac{\ln (u/b_1)}{\ln (a_1/b_1)} = \frac{\ln (b_1/u)}{\ln (b_1/a_1)} \\ d\lambda &= \frac{du}{u \ln (a_1/b_1)}, \\ \lambda - 1 &= \frac{\ln (a_1/u)}{\ln (b_1/a_1)}. \end{aligned}$$

Using these in (3.8), we have:

$$I = \frac{(\ln(b_1/x))^\alpha + (\ln(x/a_1))^\alpha}{(\ln(b_1/a_1))^\alpha} f(x) - \alpha\Gamma(\alpha) \frac{1}{\Gamma(\alpha)} \int_{b_1}^{x} \left(\frac{\ln(b_1/u)}{\ln(b_1/a_1)} \right)^{\alpha-1} \frac{f(u)}{u \ln(a_1/b_1)} du$$

$$-\alpha\Gamma(\alpha) \frac{1}{\Gamma(\alpha)} \int_{x}^{a_1} \left(\frac{\ln(a_1/u)}{\ln(b_1/a_1)} \right)^{\alpha-1} \frac{f(u)}{u \ln(a_1/b_1)} du$$

$$= \frac{(\ln(b_1/x))^{\alpha}+(\ln(x/a_1))^{\alpha}}{(\ln(b_1/a_1))^{\alpha}} f(x) - \Gamma(\alpha+1)\frac{1}{\Gamma(\alpha)}\int_{b_1}^{x}\left(\frac{\ln(b_1/u)}{\ln(b_1/a_1)}\right)^{\alpha-1}\frac{f(u)}{u\ln(a_1/b_1)}du$$

$$-\Gamma(\alpha+1)\frac{1}{\Gamma(\alpha)}\int_{x}^{a_1}\left(\frac{\ln(a_1/u)}{\ln(b_1/a_1)}\right)^{\alpha-1}\frac{f(u)}{u\ln(a_1/b_1)}du$$

$$= \frac{(\ln(b_1/x))^{\alpha}+(\ln(x/a_1))^{\alpha}}{(\ln(b_1/a_1))^{\alpha}} f(x) - \frac{\Gamma(\alpha+1)}{(\ln(b_1/x))^{\alpha}}\left[J^{\alpha}_{b_1^-}f(b_1)+J^{\alpha}_{a_1^+}f(a_1)\right]. \tag{3.9}$$

From identity (3.1) and since $|f'|$is GA-convex function,we have (3.9) as follows:

$$\begin{aligned}
I &= \int_{0}^{\ln(b_1/x)/\ln(b_1/a_1)} \lambda^{\alpha} f'(a_1^{\lambda}\ b_1^{1-\lambda}) d(a_1^{\lambda}\ b_1^{1-\lambda}) \\
&\quad + \int_{\ln(b_1/x)/\ln(b_1/a_1)}^{1} (\lambda-1)^{\alpha} f'(a_1^{\lambda}\ b_1^{1-\lambda}) d(a_1^{\lambda}\ b_1^{1-\lambda}) \\
&= \int_{0}^{\ln(b_1/x)/\ln(b_1/a_1)} \lambda^{\alpha} f'(a_1^{\lambda}\ b_1^{1-\lambda}) a_1^{\lambda}\ b_1^{1-\lambda}(\ln a_1 - \ln b_1) d\lambda \\
&\quad + \int_{\ln(b_1/x)/\ln(b_1/a_1)}^{1} \lambda^{\alpha} f'(a_1^{\lambda}\ b_1^{1-\lambda})(a_1^{\lambda}\ b_1^{1-\lambda})(\ln a_1 - \ln b_1) d\lambda \\
&= \int_{0}^{\ln(b_1/x)/\ln(b_1/a_1)} \lambda^{\alpha} f'(a_1^{\lambda}\ b_1^{1-\lambda}) \left(a_1/b_1\right)^{\lambda} b_1(\ln a_1 - \ln b_1) d\lambda \\
&\quad + \int_{\ln(b_1/x)/\ln(b_1/a_1)}^{1} (\lambda-1)^{\alpha} f'(a_1^{\lambda}\ b_1^{1-\lambda}) \left(a_1/b_1\right)^{\lambda} b_1(\ln a_1 - \ln b_1) d\lambda \\
&= b_1\ln(a_1/b_1) \int_{0}^{\ln(b_1/x)/\ln(b_1/a_1)} \lambda^{\alpha} f'(a_1^{\lambda}\ b_1^{1-\lambda}) \left(a_1/b_1\right)^{\lambda} d\lambda \\
&\quad + b_1\ln(a_1/b_1) \int_{\ln(b_1/x)/\ln(b_1/a_1)}^{1} (\lambda-1)^{\alpha} f'(a_1^{\lambda}\ b_1^{1-\lambda}) \left(a_1/b_1\right)^{\lambda} d\lambda
\end{aligned}$$

$$
\begin{aligned}
&= b_1 \ln(a_1/b_1) \int\limits_{0}^{\ln(b_1/x)/\ln(b_1/a_1)} \lambda^{\alpha} (a_1/b_1)^{\lambda} \left[\lambda|f^{'}(a_1)| + (\lambda-1)|f'(b_1)|\right] d\lambda \\
&\quad + b_1 \ln(a_1/b_1) \int\limits_{\ln(b_1/x)/\ln(b_1/a_1)}^{1} (\lambda-1)^{\alpha} (a_1/b_1)^{\lambda} \left[\lambda|f^{'}(a_1)| + (\lambda-1)|f'(b_1)|\right] d\lambda \\
&= b_1 \ln(a_1/b_1) \int\limits_{0}^{\ln(b_1/x)/\ln(b_1/a_1)} \lambda^{\alpha+1} (a_1/b_1)^{\lambda} |f^{'}(a_1)| d\lambda + b_1 \left(\ln(a_1/b_1)\right) \\
&\quad + \int\limits_{0}^{\ln(b_1/x)/\ln(b_1/a_1)} \lambda^{\alpha}(\lambda-1) (a_1/b_1)^{\lambda} |f^{'}(b_1)| d\lambda \\
&\quad + b_1 \ln(a_1/b_1) \int\limits_{\ln(b_1/x)/\ln(b_1/a_1)}^{1} (a_1/b_1)^{\lambda} \lambda(\lambda-1)^{\alpha} |f^{'}(a_1)| d\lambda \\
&\quad + b_1 \ln(a_1/b_1) \int\limits_{\ln(b_1/x)/\ln(b_1/a_1)}^{1} (a_1/b_1)^{\lambda} (\lambda-1)^{\alpha+1} |f^{'}(b_1)| d\lambda. \qquad (3.10)
\end{aligned}
$$

Now Consider the functions as follows:

$$
\begin{aligned}
B(u,m,n) &= \int\limits_{0}^{\ln(b_1/x)/\ln(b_1/a_1)} u^{\lambda} \lambda^{m} (\lambda-1)^{n} d\lambda \\
C(u,m,n) &= \int\limits_{\ln(b_1/x)/\ln(b_1/a_1)}^{1} u^{\lambda} \lambda^{m} (\lambda-1)^{n} d\lambda.
\end{aligned}
$$

Thus, we have (3.10) as follows:

$$
\begin{aligned}
|I| \leq\ & b_1 \ln(a_1/b_1) B(a_1/b_1, \alpha+1, 0) \left|f^{'}(a_1)\right| + b_1 \ln(a_1/b_1) B(a_1/b_1, \alpha, 0) \left|f^{'}(b_1)\right| \\
& + b_1 \ln(a_1/b_1) C(a_1/b_1, 1, \alpha) \left|f^{'}(a_1)\right| + b_1 \ln(a_1/b_1) C(a_1/b_1, 0, \alpha+1) \left|f^{'}(b_1)\right|.
\end{aligned}
$$

This implies

$$
\begin{aligned}
|I| \leq\ & b_1 \ln(a_1/b_1) \left[B(a_1/b_1, \alpha+1, 0) + C(a_1/b_1, 1, \alpha)\right] \left|f^{'}(a_1)\right| \\
& + b_1 \ln(a_1/b_1) \left[B(a_1/b_1, \alpha, 1) + C(a_1/b_1, 0, \alpha+1)\right] \left|f^{'}(b_1)\right|.
\end{aligned}
$$

Since f$'$is bounded, let

$$\left|f^{'}(a_1)\right| \leq M_1 \; and \; \left|f^{'}(b_1)\right| \leq M_2.$$

Putting in above and combining it with (3.9) gives the required result, i.e.,

$$\begin{aligned} |I| &= \left|\frac{(\ln(b_1/x))^{\alpha} + (\ln(x/a_1))^{\alpha}}{(\ln(b_1/a_1))^{\alpha}} f(x) - \frac{\Gamma(\alpha+1)}{(\ln(b_1/a_1))^{\alpha}}\left[J^{\alpha}_{b_1^-} f(b_1) + J^{\alpha}_{a_1^+} f(a_1)\right]\right| \\ &\leq b_1 \ln(a_1/b_1)\left[\left[B(a_1/b_1, \alpha+1, 0) + C(a_1/b_1, 1, \alpha)\right]\left|f^{'}(a_1)\right|\right. \\ &\quad \left. + \left[B(a_1/b_1, \alpha, 1) + C(a_1/b_1, 0, \alpha+1)\right]\left|f^{'}(b_1)\right|\right]. \end{aligned}$$

Hence theorem is proved. ■

Corollary 22 *For* $\alpha = 1$, *in Theorem 21 we get the basic Ostrowski type inequality(1.1).*

Chapter 4

Ostrowski Type Inequality for $\mathbf{n}^{th}$ Differentiable GA-Convex Bounded Function for Fractional Integral

New identity for a differentiable, GA-convex function is established in chapter 3. With the aid of this identity, inequalities of the Ostrowski type for fractional integral also developed. In order to account for this, we generalise Ostrowski type inequalities for GA-convex first and n^{th} differentiable bounded functions for fractional integral. As a result, some applications are also provided in the sections that follow. A power mean inequality of the Ostrowski type is also established at the conclusion.

In order to prove the Ostrowski type inequalities for n^{th} differentiable function for fractional integral, we first prove the following lemma:

Lemma 23 *Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be on n^{th} differentiable mapping on I° where $a_1, b_1 \in I$ with $a_1 < b_1$ such that $f^{(n)}(x) \in L[a_1,b_1]$,then for all $[a_1,b_1]$ and $\alpha > 0$,we have:*

$$\begin{aligned} & \frac{(\ln(b_1/x))^{\alpha} + (\ln(x/a_1))^{\alpha}}{(\ln(b_1/a_1))^{\alpha}} f^{(n-1)}(x) \\ & -\frac{\Gamma(\alpha+1)}{(\ln(b_1/a_1))^{\alpha}} \left[J_{b_1-}^{\alpha}\left(f^{(n-1)}(b_1)\right) + J_{a_1+}^{\alpha}\left(f^{(n-1)}(a_1)\right)\right] \\ = & \int_0^1 p(\lambda)\, d\left(f^{(n-1)}\left(a_1^{\lambda}\, b_1^{1-\lambda}\right)\right), \end{aligned} \tag{4.1}$$

where,

$$p(\lambda) = \begin{cases} \lambda^{\alpha}, & \lambda \in [0, \ln(b_1/x)/\ln(b_1/a_1)] \\ (\lambda-1)^{\alpha} & \lambda \in [\ln(b_1/x)/\ln(b_1/a_1), 1] \end{cases}.$$

Proof. Here,

$$\begin{aligned} I &= \int_0^1 d\left(f^{(n-1)}\left(a_1^{\lambda}\, b_1^{1-\lambda}\right)\right) \\ I &= \int_0^{\ln(b_1/x)/\ln(b_1/a_1)} \lambda^{\alpha} d\left(f^{(n-1)}\left(a_1^{\lambda}\, b_1^{1-\lambda}\right)\right) + \int_{\ln(b_1/x)/\ln(b_1/a_1)}^1 (\lambda-1)^{\alpha} d\left(f^{(n-1)}\left(a_1^{\lambda} b_1^{\lambda-1}\right)\right) \end{aligned}$$

$$\begin{aligned}
I &= \lambda^{\alpha} f^{(n-1)}\left(a_1^{\lambda} b_1^{\lambda-1}\right)\Big|_0^{\ln(b_1/x)/\ln(b_1/a_1)} - \int\limits_0^{\ln(b_1/x)/\ln(b_1/a_1)} \alpha\lambda^{(\alpha-1)} f^{(n-1)}\left(a_1^{\lambda} b_1^{\lambda-1}\right) d\lambda \\
&\quad + (\lambda-1)^{\alpha} f^{(n-1)}\left(a_1^{\lambda}\ b_1^{1-\lambda}\right)\Big|_{\ln(b_1/x)/\ln(b_1/a_1)}^{1} \\
&\quad - \int\limits_{\ln(b_1/x)/\ln(b_1/a_1)}^{1} \alpha\left(\lambda-1\right)^{\alpha-1} f^{(n-1)}\left(a_1^{\lambda}\ b_1^{1-\lambda}\right) d\lambda \\
&= \frac{\left(\ln\left(b_1/x\right)\right)^{\alpha} + \left(\ln\left(x/a_1\right)\right)^{\alpha}}{\left(\ln\left(b_1/a_1\right)\right)^{\alpha}} f^{(n-1)}\left(x\right) - \alpha \int\limits_0^{\ln(b_1/x)/\ln(b_1/a_1)} \lambda^{a_1-1} f^{(n-1)}\left(a_1^{\lambda} b_1^{\lambda-1}\right) d\lambda \\
&\quad -\alpha \int\limits_0^{\ln(b_1/x)/\ln(b_1/a_1)} \left(\lambda-1\right)^{a_1-1} f^{(n-1)}\left(a_1^{\lambda} b_1^{\lambda-1}\right) d\lambda.
\end{aligned}$$

Changing of variables,we have

$$\begin{aligned}
u &= a_1^{\lambda}\ b_1^{1-\lambda} \\
\lambda &= 0 \implies u = b_1, \\
\lambda &= 1 \implies u = a_1, \\
\lambda &= \frac{\ln\left(b_1/u\right)}{\ln\left(b_1/a_1\right)}, \\
\lambda - 1 &= \frac{\ln\left(a_1/u\right)}{\ln\left(b_1/a_1\right)},
\end{aligned}$$

and

$$d\lambda = \frac{du}{u\ln(a_1/b_1)}.$$

Therefore, we have:

$$
\begin{aligned}
I &= \left(\frac{(\ln(b_1/x))^{\alpha}+(\ln(x/a_1))^{\alpha}}{(\ln(b_1/a_1))^{\alpha}}\right) f^{(n-1)}(x) \\
&\quad -\alpha\Gamma(\alpha)\frac{1}{\Gamma(\alpha)}\int_{b_1}^{x}\left(\frac{\ln(u/b_1)}{\ln(b_1/a_1)}\right)^{\alpha-1}\frac{f^{n-1}(u)}{u\ln(a_1/b_1)}du \\
&\quad -\alpha\Gamma(\alpha)\frac{1}{\Gamma(\alpha)}\int_{x}^{a_1}\left(\frac{\ln(b_1/u)}{\ln(b_1/a_1)}\right)^{\alpha-1}\frac{f^{n-1}(u)}{u\ln(a_1/b_1)}du \\
&= \left(\frac{(\ln(b_1/x))^{\alpha}+(\ln(x/a_1))^{\alpha}}{(\ln(b_1/a_1))^{\alpha}}\right) f^{(n-1)}(x) \\
&\quad -\frac{\Gamma(\alpha+1)}{(\ln(b_1/x))^{\alpha}}\left[J^{\alpha}_{b_1^-}f^{(n-1)}(b_1)+J^{\alpha}_{a_1^+}f^{(n-1)}(a_1)\right]. \qquad (4.2)
\end{aligned}
$$

■

Theorem 24 *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an n^{th} differentiable mapping on I° (I° is interior of I) such that $f^{(n)}(x) \in L[a_1,b_1]$ for all $x \in [a_1,b_1] \subset I$, $f^{(n)}(x)$ is GA convex and bounded with $x \in [a_1,b_1]$,then the following integral inequalities for fractional integral holds:*

$$
\begin{aligned}
&\left|\frac{(\ln(b_1/x))^{\alpha}+(\ln(x/a_1))^{\alpha}}{(\ln(b_1/a_1))^{\alpha}}f^{(n-1)}(x)-\frac{\Gamma(\alpha+1)}{\ln(b_1/a_1)^{\alpha}}\left[J^{\alpha}_{b_1-}f^{(n-1)}(b_1)\left(f^{(n-1)}(b_1)\right)+J^{\alpha}_{a_1-}f^{(n-1)}(a_1)\right.\right. \\
&\leq\ b_1\ln(a_1/b_1)\left[\left(B(a_1/b_1,\alpha+1,0)+C(a_1/b_1,1,\alpha)\right)\left|f^{(n)}(a_1)\right|\right. \\
&\quad\left.+\left(B(a_1/b_1,\alpha,1)+C(a_1/b_1,0,\alpha+1)\right)\left|f^{(n)}(b_1)\right|\right]
\end{aligned}
$$

where,

$$\left|f^{(n)}(a_1)\right| \leq L_1, \quad \left|f^{(n)}(b_1)\right| \leq L_2, \qquad (4.3)$$

for some numbers L_1 and L_2,

where

$$B\left(u.m.n\right) = \int\limits_{0}^{\ln(b_1/x)/\ln(b_1/a_1)} u^{\lambda}\lambda^{m}\left(\lambda-1\right)^{n}d\lambda$$

$$C\left(u.m.n\right) = \int\limits_{\ln(b_1/x)/\ln(b_1/a_1)}^{1} u^{\lambda}\lambda^{m}\left(\lambda-1\right)^{n}d\lambda.$$

Proof. Since $\left|f^{\ (n)}\right|$ is GA-convex, we have:

$$\begin{aligned}
I &= \int\limits_{0}^{\ln(b_1/x)/\ln(b_1/a_1)} \lambda^{\alpha} f^{(n)}(a_1^{\lambda}\ b_1^{1-\lambda})d(a_1^{\lambda}\ b_1^{1-\lambda}) \\
&\quad + \int\limits_{\ln(b_1/x)/\ln(b_1/a_1)}^{1} (\lambda-1)^{\alpha} f^{(n)}(a_1^{\lambda}\ b_1^{1-\lambda})d(a_1^{\lambda}\ b_1^{1-\lambda})
\end{aligned}$$

$$\begin{aligned}
&= \int\limits_{0}^{\ln(b_1/x)/\ln(b_1/a_1)} \lambda^{\alpha} f^{(n)}(a_1^{\lambda}\ b_1^{1-\lambda})b_1\left(a_1/b_1\right)^{\lambda}\left(\ln a_1-\ln b_1\right)d\lambda \\
&\quad + \int\limits_{\ln(b_1/x)/\ln(b_1/a_1)}^{1} (\lambda-1)^{\alpha} f^{(n)}(a_1^{\lambda}\ b_1^{1-\lambda})b_1\left(a_1/b_1\right)^{\lambda}\left(\ln a_1-\ln b_1\right)d\lambda \\
&= b_1\ln\left(a_1/b_1\right) \int\limits_{0}^{\ln(b_1/x)/\ln(b_1/a_1)} \left(a_1/b_1\right)^{\lambda}\lambda^{\alpha} f^{(n)}(a_1^{\lambda}\ b_1^{1-\lambda})d\lambda \\
&\quad + b_1\ln\left(a_1/b_1\right) \int\limits_{\ln(b_1/x)/\ln(b_1/a_1)}^{1} \left(a_1/b_1\right)^{\lambda}(\lambda-1)^{\alpha} f^{(n)}(a_1^{\lambda}\ b_1^{1-\lambda})d\lambda
\end{aligned}$$

This implies,

$$\begin{aligned}
|I| &\leq b_1 \ln(a_1/b_1) \int_0^{\ln(b_1/x)/\ln(b_1/a_1)} (a_1/b_1)^\lambda \lambda^\alpha \left[\lambda \left|f^{(n)}(a_1)\right| 1 + |\lambda - 1| \left|f^{(n)}(b_1)\right|\right] d\lambda \\
&\quad + b_1 \ln(a_1/b_1) \int_{\ln(b_1/x)/\ln(b_1/a_1)}^{1} (a_1/b_1)^\lambda \left|(\lambda-1)^\alpha\right| \lambda^\alpha \left[\lambda \left|f^{(n)}(a_1)\right| 1 + |\lambda - 1| \left|f^{(n)}(b_1)\right|\right] d\lambda \\
&= b_1 \ln(a_1/b_1) \int_0^{\ln(b_1/x)/\ln(b_1/a_1)} (a_1/b_1)^\lambda \lambda^{\alpha+1} \left|f^{(n)}(a_1)\right| \\
&\quad + b_1 \ln(a_1/b_1) \int_0^{\ln(b_1/x)/\ln(b_1/a_1)} (a_1/b_1)^\lambda \lambda^\alpha |\lambda - 1| \left|f^{(n)}(b_1)\right| d\lambda \\
&\quad + b_1 \ln(a_1/b_1) \int_{\ln(b_1/x)/\ln(b_1/a_1)}^{1} (a_1/b_1)^\lambda \lambda |\lambda - 1|^\alpha \left|f^{(n)}(a_1)\right| d\lambda \\
&\quad + b_1 \ln(a_1/b_1) \int_{\ln(b_1/x)/\ln(b_1/a_1)}^{1} (a_1/b_1)^\lambda |\lambda - 1|^{\alpha+1} \left|f^{(n)}(b_1)\right| d\lambda
\end{aligned}$$

Let

$$\begin{aligned}
B(u,m,n) &= \int_0^{\ln(b_1/x)/\ln(b_1/a_1)} u^\lambda \lambda^m |\lambda - 1|^n d\lambda \\
C(u,m,n) &= \int_{\ln(b_1/x)/\ln(b_1/a_1)}^{1} u^\lambda \lambda^m |\lambda - 1|^n d\lambda,
\end{aligned}$$

then

$$\begin{aligned}
|I| &\leq b_1 \ln(a_1/b_1) B(a_1/b_1, \alpha+1, 0) \left|f^{(n)}(a_1)\right| + b_1 \ln(a_1/b_1) B(a_1/b_1, \alpha, 1) \left|f^{(n)}(b_1)\right| \\
&\quad + b_1 \ln(a_1/b_1) C(a_1/b_1, 1, \alpha) \left|f^{(n)}(a_1)\right| + b_1 \ln(a_1/b_1) C(a_1/b_1, 0, \alpha+1) \left|f^{(n)}(b_1)\right|
\end{aligned}$$

This implies

$$\begin{aligned} |I| &\leq b_1 \ln(a_1/b_1)\left[B(a_1/b_1, \alpha+1, 0)\left|f^{(n)}(a_1)\right| + C(a_1/b_1, 1, \alpha)\left|f^{(n)}(a_1)\right|\right] \\ &+ b_1 \ln(a_1/b_1)\left[B(a_1/b_1, \alpha, 1)\left|f^{(n)}(a_1)\right| + C(a_1/b_1, 0, \alpha+1)\left|f^{(n)}(b_1)\right|\right]. \end{aligned}$$

Using,

$$\left|f^{(n)}(a_1)\right| \leq L_1, \ \textit{and} \ \left|f^{(n)}(b_1)\right| \leq L_2.$$

we get the required result. ■

Corollary 25 *For $\alpha = 1$, we get the Ostrowski type inequalities for $f^{(n)}(x)$ be a GA-convex and bounded function.*

$$\begin{aligned} &\left|f^{(n-1)}(x) - \frac{1}{\ln(b_1/a_1)}\int_{a_1}^{b_1} \frac{f^{(n-1)}(x)}{x}dx\right| \\ \leq\ & b_1 \ln(a_1/b_1)\left[(B(a_1/b_1, 2, 0) + C(a_1/b_1, 1, 1))\left|f^{(n)}(a_1)\right|\right. \\ &\left. + (B(a_1/b_1, 1, 1) + C(a_1/b_1, 0, 2))\left|f^{(n)}(b_1)\right|\right], \end{aligned}$$

where

$$\left|f^{(n)}(a_1)\right| \leq L_1 \quad \textit{and} \quad \left|f^{(n)}(b_1)\right| \leq L_2$$

,for some numbers $L_1$and L_2.

Corollary 26 *For $\alpha = 1$ and $n = 1$, we get the Ostrowski type inequality for $f'(x)$ to be a GA-convex and bounded function:*

$$\begin{aligned} \left|f(x) - \frac{1}{\ln(b_1/a_1)}\int_{a_1}^{b_1} \frac{f(x)}{x}dx\right| &\leq b_1 \ln(a_1/b_1)\left[(B(a_1/b_1, 2, 0) + C(a_1/b_1, 1, 1))\left|f'(a_1)\right|\right. \\ &\left. + (B(a_1/b_1, 1, 1) + C(a_1/b_1, 0, 2))\left|f'(b_1)\right|\right]. \end{aligned} \tag{4.4}$$

where

$$\left|f^{'}\left(a_{1}\right)\right| \leq L_{1} \quad a_{1}nd \quad \left|f^{'}\left(b_{1}\right)\right| \leq L_{2}$$

for some numbers L_1 *and* L_2.

Ostrowski type inequalities is a special domain of the theory of integral inequalities with a large number of applications. Here, however, applications to spacial means are discussed.

4.1 Application to Special Means

Let a_1, b_1 be two nonnegative numbers with $b_1 > a_1$, then

Arithmetic Mean is defined as:

$$A(a_1, b_1) = \frac{a_1 + b_1}{2}.$$

Geometric Mean is defined as:

$$G\left(a_1, b_1\right) = \sqrt{a_1 b_1}.$$

Logarithmic Mean is defined as:

$$L\left(a_1, b_1\right) = \frac{b_1 - a_1}{\ln b_1 - \ln a_1}.$$

P-Logrithmic Mean is defined as:

$$Lp\left(a_1, b_1\right) = \frac{b_1^{p+1} - a_1^{p+1}}{\left(p+1\right)\left(b_1 - a_1\right)}, p \in \mathbb{R}/\left\{-1, 0\right\}.$$

Consider

$$f\left(x\right) = x^{n+1}, x > 0, n > 1.$$

Then,

$$\begin{aligned} \int_{a_1}^{b_1} \frac{f\left(x\right)}{x} dx &= \int_{a_1}^{b_1} \frac{x^{n+1}}{x} dx = \int_{a_1}^{b_1} x^n dx \\ &= \frac{x^{n+1}}{n+1} \Big|_{a_1}^{b_1} = \left(\frac{a_1^{n+1} - b_1^{n+1}}{n+1}\right), \end{aligned}$$

$$\frac{1}{\ln(b_1/a_1)}\int_{a_1}^{b_1} f(x)\,dx = \frac{a_1^{n+1}-b_1^{n+1}}{(n+1)(\ln a_1-\ln b_1)}$$
$$= \frac{b_1-a_1}{(\ln a_1-\ln b_1)}\frac{a_1^{n+1}-b_1^{n+1}}{(n+1)(b_1-a_1)}$$
$$\frac{1}{\ln(b_1/a_1)}\int_{a_1}^{b_1} f(x)\,dx = L(a_1,b_1)\,L_n(a_1,b_1).$$

Also

$$f(x)=x^{n+1},$$

$$f\left(\sqrt{a_1b_1}\right)=(a_1b_1)^{\frac{n+1}{2}}=\sqrt{a_1^{n+1}b_1^{n+1}}.$$

Thus, from Corollary 26, we have:

$$\left|G\left(a_1^{n+1}b_1^{n+1}\right)-L_n(a_1,b_1)\,L(a_1,b_1)\right| \le b_1\ln(a_1/b_1)$$
$$\times\left[\left(B(a_1/b_1,2,0)+C(a_1/b_1,1,1)\right)\left|f^{'}(a_1)\right|+\left(B(a_1/b_1,1,1)+C(a_1/b_1,0,2)\right)\left|f'(b_1)\right|\right]$$
$$\le \tfrac{b_1}{b_1-a_1}L(a_1/b_1)\left[\left(B(a_1/b_1,2,0)+C(a_1/b_1,1,1)\right)L_1+\left(B(a_1/b_1,1,1)+C(a_1/b_1,0,2)\right)L_2\right]$$

Chapter 5

Ostrowski Type Inequalities for Differentiable Fractional Integral Using Power Mean Inequality

In this dissertation, we establish a new identity for differentiable, GA-convex function. Using this identity, we develop Ostrowski type inequalities for fractional integral. Consequently, we generalize Ostrowski type inequalities for GA-convex first and n^{th} differentiable bounded function for fractional integral. Accordingly, some applications in subsequent sections are also provided. At the end, Ostrowski type inequalities using power mean inequality (1.8) are also established. After the establishment of Ostrowski type inequality we used some fractional integral to prove the new identity. We use power mean for differentiable fractional integral.

Here, we introduce some Ostrowski type inequalities using Power Mean inequality in the form of some theorems.

5.1 Main Theorem

Theorem 27 *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be differentiable mapping on* I° *where* $a_1,\ b_1 \in I$ *with* $a_1 < b_1$ *such that* $f' \in L[a_1,b_1]$. *If* $\left|f'\right|^q$ *is GA-convex on* $[a_1,b_1]$, $q>1$ *and* $x \in [a_1,b_1]$,*then the following inequality for fractional integral holds:*

$$
\begin{aligned}
&\left|\frac{\ln(b_1/x)^{\alpha}+\ln(x/a_1)^{\alpha}}{\ln(b_1/a_1)^{\alpha}}f(x)-\frac{\Gamma(\alpha+1)}{\ln(b_1/a_1)^{\alpha}}\left[J^{\alpha}_{b_1^-}f(b_1)+J^{\alpha}_{a_1^+}f(a_1)\right]\right| \\
\leq\ & b_1\ln\left(\frac{b_1}{a_1}\right)\left[\int\limits_0^{\ln(b_1/x)/\ln(b_1/a_1)}\left((a_1/b_1)^{\lambda}\lambda^{\alpha}d\lambda\right)^{1-\frac{1}{q}}\left(B(a_1/b_1,\alpha+1,0)\left|f'(a_1)\right|^q\right.\right. \\
&\left.+B(a_1/b_1,\alpha,1)\left|f'(b_1)\right|^q\right)^{\frac{1}{q}} \\
&+\int\limits_{\ln(b_1/x)/\ln(b_1/a_1)}^{1}\left((a_1/b_1)^{\lambda}(\lambda-1)^{\alpha}d\lambda\right)^{1-\frac{1}{q}}\left(C(a_1/b_1,\alpha+1,0)\left|f'(a_1)\right|^q\right. \\
&\left.\left.+C(a_1/b_1,\alpha,1)\left|f'(b_1)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

Proof. Using the power mean integral inequality (1.8) to Theorem (21), we have:

$$
\begin{aligned}
|I| &= \left|\frac{(\ln(b_1/x))^{\alpha}+(\ln(x/a_1))^{\alpha}}{(\ln(b_1/a_1))^{\alpha}}f(x)-\frac{\Gamma(\alpha+1)}{(\ln(b_1/a_1))^{\alpha}}\left[J^{\alpha}_{b_{1-}}f(b_1)+J^{\alpha}_{a_{1+}}f(a_1)\right]\right| \\
&\leq b_1\ln(a_1/b_1)\int\limits_0^{\ln(b_1/x)/\ln(b_1/a_1)}(a_1/b_1)^{\lambda}\lambda^{\alpha}\left|f'\left(a_1^{\lambda}\,b_1^{1-\lambda}\right)\right|d\lambda \\
&\quad+b_1\ln(a_1/b_1)\int\limits_{\ln(b_1/x)/\ln(b_1/a_1)}^{1}(a_1/b_1)^{\lambda}(\lambda-1)^{\alpha}\left|f'\left(a_1^{\lambda}\,b_1^{1-\lambda}\right)\right|d\lambda
\end{aligned}
$$

$$
\begin{aligned}
&\leq \quad b_1 \ln\left(a_1/b_1\right) \left[\left(\int\limits_0^{\ln(b_1/x)/\ln(b_1/a_1)} \left(a_1/b_1\right)^{\lambda} \lambda^{\alpha} d\lambda \right)^{1-\frac{1}{q}} \right. \\
&\quad \times \left(\int\limits_0^{\ln(b_1/x)/\ln(b_1/a_1)} \left(a_1/b_1\right)^{\lambda} \lambda^{\alpha} \left| f'\left(a_1^{\lambda}\, b_1^{1-\lambda}\right)\right|^q d\lambda \right)^{1/q} \\
&\quad + \left(\int\limits_{\ln(b_1/x)/\ln(b_1/a_1)}^{1} \left(a_1/b_1\right)^{\lambda} \left(\lambda-1\right)^{\alpha} d\lambda \right)^{1-\frac{1}{q}} \\
&\quad \left. \left(\int\limits_{\ln(b_1/x)/\ln(b_1/a_1)}^{1} \left(a_1/b_1\right)^{\lambda} \left(\lambda-1\right)^{\alpha} \left| f'\left(a_1^{\lambda}\, b_1^{1-\lambda}\right)\right|^q d\lambda \right)^{1/q} \right].
\end{aligned}
$$

Using,

$$
f'\left(a_1^{\lambda}\, b_1^{1-\lambda}\right) \leq \lambda f'\left(a_1\right) + \left(1-\lambda\right) f'\left(b_1\right)
$$

on the right-hand side of the inequality, we have:

$$
\begin{aligned}
|I| \quad &\leq \quad b_1 \ln\left(a_1/b_1\right) \left[\left(\int\limits_0^{\ln(b_1/x)/\ln(b_1/a_1)} \left(a_1/b_1\right)^{\lambda} \lambda^{\alpha} d\lambda \right)^{1-\frac{1}{q}} \right. \\
&\quad \times \left(\int\limits_0^{\ln(b_1/x)/\ln(b_1/a_1)} \left(a_1/b_1\right)^{\lambda} \lambda^{\alpha} \left(\lambda \left| f'\left(a_1\right)\right|^q + \left(1-\lambda\right) \left| f'\left(b_1\right)\right|^q \right) d\lambda \right)^{1/q} \\
&\quad + \left(\int\limits_{\ln(b_1/x)/\ln(b_1/a_1)}^{1} \left(a_1/b_1\right)^{\lambda} \left(\lambda-1\right)^{\alpha} d\lambda \right)^{1-\frac{1}{q}} \\
&\quad \left. \times \left(\int\limits_{\ln(b_1/x)/\ln(b_1/a_1)}^{1} \left(a_1/b_1\right)^{\lambda} \left(\lambda \left| f'\left(a_1\right)\right|^q + \left(1-\lambda\right) \left| f'\left(b_1\right)\right|^q d\lambda \right) \right)^{1/q} \right].
\end{aligned}
$$

Let,

$$B(u,m,n) = \int_0^{\ln(b_1/x)/\ln(b_1/a_1)} u^{\lambda}\lambda^{m}(\lambda-1)^{n}\,d\lambda$$

$$C(u,m,n) = \int_{\ln(b_1/x)/\ln(b_1/a_1)}^{1} u^{\lambda}\lambda^{m}(\lambda-1)^{n}\,d\lambda,$$

then,we have

$$\begin{aligned} |I| \leq\ & b_1\ln(a_1/b_1)\left(\int_0^{\ln(b_1/x)/\ln(b_1/a_1)} (a_1/b_1)^{\lambda}\lambda^{\alpha}d\lambda\right)^{1-\frac{1}{q}}\Big(B(a_1/b_1,\alpha+1,0)\left|f'(a_1)\right|^{q} \\ & +B(a_1/b_1,\alpha,1)\left|f^{\,\prime}(b_1)\right|^{q}\Big)^{\frac{1}{q}} \\ & +\left(\int_{\ln(b_1/x)/\ln(b_1/a_1)}^{1} (a_1/b_1)^{\lambda}(\lambda-1)^{\alpha}\,d\lambda\right)^{1-\frac{1}{q}}\Big(C(a_1/b_1,\alpha+1,\ 0)\left|f'(a_1)\right|^{q} \\ & +C(a_1/b_1,\alpha,1)\left|f'(b_1)\right|^{q}\Big)^{\frac{1}{q}}. \end{aligned} \tag{5.1}$$

■

Corollary 28 *For $q=1$,we get the inequality (3.5) of Theorem 20.*

Theorem 29 *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be on n^{th} differentiable mapping on I° where $a_1,\ b_1\in I$ with $a_1<b_1$ such that $f^{(n)}(x)\in L[a_1,b_1]$. If $\left|f^{(n)}(x)\right|^{q}$ is GA-convex on (a_1,b_1), $q>1$ for all $x\in[a_1,b_1]$, then the following inequality for fractional integral holds:*

Just replace $\left|f'(a_1)\right|^{q}$ and $\left|f'(b_1)\right|^{q}$ with $\left|f^{(n)}(a_1)\right|^{q}$ and $\left|f^{(n)}(b_1)\right|^{q}$ in the right-hand side of theorem 3 above and $f(x)$ with $f^{(n+1)}(x)$ on the left-hand side of theorem 3, *we have:*

$$\begin{aligned}
|I| &= \left|\frac{\ln(b_1/x)^{\alpha}+\ln(x/a_1)^{\alpha}}{\ln(b_1/a_1)^{\alpha}}f^{(n-1)}(x)-\frac{\Gamma(\alpha+1)}{\ln(b_1/a_1)^{\alpha}}\left[J_{b_1^-}^{\alpha}f^{(n-1)}(b_1)+J_{a_1^+}^{\alpha}f^{(n-1)}(a_1)\right]\right| \\
&\leq b_1\ln(a_1/b_1)\left(\left(\int_0^{\ln(b_1/x)/\ln(b_1/a_1)}(a_1/b_1)^{\lambda}\lambda^{\alpha}d\lambda\right)^{1-\frac{1}{q}}\right. \\
&\quad\times\left(B(a_1/b_1,\alpha+1,0)\left|f^{(n)}(a_1)\right|^q+B(a_1/b_1,\alpha,1)\left|f^{(n)}(b_1)\right|^q\right)^{\frac{1}{q}} \\
&\quad+\left(\int_{\ln(b_1/x)/\ln(b_1/a_1)}^{1}(a_1/b_1)^{\lambda}(\lambda-1)^{\alpha}d\lambda\right)^{1-\frac{1}{q}} \\
&\quad\left.\times\left(C(a_1/b_1,\alpha+1,0)\left|f^{(n)}(a_1)\right|^q+C(a_1/b_1,\alpha,1)\left|f^{(n)}(b_1)\right|^q\right)^{\frac{1}{q}}\right). \qquad (5.2)
\end{aligned}$$

Corollary 30 *For $q=1$, $n=1$, we get inequality (3.7) of Theorem 21 .*

Chapter 6

Conclusions and Recommendations

In this thesis, Ostrowski type inequalities for fractional integral have been studied for n^{th} differentiable GA-convex functions.

6.1 Concluding Remarks

In this thesis, first of all various types of convex functions and fractional integrals, their applications and various related identities and well-known inequalities are discussed. Then a new identity for differentiable, GA-convex function is established. Using this identity, Ostrowski type inequalities for fractional integral are developed. Then, two versions of Ostrowski type inequality for GA-convex differentiable and bounded function for Hadamard fractional integral are developed. Consequently, Ostrowski type inequalities for GA-convex nth differentiable bounded function for Hadamard fractional integral version-I and version-II are generalized. Accordingly, some applications to special means, such as arithmetic-, geometric-, logarithmic- and p-logarithmic means in subsequent sections are also provided. Further, Ostrowski type inequalities for first time differentiable and n-time differentiable GA-convex function via fractional integral are established using power mean inequality. At the end, some conclusions and recommendations for further research work are provided.

6.2 Recommendations for Future Work

Future recommendation for Development of Ostrowski type Inequality shuold be made due to complexity of fractional integral. These osrtowski type inequalities may be enhanced for some other complicated fractional integrals given in the literature. These inequalities may be generalized on differentiable mapping for various convex functions. Some variants of ostrowskis type inequalities may be established using fractional integrals alongs with other suitable technique that helps improving error estimates. Otrowski type inequalities may be appropriately modified to extend their scope by investigating classes. For example, Ostrowski type inequality for functions defined on non-Euclidean spaceses, or function with certain properties such as monotonicty, differentaibilty, or convexcity.

References

[1] Mitrinovic, D. S., Pecaric, J., & Fink, A. M. (1991).. Inequalities involving functions and their integrals and derivatives (Vol. 53). Springer Science & Business Media

[2] Hardy Little Wood, G.Polya. (1952). Inequalities (Second Edition), Combridge University Press,

[3] Farid, G. (2017). Some new Ostrowski type inequalities via fractional integrals. Int. J. Anal. Appl., 14, 64–68.

[4] Alomari, M.; Darus, M. (2010). Some Ostrowski type inequalities for convex functions with applications, RGMIA 13(1) article No 3

[5] Liu, W. (2012). Some Ostrowski type inequalities via Riemann-Liouville fractional integrals for h-convex functions. arXiv preprint arXiv:1210.3942.

[6] Tian-Yuzhang, AI-PingJi-Feng Qi,. (2013). Some inequalities of Hermite

Hadamard type for GA-convex functions with applications to means, LE MATHEMATICS, VolLXVIIL, FascI, 229-239.

[7] Yıldırım, H.; Kırtay, Z. (2014). Ostrowski inequality for generalized fractional integral and related inequalities. Malaya J. Mat. 2014, 2, 322-329.

[8] Leibniz, G.W. (1962). Letter from Hanover, Germany, to Johann Bernoulli, December 28, (1695). In Leibniz Mathematische Schriften; Olms-Verlag: Hildesheim, Germany, page.226

[9] Mir, Nazir Ahmad, Rafiq Arif and Ahmad Farooq. (2006).Generalization of integral inequalities for functions whose modulus of nth derivative are convex, General Mathematics, 14(4) 67-92.

[10] Dragomir, S.S. (2017). Ostrowski type inequalities for Riemann-Liouville fractional integrals of bounded variation, Hölder and Lipschitzian functions. RGMIA Res. Rep. Coll. , 20, 48.

[11] Dragomir, S.S. (2017). Further Ostrowski and Trapezoid type inequalities for Riemann-Liouville fractional integral of functions with bounded variation. RGMIA Res. Rep. Coll., 20, 84.

[12] Yıldız, Ç Özdemir, M.E.; Sarıkaya, M.Z. (2016). New generalizations of Ostrowski-Like type inequalities for fractional integral. Kyungpook Math. J., 56, 161–172.

[13] Hurye Kadakal. (2020). On refinements of some integral inequalities using improved power-mean integral inequalities, Number Methods partial differential Eqn,1-11.

[14] Set, E. (2012). New inequalities for Ostrowski type for mappings whose derivatives are s-convex in the second sense via fractional integrals. Comput. Math. Appl.63(7),1147-1154

[15] Magin, R.L. Ingo, C; Colon-Perez, L.; Triplett,W.; Mareci, T.H. (2013). Characterization of anomalous diffusion in porous biological tissues using fractional order derivatives an, entropy. Microporous Mesoporous Mater. 178, 39–43.

[16] Aljinević, A.A. (2014). Montgomery identity and Ostrowski type inequalities for Riemann-Liouville fractional integral. J. Math., ID 503195 6 pages.

[17] Dragomir, S.S. (2017). Ostrowski type inequalities for generalized Riemann-Liouville fractional integrals of function with bounded variation. RGMIA Res. Rep. Coll. , 20, 58.

[18] Yaldız, H.; Set, E. (2018). Some new Ostrowski type inequalities for generalized fractional integrals. AIP Conf. Proc., 1991, 020018.

[19] Farid, G. (2016) Straight forward proofs of Ostrowski inequality and some related results. Int. J. Anal. , 3918483.

[20] Leibniz, G.W. (1962) Letter from Hanover, Germany, to John Wallis, May 30,

1697. In Leibniz Mathematatische Schriften; Olms-Verlag: Hildesheim, Germany, page 25.

[21] Liu, Z. (2009). Some companions of an Ostrowski type inequality and application, J. Inequal. Pure Appl. Math 10 (2) 12. Art. 52.

[22] Magin, R.L. (2010). Fractional calculus models of complex dynamics in biological tissues. Comput. Math. Appl.,59, 1586–1593.

[23] Ostrowski, A.(1937). Über die Absolutabweichung einer differentiierbaren function vonihrem Integralmittelwert,Comment. Math. Helv., 10„ 226-227.

[24] Özdemir, M.E. Kavurmacı, H. Set, E. (1938). Ostrowski's type inequalities for (α, m)-convex functions, Kyungpook Math. J. 50 371–378.

[25] Pachpatte, B.G. (2000). On an inequality of Ostrowski type in three independent variables, J. Math. Anal. Appl. 249 583–591.

[26] Pachpatte, B.G. (2001). On a new Ostrowski type inequality in two independent variables, Tamkang J. Math. 32 (1) 45–49.

[27] Rafiq, A. Mir, N.A. Ahmad, F. (2007). Weighted Čebyšev–Ostrowski type inequalities, Applied Math. Mechanics 28 (7) 901–906. English Edition.

[28] Sarikaya, M.Z. (2010). On the Ostrowski type integral inequality, Acta Math. Univ. Comenianae LXXIX (1) 129–134.

[29] Tunç, M. (2013). Ostrowski type inequalities via h-convex functions with applications for special means.J. Inequal. Appl., 326.

[30] Ujević, N. (2004). Sharp inequalities of Simpson type and Ostrowski type, Comput. Math. Appl. 48 145–151.

[31] Yue, Hu. (2013). Ostrowski inequality for fractional integrals and related fractional inequalities. Transylv.Math.Mech. 5 , No. 1, 85-89.

[32] Zhongxue, L. (2008). On sharp inequalities of Simpson type and Ostrowski type in two independent variables, Comput. Math. Appl. 56 2043–2047.