DECISION ANALYSIS OF GENERAL LINGUISTIC INTERVAL VALUED INTUITIONISTIC FUZZY SOFT EXPERT SETS

By

TASADUQ MEHMOOD

NATIONAL UNIVERSITY OF MODERN LANGUAGES

ISLAMABAD

May, 2023

Decision Analysis of General Linguistic Interval Valued Intuitionistic Fuzzy Soft Expert Sets

By

TASADUQ MEHMOOD

MS Mathematics, National University of Modern Languages, Islamabad, 2023

A THESIS SUBMITTED IN PARTIAL FULFILMENT OF
THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

Mathematics

To

FACULTY OF ENGINEERING & COMPUTER SCIENCE

NATIONAL UNIVERSITY OF MODERN LANGUAGES ISLAMABAD

NATIONAL UNIUVERSITY OF MODERN LANGUAGES FACULTY OF ENGINEERIING & COMPUTER SCIENCE

THESIS AND DEFENSE APPROVAL FORM

The undersigned certify that they have read the following thesis, examined the defense, are satisfied with overall exam performance, and recommend the thesis to the Faculty of Engineering and Computer Sciences for acceptance.

Thesis Title: Decision Analysis of General Linguistic Interval Valued Intuitionistic Fuzzy Soft Expert Sets

Submitted By: Tasaduq Mehmood **Registration #:** 14 MS/MATH/F20

Master of Science in Mathematics
Title of the Degree

Mathematics
Name of Discipline

Dr. Afshan Qayyum
Name of Research Supervisor

Signature of Research Supervisor

Dr. Muhammad Noman Malik
Name of Dean (FE&CS)

Signature of Dean (FE&CS)

Brig. Syed Nadir Ali
Name Director General

Signature of Director General

May 22nd, 2023

AUTHOR'S DECLARATION

I Tasaduq Mehmood

Daughter of Arshad Mehmood

Registration # 14 MS/Maths/F20

Discipline Mathematics

Candidate of **Master of Science in Mathematics (MS Mathematics)** at the National University of Modern Languages do hereby declare that the thesis **"Decision Analysis of General Linguistic Interval Valued Intuitionistic Fuzzy Soft Expert Sets"** submitted by me in partial fulfillment of MS Mathematics degree, is result of my own research except as cited in references. This thesis has not been submitted or published earlier. I also solemnly declarethat it shall not, in the future, be submitted by me for getting any other degree from this or anyother university or institution. I also understand that if evidence of plagiarism is found in my thesis at any stage, even after the award of a degree, the work may be canceled and the degree revoked.

Signature of Candidate

Tasaduq Mehmood

Name of Candidate

22 May, 2023

Date

ABSTRACT

Title: Decision Analysis of General Linguistic Interval-Valued Intuitionistic Fuzzy Soft Expert Sets

The 2-Dimensional Linguistic Intuitionistic Fuzzy Variables (2-DLIFVs) add a subjective estimation of the trustworthiness of the evaluated results provided by experts, so Two-Dimensional Linguistic Intuitionistic Fuzzy Variables (2-DLIFVs) are very valuable instruments for describing uncertain or fuzzy information. This work extends the notion of 2-DLIVs by introducing General Linguistic Interval-Valued Intuitionistic Fuzzy Soft Expert Sets (GLIVIFSESs) in which the two terms are contained, the first term describes the subjective estimation of the objects under observation or discussions, second term describes the subjective evaluations of the reliability of the valuated results provided by experts. In this thesis we construct few operations on the structure (GLIVIFSESs) and then defines operational laws, scores, and accuracy functions for GLIVIFSESs. We illustrate some examples for the described operations. Further, we progress some arithmetical and geometrical aggregation operators for aggregating GLIVIFSE information and prove so many important properties related with them.

ACKNOWLEDGEMENT

Firstly, In the name of Almighty Allah, the most Merciful and Compassionate, All theworship and honor is for Him, Who always helped me to stick to the right path and always provided me the courage and strength to complete my degree with hard work and dedication.

I offer my humblest and sincerest words of thanks to the Holy Prophet Muhammad (Peace Be upon Him) who is, forever, a torch of guidance and knowledge for humanity.

Then I would like to express a deep sense of gratitude and appreciation to my most respectable supervisor Dr. Afshan Qayyum, during my studies she helped me and encouraged me to complete my assigned tasks on time. Her guidance and advice carried me through all the stages of writing my thesis. Her kind efforts and motivations would never be forgotten, and she will always be in my prayers. The aim of this dissertation would not have been realized without her persistent support.

I would also like to warmest thanks to Dr. Saffyan, In spite of his extremely busy schedule he always uses to take his precious time for me, he always motivates me, give me suggestions and for his selfless support. I wish to express my warmest thanks to my Maths Teacher Ma'am Kaneez (Late) may her soul rest in peace, whose efforts actually developed an interest in me for mathematics.

My thanks also go to all staff and faculty members at the Department of Mathematics, National University of Modern Languages, Islamabad, for conveying their knowledge in many effective ways.

I acknowledge with sincere thanks for the help, encouragement, and moral support extended by my parent and siblings. They are my strength, their prayers for me was what sustained me this far.

Finally, I am thankful to the faculty of the National University of Modern Languages for providing me with a comfortable and friendly environment.

I pray that Allah Almighty gives me success in life, and makes me able to work for the betterment of the whole world and mankind. (Ameen)

Tasaduq Mehmood

DEDICATION

My dissertation work is dedicated to my family and supervisor. A special feeling of gratitude is for my parents who always taught me to work hard with humility, dignity and never lose faith and hope. My Supervisor, who is ideal for me, always motivates me, encourages me, and supports me to follow right path to complete my research work on time.

Contents

Chapter 1

Introduction

Researchers and mathematicians have developed analytical skills and problem-solving strategies to address a wide range of issues in commerce, science, and the arts. However, dealing with uncertain and vague situations has been a challenge, leading to the development of theories like fuzzy set theory. Fuzzy sets handle possibilistic uncertainty related to imprecision of states, perceptions, and preferences. Over time, extensions have been made to fuzzy set theory, such as interval-valued fuzzy sets, intuitionistic fuzzy sets, soft set theory, Interval valued intuitionistic fuzzy sets, among others. These theories have been successfully applied in various fields, including medicine and decision analysis. Soft expert sets are a recent addition that allows users to access the opinions of all experts in one model.

The evolution of set theory and how the introduction of fuzzy set theory by Zadeh in 1965 overcame the limitations of crisp set theory in dealing with imprecise and indeterministic information sets. Atanasov generalized fuzzy sets in 1983 by assigning each element a non-membership degree along with their membership degree and defined Intuitionistic Fuzzy Sets. Later on, R. Verma worked on intuitionistic fuzzy sets in multiple directions from 2011-2015. The article further explains that Zadeh broadened the notion of fuzzy sets by introducing Interval-valued fuzzy sets, and Atanasov and Gargov extended the idea of IFSs by defining Interval-Valued Intuitionistic Fuzzy Sets (IVIFSs). The article also highlights the importance of linguistic variables in modeling vague and uncertain information and the use of linguistic intuitionistic fuzzy sets to deal with these variables. Finally, this thesis describes the work of various researchers who have proposed different weighted averaging and aggregation operators for LIFNs and UL knowledge.

This thesis is divided into six chapters, each with its own worth. The first Chapter serves as an introduction to the research, outlining its motivations research questions,reseach objectives significance of the study and procedure of research. The second Chapter covers literature review, in this chapter several dimensions of fuzzy sets with refrences are discussed. In third Chapter fundamental concepts, including Crisp Set, fuzzy sets, interval-valued fuzzy sets, intuitionistic fuzzy sets, interval-valued intuitionistic fuzzy sets, soft sets, soft expert sets, fuzzy soft expert sets, generalized interval valed fuzzy soft set, linguistic variables, linguistic intuitionistic fuzzy numbers, 2-dimensional linguistic variables. By explaining these concepts and their properties and operations, the third Chapter lays a foundation for understanding the rest of the thesis.

In fourth Chapter General Linguistic Interval Valued Intuitionistic Fuzzy Soft Expert Sets (GLIVIFSESs) is defined. which is the combination and modification of 2-Dimensional Linguistic Variables and Intuitionistic Fuzzy Soft Expert Sets introduced by many researchers especially by Verma , Zadeh, Attanassov, Alkhazaleh and Salleh. GLIVIFSESs play a vital role in decision analysis. The aim of General Linguistic Interval Valued Intuitionistic Fuzzy soft expert sets is to present opinions of expert in the form of interval valued fuzzy sets also verify the reliability of the experts opinions by using aggregation operators defined on the structure. In this chapter, some other basic definitions related to the structure are described. Algebraic operations, Algebraic operational laws, theorems are constructed and proved. Additionally, examples are provided to enhance understanding.

In fifth Chapter, firstly Weighted aggregated operators are defined such as General Linguistic Interval Valued Intuitionistic Fuzzy Soft Expert Weighted Averaging Operator (GLIVIFSEWA operator), General Linguistic Interval Valued Intuitionistic Fuzzy Soft Expert Ordered Weighted Averaging Operator (GLIVIFSEOWA operator), General Linguistic Interval Valued Intuitionistic Fuzzy Soft Expert Weighted Geometric Operator (GLIVIFSEWG operator), General Linguistic Interval Valued Intuitionistic Fuzzy Soft Expert Ordered Weighted Geometric Operator (GLIVIFSEOWG operator). Furthermore, certain properties such as Idempotency, Monotonicity, Boundedness and few more are provided for each aggregated operator to elucidate the concepts. At the end of this chapter, a decision analysis problem is formulated based on GLIVIFESESs and solved by applying the defined aggregated operations on GLIVIFSESs.

In sixth Chapter, the thesis concludes by discussing the key findings and conclusions of the research. It also provides an overview of future work that can be pursued. The algebraic operations, examples and decision analysis problem conducted throughout the thesis are summarized and presented, revealing insights into the research questions and objectives stated at the beginning.

1.1 Research Objectives:

2-Dimensional Linguistic Intuitionistic fuzzy variables can be modified as Generalized Linguistic Interval Valued Intuitionistic Fuzzy Soft Expert Sets (GLIVIFSESs) in which intervals have been added to make decision analysis easier to handle and broader their range.

1.2 Problem statement:

We studied [56] Verma's paper, In this paper 2-Dimensional Linguistic Variables and their applications have been discussed. We aim to modify and generalize the results of the above-mentioned paper. We also construct a decision analysis problem in this work by using operator theory.

1.3 Research Questions:

In this work, we modified the concept of 2-DLIF variables.

• How to make a sweeping statement about the idea of Two-Dimensional Linguistic Variables?

• How to examine and generalize the notion of Two-Dimensional LIFVs?

• Since IFSs provide existence and non-existence degrees as singular values from [0,1], How to over simplify this concept by exhibiting expert opinions as intervals?

•How to generalize and develop the idea of Two-dimension LIF aggregations operators to more general concepts of general linguistics IVIFSE operators?

1.4 Significance of the study:

Generalized Linguistic Interval Valued Intuitionistic Fuzzy Soft Expert Sets (GLIV-IFSESs) have the benefit over the current theories in that it gives better results for each interval separately. To deal with vague and unclear information in decision-making, the definition of (LIFSSs) is presented. In our work, we aim to generalize the concept by using IVIFSESs. Also we define GLIVIFSESs by combining the notion of LVs and IFSESs also modified these concepts by adding intervals to make decision analysis easier to handle and broader their range.

1.5 Procedure:

In this thesis, first of all we identify the research problem by reading the related papers to the fuzzy sets, linguistic variables, IVIF sets, IF sets, Two-dimensional LIF sets etc, After the problem analysis, we write the review of related literature, then do some theoretical work, as we give the proper definition of our new structure i.e. GLIVIFSE Sets, define operations of GLIVIFSESs, give examples for better understanding, then defined operationals laws also proved them and disproved them with counter examples where needed, then we define aggregational operators such as GLIVIFSEWA, GLIVIFSEOWA, GLIVIFSEWG, and GLIVIFSEOWA. At the end we construct a decision analysis real life problem to check the authenticity of newly constructed structure.

Chapter 2

Literature Review

The set theory or crisp set theory in 1870, [11] was introduced by German mathematicians Cantor and Dedekind, which plays an important role in some mathematical concepts. Most of our day by day life obstacles in commerce, engineering medical sciences, environmental sciences, management and social sciences are repeatedly knotted with information sets, that are imprecise and indeterministic rather than crisp and precise. These complexities were overcome by a meaningful notion of fuzzy set theory which Zadeh [63] who was appointed as a computer science's professor in Berkeley, at the Unversity of California, proposed in 1965 in his first seminal paper. The fuzzy set theory has implementation in many areas/fields of engineering and mathematical sciences, inclusive of artificial intelligence, computer sciences, decision-making, management sciences and operational research. Numerous domains have benefited greatly from FS theory. In FS theory only the degree of membership or association was defined. Although many researchers and mathematicians from all over the world put their effort to build decision analysis problems using fuzzy sets,

[13] As in 2014 Cabbrerizo et al. use fuzzy sets in their decision making problems.

Krasimir Atanasov in 1983, [6] generalize the Zadeh's fuzzy set by assigning each element a non-membership degree along with their membership degree. Atanasov named his set as Intuitionistic fuzzy set, in which he defined the membership degree MD and non-membership degree. Latter on R. Verma [47, 48, 49, 50, 51, 52, 53, 54, 55] from 2011- 2015 works on intuitionistic fuzzy sets in multi-directions.

Zadeh [64] broaden the nation of Fuzzy sets by Interval-valued fuzzy sets in (1970s).Further Atanasov and Gargov in 1989 [7] widen the abstraction or conviction of IFSs IVIFSs, in which belongingness and non-belongingness were defined in terms of interval. In 1994 Attanassov,[8] defined some new operations on Intuionistic Fuzzy Sets. In 1996 Burillo, et al. [10] define entropy intuitionistic fuzzy sets (IFSs.) and they also do their efforts on interval-valued fuzzy sets. In 1999 Attanassov modified his work on the same structure IVIFSs. For its admirable resilience and swiftness in coping with ambiguity or vagueness the view of FSs/IVFSs/IFSs. IVIFSs. has been extensively inspected and practiced in various fields.

It is essential to note down that the already existing theories in accordance with the intuitionistic fuzzy sets have examined only quantitative features. Even though, in many real-life circumstances, the IFSs can not signify the ambiguous information broadly because the expert or decision maker may anticipate using the linguistic variables to show their preferred values towards the object. For example, if a person requests his friend please bring 2 kg Red apples for him, the person may use some linguistic terms, such as Perfectly red, Almost red, Slightly red, or Not red etc. In 1975 Zadeh [65, 66], gives the notion of

Linguistic variables, also put his efforts on defining its aplications and he also utilized its interpretation on FS. Definitely, the linguistic variables present more flexibility for modeling vague and uncertain information than do other variables.

The work of Zeshui [59] on linguistic assessments for MP-MADM was completed in 2009. The concept of linguistic intuitionistic FSs has been explained by Zhang in 2014, [67] while keeping these advantages in mind. They involve the encapsulation of the NMD and the MD in lLVs, instead of numerical values. He takes his task to solve the linguistic intuitionistic FSs that were being faced by MAGDM students in order to solve problems. Also, in the same year Zhang defined some aggregational operators for the LIFNs and talk about their properties or applications and purposes in decision-making. [16] thesis, introducing some new weighted averaging operators for LIFNs was introduced by Chen et al. in 2015. Liu. P. Y. Wang, [35] ,in 2016, In addition to improving operational laws in 2-dimension uncertain linguistic information, they worked on a weighted average operator for 2DULIFS, a WA operator for 2-dimension UL knowledge, as well as a weighted average operator for 2-dimension UL knowledge. [33] , In the very next year i.e. in 2017 Liu and Q in recognized the aggregation operators of linguistic intuitionistic fuzzy information such as the Maclaurin symmetric mean. Peng et al. in 2018 proposed the Frank Heronian means operators to estimate some real world problems in the linguistic intuitionist fuzzy environment by conducting Frank mean operations on LIFSs and defining Frank Heronian means operators for these real world problems. Later on Cabrerizo, Pedrycz, et al. [14] , they use linguistic informations to solve the problems in decision making. Not long ago, in 2020, [56, 57], Verma and Marigo, works on linguistic variables in different directions such as they established its application on MAGDM.There is no doubt that linguistic decision analysis (LDA) is a suitable, appropriate and also is an applicable method for solving many complex real life decision analysis problems. However, the LDA approach always includes the loss of information. This leads to a significant error in the final result as a result of biased selection.

In addition to what we discussed earlier, LIFNs allow us to demonstrate real-life information in a more comprehensive manner by providing a supplementary degree of freedom. The 2-DL information interpretation model, in contrast, focuses exclusively on the degree of satisfaction to alternatives or attributes and the consistency of the assessment knowledge or information in order to represent the knowledge or information. In the context of real life applications, both information interpretation models are subject to certain limitations. By combining the 2 LD and LIFNs into a particular or specific formulation, Verma in 2020 proposed a fusion information or knowledge interpretation model to overcome these limitations. In this model described by Verma, is an information model based on linguistic intuitionistic fuzzy information in two dimensions (2 DLIF). In many real-life circumstances, it serves uncertain high quality decision information or knowledge in the shape of LIFNs that represent both one-dimensional and two-dimensional knowledge or information, which makes it stronger and more effective.

In 1999 Molodtsov [40] developed the motion of soft set theory which was totally a recent approach for modeling obscurity and imprecisions, also since the theory is well suited to parametric measurements, it attracts the attention or thinking of a large number of researchers. Recently, SS theory work is progressing promptly. Biswas, Maji, and Roy in 2003 [38] they talk about decision analysis problems through the soft sets and fuzzy soft sets, they characterized some operations on soft sets e.g. union and intersection. De, et al (2001) [17] They also familiarized the soft set with the decision-analysis problems. In both pure and applied mathematics and sciences, fuzzy soft set theory plays a considerable role in developing knowledge. Later on Jaing, Y. et al (2010) [24] works on IVIFSESs and their properties.

In this thesis, we have explored the decision-analysis problems under the GLIV-IFSESs. Firstly, this thesis has represented the notion of recent structure Generalized Linguistic Interval Valued Intuitionistic Fuzzy Soft Expert (GLIVIFSESs) by combining the concetpts of LVs and IFSESs also modified these concepts by adding intervals to make decision analysis easier to handle and broader their range, also gives the proper definition of GLIVIFSESs and then described some basic operations and then operational laws on them. Which is encouraged and motivated by the research being made in this field. One of the central features of GLIVIFSESs is that it associates the advantages of 2-DLIFVs and IVFSESs in a particular formulation. So, we are able to characterize ambiguous or in more pure and reasonable approach, fuzzy knowledge can be expressed. GLIVIFSESs have the benefit over the current theories in that it gives better results for each interval separately.

Moreover, we demonstrate the operations for generalized Linguistic interval-valued intuitionistic FSE sets. We characterize and prove the aggregational operational laws and explain them with associated examples, effectively using the resolvent technique. As we have established numerous aggregational operators such as the GLIVIFSEWA operator, GLIVIFSEOWA operator, GLIVIFSEWG operator, GLIVIFSEOWG operator for accumulating GLIVIFSESs information attained from several sources.Atanassov, [6] Generalized Fuzzy Sets which were proposed by Zadeh [63] by introducing Intuitionistic Fuzzy Sets. In this thesis, Zadeh also discussed its applications. Atanassov, Gargov [7] introduced the Intuitionistic Interval Valued Fuzzy Sets by defining the membership degree (MD) and non-membership degree (NMD) on closed unit subintervals. In 2011, [12] N. Cagman, Enginoglu, Citak, Described the FS set theory in multiple directions. In this paper, he also defined the fuzzy soft aggregation operator and applied it to some examples to generalize the results. In 2001 [22] F.Herrera, L.Martı´nez developed the model which was related to linguistic variables in multiple directions, In this paper, they explain finite linguistic terms with different examples. [25], In 2012, Jun et al. discussed the ambiguous or fuzzy information.

R.Verma and Sharma, put their efforts on fuzzy sets in many directions,[47] In 2011, they works on Intuitionistic Fuzzy Sets, which were introduced by Attanasov in 1983 [6] they defined some new properties and results on intuitionistic fuzzy sets. They also generalized the intuitionistic fuzzy relative knowledge and discuss its properties in 2012, in their paper [48]. In their different paper they explain the notion of fuzzy sets (FS), IVF Sets. In the start of 2013, [49] they construct the applications to the multiple attribute decision analysis on intuitionistic fuzzy sets. Later on in 2013,[50] they defined some exponential function on intuitionistic fuzzy sets. In their paper they defined, and explained some fundamental concepts of fuzzy entropy, and an exponential intuitionistic fuzzy entropy measure, this measure is a generalization of exponential fuzzy entropy proposed by Pal and Pal [42] which is also mentioned in Verma and Sharma's 2011 paper [47] based on the concepts of Attanassov's intuitionistic fuzzy sets [6] they also proposed a relation between exponential and exponential fuzzy entropy. They also introduced few interesting properties on the same structure. Lastly they give a numerical example to show that the proposed entropy . In the same year Verma and Sharma works together on intuitionistic fuzzy sets to propose some new consequences.In their paper [51] they defiend also proved the newly proposed properties. In the start of 2014 [52] Verma and Sharma proposed some new ordered measures by using the notion of Rènyi's entropy, they proposed intuitionistic fuzzy entropy having on the intuitionistic fuzzy sets theory. This measure is a generalization of fuzzy entropy of order [42] which was proposed by Pal and Bhandari and intuitionistic fuzzy entropy. In 2010, defined and explained by Vlachos and Sergiadis in thier mentioned [58] paper, they studied the four important and few other properties of the defined measure distinctly establishes the sustainability of the measure as an intuitionistic fuzzy entropy. At the end of their paper they illustrate a numerical example to describe the proposed or defined entropy measure.

At the end of 2014 Verma and Sharma, proposed a new accuracy measure theory, also works on its applications on MCDM. In the start of 2015, [54, 55] Verma and Sharma works on different directions of intuitionistic fuzzy sets. R.Verma, [56] in this paper, real-life decision analysis problems are addressed by using linguistic decision-making tools, also developed several aggregation operators, and discussed their properties and lastly also illustrate the examples with them to make to the concept more clear.

Zadeh, [63] defines give the basic concepts of the Fuzzy sets and discussed the degree of membership. Zadeh, [64] in this paper, he defines the interval-valued fuzzy sets. In the start of 1975 Lotfi Aliasker Zadeh, [65] modifies his work by introducing the linguistic variables and linguistic terms.

Chapter 3

Preliminaries

Throughout this section, we will be reclaiming some of the fundamental notions associated with this work, such as Crisp set, Fuzzy set (FS), Intuitionistic fuzzy set (IFS), Interval valued fuzzy set (IVFS), Linguistic intuitionistic fuzzy set, Linguistic variables, Linguistic intuitionistic fuzzy set, Linguistic variables, Soft set, Soft expert set (SES), Generalized interval valued fuzzy soft set, in order to build a new structure from previous researches.

3.1 Basic Definitions

Definition 1 *[11] The set theory or crisp set theory in 1870 was introduced by German mathematicians Cantor and Dedekind, which plays vital role in some mathematical concepts, "A crisp set is a collection of an object that consist of fixed, strict, well-defined, and clear boundaries to show the object's belongingness to (or non belongingness) to a set".*

Consequently, frequent real-life problems cannot be managed by classical set theory as the sets are evaluated only by the numbers $\{0,1\}$.This difficulty was overcome by presenting fuzzy sets by Zadeh.

Definition 2 *[63] In 1965, Zadeh who was appointed as a computer science professor in Berkeley, at the University of California introduced the notion of fuzzy set which is defined as "Fuzzy set is a set, which is based on the ambiguous or imprecise boundary"*

Limitations of the fuzzy sets are, as these sets deal only with $[0,1]$-valued mapping generally recognized as membership functions.This difficulty was overwhelmed by Atanasov in 1983, by giving the notion of Intuitionistic Fuzzy Sets.

Definition 3 *[6] Atanasov in 1983, generalize the Zadeh's fuzzy set by assigning each element a non-membership degree along with their membership degree. Atanasov named his set as Intuitionistic Fuzzy Set (IFS), which is defined as "Let* $\hat{X} = \left\{\hat{x}_1, \hat{x}_2, ..., \hat{x}_n\right\}$ *be a non-empty finite set. An Intuitionistie fuzzy set* $\tilde{I}$ *in*$\hat{X}$ *is represented as*

$$\tilde{I} = \left\{\left\langle \hat{x}, \breve{\xi}\left(\hat{x}\right), \tilde{\Psi}\left(\hat{x}\right)\right\rangle \mid \hat{x} \in \hat{X}\right\}$$

and $\breve{\xi} : \hat{X} \longrightarrow [0\ ,\ 1]$, *denoted the membership degree and* $\tilde{\Psi} : \hat{X} \longrightarrow [0\ ,\ 1]$, *denoted the non-membership degree (NMD), of an element* $\hat{x} \in \hat{X}$ *to the set* $\tilde{I}$, *with the condition that* $0 \leq \breve{\xi}\left(\hat{x}\right) + \tilde{\Psi}\left(\hat{x}\right) \leq 1$, *futher* $1 - \breve{\xi}\left(\hat{x}\right) - \tilde{\Psi}\left(\hat{x}\right)$ *is called degree of reluctance or hesitancy for any number* $\hat{x} \in \hat{X}$, *The pair* $\left(\breve{\xi}\left(\hat{x}\right), \tilde{\Psi}\left(\hat{x}\right)\right)$ *is called Intuitionistic fuzzy number (IFN) and represented as* $\hat{X} = \left(\breve{\xi}\ ,\ \tilde{\Psi}\right)$.

Example 4 *One of the best example from our dauly life to go throuh the nation of Intuitionistic fuzzy set is, when a person took a part in election, stand for his/her desired seat* *in his/her favourite department then in intuistionistic fuzzy set, MD, NMD, and degree of hesitancy. is defnied as*

.***Degree of Membership****: He/She gets a vote in his/her favour.*

.***Degree of Non-Membership****: He/She gets vote against them.*

.***Degree of Hesitancy****: He/She gets reluctant or undecided vote.*

Definition 5 *"The notion of Linguistic varibles [65, 66, 67] is defined such as"By a linguistic variable, we mean a variable whose values are words or sentences in a natural or artificial language".*

Example 6 *Height is a linguistic variable if its values are linguistic rather than numerical, i.e., tall, not tall, very tall, quite tall, small,not small, not very small and not very tall, etc., rather than 4'9", 5'2", 5'6", 6', 6'4" etc.*

Definition 7 *In 2014 Zhang [67] defined the Linguistic Intuitionistic Fuzzy Number as"Let* $\tilde{X} = \{\tilde{x}_1, \tilde{x}_2, ...\tilde{x}_n\}$ *be a finite set which is also non-empty set, and* $\hat{S}_{[o,t]}$ *is a linguistic term set (continuous).A LIFS* $\tilde{A}$ *in* $\tilde{X}$ *is given as* $\tilde{A} = \left\{\left\langle \tilde{x}, S\zeta_{\bar{A}}(\tilde{x}), S\tilde{\Psi}_{\bar{A}}(\tilde{x})\right\rangle | \tilde{x} \in \tilde{X}\right\}$*, where* $S_{\zeta_{\bar{A}}(\tilde{x})}, S_{\tilde{\Psi}_{\bar{A}}(\tilde{x})} \in \hat{S}_{[0,t]}$ *describes the membership degree (MD) and the non-membership degree (NMD) of the element* $\tilde{x} \in \tilde{X}$ *to the set*$\tilde{A}$*, accordingly.For any* $\tilde{x} \in \tilde{X}$ $0 \leq \zeta_{\bar{A}} + \tilde{\Psi}_{\bar{A}} \leq t$ *is always satisfied, and the linguistic intuitionistic index of* $\tilde{x}$ *to* $\tilde{A}$ *is defined as* $S_{\eta_{\bar{A}}}(\tilde{x}) = S_t - \zeta_{\bar{A}}(\tilde{x})$ *.For a given element* $\tilde{x}$*, the pair* $\left(\zeta_{\bar{A}}(\tilde{x}), \tilde{\Psi}_{\bar{A}}(\tilde{x})\right)$ *is represented as* ***Linguistic Intuitionistic Fuzzy Number*** *(LIFN), and it is simplified as* $\tilde{\alpha} = \left(S_{\zeta_{\bar{x}}}, S_{\tilde{\Psi}_{\bar{x}}}\right)$*".*

Definition 8 *[68] In 2009, Zhu et al. They defined the two dimensional linguistic variable as "Let* $\hat{S}$=$\left\{\dot{S}_d | d = 0, 1, ..., t\right\}$ *and* $\hat{S}^{(2)} = \hat{S} = \left\{\ddot{S}_{d'} | d' = 0, 1, ..., t\right\}$ *are two linguistic term* *sets having the odd cardinalities, where t and* t' *are natural numbers. A variable* $\hat{S} = \left(\dot{S}_a, \ddot{S}_a\right)$ *is represented as two dimensional linguistic variable, where* $\dot{s}_a \in \hat{S}^{(1)}$ *is 1-dimension linguistic knowledge /information describing the expert's or decison maker's prefrence value for an estimated attribute/object, while* $\ddot{s}_b \in \hat{S}^{(2)}$ *is 2-dimension linguistic knowledge/ information representing the subjective judgment on the reliableness of their given consequences".*

Definition 9 *[14]"A pair* $(F, \hat{S})$ *is called a* ***Soft Expert Set*** *over* $\hat{U}$*, where F is a mapping given by* $F : \hat{S} \to P(\hat{U})$*. Thus, a soft expert set can be considered as a soft set in which parameter set is replaced with Cartesian product of set of parameters, set of experts and set of opinions i-e* $Z = E \times X \times O$*".*

Definition 10 *[3], [4]"A pair* $(F, \hat{S})$ *is called a* ***Fuzzy Soft Expert Set*** *over* $\hat{U}$*, where F is a mapping given by* $F : \hat{S} \to I^{\hat{U}}$ *where* $I^{\hat{U}}$ *denoted all fuzzy subsets of* $\hat{U}$*".*

Definition 11 *[5]"Let* $\hat{U} = \{\hat{u}_1, \hat{u}_2, ..., \hat{u}_n\}$ *be the universal set of elements and* $\tilde{P} = \{\tilde{v}_1, \tilde{v}_2, ..., \tilde{v}_n\}$ *be the universal set of parameters. The pair* $\left(\hat{U}, \tilde{P}\right)$ *wll be called a soft universe. Let* $F^* : \tilde{P} \to Int\left(\hat{U}\right)$ *and S be a fuzzy set of* $\hat{E}$*, i.e.* $S : \tilde{P} \to I = [0, 1]$*, where* $Int\left(\hat{U}\right)$ *is the set of all interval-valued fuzzy subsets on* $\hat{U}$*. Let* $F^*_\eta : \tilde{P} \to Int\left(\hat{U}\right) \times I$ *be a function defined as follows:*

$$F^*_s(\tilde{v}) = \left(F^*(\tilde{v})(\hat{u}), S(\tilde{v})\right), \forall \hat{u} \in \hat{U}$$

Then F^*_s *is called a* ***Generalised Interval Valued Fuzzy Soft Set*** *(GIVFS set) over the soft set* $\left(\hat{U}, \tilde{P}\right)$*".*

Chapter 4

General Linguistic Interval Valued Intuitionistic Fuzzy Soft Expert Sets (GLIVIFSESs)

4.1 Introduction

It is a combination of the concepts of linguistic value sets and interval valued intuitionistic fuzzy soft expert sets that we present here, a General Linguistic Interval Valued Intuitionistic Fuzzy Soft Expert Set (GLIVIFSES), is a linguistic value set and an interval valued intuitionistic fuzzy soft expert set doing the same job that is presented in this chapter also modified these concepts by adding intervals to make decision analysis easier to handle and broader their range, also described the proper definition of GLIVIFSESs and give some basic operations and then operational laws on them. Which is encouraged and motivated by the research being made in this field.

4.2 Definitions

Definition 12 *Let* $\hat{S}^{[1]}$ *and* $\hat{S}^{[2]}$ *be two General Linguistic Interval Valued Intuitisnistic Fuzzy Soft Expert Terms, with odd cardinality, these terms are defined as:*

$$\begin{aligned}\hat{S}^{[1]} &= \Big\{\dot{S}_z \mid z = \Big\{d_1, d_1^{'}\Big\} \wedge\ d_1 = \Big[\breve{\xi}, \breve{\xi}^{'}\Big],\ d_1^{'} = \Big[\tilde{\psi}, \tilde{\psi}^{'}\Big]; \\ &\dot{S}_{[0,0]} \subseteq \dot{S}_{[\breve{\xi},\breve{\xi}^{'}]} + \dot{S}_{[\tilde{\Psi},\tilde{\Psi}']} \subseteq \dot{S}_{[t,t]}\Big\}\end{aligned}$$

and

$$\begin{aligned}\hat{S}^{[2]} &= \Big\{\ddot{S}_{z'} \mid z^{'} = \Big\{d_2, d_2^{'}\Big\} \wedge\ d_2 = \Big[\breve{\mu}, \breve{\mu}^{'}\Big],\ d_2^{'} = \Big[\breve{\nu}, \breve{\nu}^{'}\Big]; \\ &\ddot{S}_{[0,\ 0]} \subseteq \ddot{S}_{[\breve{\mu},\breve{\mu}']} + \ddot{S}_{[\breve{\nu},\breve{\nu}']} \subseteq \ddot{S}_{[t',t']}\Big\}\end{aligned}$$

Where $t, t^{'} \in 2n$, $n \in N$ *(Natural Numbers)*

$$\begin{aligned}A\ Set\quad \hat{S} &= \Big(\Big\langle \dot{S}_{[\breve{\xi},\breve{\xi}^{'}]}, \dot{S}_{[\tilde{\Psi},\tilde{\Psi}']}\Big\rangle, \Big\langle \ddot{S}_{[\breve{\mu},\breve{\mu}']}, \ddot{S}_{[\breve{\nu},\breve{\nu}']}\Big\rangle; \\ &\Big[\breve{\xi}, \breve{\xi}^{'}\Big], \Big[\tilde{\Psi}, \tilde{\Psi}'\Big] \subseteq [0,t]; [\breve{\mu}, \breve{\mu}'], [\breve{\nu}, \breve{\nu}'] \subseteq \Big[0, t^{'}\Big]\Big)\end{aligned} \tag{4.1}$$

is called General Linguistic Interval Valued Intuitionistic Fuzzy Soft Expert Sets (GLIVIFSESs).

$$Where\quad \Big(\Big\langle \dot{S}_{[\breve{\xi},\breve{\xi}']}, \dot{S}_{[\tilde{\Psi},\tilde{\Psi}']}\Big\rangle; \dot{S}_{[0,0]} \subseteq \dot{S}_{[\breve{\xi},\breve{\xi}^{'}]} + \dot{S}_{[\tilde{\Psi},\tilde{\Psi}']} \subseteq \dot{S}_{[t,t]}\Big) \in \hat{S}^{[1]}$$

is representing decision maker's (expert's) assessment value for an evaluated objects(Attributes),

$$While\quad \Big(\Big\langle \ddot{S}_{[\breve{\mu},\breve{\mu}']}, \ddot{S}_{[\breve{\nu},\breve{\nu}']}\Big\rangle;\ \ddot{S}_{[0.0,\ 0.0]} \subseteq\ \ddot{S}_{[\breve{\mu},\breve{\mu}']} +\ \ddot{S}_{[\breve{\nu},\breve{\nu}']} \subseteq \ddot{S}_{[t',t']}\Big) \in \hat{S}^{[2]}$$

represents GLIVIFSE informations describing the subjective evaluation on the reliability of expert's given results. Also $\dot{S}_{\left[\breve{\xi},\breve{\xi}'\right]}, \ddot{S}_{[\breve{\mu},\breve{\mu}']}$ *defines degree of membership and* $\dot{S}_{\left[\tilde{\Psi},\tilde{\Psi}'\right]}, \ddot{S}_{[\breve{\nu},\breve{\nu}']}$ *defines degree of non-membership in* $\hat{S}^{[1]}$ *and* $\hat{S}^{[2]}$.

Definition 13 *GLIVIFSESs is called Original GLIVIFSESs, If*

$$\dot{S}_{[\breve{\xi},\breve{\xi}']}, \dot{S}_{[\tilde{\Psi},\tilde{\Psi}']} \in \hat{S}^{[1]} and \ddot{S}_{[\breve{\mu},\breve{\mu}']}, \ddot{S}_{[\breve{\nu},\breve{\nu}']} \in \hat{S}^{[2]}$$

GLIVIFSESs is called Virtual GLIVIFSESs, If

$$\dot{S}_{[\breve{\xi},\breve{\xi}']}, \dot{S}_{[\tilde{\Psi},\tilde{\Psi}']} \notin \hat{S}^{[1]} or \ddot{S}_{[\breve{\mu},\breve{\mu}']}, \ddot{S}_{[\breve{\nu},\breve{\nu}']} \in \hat{S}^{[2]}$$

or

$$\dot{S}_{[\breve{\xi},\breve{\xi}']}, \dot{S}_{[\tilde{\Psi},\tilde{\Psi}']} \in \hat{S}^{[1]} or \ddot{S}_{[\breve{\mu},\breve{\mu}']}, \ddot{S}_{[\breve{\nu},\breve{\nu}']} \notin \hat{S}$$

or

$$\dot{S}_{[\breve{\xi},\breve{\xi}']}, \dot{S}_{[\tilde{\Psi},\tilde{\Psi}']} \notin \hat{S}^{[1]} and \ddot{S}_{[\breve{\mu},\breve{\mu}']}, \ddot{S}_{[\breve{\nu},\breve{\nu}']} \notin \hat{S}^{[2]}$$

Definition 14 *For the comparision of any two GLIVIFSESs the score function and accuracy rule / function are described as follow:*

Score Function of GLIVIFSESs denoted by $\tilde{\sigma}$ *and defined as:*

$$\tilde{\sigma}\left(\hat{S}\right) = \tilde{\sigma}_{\left[\left(\frac{t+\breve{\xi}+\breve{\xi}'-\tilde{\Psi}-\tilde{\Psi}'}{3t}\right)\times\left(\frac{t+\breve{\mu}+\breve{\mu}'-\breve{\nu}-\breve{\nu}'}{3t'}\right)\right]}$$

As

$$\begin{aligned}\hat{S} &= \left(\left\langle \dot{S}_{[\breve{\xi},\breve{\xi}']}, \dot{S}_{[\tilde{\Psi},\tilde{\Psi}']}\right\rangle, \left\langle \ddot{S}_{[\breve{\mu},\breve{\mu}']}, \ddot{S}_{[\breve{\nu},\breve{\nu}']}\right\rangle;\right.\\ &\left.\left[\breve{\xi},\breve{\xi}'\right], \left[\tilde{\Psi},\tilde{\Psi}'\right] \subseteq [0,t]; \left[\breve{\mu},\breve{\mu}'\right], \left[\breve{\nu},\breve{\nu}'\right] \subseteq \left[0,t'\right]\right)\end{aligned}$$

Where

$$\left(\left\langle \dot{S}_{\left[\breve{\xi},\breve{\xi}'\right]},\dot{S}_{\left[\tilde{\Psi},\tilde{\Psi}'\right]}\right\rangle ;\dot{S}_{[0,0]}\subseteq\dot{S}_{\left[\breve{\xi},\breve{\xi}'\right]}+\dot{S}_{\left[\tilde{\Psi},\tilde{\Psi}'\right]}\subseteq\dot{S}_{[t,t]}\right)\in\hat{S}^{[1]}$$
$$\left(\left\langle \ddot{S}_{[\breve{\mu},\breve{\mu}']},\ddot{S}_{[\breve{\nu},\breve{\nu}']}\right\rangle ;\ddot{S}_{[0,\ 0]}\subseteq\ddot{S}_{\left[\breve{\mu},\breve{\mu}'\right]}+\ddot{S}_{[\breve{\nu},\breve{\nu}']}\subseteq\ddot{S}_{\left[t',t'\right]}\right)\in\hat{S}^{[2]}$$

$\hat{S}^{[1]}$ and $\hat{S}^{[2]}$ are two gernalized interval valued intuitionistic fuzzy soft expert terms, where score function lies between the interval $[-1,1]$ that is $\tilde{\sigma}\left(\hat{S}\right)\in[-1,1]$.

Example 15 *let* $\hat{S}_1,\hat{S}_2,\hat{S}_3$ *and* $\hat{S}_4$ *are four General Linguistic Interval Valued Intuitionistic Fuzzy Soft Expert Sets.*

$$\hat{S}_1=\left(\left\langle \dot{S}_{[0.2\ ,\ 0.5]},\ \dot{S}_{[0.3\ ,\ 0.5]}\right\rangle ,\ \left\langle \ddot{S}_{[0.2\ ,\ 0.7]},\ \ddot{S}_{[0.1\ ,\ 0.3]}\right\rangle ,\textit{where } t=6,\textit{ and } t'=4\right)$$

$$\hat{S}_2=\left(\left\langle \dot{S}_{[0.5\ ,\ 0.7]},\ \dot{S}_{[0.1\ ,\ 0.3]}\right\rangle ,\ \left\langle \ddot{S}_{[0.6\ ,\ 0.8]},\ \ddot{S}_{[0.1\ ,\ 0.2]}\right\rangle ,\textit{where } t=6,\textit{ and } t'=4\right),$$

$$\hat{S}_3=\left(\left\langle \dot{S}_{[0.5\ ,\ 0.7]},\ \dot{S}_{[0.1\ ,\ 0.3]}\right\rangle ,\ \left\langle \ddot{S}_{[0.4\ ,\ 0.6]},\ \ddot{S}_{[0.2\ ,\ 0.4]}\right\rangle ,\textit{where } t=6,\textit{ and } t'=4\right),$$

$$\hat{S}_4=\left(\left\langle \dot{S}_{[0.4\ ,\ 0.8]},\ \dot{S}_{[0.1\ ,\ 0.2]}\right\rangle ,\ \left\langle \ddot{S}_{[0.5\ ,\ 0.8]},\ \ddot{S}_{[0.1\ ,\ 0.2]}\right\rangle ,\textit{where } t=6,\textit{ and } t'=4\right).$$

$$\begin{aligned}\tilde{\sigma}\left(\hat{S}\right) &= \tilde{\sigma}_{\left[\left(\frac{t+\breve{\xi}+\breve{\xi}'-\tilde{\Psi}-\tilde{\Psi}'}{3t}\right)\times\left(\frac{t+\breve{\mu}+\breve{\mu}'-\breve{\nu}-\breve{\nu}'}{3t'}\right)\right]}\\ \tilde{\sigma}\left(\hat{S}_1\right) &= \tilde{\sigma}_{\left[\left(\frac{6+0.2+0.5-0.3-0.5}{3(6)}\right)\times\left(\frac{4+0.2+0.7-0.1-0.3}{3(4)}\right)\right]}\\ &= \tilde{\sigma}_{[0.12292]}\end{aligned}$$

$$\begin{aligned}\tilde{\sigma}\left(\hat{S}_2\right) &= \tilde{\sigma}_{\left[\left(\frac{6+0.5+0.7-0.1-0.3}{3(6)}\right)\times\left(\frac{4+0.6+0.8-0.1-0.2}{3(4)}\right)\right]}\\ &= \tilde{\sigma}_{[0.16056]}\end{aligned}$$

$$\tilde{\sigma}\left(\hat{S}_3\right) = \tilde{\sigma}_{\left[\left(\frac{6+0.5+07-0.1-0.3}{3(6)}\right)\times\left(\frac{4+(0.4)+0.6-0.2-0.4}{3(4)}\right)\right]}$$
$$= \tilde{\sigma}_{[0.13852]}$$

$$\tilde{\sigma}\left(\hat{S}_4\right) = \tilde{\sigma}_{\left[\left(\frac{6+0.4+0.8-0.1-0.2}{3(6)}\right)\times\left(\frac{4+0.5+0.8-0.1-0.2}{3(4)}\right)\right]}$$
$$= \tilde{\sigma}_{[0.15972]}$$

Thus,

$$\tilde{\sigma}\left(\hat{S}_2\right) > \tilde{\sigma}\left(\hat{S}_4\right) > \tilde{\sigma}\left(\hat{S}_3\right) > \tilde{\sigma}\left(\hat{S}_1\right)$$

Which implies that $\hat{S}_2 > \hat{S}_4 > \hat{S}_3 > \hat{S}_1$ *is required ranking order of gernalized linguistic interval valued intuitionistic fuzzy soft expert sets.*

Definition 16 *Accuracy Function* $\hat{\rho}$ *of gernalized linguistic interval valued intuitionistic fuzzy soft expert sets (GLIVIFSESs) is defined as:*

$$\hat{\rho}\left(\hat{S}\right) = \hat{\rho}_{\left[\left(\frac{\natural+\check{\xi}+\check{\xi}'+\tilde{\Psi}+\tilde{\Psi}'}{3\natural}\right)\times\left(\frac{\natural+\check{\mu}+\check{\mu}'+\check{\nu}+\check{\nu}'}{3\natural'}\right)\right]}$$

The accuracy function lies among the interval $[0,1]$ *that is* $\hat{\rho}\left(\hat{S}\right) \in [0,1]$.

Example 17 *let* $\hat{S}_1, \hat{S}_2, \hat{S}_3$ *and* $\hat{S}_4$ *are four General Linguistic Interval Valued Intuitionistic Fuzzy Soft Expert Sets.*

$$\hat{S}_1 = \left(\left\langle \dot{S}_{[0.2\ ,\ 0.5]},\ \dot{S}_{[0.3\ ,\ 0.5]}\right\rangle,\ \left\langle \ddot{S}_{[0.2\ ,\ 0.7]},\ \ddot{S}_{[0.1\ ,\ 0.3]}\right\rangle,\ where\ \natural = 8, and\ \natural' = 6\right)$$

$$\hat{S}_2 = \left(\left\langle \dot{S}_{[0.5\ ,\ 0.7]},\ \dot{S}_{[0.1\ ,\ 0.3]}\right\rangle,\ \left\langle \ddot{S}_{[0.6\ ,\ 0.8]},\ \ddot{S}_{[0.1\ ,\ 0.2]}\right\rangle,\ where\ \natural = 8, and\ \natural' = 6\right),$$

$$\hat{S}_3 = \left(\left\langle \dot{S}_{[0.5\ ,\ 0.7]},\ \dot{S}_{[0.1\ ,\ 0.3]}\right\rangle,\ \left\langle \ddot{S}_{[0.4\ ,\ 0.6]},\ \ddot{S}_{[0.2\ ,\ 0.4]}\right\rangle,\ where\ \natural = 8, and\ \natural' = 6\right),$$

$$\hat{S}_4 = \left(\left\langle \dot{S}_{[0.4\ ,\ 0.8]},\ \dot{S}_{[0.1\ ,\ 0.2]}\right\rangle,\ \left\langle \ddot{S}_{[0.5\ ,\ 0.8]},\ \ddot{S}_{[0.1\ ,\ 0.2]}\right\rangle,\ where\ \natural = 8, and\ \natural' = 6\right).$$

then accuracy functions of above mentioned GLIVIFSES sets are as follow:

$$\hat{\rho}\left(\hat{S}_1\right) = \hat{\rho}_{\left[\left(\frac{8+0.2+0.5+0.3+0.5}{3(8)}\right)\times\left(\frac{6+0.2+0.7+0.1+0.3}{3(6)}\right)\right]}$$
$$\hat{\rho}\left(\hat{S}_1\right) = \hat{\rho}_{[0.1605]}$$

$$\hat{\rho}\left(\hat{S}_2\right) = \hat{\rho}_{\left[\left(\frac{8+0.5+0.7+0.1+0.3}{3(8)}\right)\times\left(\frac{6+0.6+0.8+0.1+0.2}{3(6)}\right)\right]}$$
$$\hat{\rho}\left(\hat{S}_2\right) = \delta_{[0.1711]}$$

$$\hat{\rho}\left(\hat{S}_3\right) = \hat{\rho}_{\left[\left(\frac{8+0.5+0.7+0.1+0.3}{3(8)}\right)\times\left(\frac{6+0.4+0.6+0.2+0.4}{3(6)}\right)\right]}$$
$$\hat{\rho}\left(\hat{S}_3\right) = \hat{\rho}_{[0.1689]}$$

$$\hat{\rho}\left(\hat{S}_4\right) = \hat{\rho}_{\left[\left(\frac{8+0.4+0.8+0.1+0.2}{3(8)}\right)\times\left(\frac{6+0.5+0.8+0.1+0.2}{3(6)}\right)\right]}$$
$$\hat{\rho}\left(\hat{S}_4\right) = \hat{\rho}_{[0.1671]}$$

Hence by using accuracy function $\hat{\rho}\left(\hat{S}\right)$ *on General linguistic interval valued intuitionistc fuzzy soft expert sets we get the following ranking order*

$$\hat{\rho}\left(\hat{S}_2\right) > \hat{\rho}\left(\hat{S}_3\right) > \hat{\rho}\left(\hat{S}_4\right) > \hat{\rho}\left(\hat{S}_1\right)$$

which implies that:

$$\hat{S}_2 > \hat{S}_3 > \hat{S}_4 > \hat{S}_1 \quad \textit{is required ranking order.}$$

The law of comparison is based on these two functions for two different GLIVIFSESs $\hat{S}_1$ *and* $\hat{S}_2$ *can be defined as follows:*

i) If $\tilde{\sigma}\left(\overset{\wedge}{S_1}\right) > \tilde{\sigma}\left(\overset{\wedge}{S_2}\right)$ *then* $\overset{\wedge}{S_1} > \overset{\wedge}{S_2}$*(order will remaing preserved)*
ii) If $\tilde{\sigma}\left(\overset{\wedge}{S_1}\right) > \tilde{\sigma}\left(\overset{\wedge}{S_2}\right)$ *then* $\hat{\rho}\left(\overset{\wedge}{S_1}\right) > \hat{\rho}\left(\overset{\wedge}{S_2}\right)$ *then* $\overset{\wedge}{S_1} > \overset{\wedge}{S_2}$
iii) If $\tilde{\sigma}\left(\overset{\wedge}{S_1}\right) = \tilde{\sigma}\left(\overset{\wedge}{S_2}\right)$ *and* $\hat{\rho}\left(\overset{\wedge}{S_1}\right) = \hat{\rho}\left(\overset{\wedge}{S_2}\right)$ *then* $\overset{\wedge}{S_1} = \overset{\wedge}{S_2}$

4.3 Algebraic Operations on General Linguistic Interval Valued Intuitionistc Fuzzy Soft Expert Sets GLIVIFSESs

Let $\hat{S}$, $\overset{\wedge}{S_1}$ and $\overset{\wedge}{S_2}$ be any three General Linguistic Interval Valued Intuitionistc Fuzzy Soft Expert Sets (GLIVIFSESs), with odd cardinality.

$$\begin{aligned}
\hat{S} &= \left(\left\langle \dot{S}_{[\breve{\xi},\breve{\xi}']}, \dot{S}_{[\tilde{\Psi},\tilde{\Psi}']}\right\rangle, \left\langle \ddot{S}_{[\breve{\mu},\breve{\mu}']}, \ddot{S}_{[\breve{\nu},\breve{\nu}']}\right\rangle;\right.\\
&\qquad \left.\left[\breve{\xi},\breve{\xi}'\right], \left[\tilde{\Psi},\tilde{\Psi}'\right] \subseteq [0,\mathfrak{t}];\ \left[\breve{\mu},\breve{\mu}'\right], \left[\breve{\nu},\breve{\nu}'\right] \subseteq \left[0,\mathfrak{t}'\right]\right),\\
\overset{\wedge}{S_1} &= \left(\left\langle \dot{S}_{[\breve{\xi}_1,\breve{\xi}_1']}, \dot{S}_{[\tilde{\Psi}_1,\tilde{\Psi}_1']}\right\rangle, \left\langle \ddot{S}_{[\breve{\mu}_1,\breve{\mu}_1']}, \ddot{S}_{[\breve{\nu}_1,\breve{\nu}_1']}\right\rangle;\right.\\
&\qquad \left.\left[\breve{\xi}_1,\breve{\xi}_1'\right], \left[\tilde{\psi}_1,\tilde{\psi}_1'\right] \subseteq [0,\mathfrak{t}];\ \left[\breve{\mu}_1,\breve{\mu}_1'\right], \left[\breve{\nu}_1,\breve{\nu}_1'\right] \subseteq \left[0,\mathfrak{t}'\right]\right),\\
\overset{\wedge}{S_2} &= \left(\left\langle \dot{S}_{[\breve{\xi}_2,\breve{\xi}_2']}, \dot{S}_{[\tilde{\Psi}_2,\tilde{\Psi}_2']}\right\rangle, \left\langle \ddot{S}_{[\breve{\mu}_2,\breve{\mu}_2']}, \ddot{S}_{[\breve{\nu}_2,\breve{\nu}_2']}\right\rangle;\right.\\
&\qquad \left.\left[\breve{\xi}_2,\breve{\xi}_2'\right], \left[\tilde{\psi}_2,\tilde{\psi}_2'\right] \subseteq [0,\mathfrak{t}];\ \left[\breve{\mu}_2,\breve{\mu}_2'\right], \left[\breve{\nu}_2,\breve{\nu}_2'\right] \subseteq \left[0,\mathfrak{t}'\right]\right).
\end{aligned}$$

Where

$$\begin{aligned}
\left(\left\langle \dot{S}_{[\breve{\xi}_i,\breve{\xi}_i']}, \dot{S}_{[\tilde{\Psi}_i,\tilde{\Psi}_i']}\right\rangle; \dot{S}_{[0,0]} \subseteq \dot{S}_{[\breve{\xi}_i,\breve{\xi}_i']} + \dot{S}_{[\tilde{\Psi}_i,\tilde{\Psi}_i']} \subseteq \dot{S}_{[\mathfrak{t},\mathfrak{t}]}\right) &\in \hat{S}^{[1]}\\
\left(\left\langle \ddot{S}_{[\breve{\mu}_i,\breve{\mu}_i']}, \ddot{S}_{[\breve{\nu}_i,\breve{\nu}_i']}\right\rangle; \ddot{S}_{[0,\,0]} \subseteq \ddot{S}_{[\breve{\mu}_i,\breve{\mu}_i']}, \ddot{S}_{[\breve{\nu}_i,\breve{\nu}_i']} \subseteq \ddot{S}_{[\mathfrak{t}',\mathfrak{t}']}\right) &\in \hat{S}^{[2]}
\end{aligned}$$

then following operations are defined as:

Definition 18 ***Sum*** *of two General Linguistic Interval Valued Intuitionistc Fuzzy Soft Expert Sets (**GLIVIFSESs**) is defined as:*

$$\hat{S}_1 \oplus \hat{S}_2 = \left(\begin{array}{c} \left\langle \dot{S}_{\left[\breve{\xi}_1+\breve{\xi}_2-\frac{\breve{\xi}_1\breve{\xi}_2}{\text{ţ}}, \breve{\xi}_1'+\breve{\xi}_2'-\frac{\breve{\xi}_1'\breve{\xi}_2'}{\text{ţ}}\right]}, \dot{S}_{\left[\frac{\tilde{\Psi}_1\tilde{\Psi}_2}{\text{ţ}}, \frac{\tilde{\Psi}_1'\tilde{\Psi}_2'}{\text{ţ}}\right]} \right\rangle, \left\langle \ddot{S}_{\left[\breve{\mu}_1+\breve{\mu}_2-\frac{\breve{\mu}_1\breve{\mu}_2}{\text{ţ}'}, \breve{\mu}_1'+\breve{\mu}_2'-\frac{\breve{\mu}_1'\breve{\mu}_2'}{\text{ţ}'}\right]}, \ddot{S}_{\left[\frac{\breve{\nu}_1\breve{\nu}_2}{\text{ţ}'}, \frac{\breve{\nu}_1'\breve{\nu}_2'}{\text{ţ}'}\right]} \right\rangle, \\ \left[\breve{\xi}_i, \breve{\xi}_i'\right], \left[\tilde{\Psi}_i, \tilde{\Psi}_i'\right] \subseteq [0, \text{ţ}] ; [\breve{\mu}_i, \breve{\mu}_i'], \left[\breve{\nu}_i, \breve{\nu}_i'\right] \subseteq \left[0, \text{ţ}'\right], \text{ where } i = 1, 2 \end{array} \right)$$

Example 19 *Let* $\hat{S}_1$ *and* $\hat{S}_2$ *be the two General Linguistic Interval Valued Intuitionistc Fuzzy Soft Expert Sets GLIVIFSESs.*

$$\hat{S}_1 = \left(\left\langle \dot{S}_{[0.4\ ,\ 0.5]}, \dot{S}_{[0.2\ ,\ 0.3]} \right\rangle, \left\langle \dot{S}_{[0.4\ ,\ 0.6]}, \dot{S}_{[0.2\ ,\ 0.4]} \right\rangle, \text{where ţ} = 6, \text{and ţ}' = 4 \right)$$
$$\hat{S}_2 = \left(\left\langle \dot{S}_{[0.5\ ,\ 0.7]}, \dot{S}_{[0.1\ ,\ 0.3]} \right\rangle, \left\langle \dot{S}_{[0.6\ ,\ 0.8]}, \dot{S}_{[0.1\ ,\ 0.2]} \right\rangle, \text{where ţ} = 6, \text{and ţ}' = 4 \right)$$

$$\hat{S}_1 \oplus \hat{S}_2 = \left(\begin{array}{c} \left\langle \dot{S}_{\left[0.4+0.5-\frac{(0.4)(0.5)}{6}, 0.5+0.7-\frac{(0.5)(0.7)}{6}\right]}, \dot{S}_{\left[\frac{(0.2)(0.1)}{6}, \frac{(0.3)(0.3)}{6}\right]} \right\rangle \\ \left\langle \ddot{S}_{\left[0.4+0.6-\frac{(0.4)(0.6)}{4}, 0.6+0.8-\frac{(0.6)(0.8)}{4}\right]}, \ddot{S}_{\left[\frac{(0.2)(0.1)}{4}, \frac{(0.4)(0.2)}{4}\right]} \right\rangle \end{array} \right)$$

$$\hat{S}_1 \oplus \hat{S}_2 = \left(\left\langle \dot{S}_{[0.8667,\ 1.142]}, \dot{S}_{[0.0033,\ 0.015]}, \ddot{S}_{[0.94,\ 1.28]}, \ddot{S}_{[0.005,\ 0.02]} \right\rangle \right)$$

Definition 20 ***Product*** *of two General Linguistic Interval Valued Intuitionistc Fuzzy Soft Expert Sets **GLIVIFSESs** is defined as:*

$$\hat{S}_1 \otimes \hat{S}_2 = \left(\begin{array}{c} \left\langle \dot{S}_{\left[\frac{\breve{\xi}_1\breve{\xi}_2}{\text{ţ}}, \frac{\breve{\xi}_1'\breve{\xi}_2'}{\text{ţ}}\right]}, \dot{S}_{\left[\tilde{\Psi}_1+\tilde{\Psi}_2-\frac{\tilde{\Psi}_1\tilde{\Psi}_2}{\text{ţ}}, \tilde{\Psi}_1'+\tilde{\Psi}_2'-\frac{\tilde{\Psi}_1'\tilde{\Psi}_2'}{\text{ţ}}\right]} \right\rangle, \\ \left\langle \ddot{S}_{\left[\frac{\breve{\mu}_1\breve{\mu}_2}{\text{ţ}'}, \frac{\breve{\mu}_1'\breve{\mu}_2'}{\text{ţ}'}\right]}, \ddot{S}_{\left[\breve{\nu}_1+\breve{\nu}_2-\frac{\breve{\nu}_1\breve{\nu}_2}{\text{ţ}'}, \breve{\nu}_1'+\breve{\nu}_2'-\frac{\breve{\nu}_1'\breve{\nu}_2'}{\text{ţ}'}\right]} \right\rangle \end{array} \right)$$

Example 21 *Let* $\hat{S}_1$ *and* $\hat{S}_2$ *be the two General Linguistic Interval Valued Intuitionistc Fuzzy Soft Expert Sets i.e. GLIVIFSESs.*

$$\hat{S}_1 = \left(\left\langle \dot{S}_{[0.4\ ,\ 0.5]}, \dot{S}_{[0.2\ ,\ 0.3]} \right\rangle, \left\langle \ddot{S}_{[0.4\ ,\ 0.6]}, \ddot{S}_{[0.2\ ,\ 0.4]} \right\rangle, \text{where ţ} = 6, \text{and ţ}' = 4 \right)$$
$$\hat{S}_2 = \left(\left\langle \dot{S}_{[0.5\ ,\ 0.7]}, \dot{S}_{[0.1\ ,\ 0.3]} \right\rangle, \left\langle \ddot{S}_{[0.6\ ,\ 0.8]}, \ddot{S}_{[0.1\ ,\ 0.2]} \right\rangle, \text{where ţ} = 6, \text{and ţ}' = 4 \right)$$

then

$$\hat{S}_1 \otimes \hat{S}_2 = \begin{pmatrix} \left\langle \dot{S}_{\left[\frac{0.4(0.5)}{6},\ \frac{0.5(0.7)}{6}\right]}, \dot{S}_{\left[0.2+0.1-\frac{(0.2)(0.1)}{6},\ 0.3+0.3-\frac{(0.3)(0.3)}{6}\right]} \right\rangle, \\ \left\langle \ddot{S}_{\left[\frac{0.4(0.6)}{4}, \frac{0.6(0.8)}{4}\right]}, \ddot{S}_{\left[0.2+0.1-\frac{(0.2)(0.1)}{4},\ 0.4+0.2-\frac{(0.4)(0.2)}{4}\right]} \right\rangle \end{pmatrix}$$

$$\hat{S}_1 \otimes \hat{S}_2 = \left(\left\langle \dot{S}_{[0.0333,\ 0.0583]}, \dot{S}_{[0.29667,\ 0.585]} \right\rangle, \left\langle \ddot{S}_{[0.06,\ 0.12]}, \ddot{S}_{[0.295,\ 0.58]} \right\rangle\right)$$

Definition 22 ***Scalar Product*** *of any General Linguistic Interval Valued Intuitionistc Fuzzy Soft Expert Sets (**GLIVIFSESs**) is defined as:*

$$\lambda\hat{S} = \left(\left\langle \dot{S}_{\mathfrak{t}[(1-(1-\frac{\breve{\xi}}{\mathfrak{t}})^{\lambda},(1-(1-\frac{\breve{\xi}'}{\mathfrak{t}})^{\lambda}]}, \dot{S}_{\mathfrak{t}[(\frac{\tilde{\Psi}}{\mathfrak{t}})^{\lambda},(\frac{\tilde{\Psi}'}{\mathfrak{t}})^{\lambda}]} \right\rangle, \left\langle \ddot{S}_{\mathfrak{t}'[(1-(1-\frac{\ddot{\mu}}{\mathfrak{t}'})^{\lambda},(1-(1-\frac{\ddot{\mu}'}{\mathfrak{t}'})^{\lambda}]}, \ddot{S}_{\mathfrak{t}'[(\frac{\breve{\nu}}{\mathfrak{t}'})^{\lambda},(\frac{\nu'}{\mathfrak{t}'})^{\lambda}]} \right\rangle, (\lambda \geq 0)\right)$$

Example 23 *Let $\hat{S}$ be any General Linguistic Interval Valued Intuitionistc Fuzzy Soft Expert Set (GLIVIFSES) given below and for any scalar $\lambda = 3$ then*

$$\hat{S} = \left(\left\langle \dot{S}_{[0.4\ ,\ 0.7]},\ \dot{S}_{[0.1,\ 03]} \right\rangle, \left\langle \ddot{S}_{[0.4\ ,\ 0.6]}\ ,\ \ddot{S}_{[0.2\ ,\ 0.4]} \right\rangle, \textit{where } \mathfrak{t} = 6, \textit{and } \mathfrak{t}' = 4\right)$$

$$\lambda\hat{S} = \begin{pmatrix} \left\langle \dot{S}_{6\left[1-\left(1-\frac{0.4}{6}\right)^3,\left(1-\left(1-\frac{0.7}{6}\right)^3\right)\right]}, \dot{S}_{6\left[\left(\frac{0.1}{6}\right)^3,\left(\frac{0.3}{6}\right)^3\right]} \right\rangle, \\ \left\langle \ddot{S}_{4\left[1-\left(1-\frac{0.4}{4}\right)^3,\left(1-\left(1-\frac{0.6}{4}\right)^3\right)\right]}, \ddot{S}_{4\left[\left(\frac{0.2}{4}\right)^3,\left(\frac{0.4}{4}\right)^3\right]} \right\rangle \end{pmatrix}$$

$$\lambda\hat{S} = \begin{pmatrix} \left\langle \dot{S}_{6[0.1870\ ,\ 03108]}\ ,\ \dot{S}_{6[0.0000046\ ,\ 0.000125]} \right\rangle, \\ \left\langle \ddot{S}_{4[0.271\ ,\ 0.385875]},\ \ddot{S}_{4[0.000125,\ 0.001]} \right\rangle \end{pmatrix}$$

$$\lambda\hat{S} = \begin{pmatrix} \left\langle \dot{S}_{[1.122\ ,\ 1.8648]}\ ,\ \dot{S}_{[0.00002\ ,\ 0.00075]} \right\rangle, \\ \left\langle \ddot{S}_{[1.084\ ,\ 1.5435]}\ ,\ \ddot{S}_{[0.00075,\ 0.004]} \right\rangle \end{pmatrix}$$

Definition 24 ***Exponent*** *of any General Linguistic Interval Valued Intuitionistc Fuzzy Soft Expert Set **GLIVIFSES** is defined as:*

$$\hat{S}^{\lambda} = \begin{pmatrix} \left\langle \dot{S}_{\mathfrak{t}[(\frac{\breve{\xi}}{\mathfrak{t}})^{\lambda},(\frac{\breve{\xi}'}{\mathfrak{t}})^{\lambda}]}, \dot{S}_{\mathfrak{t}[(1-(1-\frac{\tilde{\psi}}{\mathfrak{t}})^{\lambda},(1-(1-\frac{\tilde{\Psi}'}{\mathfrak{t}})^{\lambda}]} \right\rangle, \\ \left\langle \ddot{S}_{\mathfrak{t}'[(\frac{\ddot{\mu}}{\mathfrak{t}'})^{\lambda},(\frac{\ddot{\mu}'}{\mathfrak{t}'})^{\lambda}]}, \ddot{S}_{\mathfrak{t}'[(1-(1-\frac{\breve{\nu}}{\mathfrak{t}'})^{\lambda},(1-(1-\frac{\breve{\nu}'}{\mathfrak{t}'})^{\lambda}]} \right\rangle, (\lambda \geq 0) \end{pmatrix}$$

Example 25 *For*

$$\hat{S} = \begin{pmatrix} \left\langle \dot{S}_{[0.4\ ,\ 0.7]},\ \dot{S}_{[0.1,\ 03]} \right\rangle, \left\langle \ddot{S}_{[0.4\ ,\ 0.6]}\ ,\ \ddot{S}_{[0.2\ ,\ 0.4]} \right\rangle, \\ where\ \mathfrak{t} = 6, and\ \mathfrak{t}' = 4 \end{pmatrix}$$

then

$$\overset{\wedge\lambda}{S} = \begin{pmatrix} \left\langle \dot{S}_{6\left[\left(\frac{0.4}{6}\right)^3\ ,\ \left(\frac{0.7}{6}\right)^3\right]}\ ,\ \dot{S}_{6\left[1-\left(1-\frac{0.1}{6}\right)^3\ ,\ \left(1-\left(1-\frac{0.3}{6}\right)^3\right)\right]} \right\rangle, \\ \left\langle \ddot{S}_{4\left[\left(\frac{0.4}{4}\right)^3\ ,\ \left(\frac{0.6}{4}\right)^3\right]}\ ,\ \ddot{S}_{4\left[1-\left(1-\frac{0.2}{4}\right)^3,\ 1-\left(1-\frac{0.4}{4}\right)^3\right]} \right\rangle \end{pmatrix}$$

$$\overset{\wedge\lambda}{S} = \begin{pmatrix} \left\langle \dot{S}_{6[0.0003\ ,\ 0.001588]}\ ,\ \dot{S}_{6[0.04917\ ,\ 0.14263]} \right\rangle, \\ \left\langle \ddot{S}_{4[0.001\ ,\ 0.003375]}\ ,\ \ddot{S}_{4[0.14263\ ,\ 0.271]} \right\rangle \end{pmatrix}$$

$$= \begin{pmatrix} \left\langle \dot{S}_{[0.0018\ ,\ 0.0095]}\ ,\ \dot{S}_{[0.2950\ ,\ 1.626]} \right\rangle, \\ \left\langle \ddot{S}_{[0.004\ ,\ 0.0135]}\ ,\ \ddot{S}_{[0.57052\ ,\ 1.084]} \right\rangle \end{pmatrix}$$

Definition 26 *Let* $\hat{S}_1$ *and* $\hat{S}_2$ *be any two General Linguistic Interval Valued Intuitionistc Fuzzy Soft Expert Sets GLIVIFSESs, then Subtraction of* $\hat{S}_1$ *and* $\hat{S}_2$ *is defined as:*

$$\hat{S}_1 \ominus \hat{S}_2 = \begin{pmatrix} \left\langle \dot{S}_{\left[Inf\left\{\breve{\xi}_1,\ \mathfrak{t}-\breve{\xi}_2'\right\},\ Inf\left\{\breve{\xi}_1',\ \mathfrak{t}-\breve{\xi}_2\right\}\right]}, \dot{S}_{\left[Inf\left\{\tilde{\Psi}_1,\ \mathfrak{t}-\tilde{\Psi}_2'\right\},\ Inf\left\{\tilde{\Psi}_1',\ \mathfrak{t}-\tilde{\Psi}_2\right\}\right]} \right\rangle, \\ \left\langle \ddot{S}_{\left[Inf\left\{\breve{\mu}_1,\ \mathfrak{t}-\breve{\mu}_2'\right\},\ Inf\left\{\breve{\mu}_1',\ \mathfrak{t}-\breve{\mu}_2\right\}\right]}, \ddot{S}_{\left[Inf\left\{\breve{\nu}_1,\ \mathfrak{t}-\breve{\nu}_2'\right\},\ Inf\left\{\breve{\nu}_1',\ \mathfrak{t}-\breve{\nu}_2\right\}\right]} \right\rangle; \\ \left[\breve{\xi}_i, \breve{\xi}_i'\right], \left[\tilde{\Psi}_i, \tilde{\Psi}_i'\right] \subseteq [0, \mathfrak{t}]\,; [\breve{\mu}_i, \breve{\mu}_i'], [\breve{\nu}_i, \breve{\nu}_i'] \subseteq \left[0, \mathfrak{t}'\right]\ \ where\ i = 1, 2 \end{pmatrix}$$

Definition 27 *Let* $\hat{S}$ *be any General Linguistic Interval Valued Intuitionistc Fuzzy Soft Expert Set GLIVIFSES then its Complement is denoted by* $\hat{S}^c$ *and defined as:*

$$\left(\hat{S}\right)^c = \begin{pmatrix} \left\langle \dot{S}_{[\tilde{\Psi}, \tilde{\Psi}']}, \dot{S}_{[\breve{\xi}, \breve{\xi}']} \right\rangle, \left\langle \ddot{S}_{[\breve{\nu}, \breve{\nu}']}, \ddot{S}_{[\breve{\mu}, \breve{\mu}']} \right\rangle; \\ \left[\breve{\xi}, \breve{\xi}'\right], \left[\tilde{\Psi}, \tilde{\Psi}'\right] \subseteq [0, \mathfrak{t}]\,; [\breve{\mu}, \breve{\mu}'], [\breve{\nu}, \breve{\nu}'] \subseteq \left[0, \mathfrak{t}'\right] \end{pmatrix}$$

Example 28 *let*

$$\hat{S} = \left(\begin{array}{c} \left\langle \dot{S}_{[0.4\ ,\ 0.7]},\ \dot{S}_{[0.1,\ 03]} \right\rangle, \left\langle \ddot{S}_{[0.4\ ,\ 0.6]}\ ,\ \ddot{S}_{[0.2\ ,\ 0.4]} \right\rangle; \\ where\ \mathfrak{t} = 6,\ \ and\ \mathfrak{t}^{'} = 4 \end{array} \right)$$

then

$$\left(\hat{S} \right)^{c} = \left(\begin{array}{c} \left\langle \dot{S}_{[0.1,\ 03]}\ , \dot{S}_{[0.4\ ,\ 0.7]} \right\rangle, \left\langle\ \ddot{S}_{[0.2\ ,\ 0.4]}\ , \ddot{S}_{[0.4\ ,\ 0.6]} \right\rangle; \\ where\ \mathfrak{t} = 6,\ \ and\ \mathfrak{t}^{'} = 4 \end{array} \right)$$

Theorem 29 *Let*

$$\hat{S} = \left(\begin{array}{c} \left\langle \dot{S}_{[\breve{\xi},\breve{\xi}^{'}]}, \dot{S}_{[\tilde{\Psi},\tilde{\Psi}']} \right\rangle, \left\langle \ddot{S}_{[\breve{\mu},\breve{\mu}']}, \ddot{S}_{[\nu,\nu']} \right\rangle; \left[\breve{\xi}, \breve{\xi}^{'} \right], \\ \left[\tilde{\Psi}, \tilde{\Psi}' \right] \subseteq [0, \mathfrak{t}]; [\breve{\mu}, \breve{\mu}'], [\breve{\nu}, \breve{\nu}'] \subseteq \left[0, \mathfrak{t}^{'} \right] \end{array} \right),$$

where

$$\left(\left\langle \dot{S}_{\left[\breve{\xi},\breve{\xi}'\right]}, \dot{S}_{\left[\tilde{\Psi},\tilde{\Psi}'\right]} \right\rangle; \dot{S}_{[0,0]} \subseteq \dot{S}_{[\breve{\xi},\breve{\xi}^{'}]} + \dot{S}_{[\tilde{\Psi},\tilde{\Psi}']} \subseteq \dot{S}_{[\mathfrak{t},\mathfrak{t}]} \right) \in \hat{S}^{[1]}$$
$$\left(\left\langle \ddot{S}_{[\breve{\mu},\breve{\mu}']}, \ddot{S}_{[\breve{\nu},\breve{\nu}']} \right\rangle; \ddot{S}_{[0,\ 0]} \subseteq \ddot{S}_{\left[\breve{\mu},\breve{\mu}'\right]} + \ddot{S}_{\left[\breve{\mu},\breve{\mu}'\right]} \subseteq \ddot{S}_{\left[\mathfrak{t}^{'},\mathfrak{t}'\right]} \right) \in \hat{S}^{[2]}$$

then

$$\left(\left(\hat{S} \right)^{c} \right)^{c} = \hat{S}$$

Proof. L.H.S.

$$\left(\left(\hat{S} \right)^{c} \right)^{c} = \left(\begin{array}{c} \left\langle \dot{S}_{[\tilde{\Psi},\tilde{\Psi}']}, \dot{S}_{[\breve{\xi},\breve{\xi}^{'}]} \right\rangle, \left\langle \ddot{S}_{[\nu,\nu']}, \ddot{S}_{[\breve{\mu},\breve{\mu}']} \right\rangle; \\ \left[\breve{\xi}, \breve{\xi}^{'} \right], \left[\tilde{\Psi}, \tilde{\Psi}' \right] \subseteq [0, \mathfrak{t}]; [\breve{\mu}, \breve{\mu}'], [\breve{\nu}, \breve{\nu}'] \subseteq \left[0, \mathfrak{t}^{'} \right] \end{array} \right)^{c}$$

$$= \left(\begin{array}{c} \left\langle \dot{S}_{\left[\breve{\xi},\ \breve{\xi}'\right]}, \dot{S}_{[\tilde{\Psi},\ \tilde{\Psi}']} \right\rangle, \left\langle \ddot{S}_{[\breve{\mu},\ \breve{\mu}']}, \ddot{S}_{\left[\breve{\nu},\ \breve{\nu}'\right]} \right\rangle; \\ \left[\breve{\xi}, \breve{\xi}^{'} \right], \left[\tilde{\Psi}, \tilde{\Psi}' \right] \subseteq [0, \mathfrak{t}]; [\breve{\mu}, \breve{\mu}'], [\breve{\nu}, \breve{\nu}'] \subseteq [0, \mathfrak{t}'] \end{array} \right) = \hat{S}$$

$= \ $ R.H.S.

Hence proved. ■

Definition 30 *Let* $\hat{S}$ *be any General Linguistic Interval Valued Intuitionistc Fuzzy Soft Expert Sets GLIVIFSES then its negation is defined as:*

$$\left(\hat{S}\right)^{\sim}=\left(\begin{array}{c}\left\langle \dot{S}_{\left[\mathfrak{t}-\breve{\xi}',\ \mathfrak{t}-\breve{\xi}\right]},\dot{S}_{[\mathfrak{t}-\tilde{\Psi}',\ \mathfrak{t}-\tilde{\Psi}]}\right\rangle,\left\langle \ddot{S}_{[\mathfrak{t}'-\breve{\mu}',\ \mathfrak{t}'-\breve{\mu}]},\ddot{S}_{\left[\mathfrak{t}'-\breve{\nu}',\ \mathfrak{t}'-\breve{\nu}\right]}\right\rangle; \\ \left[\breve{\xi},\breve{\xi}'\right],\left[\tilde{\Psi},\tilde{\Psi}'\right]\subseteq[0,\mathfrak{t}];[\breve{\mu},\breve{\mu}'],\left[\breve{\nu},\breve{\nu}'\right]\subseteq\left[0,\mathfrak{t}'\right]\end{array}\right)$$

Theorem 31 *Let*

$$\hat{S}=\left(\begin{array}{c}\left\langle \dot{S}_{[\breve{\xi},\breve{\xi}']},\dot{S}_{[\tilde{\Psi},\tilde{\Psi}']}\right\rangle,\left\langle \ddot{S}_{[\breve{\mu},\breve{\mu}']},\ddot{S}_{[\nu,\nu']}\right\rangle; \\ \left[\breve{\xi},\breve{\xi}'\right],\left[\tilde{\Psi},\tilde{\Psi}'\right]\subseteq[0,\mathfrak{t}];[\breve{\mu},\breve{\mu}'],\left[\breve{\nu},\breve{\nu}'\right]\subseteq\left[0,\mathfrak{t}'\right]\end{array}\right),$$

where

$$\left(\left\langle \dot{S}_{\left[\breve{\xi},\breve{\xi}'\right]},\dot{S}_{\left[\tilde{\Psi},\tilde{\Psi}'\right]}\right\rangle;\dot{S}_{[0,0]}\subseteq\dot{S}_{[\breve{\xi},\breve{\xi}']}+\dot{S}_{[\tilde{\Psi},\tilde{\Psi}']}\subseteq\dot{S}_{[\mathfrak{t},\mathfrak{t}]}\right)\in\hat{S}^{[1]}$$
$$\left(\left\langle \ddot{S}_{[\breve{\mu},\breve{\mu}']},\ddot{S}_{[\breve{\nu},\breve{\nu}']}\right\rangle;\ddot{S}_{[0.0,\ 0.0]}\subseteq\ddot{S}_{[\breve{\mu},\breve{\mu}']}+\ddot{S}_{[\breve{\nu},\breve{\nu}']}\subseteq\ddot{S}_{\left[\mathfrak{t}',\mathfrak{t}'\right]}\right)\in\hat{S}^{[2]}$$

then $\left(\left(\hat{S}\right)^{\sim}\right)^{\sim}=\hat{S}$

Proof. L.H.S.

$$\left(\left(\hat{S}\right)^{\sim}\right)^{\sim}=\left(\begin{array}{c}\left\langle \dot{S}_{\left[\mathfrak{t}-\ (\mathfrak{t}-\breve{\xi}),\ \mathfrak{t}-\left(\mathfrak{t}-\breve{\xi}'\right)\right]},\dot{S}_{[\mathfrak{t}-\ (\mathfrak{t}-\tilde{\Psi}),\ (\mathfrak{t}-(\mathfrak{t}-\tilde{\Psi}'))]}\right\rangle, \\ \left\langle \ddot{S}_{[\mathfrak{t}'-(\mathfrak{t}'-\breve{\mu}),\ \mathfrak{t}'-(\mathfrak{t}'-\breve{\mu}')]},\ddot{S}_{\left[\mathfrak{t}'-(\mathfrak{t}'-\breve{\nu}),\ \mathfrak{t}'-\left(\mathfrak{t}'-\breve{\nu}'\right)\right]}\right\rangle; \\ \left[\breve{\xi},\breve{\xi}'\right],\left[\tilde{\Psi},\tilde{\Psi}'\right]\subseteq[0,\mathfrak{t}];[\breve{\mu},\breve{\mu}'],\left[\breve{\nu},\breve{\nu}'\right]\subseteq\left[0,\mathfrak{t}'\right]\end{array}\right)$$

$$=\left(\begin{array}{c}\left\langle \dot{S}_{\left[\breve{\xi},\ \breve{\xi}'\right]},\dot{S}_{[\tilde{\Psi},\ \tilde{\Psi}']}\right\rangle,\left\langle \ddot{S}_{[\breve{\mu},\ \breve{\mu}']},\ddot{S}_{\left[\breve{\nu},\ \breve{\nu}'\right]}\right\rangle; \\ \left[\breve{\xi},\breve{\xi}'\right],\left[\tilde{\Psi},\tilde{\Psi}'\right]\subseteq[0,\mathfrak{t}];[\breve{\mu},\breve{\mu}'],\left[\breve{\nu},\breve{\nu}'\right]\subseteq\left[0,\mathfrak{t}'\right]\end{array}\right)=\hat{S}$$

$$=\ \text{R.H.S.}$$

Hence proved. ■

Definition 32 *Let $\overset{\wedge}{S_1}$ and $\overset{\wedge}{S_2}$ are any two General Linguistic Interval Valued Intuitionistc Fuzzy Soft Expert Sets GLIVIFSESs, then Superimum and Infimum of GLIVIFSESs is defined as:*

$$\overset{\wedge}{S_1} = \begin{pmatrix} \left\langle \dot{S}_{\left[\breve{\xi}_1,\ \breve{\xi}_1'\right]}, \dot{S}_{[\tilde{\Psi}_1,\ \tilde{\Psi}_1']} \right\rangle, \left\langle \ddot{S}_{[\breve{\mu}_1,\ \breve{\mu}_1']}, \ddot{S}_{[\breve{\nu}_1,\ \breve{\nu}_1']} \right\rangle; \\ \left[\breve{\xi}_1, \breve{\xi}_1'\right], \left[\tilde{\Psi}_1, \tilde{\Psi}_1'\right] \subseteq [0, t]; [\breve{\mu}_1, \breve{\mu}_1'], [\breve{\nu}_1, \breve{\nu}_1'] \subseteq \left[0, t'\right] \end{pmatrix}$$

$$\overset{\wedge}{S_2} = \begin{pmatrix} \left\langle \dot{S}_{\left[\breve{\xi}_2,\ \breve{\xi}_2'\right]}, \dot{S}_{[\tilde{\Psi}_2,\ \tilde{\Psi}_2']} \right\rangle, \left\langle \ddot{S}_{[\breve{\mu}_2,\ \breve{\mu}_2']}, \ddot{S}_{[\breve{\nu}_2,\ \breve{\nu}_2']} \right\rangle; \\ \left[\breve{\xi}_2, \breve{\xi}_2'\right], \left[\tilde{\Psi}_2, \tilde{\Psi}_2'\right] \subseteq [0, t]; [\breve{\mu}_2, \breve{\mu}_2'], [\breve{\nu}_2, \breve{\nu}_2'] \subseteq \left[0, t'\right] \end{pmatrix}$$

$$Sup\left\{\overset{\wedge}{S_1}, \overset{\wedge}{S_2}\right\} = \begin{pmatrix} \left\langle \dot{S}_{\left[Sup\{\breve{\xi}_1,\ \breve{\xi}_2\}\ ,\ Sup\left\{\breve{\xi}_1',\ \breve{\xi}_2'\right\}\right]}, \dot{S}_{\left[Sup\{\tilde{\Psi}_1,\ \tilde{\Psi}_2\ \},\ Sup\{\tilde{\Psi}_1',\ \tilde{\Psi}_2'\}\right]} \right\rangle, \\ \left\langle \ddot{S}_{\left[Sup\{\breve{\mu}_1,\ \breve{\mu}_2\}\ ,\ Sup\left\{\breve{\mu}_1',\ \breve{\mu}_2'\right\}\right]}, \ddot{S}_{[Sup\{(\breve{\mu}_1,\ \breve{\mu}_2)\}\ ,\ Sup\{\breve{\nu}_1',\ \breve{\nu}_2'\}]} \right\rangle; \\ \left[\breve{\xi}_i, \breve{\xi}_i'\right], \left[\tilde{\Psi}_i, \tilde{\Psi}_i'\right] \subseteq [0, t]; [\breve{\mu}_i, \breve{\mu}_i'], [\breve{\nu}_i, \breve{\nu}_i'] \subseteq \left[0, t'\right] \text{ where } i = 1, 2 \end{pmatrix}$$

$$Inf\left\{\overset{\wedge}{S_1}, \overset{\wedge}{S_2}\right\} = \begin{matrix} \left(\left\langle \dot{S}_{\left[Inf\{\breve{\xi}_1,\ \breve{\xi}_2\}\ ,\ Inf\left\{\breve{\xi}_1',\ \breve{\xi}_2'\right\}\right]}, \dot{S}_{\left[Inf\{\tilde{\Psi}_1,\ \tilde{\Psi}_2\ \},\ Inf\{\tilde{\Psi}_1',\ \tilde{\Psi}_2'\}\right]} \right\rangle,\right. \\ \left\langle \ddot{S}_{\left[Inf\{\breve{\mu}_1,\ \breve{\mu}_2\}\ ,Inf\left\{\breve{\mu}_1',\ \breve{\mu}_2'\right\}\right]}, \ddot{S}_{[Inf\{\breve{\mu}_1,\ \breve{\mu}_2\}\ ,\ Inf\{\breve{\nu}_1',\ \breve{\nu}_2'\}]} \right\rangle; \\ \left.\left[\breve{\xi}_i, \breve{\xi}_i'\right], \left[\tilde{\Psi}_i, \tilde{\Psi}_i'\right] \subseteq [0, t]; [\breve{\mu}_i, \breve{\mu}_i'], [\breve{\nu}_i, \breve{\nu}_i'] \subseteq \left[0, t'\right] \text{ where } i = 1, 2\right) \end{matrix}$$

4.4 Operational Laws of General Linguistic Interval Valued Intuitionistc Fuzzy Soft Expert Sets (GLIVIFSESs)

Theorem 33 *let*

$$\begin{aligned} \overset{\wedge}{S_i} &= \left(\left\langle \dot{S}_{[\breve{\xi}_i, \breve{\xi}_i']}, \dot{S}_{[\tilde{\Psi}_i, \tilde{\Psi}_i']} \right\rangle, \left\langle \ddot{S}_{[\breve{\mu}_i, \breve{\mu} i]}, \ddot{S}_{[\nu_i, \nu_i']} \right\rangle;\right. \\ &\quad \left.\left[\breve{\xi}_i, \breve{\xi}_i'\right], \left[\tilde{\Psi}_i, \tilde{\Psi}_i'\right] \subseteq [0, t]; \ \ [\breve{\mu}_i, \breve{\mu}_i'], [\breve{\nu}_i, \breve{\nu}_i'] \subseteq \left[0, t'\right], \text{where } i = 1, 2, 3\right) \end{aligned}$$

*be any three **GLIVIFSESs,** where* $\dot{S}_{[\breve{\xi},\breve{\xi}']}, \dot{S}_{[\breve{\xi}_1,\breve{\xi}_1']}, \dot{S}_{[\breve{\xi}_2,\breve{\xi}_2']}$ *defines the degree of belongingness and* $\dot{S}_{[\tilde{\Psi},\tilde{\Psi}']}, \dot{S}_{[\tilde{\Psi}_1,\tilde{\Psi}_1']}, \dot{S}_{[\tilde{\Psi}_2,\tilde{\Psi}_2']}$ *describes degree of non-belongingness in* $\hat{S}_i$ *and* $\left\langle \dot{S}_{[\breve{\xi}_i,\breve{\xi}_i']}, \dot{S}_{[\tilde{\Psi}_i,\tilde{\Psi}_i']} \right\rangle$, *which are representing experts opinion about evaluated objects (attributes), while* $\ddot{S}_{[\breve{\mu},\breve{\mu}']}, \ddot{S}_{[\breve{\mu}_1,\breve{\mu}_1']}$ *and* $\ddot{S}_{[\breve{\mu}_2,\breve{\mu}_2']}$ *defines the MD* $\ddot{S}_{[\nu,\nu']}$, $\ddot{S}_{[\nu_1,\nu_1']}$ *and* $\ddot{S}_{[\nu_2,\nu_2']}$ *are describing the NMD in* $\hat{S}_i$ *and* $\left\langle \ddot{S}_{[\breve{\mu}_i,\breve{\mu}_i']}, \ddot{S}_{[\breve{\nu}_i,\nu_i']} \right\rangle$ *are describing the subjective evaluation on the reliability of the expert′s opinon then the operational laws of GLIVIFSESs are interpreted as:*

i) Commutative Law with respect to Addition holds.

$$\overset{\wedge}{\hat{S}}_1 \oplus \hat{S}_2 = \hat{S}_2 \oplus \hat{S}_1$$

ii) Commutative Law with respect to Multiplication holds.

$$\hat{S}_1 \otimes \hat{S}_2 = \hat{S}_2 \otimes \hat{S}_1$$

iii) Associative Law with respect to Addition holds.

$$(\hat{S}_1 \oplus \hat{S}_2) \oplus \hat{S}_3 = \hat{S}_1 \oplus (\hat{S}_2 \oplus \hat{S}_3)$$

iv) Associative Law with respect to Addition holds.

$$(\hat{S}_1 \otimes \hat{S}_2) \otimes \hat{S}_3 = \hat{S}_1 \otimes (\hat{S}_2 \otimes S)$$

v) Distributive Law with respect to Multiplication over Addition does not holds.

$$\hat{S}_1 \otimes (\hat{S}_2 \oplus \hat{S}_3) \neq (\hat{S}_1 \otimes \hat{S}_2) \oplus (\hat{S}_1 \otimes \hat{S}_3)$$

vi) De Morgan's for any two GLIVIFSESs holds.

$$\text{i)} \quad \left(\hat{S}_1 \oplus \hat{S}_2\right)^c = \hat{S}{}_1^c \otimes \hat{S}{}_2^c$$

$$\text{ii)} \quad \left(\hat{S}_1 \otimes \hat{S}_2\right)^c = \hat{S}{}_1^c \oplus \hat{S}{}_2^c$$

Proof. The above mentioned operational laws can be proved as follow:

$$\text{(i)} \qquad \hat{S}_1 \oplus \hat{S}_2 = \hat{S}_2 \oplus \hat{S}_1$$

L.H.S , By using definition 18 of Sum,

$$\hat{S}_1 \oplus \hat{S}_2 = \begin{pmatrix} \left\langle \dot{S}_{\left[\breve{\xi}_1+\breve{\xi}_2-\frac{\breve{\xi}_1\breve{\xi}_2}{\mathfrak{t}},\breve{\xi}'_1+\breve{\xi}'_2-\frac{\breve{\xi}'_1\breve{\xi}'_2}{\mathfrak{t}}\right]}, \dot{S}_{\left[\frac{\tilde{\Psi}_1\tilde{\Psi}_2}{\mathfrak{t}},\frac{\tilde{\Psi}'_1\tilde{\Psi}'_2}{\mathfrak{t}}\right]} \right\rangle, \\ \left\langle \ddot{S}_{\left[\breve{\mu}_1+\breve{\mu}_2-\frac{\breve{\mu}_1\breve{\mu}_2}{\mathfrak{t}'},\breve{\mu}'_1+\breve{\mu}'_2-\frac{\breve{\mu}'_1\breve{\mu}'_2}{\mathfrak{t}'}\right]}, \ddot{S}_{\left[\frac{\breve{\nu}_1\breve{\nu}_2}{\mathfrak{t}'},\frac{\breve{\nu}'_1\breve{\nu}'_2}{\mathfrak{t}'}\right]} \right\rangle; \\ \left[\breve{\xi}_i,\breve{\xi}_i^{'}\right], \left[\tilde{\Psi}_i,\tilde{\Psi}'_i\right] \subseteq [0,\mathfrak{t}]\,; [\breve{\mu}_i,\breve{\mu}'_i], \left[\breve{\nu}_i,\breve{\nu}'_i\right] \subseteq \left[0,\mathfrak{t}^{'}\right] \text{ where } i=1,2 \end{pmatrix} \tag{4.2}$$

$$\hat{S}_2 \oplus \hat{S}_1 = \begin{pmatrix} \left\langle \dot{S}_{\left[\breve{\xi}_2+\breve{\xi}_1-\frac{\breve{\xi}_2\breve{\xi}_1}{\mathfrak{t}},\breve{\xi}'_2+\breve{\xi}'_1-\frac{\breve{\xi}'_2\breve{\xi}'_1}{\mathfrak{t}}\right]}, \dot{S}_{\left[\frac{\tilde{\Psi}_2\tilde{\Psi}_1}{\mathfrak{t}},\frac{\tilde{\Psi}'_2\tilde{\Psi}'_1}{\mathfrak{t}}\right]} \right\rangle, \\ \left\langle \ddot{S}_{\left[\breve{\mu}_2+\breve{\mu}_1-\frac{\breve{\mu}_2\breve{\mu}_1}{\mathfrak{t}'},\breve{\mu}'_2+\breve{\mu}'_1-\frac{\breve{\mu}'_2\breve{\mu}'_1}{\mathfrak{t}'}\right]}, \ddot{S}_{\left[\frac{\breve{\nu}_2\breve{\nu}_1}{\mathfrak{t}'},\frac{\breve{\nu}'_2\breve{\nu}'_1}{\mathfrak{t}'}\right]} \right\rangle; \\ \left[\breve{\xi}_i,\breve{\xi}_i^{'}\right], \left[\tilde{\Psi}_i,\tilde{\Psi}'_i\right] \subseteq [0,\mathfrak{t}]\,; [\breve{\mu}_i,\breve{\mu}'_i], \left[\breve{\nu}_i,\breve{\nu}'_i\right] \subseteq \left[0,\mathfrak{t}^{'}\right] \text{ where } i=1,2 \end{pmatrix}$$

$$\hat{S}_2 \oplus \hat{S}_1 = \begin{pmatrix} \left\langle \dot{S}_{\left[\breve{\xi}_1+\breve{\xi}_2-\frac{\breve{\xi}_1\breve{\xi}_2}{\mathfrak{t}},\breve{\xi}'_1+\breve{\xi}'_2-\frac{\breve{\xi}'_1\breve{\xi}'_2}{\mathfrak{t}}\right]}, \dot{S}_{\left[\frac{\tilde{\Psi}_1\tilde{\Psi}_2}{\mathfrak{t}},\frac{\tilde{\Psi}'_1\tilde{\Psi}'_2}{\mathfrak{t}}\right]} \right\rangle, \\ \left\langle \ddot{S}_{\left[\breve{\mu}_1+\breve{\mu}_2-\frac{\breve{\mu}_1\breve{\mu}_2}{\mathfrak{t}'},\breve{\mu}'_1+\breve{\mu}'_2-\frac{\breve{\mu}'_1\breve{\mu}'_2}{\mathfrak{t}'}\right]}, \ddot{S}_{\left[\frac{\breve{\nu}_1\breve{\nu}_2}{\mathfrak{t}'},\frac{\breve{\nu}'_1\breve{\nu}'_2}{\mathfrak{t}'}\right]} \right\rangle; \\ \left[\breve{\xi}_i,\breve{\xi}_i^{'}\right], \left[\tilde{\Psi}_i,\tilde{\Psi}'_i\right] \subseteq [0,\mathfrak{t}]\,; [\breve{\mu}_i,\breve{\mu}'_i], \left[\breve{\nu}_i,\breve{\nu}'_i\right] \subseteq \left[0,\mathfrak{t}^{'}\right] \text{ where } i=1,2 \end{pmatrix} \tag{4.3}$$

By 4.2 & 4.3 we have $\hat{S}_1 \oplus \hat{S}_2 = \hat{S}_2 \oplus \hat{S}_1$

Hence proved.

(ii) By using Definition (20) of Product

$$\hat{S}_1 \otimes \hat{S}_2 = \hat{S}_2 \otimes \hat{S}_1$$

Consider L.H.S $\overset{\wedge}{S_1} \otimes \overset{\wedge}{S_2}$

$$= \left(\begin{array}{c} \left\langle \dot{S}_{\left[\frac{\breve{\xi}_1\breve{\xi}_2}{\mathfrak{t}}, \frac{\breve{\xi}_1'\breve{\xi}_2'}{\mathfrak{t}}\right]}, \dot{S}_{\left[\tilde{\Psi}_1+\tilde{\Psi}_2-\frac{\tilde{\Psi}_1\tilde{\Psi}_2}{\mathfrak{t}}, \tilde{\Psi}_1'+\tilde{\Psi}_2'-\frac{\tilde{\Psi}_1'\tilde{\Psi}_2'}{\mathfrak{t}}\right]} \right\rangle, \\ \left\langle \ddot{S}_{\left[\frac{\breve{\mu}_1\breve{\mu}_2}{\mathfrak{t}'}, \frac{\breve{\mu}_1'\breve{\mu}_2'}{\mathfrak{t}'}\right]}, \ddot{S}_{\left[\breve{\nu}_1+\breve{\nu}_2-\frac{\breve{\nu}_1\breve{\nu}_2}{\mathfrak{t}'}, \breve{\nu}_1'+\breve{\nu}_2'-\frac{\breve{\nu}_1'\breve{\nu}_2'}{\mathfrak{t}'}\right]} \right\rangle; \\ \left[\breve{\xi}_i, \breve{\xi}_i'\right], \left[\tilde{\Psi}_i, \tilde{\Psi}_i'\right] \subseteq [0, \mathfrak{t}]; [\breve{\mu}_i, \breve{\mu}_i'], [\breve{\nu}_i, \breve{\nu}_i'] \subseteq \left[0, \mathfrak{t}'\right] \text{ where } i = 1, 2 \end{array} \right) \tag{4.4}$$

Consider R.H.S $\overset{\wedge}{S_2} \otimes \overset{\wedge}{S_1}$

$$= \left(\begin{array}{c} \left\langle \dot{S}_{\left[\frac{\breve{\xi}_2\breve{\xi}_1}{\mathfrak{t}}, \frac{\breve{\xi}_2'\breve{\xi}_1'}{\mathfrak{t}}\right]}, \dot{S}_{\left[\tilde{\Psi}_2+\tilde{\Psi}_1-\frac{\tilde{\Psi}_2\tilde{\Psi}_1}{\mathfrak{t}}, \tilde{\Psi}_2'+\tilde{\Psi}_1'-\frac{\tilde{\Psi}_2'\tilde{\Psi}_1'}{\mathfrak{t}}\right]} \right\rangle, \\ \left\langle \ddot{S}_{\left[\frac{\breve{\mu}_2\breve{\mu}_1}{\mathfrak{t}'}, \frac{\breve{\mu}_2'\breve{\mu}_1'}{\mathfrak{t}'}\right]}, \ddot{S}_{\left[\breve{\nu}_2+\breve{\nu}_1-\frac{\breve{\nu}_2\breve{\nu}_1}{\mathfrak{t}'}, \breve{\nu}_2'+\breve{\nu}_1'-\frac{\breve{\nu}_2'\breve{\nu}_1'}{\mathfrak{t}'}\right]} \right\rangle; \\ \left[\breve{\xi}_i, \breve{\xi}_i'\right], \left[\tilde{\Psi}_i, \tilde{\Psi}_i'\right] \subseteq [0, \mathfrak{t}]; [\breve{\mu}_i, \breve{\mu}_i'], [\breve{\nu}_i, \breve{\nu}_i'] \subseteq \left[0, \mathfrak{t}'\right] \text{ where } i = 1, 2 \end{array} \right)$$

$$\overset{\wedge}{S_2} \otimes \overset{\wedge}{S_1} = \left(\begin{array}{c} \left\langle \dot{S}_{\left[\frac{\breve{\xi}_1\breve{\xi}_2}{\mathfrak{t}}, \frac{\breve{\xi}_1'\breve{\xi}_2'}{\mathfrak{t}}\right]}, \dot{S}_{\left[\tilde{\Psi}_1+\tilde{\Psi}_2-\frac{\tilde{\Psi}_1\tilde{\Psi}_2}{\mathfrak{t}}, \tilde{\Psi}_1'+\tilde{\Psi}_2'-\frac{\tilde{\Psi}_1'\tilde{\Psi}_2'}{\mathfrak{t}}\right]} \right\rangle, \\ \left\langle \ddot{S}_{\left[\frac{\breve{\mu}_1\breve{\mu}_2}{\mathfrak{t}'}, \frac{\breve{\mu}_1'\breve{\mu}_2'}{\mathfrak{t}'}\right]}, \ddot{S}_{\left[\breve{\nu}_1+\breve{\nu}_2-\frac{\breve{\nu}_1\breve{\nu}_2}{\mathfrak{t}'}, \breve{\nu}_1'+\breve{\nu}_2'-\frac{\breve{\nu}_1'\breve{\nu}_2'}{\mathfrak{t}'}\right]} \right\rangle; \\ \left[\breve{\xi}_i, \breve{\xi}_i'\right], \left[\tilde{\Psi}_i, \tilde{\Psi}_i'\right] \subseteq [0, \mathfrak{t}]; [\breve{\mu}_i, \breve{\mu}_i'], [\breve{\nu}_i, \breve{\nu}_i'] \subseteq \left[0, \mathfrak{t}'\right] \text{ where } i = 1, 2 \end{array} \right) \tag{4.5}$$

By equation 4.4 and equation 4.5, we get the desired result,

$$\overset{\wedge}{S_1} \otimes \overset{\wedge}{S_2} = \overset{\wedge}{S_2} \otimes \overset{\wedge}{S_1}$$

Hence proved.

(iii). Consider L.H.S $(\overset{\wedge}{S_1} \oplus \overset{\wedge}{S_2}) \oplus \overset{\wedge}{S_3}$

$$(\overset{\wedge}{S_1} \oplus \overset{\wedge}{S_2}) = \left(\begin{array}{c} \left\langle \dot{S}_{\left[\breve{\xi}_1+\breve{\xi}_2-\frac{\breve{\xi}_2\breve{\xi}_1}{\mathfrak{t}}, \breve{\xi}_1'+\breve{\xi}_2'-\frac{\breve{\xi}_1'\breve{\xi}_2'}{\mathfrak{t}}\right]}, \dot{S}_{\left[\frac{\tilde{\Psi}_1\tilde{\Psi}_2}{\mathfrak{t}}, \frac{\tilde{\Psi}_1'\tilde{\Psi}_2'}{\mathfrak{t}}\right]} \right\rangle, \\ \left\langle \ddot{S}_{\left[\breve{\mu}_1+\breve{\mu}_2-\frac{\breve{\mu}_1\breve{\mu}_2}{\mathfrak{t}'}, \breve{\mu}_1'+\breve{\mu}_2'-\frac{\breve{\mu}_1'\breve{\mu}_2'}{\mathfrak{t}'}\right]}, \ddot{S}_{\left[\frac{\nu_1\breve{\nu}_2}{\mathfrak{t}'}, \frac{\breve{\nu}_1'\breve{\nu}_2'}{\mathfrak{t}'}\right]} \right\rangle; \\ \left[\breve{\xi}_i, \breve{\xi}_i'\right], \left[\tilde{\Psi}_i, \tilde{\Psi}_i'\right] \subseteq [0, \mathfrak{t}]; [\breve{\mu}_i, \breve{\mu}_i'], [\breve{\nu}_i, \breve{\nu}_i'] \subseteq \left[0, \mathfrak{t}'\right] \text{ where } i = 1, 2 \end{array} \right)$$

By using Definition 18 of Sum of any two GLIVIFSE Sets

Now

$$(\overset{\wedge}{S_1}\oplus\overset{\wedge}{S_2})\oplus\overset{\wedge}{S_3} = \left(\begin{array}{c} \left\langle \begin{array}{c} \dot{S}_{[\breve{\xi}_1+\breve{\xi}_2-\frac{\breve{\xi}_1\breve{\xi}_2}{\text{ţ}}+\breve{\xi}_3 \,-\, \left(\breve{\xi}_1+\breve{\xi}_2-\frac{\breve{\xi}_1\breve{\xi}_2}{\text{ţ}}\right)\frac{\breve{\xi}_3}{\text{ţ}},\breve{\xi}'_1+\breve{\xi}'_2-\frac{\breve{\xi}'_1\breve{\xi}'_2}{\text{ţ}}+\breve{\xi}'_3 \,-\, \left(\breve{\xi}'_1+\breve{\xi}'_2-\frac{\breve{\xi}'_1\breve{\xi}'_2}{\text{ţ}}\right)\frac{\breve{\xi}'_3}{\text{ţ}}],} \\ \dot{S}_{[\frac{(\frac{\tilde{\Psi}_1\tilde{\Psi}_2}{\text{ţ}})\tilde{\Psi}_3}{\text{ţ}},\frac{(\frac{\tilde{\Psi}'_1\tilde{\Psi}'_2}{\text{ţ}})\tilde{\Psi}'_3}{\text{ţ}}]} \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \ddot{S}_{[\breve{\mu}_2+\breve{\mu}_1-\frac{\breve{\mu}_2\breve{\mu}_1}{\text{ţ}}+\breve{\mu}_3 \,-\, \left(\breve{\mu}_1+\breve{\mu}_2-\frac{\breve{\mu}_1\breve{\mu}2}{\text{ţ}}\right)\frac{\breve{\mu}_3}{\text{ţ}},\breve{\mu}'_2+\breve{\mu}'_1-\frac{\breve{\mu}'_2\breve{\mu}'_1}{\text{ţ}}+\breve{\mu}'_3 \,-\, \left(\breve{\mu}'_1+\breve{\mu}'_2-\frac{\breve{\mu}'_1\breve{\mu}'2}{\text{ţ}}\right)\frac{\breve{\mu}'_3}{\text{ţ}}]} \\ ,\ddot{S}_{[\frac{(\frac{\nu_1\breve{\nu}_2}{\text{ţ}'})\nu_3}{\text{ţ}'},\frac{(\frac{\breve{\nu}'_1\breve{\nu}'_2}{\text{ţ}'})\breve{\nu}'_3}{\text{ţ}'})]} \end{array} \right\rangle; \\ \left[\breve{\xi}_i,\breve{\xi}_i^{'}\right],\left[\tilde{\Psi}_i,\tilde{\Psi}'_i\right] \subseteq [0,\text{ţ}]\,;[\breve{\mu}_i,\breve{\mu}'_i]\,,[\breve{\nu}_i,\breve{\nu}'_i] \subseteq \left[0,\text{ţ}^{'}\right] \text{ where } i=1,2,3 \end{array}\right)$$

$$= \left(\begin{array}{c} \left\langle \begin{array}{c} \dot{S}_{[\breve{\xi}_1+\breve{\xi}_{2+}+\breve{\xi}_3-\frac{\breve{\xi}_1\breve{\xi}_2}{\text{ţ}}-\frac{\breve{\xi}_1\breve{\xi}_3}{\text{ţ}}-\frac{\breve{\xi}2\breve{\xi}_3}{\text{ţ}}+\frac{\breve{\xi}_1\breve{\xi}_2\breve{\xi}_3}{\text{ţ}}\;,\breve{\xi}'_1+\breve{\xi}'_2+\breve{\xi}'_3-\frac{\breve{\xi}'_1\breve{\xi}'_2}{\text{ţ}}-\frac{\breve{\xi}'_1\breve{\xi}'_3}{\text{ţ}}-\frac{\breve{\xi}'_2\breve{\xi}'_3}{\text{ţ}}+\frac{\breve{\xi}'_1\breve{\xi}'_2\breve{\xi}'_3}{\text{ţ}^2}],} \\ \dot{S}_{[\frac{\tilde{\Psi}_1\tilde{\Psi}_2\tilde{\Psi}_3}{\text{ţ}^2},\frac{\tilde{\Psi}'_1\tilde{\Psi}'_2\tilde{\Psi}'_3}{\text{ţ}^2}]} \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \ddot{S}_{[\breve{\mu}_2+\breve{\mu}_1+\breve{\mu}_3-\frac{\breve{\mu}_1\breve{\mu}_2}{\text{ţ}}-\frac{\breve{\mu}_1\breve{\mu}_3}{\text{ţ}}-\frac{\breve{\mu}_2\breve{\mu}_3}{\text{ţ}}+\frac{\breve{\mu}_1\breve{\mu}_2\breve{\mu}_3}{\text{ţ}^2}\;,\breve{\mu}'_2+\breve{\mu}'_1+\breve{\mu}'_3-\frac{\breve{\mu}'_1\breve{\mu}'_2}{\text{ţ}}-\frac{\breve{\mu}'_1\breve{\mu}'_3}{\text{ţ}}-\frac{\breve{\mu}'_2\breve{\mu}'_3}{\text{ţ}}+\frac{\breve{\mu}'_1\breve{\mu}'_2\breve{\mu}'_3}{\text{ţ}^2}]} \\ ,\ddot{S}_{[\frac{\breve{\nu}_1\breve{\nu}_2\breve{\nu}_3}{\text{ţ}'^2},\frac{\breve{\nu}'_1\breve{\nu}'_2\breve{\nu}'_3}{\text{ţ}'^2}]} \end{array} \right\rangle; \\ \left[\breve{\xi}_i,\breve{\xi}_i^{'}\right],\left[\tilde{\Psi}_i,\tilde{\Psi}'_i\right] \subseteq [0,\text{ţ}]\,;[\breve{\mu}_i,\breve{\mu}'_i]\,,[\breve{\nu}_i,\breve{\nu}'_i] \subseteq \left[0,\text{ţ}^{'}\right] \text{ where } i=1,2,3 \end{array}\right) \tag{4.6}$$

Consider R.H.S. $\overset{\wedge}{S_1} \oplus (\overset{\wedge}{S_2} \oplus \overset{\wedge}{S_3})$

$$\overset{\wedge}{S_2} \oplus \overset{\wedge}{S_3} = \left(\begin{array}{c} \left\langle \dot{S}_{[\breve{\xi}_2+\breve{\xi}_3-\frac{\breve{\xi}_2\breve{\xi}_3}{\text{ţ}},\breve{\xi}'_2+\breve{\xi}'_3-\frac{\breve{\xi}'_2\breve{\xi}'_3}{\text{ţ}}]},\dot{S}_{[\frac{\tilde{\Psi}_2\tilde{\Psi}_3}{\text{ţ}},\frac{\tilde{\Psi}'_2\tilde{\Psi}'_3}{\text{ţ}}]} \right\rangle, \\ \left\langle \ddot{S}_{[\breve{\mu}_2+\breve{\mu}_3-\frac{\breve{\mu}_2\breve{\mu}_3}{\text{ţ}'},\breve{\mu}'_2+\breve{\mu}'_3-\frac{\breve{\mu}'_2\breve{\mu}'_3}{\text{ţ}'}]},\ddot{S}_{[\frac{\breve{\nu}_2\breve{\nu}_3}{\text{ţ}'},\frac{\breve{\nu}'_2\breve{\nu}'_3}{\text{ţ}'}]} \right\rangle; \\ \left[\breve{\xi}_i,\breve{\xi}_i^{'}\right],\left[\tilde{\Psi}_i,\tilde{\Psi}'_i\right] \subseteq [0,\text{ţ}]\,;[\breve{\mu}_i,\breve{\mu}'_i]\,,[\breve{\nu}_i,\breve{\nu}'_i] \subseteq \left[0,\text{ţ}^{'}\right] \text{ where } i=2,3 \end{array}\right)$$

Now

$$\overset{\wedge}{S_1}\oplus(\overset{\wedge}{S_2}\oplus\overset{\wedge}{S_3}) = \left(\begin{array}{c} \left\langle \begin{array}{c} \dot{S}_{[\breve{\xi}_1+(\breve{\xi}_2+\breve{\xi}_3-\frac{\breve{\xi}_1\breve{\xi}_2}{\mathfrak{t}}) \;-\; \frac{\breve{\xi}_1}{\mathfrak{t}}\left(\breve{\xi}_2+\breve{\xi}_3-\frac{\breve{\xi}_2\breve{\xi}_3}{\mathfrak{t}}\right),\breve{\xi}'_1+(\breve{\xi}'_2+\breve{\xi}'_3-\frac{\breve{\xi}'_1\breve{\xi}'_2}{\mathfrak{t}}) \;-\; \frac{\breve{\xi}'_1}{\mathfrak{t}}\left(\breve{\xi}'_2+\breve{\xi}'_3-\frac{\breve{\xi}'_2\breve{\xi}'_3}{\mathfrak{t}}\right)],} \\ \dot{S}_{\left[\frac{\tilde{\Psi}_1(\frac{\tilde{\Psi}_2\tilde{\Psi}_3}{\mathfrak{t}})}{\mathfrak{t}},\frac{\tilde{\Psi}'_1(\frac{\tilde{\Psi}'_2\tilde{\Psi}'_3}{\mathfrak{t}})}{\mathfrak{t}}\right]} \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \ddot{S}_{[\breve{\mu}_1+(\breve{\mu}_2+\breve{\mu}_3-\frac{\breve{\mu}_2\breve{\mu}_3}{\mathfrak{t}}) \;-\; \frac{\breve{\mu}_1}{\mathfrak{t}}\left(\breve{\mu}_2+\breve{\mu}_3-\frac{\breve{\mu}_2\breve{\mu}_3}{\mathfrak{t}}\right),\breve{\mu}'_1+(\breve{\mu}'_2+\breve{\mu}'_3-\frac{\breve{\mu}'_2\breve{\mu}'_1}{\mathfrak{t}}) \;-\frac{\breve{\mu}'_1}{\mathfrak{t}}\left(\breve{\mu}'_2+\breve{\mu}'_3-\frac{\breve{\mu}'_2\breve{\mu}'_3}{\mathfrak{t}}\right)],} \\ \ddot{S}_{\left[\frac{\breve{\nu}_1(\frac{\breve{\nu}_2\breve{\nu}_3}{\mathfrak{t}'})}{\mathfrak{t}'},\frac{\breve{\nu}'_1(\frac{\breve{\nu}'_2\breve{\nu}'_3}{\mathfrak{t}'})}{\mathfrak{t}'}\right]} \end{array} \right\rangle; \\ \left[\breve{\xi}_i,\breve{\xi}_i^{'}\right],[\Psi_i,\Psi'_i] \subseteq [0,\mathfrak{t}];[\breve{\mu}_i,\breve{\mu}'_i],\left[\breve{\nu}_i,\breve{\nu}'_i\right] \subseteq \left[0,\mathfrak{t}^{'}\right] \text{ where } i=1,2,3 \end{array}\right)$$

$$\overset{\wedge}{S_1}\oplus(\overset{\wedge}{S_2}\oplus\overset{\wedge}{S_3}) = \left(\begin{array}{c} \left\langle \begin{array}{c} \dot{S}_{[\breve{\xi}_1+\breve{\xi}_{2+}+\breve{\xi}_3-\frac{\breve{\xi}_1\breve{\xi}_2}{\mathfrak{t}}-\frac{\breve{\xi}_1\breve{\xi}_3}{\mathfrak{t}}-\frac{\breve{\xi}2\breve{\xi}_3}{\mathfrak{t}}+\frac{\breve{\xi}_1\breve{\xi}_2\breve{\xi}_3}{\mathfrak{t}} \;,\breve{\xi}'_1+\breve{\xi}'_2+\breve{\xi}'_3-\frac{\breve{\xi}'_1\breve{\xi}'_2}{\mathfrak{t}}-\frac{\breve{\xi}'_1\breve{\xi}'_3}{\mathfrak{t}}-\frac{\breve{\xi}'_2\breve{\xi}'_3}{\mathfrak{t}}+\frac{\breve{\xi}'_1\breve{\xi}'_2\breve{\xi}'_3}{\mathfrak{t}^2}],} \\ \dot{S}_{[\frac{\Psi_1\Psi_2\Psi_3}{\mathfrak{t}^2},\frac{\Psi'_1\Psi'_2\Psi'_3}{\mathfrak{t}^2}]} \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \ddot{S}_{[\breve{\mu}_2+\breve{\mu}_1+\breve{\mu}_3-\frac{\breve{\mu}_1\breve{\mu}_2}{\mathfrak{t}}-\frac{\breve{\mu}_1\breve{\mu}_3}{\mathfrak{t}}-\frac{\breve{\mu}_2\breve{\mu}_3}{\mathfrak{t}}+\frac{\breve{\mu}_1\breve{\mu}_2\breve{\mu}_3}{\mathfrak{t}^2} \;,\breve{\mu}'_2+\breve{\mu}'_1+\breve{\mu}'_3-\frac{\breve{\mu}'_1\breve{\mu}'_2}{\mathfrak{t}}-\frac{\breve{\mu}'_1\breve{\mu}'_3}{\mathfrak{t}}-\frac{\breve{\mu}'_2\breve{\mu}'_3}{\mathfrak{t}}+\frac{\breve{\mu}'_1\breve{\mu}'_2\breve{\mu}'_3}{\mathfrak{t}^2}],} \\ \ddot{S}_{[\frac{\breve{\nu}_1\breve{\nu}_2\breve{\nu}_3}{\mathfrak{t}'^2},\frac{\breve{\nu}'_1\breve{\nu}'_2\breve{\nu}'_3}{\mathfrak{t}'^2}]} \end{array} \right\rangle; \end{array}\right) \tag{4.7}$$

By 4.6 and 4.7 we get:

$$(\overset{\wedge}{S_1}\oplus\overset{\wedge}{S_2})\oplus\overset{\wedge}{S_3} = \overset{\wedge}{S_1}\oplus(\overset{\wedge}{S_2}\oplus\overset{\wedge}{S_3})$$

Hence proved.

iv)

$$(\overset{\wedge}{S_1}\otimes\overset{\wedge}{S_2})\otimes\overset{\wedge}{S_3} = \overset{\wedge}{S_1}\otimes(\overset{\wedge}{S_2}\otimes\overset{\wedge}{S_3})$$

Consider L.H.S. $(\overset{\wedge}{S_1}\otimes\overset{\wedge}{S_2})\otimes\overset{\wedge}{S_3}$

$$\overset{\wedge}{S_1}\otimes\overset{\wedge}{S_2} = \left(\begin{array}{c} \left\langle \dot{S}_{[\frac{\breve{\xi}_1\breve{\xi}_2}{\mathfrak{t}},\frac{\breve{\xi}'_1\breve{\xi}'_2}{\mathfrak{t}}]},\dot{S}_{[\Psi_1+\Psi_2-\frac{\Psi_1\Psi_2}{\mathfrak{t}},\Psi'_1+\Psi'_2-\frac{\Psi'_1\tilde{\Psi}'_2}{\mathfrak{t}}]}\right\rangle, \\ \left\langle \ddot{S}_{[\frac{\breve{\mu}_1\breve{\mu}_2}{\mathfrak{t}'},\frac{\breve{\mu}'_1\breve{\mu}'_2}{\mathfrak{t}'}]},\ddot{S}_{[\breve{\nu}_1+\breve{\nu}_2-\frac{\breve{\nu}_1\breve{\nu}_2}{\mathfrak{t}'},\breve{\nu}'_1+\breve{\nu}'_2-\frac{\breve{\nu}'_1\breve{\nu}'_2}{\mathfrak{t}'}]}\right\rangle \end{array}\right)$$

By using Definition 20 of Product of any two GLIVIFSE Sets

Now

$$(\overset{\wedge}{S_1} \otimes \overset{\wedge}{S_2}) \otimes \overset{\wedge}{S_3}$$

$$= \left(\left\langle \begin{array}{c} \dot{S}_{\left[\frac{(\frac{\breve{\xi}_1\breve{\xi}_2}{\mathfrak{t}})\breve{\xi}_3}{\mathfrak{t}}, \frac{(\frac{\breve{\xi}'_1\breve{\xi}'_2}{\mathfrak{t}})\breve{\xi}'_3}{\mathfrak{t}}\right]}, \\ \dot{S}_{\left[(\Psi_1+\Psi_2-\frac{\Psi_1\Psi_2}{\mathfrak{t}})+\Psi_3-(\Psi_1+\Psi_2-\frac{\Psi_1\Psi_2}{\mathfrak{t}})\frac{\Psi_3}{\mathfrak{t}}, (\Psi'_1+\Psi'_2-\frac{\Psi'_1\Psi'_2}{\mathfrak{t}})+\Psi'_3-(\Psi'_1+\Psi'_2-\frac{\Psi'_1\Psi'_2}{\mathfrak{t}})\frac{\Psi'_3}{\mathfrak{t}}\right]} \end{array} \right\rangle, \left\langle \begin{array}{c} \ddot{S}_{\left[\frac{(\frac{\breve{\mu}_1\breve{\mu}_2}{\mathfrak{t}'})\breve{\mu}_3}{\mathfrak{t}'}, \frac{(\frac{\breve{\mu}'_1\breve{\mu}'_2}{\mathfrak{t}'})\breve{\mu}'_3}{\mathfrak{t}'}\right]}, \\ \ddot{S}_{\left[(\breve{\nu}_1+\breve{\nu}_2-\frac{\breve{\nu}_1\breve{\nu}_2}{\mathfrak{t}'})+\breve{\nu}_3-(\breve{\nu}_1+\breve{\nu}_2-\frac{\breve{\nu}_1\breve{\nu}_2}{\mathfrak{t}'})\frac{\breve{\nu}_3}{\mathfrak{t}'}, (\breve{\nu}'_1+\breve{\nu}'_2-\frac{\breve{\nu}'_1\breve{\nu}'_2}{\mathfrak{t}'})+\breve{\nu}'_3-(\breve{\nu}'_1+\breve{\nu}'_2-\frac{\breve{\nu}'_1\breve{\nu}'_2}{\mathfrak{t}'})\frac{\breve{\nu}'_3}{\mathfrak{t}'}\right]} \end{array} \right\rangle; \\ \left[\breve{\xi}_i, \breve{\xi}'_i\right], [\Psi_i, \Psi'_i] \subseteq [0, \mathfrak{t}]; [\breve{\mu}_i, \breve{\mu}'_i], [\breve{\nu}_i, \breve{\nu}'_i] \subseteq \left[0, \mathfrak{t}'\right] \text{ where } i = 1, 2, 3 \right)$$

Which implies that:

$$(\overset{\wedge}{S_1} \otimes \overset{\wedge}{S_2}) \otimes \overset{\wedge}{S_3} = \left(\left\langle \begin{array}{c} \dot{S}_{\left[\frac{\breve{\xi}_1\breve{\xi}_2\breve{\xi}_3}{\mathfrak{t}^2}, \frac{\breve{\xi}'_1\breve{\xi}'_2\breve{\xi}'_3}{\mathfrak{t}^2}\right]}, \\ \dot{S}_{\left[(\Psi_1+\Psi_2+\Psi_3-\frac{\Psi_1\Psi_2}{\mathfrak{t}}-\frac{\Psi_1\Psi_3}{\mathfrak{t}}-\frac{\Psi_2\Psi_3}{\mathfrak{t}}+\frac{\Psi_1\Psi_2\Psi_3}{\mathfrak{t}^2}, \Psi'_1+\Psi'_2+\Psi'_3-\frac{\Psi'_1\Psi'_2}{\mathfrak{t}}-\frac{\Psi'_1\Psi'_3}{\mathfrak{t}}-\frac{\Psi'_2\Psi'_3}{\mathfrak{t}}+\frac{\Psi'_1\Psi'_2\Psi'_3}{\mathfrak{t}^2}\right.} \end{array} \right\rangle, \left\langle \begin{array}{c} \ddot{S}_{\left[\frac{\breve{\mu}_1\breve{\mu}_2\breve{\mu}_3}{\mathfrak{t}'^2}, \frac{\breve{\mu}'_1\breve{\mu}'_2\breve{\mu}'_3}{\mathfrak{t}'^2}\right]}, \\ \ddot{S}_{\left[\breve{\nu}_1+\breve{\nu}_2+\breve{\nu}_3-\frac{\breve{\nu}_1\breve{\nu}_2}{\mathfrak{t}'}-\frac{\breve{\nu}_1\breve{\nu}_3}{\mathfrak{t}'}-\frac{\breve{\nu}_2\breve{\nu}_3}{\mathfrak{t}'}+\frac{\breve{\nu}_1\breve{\nu}_2\breve{\nu}_3}{\mathfrak{t}'^2}, \breve{\nu}'_1+\breve{\nu}'_2+\breve{\nu}'_3-\frac{\breve{\nu}'_1\breve{\nu}'_2}{\mathfrak{t}'}-\frac{\breve{\nu}'_1\breve{\nu}'_3}{\mathfrak{t}'}-\frac{\breve{\nu}'_2\breve{\nu}'_3}{\mathfrak{t}'}+\frac{\breve{\nu}'_1\breve{\nu}'_2\breve{\nu}'_3}{\mathfrak{t}'^2}\right]} \end{array} \right\rangle; \\ \left[\breve{\xi}_i, \breve{\xi}'_i\right], [\Psi_i, \Psi'_i] \subseteq [0, \mathfrak{t}]; [\breve{\mu}_i, \breve{\mu}'_i], [\breve{\nu}_i, \breve{\nu}'_i] \subseteq \left[0, \mathfrak{t}'\right] \text{ where } i = 1, 2, 3 \right) \tag{4.8}$$

Consider R.H.S.

$$\overset{\wedge}{S_1} \otimes (\overset{\wedge}{S_2} \otimes \overset{\wedge}{S_3})$$

$$\overset{\wedge}{S_2} \otimes \overset{\wedge}{S_3} = \left(\left\langle \dot{S}_{\left[\frac{\breve{\xi}_2\breve{\xi}_3}{\mathfrak{t}}, \frac{\breve{\xi}'_2\breve{\xi}'_3}{\mathfrak{t}}\right]}, \dot{S}_{\left[\Psi_2+\Psi_3-\frac{\Psi_2\Psi_3}{\mathfrak{t}}, \Psi'_2+\Psi'_3-\frac{\Psi'_2\Psi'_3}{\mathfrak{t}}\right]} \right\rangle, \left\langle \ddot{S}_{\left[\frac{\breve{\mu}_2\breve{\mu}_3}{\mathfrak{t}'}, \frac{\breve{\mu}'_2\breve{\mu}'_3}{\mathfrak{t}'}\right]}, \ddot{S}_{\left[\breve{\nu}_2+\breve{\nu}_3-\frac{\breve{\nu}_2\breve{\nu}_3}{\mathfrak{t}'}, \breve{\nu}'_1+\breve{\nu}'_2-\frac{\breve{\nu}'_2\breve{\nu}'_3}{\mathfrak{t}'}\right]} \right\rangle \right)$$

By using the Definition (20) of Product of two GLIVIFSE Sets,

Now

$$\hat{S}_1 \otimes (\hat{S}_2 \otimes \hat{S}_3)$$

$$\hat{S}_1 \otimes (\hat{S}_2 \otimes \hat{S}_3) = \left(\begin{array}{c} \left\langle \begin{array}{c} \dot{S}_{\left[\frac{\breve{\xi}_1(\frac{\breve{\xi}_2\breve{\xi}_3}{\mathfrak{t}})}{\mathfrak{t}}, \frac{\breve{\xi}'_1(\frac{\breve{\xi}'_2\breve{\xi}'_3}{\mathfrak{t}})}{\mathfrak{t}}\right]}, \\ \dot{S}_{\left[\Psi_3+(\Psi_2+\Psi_3-\frac{\Psi_2\Psi_3}{\mathfrak{t}})-\frac{\Psi_1}{\mathfrak{t}}(\Psi_2+\Psi_3-\frac{\Psi_2\Psi_3}{\mathfrak{t}}), \Psi'_1+(\Psi'_2+\Psi'_3-\frac{\Psi'_2\Psi'_3}{\mathfrak{t}})-\frac{\Psi'_1}{\mathfrak{t}}(\Psi'_2+\Psi'_3-\frac{\Psi'_2\Psi'_3}{\mathfrak{t}})\right]} \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \ddot{S}_{\left[\frac{\breve{\mu}_1(\frac{\breve{\mu}_2\breve{\mu}_3}{\mathfrak{t}'})}{\mathfrak{t}'}, \frac{\breve{\mu}'_1(\frac{\breve{\mu}'_2\breve{\mu}'_3}{\mathfrak{t}'})}{\mathfrak{t}'}\right]}, \\ \ddot{S}_{\left[\breve{\nu}_1+(\breve{\nu}_2+\breve{\nu}_3-\frac{\breve{\nu}_1\breve{\nu}_2}{\mathfrak{t}'})-\frac{\breve{\nu}_1}{\mathfrak{t}'}(\breve{\nu}_2+\breve{\nu}_3-\frac{\breve{\nu}_2\breve{\nu}_3}{\mathfrak{t}'}), \breve{\nu}'_1+(\breve{\nu}'_2+\breve{\nu}'_3-\frac{\breve{\nu}'_1\breve{\nu}'_2}{\mathfrak{t}'})-\frac{\breve{\nu}'_1}{\mathfrak{t}'}(\breve{\nu}'_2+\breve{\nu}'_3-\frac{\breve{\nu}'_2\breve{\nu}'_3}{\mathfrak{t}'})\right]} \end{array} \right\rangle \end{array} \right)$$

$$\hat{S}_1 \otimes (\hat{S}_2 \otimes \hat{S}_3) = \left(\begin{array}{c} \left\langle \begin{array}{c} \dot{S}_{\left[\frac{\breve{\xi}_1\breve{\xi}_2\breve{\xi}_3}{\mathfrak{t}^2}, \frac{\breve{\xi}'_1\breve{\xi}'_2\breve{\xi}'_3}{\mathfrak{t}^2}\right]}, \\ \dot{S}_{[(\Psi_1+\Psi_2+\Psi_3-\frac{\Psi_1\Psi_2}{\mathfrak{t}}-\frac{\Psi_1\Psi_3}{\mathfrak{t}}-\frac{\Psi_2\Psi_3}{\mathfrak{t}}+\frac{\Psi_1\Psi_2\Psi_3}{\mathfrak{t}}, \Psi'_1+\Psi'_2+\Psi'_3-\frac{\Psi'_1\Psi'_2}{\mathfrak{t}}-\frac{\Psi'_1\Psi'_3}{\mathfrak{t}}-\frac{\Psi'_2\Psi'_3}{\mathfrak{t}}+\frac{\Psi'_1\Psi'_2\Psi'_3}{\mathfrak{t}^2}} \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \ddot{S}_{\left[\frac{\breve{\mu}_1\breve{\mu}_2\breve{\mu}_3}{\mathfrak{t}'^2}, \frac{\breve{\mu}'_1\breve{\mu}'_2\breve{\mu}'_3}{\mathfrak{t}'^2}\right]}, \\ \ddot{S}_{\left[\breve{\nu}_1+\breve{\nu}_2+\breve{\nu}_3-\frac{\breve{\nu}_1\breve{\nu}_2}{\mathfrak{t}'}-\frac{\breve{\nu}_1\breve{\nu}_3}{\mathfrak{t}'}-\frac{\breve{\nu}_2\breve{\nu}_3}{\mathfrak{t}'}+\frac{\breve{\nu}_1\breve{\nu}_2\breve{\nu}_3}{\mathfrak{t}'^2}, \breve{\nu}'_1+\breve{\nu}'_2+\breve{\nu}'_3-\frac{\breve{\nu}'_1\breve{\nu}'_2}{\mathfrak{t}'}-\frac{\breve{\nu}'_1\breve{\nu}'_3}{\mathfrak{t}'}-\frac{\breve{\nu}'_2\breve{\nu}'_3}{\mathfrak{t}'}+\frac{\breve{\nu}'_1\breve{\nu}'_2\breve{\nu}'_3}{\mathfrak{t}'^2}\right]} \end{array} \right\rangle; \\ \left[\breve{\xi}_i, \breve{\xi}'_i\right], [\Psi_i, \Psi'_i] \subseteq [0, \mathfrak{t}]; [\breve{\mu}_i, \breve{\mu}'_i], [\breve{\nu}_i, \breve{\nu}'_i] \subseteq \left[0, \mathfrak{t}'\right] \text{ where } i = 1, 2, 3 \end{array} \right) \tag{4.9}$$

By 4.8 and 4.9 We get the desired result:

$$(\hat{S}_1 \otimes \hat{S}_2) \otimes \hat{S}_3 = \hat{S}_1 \otimes (\hat{S}_2 \otimes \hat{S}_3)$$

Hence proved. ■

Remark 34 *Distributive law of multiplication over addition does not hold in General Linguistic Interval Valued Intuitionistc Fuzzy Soft Expert Sets GLIVIFSESs.*

Counter Example to disprove the above mentioned law is given below:

Example 35

$$\hat{S}_1 \otimes \left(\hat{S}_2 \oplus \hat{S}_3 \right) \neq \left(\hat{S}_1 \otimes \hat{S}_2 \right) \oplus \left(\hat{S}_1 \otimes \hat{S}_3 \right)$$

Let

$$\hat{S}_1,\ \hat{S}_2 \text{ and } \hat{S}_3 \text{be three GLIVIFSESs with odd cardinality.}$$

Where

$$\hat{S}_1 = \left(\left\langle \dot{S}_{[0.4\ ,\ 0.6]}, \dot{S}_{[0.2\ ,\ 0.4]} \right\rangle, \left\langle \ddot{S}_{[0.3\ ,\ 0.6]}, \ddot{S}_{[0.2\ ,\ 0.4]} \right\rangle \right)$$

$$\hat{S}_2 = \left(\left\langle \dot{S}_{[0.5\ ,\ 0.7]}, \dot{S}_{[0.1\ ,\ 0.3]} \right\rangle, \left\langle \ddot{S}_{[0.6\ ,\ 0.8]}, \ddot{S}_{[0.1\ ,\ 0.2]} \right\rangle \right)$$

$$\hat{S}_3 = \left(\left\langle \dot{S}_{[0.4\ ,\ 0.7]}, \dot{S}_{[0.2\ ,\ 0.3]} \right\rangle, \left\langle \ddot{S}_{[0.5\ ,\ 0.7]}, \ddot{S}_{[0.1\ ,\ 0.3]} \right\rangle \right)$$

Also

$$\text{ţ} = 4, \qquad \text{ţ}' = \ 6, \quad \text{For all } S_i \quad (i = 1, 2, 3)$$

Consider L.H.S.

$$\hat{S}_1 \otimes \left(\hat{S}_2 \oplus \hat{S}_3 \right)$$

$$\hat{S}_2 \oplus \hat{S}_3 = \begin{pmatrix} \left\langle \dot{S}_{\left[0.9 - \frac{(0.4)(0.5)}{4},\ 1.4 - \frac{(0.7)(0.7)}{4}\right]}, \dot{S}_{\left[\frac{(0.1)(0.2)}{4},\ \frac{(0.3)(0.3)}{4}\right]} \right\rangle, \\ \left\langle \ddot{S}_{\left[1.1 - \frac{(0.5)(0.6)}{6},\ 1.5 \frac{(0.7)(0.8)}{6}\right]}, \ddot{S}_{\left[\frac{(0.1)(0.1)}{6},\ \frac{(0.2)(0.3)}{6}\right]} \right\rangle \end{pmatrix}$$

$$\hat{S}_2 \oplus \hat{S}_3 = \begin{pmatrix} \left\langle \dot{S}_{[0.85\ ,\ 1.2775]}, \dot{S}_{[0.005\ ,\ 0.0225]} \right\rangle, \\ \left\langle \ddot{S}_{[1.05\ ,\ 1.40667]}, \ddot{S}_{[0.001667\ ,\ 0.01]} \right\rangle \end{pmatrix}$$

$$\hat{S}_1 \otimes \left(\hat{S}_2 \oplus \hat{S}_3 \right) = \begin{pmatrix} \left\langle \dot{S}_{\left[\frac{(0.4)(0.85)}{4},\ \frac{(0.6)(1.2775)}{4}\right]}, \dot{S}_{\left[0.205 - \frac{(0.2)(0.005)}{4},\ 0.4225 - \frac{(0.4)(0.0225)}{4}\right]} \right\rangle, \\ \left\langle \ddot{S}_{\left[\frac{(0.3)(1.05)}{6},\ \frac{(0.6)(1.40667)}{6}\right]}, \ddot{S}_{\left[0.201667 - \frac{0.2(0.001667)}{6},\ 0.41 - \frac{(0.4)(0.01)}{6}\right]} \right\rangle \end{pmatrix}$$

$$\hat{S}_1 \otimes \left(\hat{S}_2 \oplus \hat{S}_3 \right) = \begin{pmatrix} \left\langle \dot{S}_{[0.85\ ,\ 0.1916]}, \dot{S}_{[0.2048\ ,\ 0.4203]} \right\rangle, \\ \left\langle \ddot{S}_{[0.0525\ ,\ 0.14067]}, \ddot{S}_{[0.2016\ ,\ 0.4093]} \right\rangle \end{pmatrix} \tag{4.10}$$

Consider R.H.S.

$$\overset{\wedge}{S_1} \otimes \overset{\wedge}{S_2} = \begin{pmatrix} \left\langle \dot{S}_{\left[\frac{0.4\ (0.5)}{4}, \frac{0.6\ (0.7)}{4}\right]}, \dot{S}_{\left[0.3 - \frac{(0.2)\ (0.1)}{4}, 0.7 - \frac{(0.4)\ (0.3)}{4}\right]} \right\rangle, \\ \left\langle \ddot{S}_{\left[\frac{0.3\ (0.6)}{6}, \frac{0.6\ (0.8)}{6}\right]}, \ddot{S}_{\left[0.3 - \frac{(0.2)\ (0.1)}{6}, (0.6) - \frac{(0.2)\ (0.4)}{6}\right]} \right\rangle \end{pmatrix}$$

$$= \begin{pmatrix} \left\langle \dot{S}_{[0.05\ ,\ 0.105]}, \dot{S}_{[0.295\ ,\ 0.67]} \right\rangle, \\ \left\langle \ddot{S}_{[0.03\ ,\ 0.08]}, \ddot{S}_{[0.29667\ ,\ 0.58667]} \right\rangle \end{pmatrix}$$

$$\overset{\wedge}{S_1} \otimes \overset{\wedge}{S_3} = \begin{pmatrix} \left\langle \dot{S}_{\left[\frac{0.4(0.4)}{4}\ ,\ \frac{0.6(0.7)}{4}\right]}, \dot{S}_{\left[0.4 - \frac{0.2(0.2)}{4}\ ,\ 0.7 - \frac{(0.3)(0.4)}{4}\right]} \right\rangle, \\ \left\langle \ddot{S}_{\left[\frac{0.3(0.5)}{6}\ ,\ \frac{0.6(0.7)}{6}\right]}, \ddot{S}_{\left[0.3 - \frac{(0.2)(0.1)}{6}\ ,\ 0.7 - \frac{(0.3)(0.4)}{6}\right]} \right\rangle \end{pmatrix}$$

$$= \left(\left\langle \dot{S}_{[0.04\ ,\ 0.105]}, \dot{S}_{[0.39\ ,\ 0.67]} \right\rangle, \left\langle \ddot{S}_{[0.025\ ,\ 0.07]}, \ddot{S}_{[0.29667\ ,\ 0.68]} \right\rangle \right)$$

Now

$$\left(\overset{\wedge}{S_1} \otimes\ \overset{\wedge}{S_2}\right) \oplus \left(\overset{\wedge}{S_1} \otimes \overset{\wedge}{S_3}\right) = \begin{pmatrix} \left\langle \begin{matrix} \dot{S}_{\left[0.07 - \frac{(0.03)\ (0.04)}{4}, 0.185 - \frac{(0.08)\ (0.105)}{4}\right]}, \\ \dot{S}_{\left[\frac{(0.295)\ (0.39)}{4}, \frac{(0.67)\ (0.67)}{4}\right]} \end{matrix} \right\rangle, \\ \left\langle \begin{matrix} \ddot{S}_{\left[0.055 - \frac{0.03\ (0.025)}{6}, 0.15 - \frac{(0.08)\ (0.7)}{6}\right]}, \\ \ddot{S}_{\left[\frac{(0.29667)\ (0.29667)}{6}, \frac{(0.58667)\ (0.68)}{6}\right]} \end{matrix} \right\rangle \end{pmatrix}$$

$$= \begin{pmatrix} \left\langle \dot{S}_{[0.067\ ,\ 0.1829]}, \dot{S}_{[0.2876\ ,\ 0.11223]} \right\rangle, \\ \left\langle \ddot{S}_{[0.05425\ ,\ 0.056667]}, \ddot{S}_{[0.01466885\ ,\ 0.06649]} \right\rangle \end{pmatrix} \tag{4.11}$$

By 4.10 and 4.11 we get

$$\overset{\wedge}{S_1} \otimes \left(\overset{\wedge}{S_2}\ \oplus\ \overset{\wedge}{S_3}\right) \neq \left(\overset{\wedge}{S_1} \otimes\ \overset{\wedge}{S_2}\right)\ \oplus\ \left(\overset{\wedge}{S_1} \otimes\ \overset{\wedge}{S_3}\right) \text{ As required.}$$

Remark 36 *Similarly, we can disprove by counter example that Distributive law of addition over multiplication does not hold in General Linguistic Interval Valued Intuitionistc Fuzzy Soft Expert Sets (GLIVIFSESs)*

Proof. vi)

$$\text{i)}\quad \left(\hat{S}_1 \oplus \hat{S}_2\right)^c = \hat{S}{}_1^{c} \otimes \hat{S}{}_2^{c}$$

Consider L.H.S.

$$\hat{S}_1 \oplus \hat{S}_2 = \left(\begin{array}{c} \left\langle \dot{S}_{\left[\breve{\xi}_1 + \breve{\xi}_2 - \frac{\breve{\xi}_1\breve{\xi}_2}{\mathfrak{t}}, \breve{\xi}'_1 + \breve{\xi}'_2 - \frac{\breve{\xi}'_1\breve{\xi}'_2}{\mathfrak{t}}\right]}, \dot{S}_{\left[\frac{\Psi_1\ \Psi_2}{\mathfrak{t}'}, \frac{\Psi'_1\Psi'_2}{\mathfrak{t}'}\right]} \right\rangle, \\ \left\langle \ddot{S}_{\left[\breve{\mu}_1 + \breve{\mu}_2 - \frac{\breve{\mu}_1\breve{\mu}_2}{\mathfrak{t}'}, \breve{\mu}'_1 + \breve{\mu}'_2 - \frac{\breve{\mu}'_1\breve{\mu}'_2}{\mathfrak{t}'}\right]}, \ddot{S}_{\left[\frac{\breve{\nu}_1\ \breve{\nu}_2}{\mathfrak{t}'}, \frac{\breve{\nu}'_1\breve{\nu}'2}{\mathfrak{t}}\right]} \right\rangle; \end{array}\right)$$

By using Definiton (18) of Sum of any two General lLinguistic Interval Valued Intuitionistc Fuzzy Soft Expert Sets (GLIVIFSESs).

Now

$$\left(\hat{S}_1 \oplus \hat{S}_2\right)^c = \left(\begin{array}{c} \left\langle \dot{S}_{\left[\frac{\Psi_1\ \Psi_2}{\mathfrak{t}}, \frac{\Psi'_1\Psi'_2}{\mathfrak{t}}\right]}, \dot{S}_{\left[\breve{\xi}_1 + \breve{\xi}_2 - \frac{\breve{\xi}_1\breve{\xi}_2}{\mathfrak{t}}, \breve{\xi}'_1 + \breve{\xi}'_2 - \frac{\breve{\xi}'_1\breve{\xi}'_2}{\mathfrak{t}}\right]} \right\rangle, \\ \left\langle \ddot{S}_{\left[\frac{\breve{\nu}_1\ \breve{\nu}_2}{\mathfrak{t}'}, \frac{\breve{\nu}'_1\breve{\nu}'2}{\mathfrak{t}'}\right]}, \ddot{S}_{\left[\breve{\mu}_1 + \breve{\mu}_2 - \frac{\breve{\mu}_1\breve{\mu}_2}{\mathfrak{t}'}, \breve{\mu}'_1 + \breve{\mu}'_2 - \frac{\breve{\mu}'_1\breve{\mu}'_2}{\mathfrak{t}'}\right]} \right\rangle; \\ \left[\breve{\xi}_i, \breve{\xi}_i^{'}\right], [\Psi_i, \Psi'_i] \subseteq [0, \mathfrak{t}]; [\breve{\mu}_i, \breve{\mu}'_i], [\breve{\nu}_i, \breve{\nu}'_i] \subseteq \left[0, \mathfrak{t}^{'}\right] \text{ where } i = 1, 2 \end{array}\right) \tag{4.12}$$

Consider R.H.S.

$$\hat{S}{}_1^{c} \otimes \hat{S}{}_2^{c}$$

Where

$$\hat{S}{}_1^{c} = \left(\begin{array}{c} \left\langle \dot{S}_{[\Psi_1, \Psi'_1]}, \dot{S}_{\left[\breve{\xi}_1, \breve{\xi}'_1\right]} \right\rangle, \left\langle \ddot{S}_{[\breve{\mu}_1, \breve{\mu}'_1]}, \ddot{S}_{[\breve{\nu}_1, \breve{\nu}'_1]} \right\rangle; \\ \left[\breve{\xi}_1, \breve{\xi}_1^{'}\right], [\Psi_1, \Psi'_1] \subseteq [0, \mathfrak{t}]; [\breve{\mu}_1, \breve{\mu}'_1], [\breve{\nu}_1, \breve{\nu}'_1] \subseteq \left[0, \mathfrak{t}^{'}\right] \end{array}\right)$$

$$\hat{S}{}_2^{c} = \left(\begin{array}{c} \left\langle \dot{S}_{[\Psi_2, \Psi'_2]}, \dot{S}_{\left[\breve{\xi}_2, \breve{\xi}'_2\right]} \right\rangle, \left\langle \ddot{S}_{[\breve{\mu}_2, \breve{\mu}'_2]}, \ddot{S}_{[\breve{\nu}_2, \breve{\nu}'_2]} \right\rangle; \\ \left[\breve{\xi}_2, \breve{\xi}_2^{'}\right], [\Psi_2, \Psi'_2] \subseteq [0, \mathfrak{t}]; [\breve{\mu}_2, \breve{\mu}'_2], [\breve{\nu}_2, \breve{\nu}'_2] \subseteq \left[0, \mathfrak{t}^{'}\right] \end{array}\right)$$

By using the Definition (27) Complement of any two General linguistic interval valued intuitionistc fuzzy soft expert sets (GLIVIFSESs)

Now

$$\hat{S}_1^c \otimes \hat{S}_2^c = \left(\begin{array}{c} \left\langle \dot{S}_{\left[\frac{\Psi_1\Psi_2}{\mathfrak{t}}, \frac{\Psi_1'\Psi_2'}{\mathfrak{t}}\right]}, \dot{S}_{\left[\breve{\xi}_1 + \breve{\xi}_2 - \frac{\breve{\xi}_1\breve{\xi}_2}{\mathfrak{t}}, \breve{\xi}_1' + \breve{\xi}_2' - \frac{\breve{\xi}_1'\breve{\xi}_2'}{\mathfrak{t}}\right]} \right\rangle, \\ \left\langle \ddot{S}_{\left[\frac{\breve{\nu}_1 \breve{\nu}_2}{\mathfrak{t}'}, \frac{\breve{\nu}_1'\breve{\nu}'2}{\mathfrak{t}'}\right]}, \ddot{S}_{\left[\breve{\mu}_1 + \breve{\mu}_2 - \frac{\breve{\mu}_1\breve{\mu}_2}{\mathfrak{t}'}, \breve{\mu}_1' + \breve{\mu}_2' - \frac{\breve{\mu}_1'\breve{\mu}_2'}{\mathfrak{t}'}\right]} \right\rangle; \\ \left[\breve{\xi}_i, \breve{\xi}_i'\right], [\Psi_i, \Psi_i'] \subseteq [0, \mathfrak{t}]; [\breve{\mu}_i, \breve{\mu}_i'], [\breve{\nu}_i, \breve{\nu}_i'] \subseteq \left[0, \mathfrak{t}'\right] \text{ where } i = 1, 2 \end{array} \right) \quad (4.13)$$

By 4.12 and 4.13, we get the desired result such as:

$$\left(\hat{S}_1 \oplus \hat{S}_2\right)^c = \left(\hat{S}_1\right)^c \otimes \left(\hat{S}_2\right)^c$$

Hence Proved ■

Proof. *ii)*

$$\left(\hat{S}_1 \otimes \hat{S}_2\right)^c = \left(\hat{S}_1\right)^c \oplus \left(\hat{S}_2\right)^c$$

Consider L.H.S.

$$\hat{S}_1 \otimes \hat{S}_2 = \left(\begin{array}{c} \left\langle \dot{S}_{\left[\frac{\breve{\xi}_1 \breve{\xi}_2}{\mathfrak{t}}, \frac{\breve{\xi}_1'\breve{\xi}_2'}{\mathfrak{t}}\right]}, \dot{S}_{\left[\Psi_1 + \Psi_2 - \frac{\Psi_1\Psi_2}{\mathfrak{t}}, \Psi_1' + \Psi_2' - \frac{\Psi_1'\Psi_2'}{\mathfrak{t}}\right]} \right\rangle, \\ \left\langle \ddot{S}_{\left[\frac{\breve{\mu}_1\breve{\mu}_2}{\mathfrak{t}'}, \frac{\breve{\mu}_1'\breve{\mu}_2'}{\mathfrak{t}'}\right]}, \ddot{S}_{\left[\breve{\nu}_1 + \breve{\nu}_2 - \frac{\breve{\nu}_1 \breve{\nu}_2}{\mathfrak{t}'}, \breve{\nu}_1' + \breve{\nu}_2' - \frac{\breve{\nu}_1' \breve{\nu}_2'}{\mathfrak{t}'}\right]} \right\rangle; \\ \left[\breve{\xi}_i, \breve{\xi}_i'\right], [\Psi_i, \Psi_i'] \subseteq [0, \mathfrak{t}]; [\breve{\mu}_i, \breve{\mu}_i'], [\breve{\nu}_i, \breve{\nu}_i'] \subseteq \left[0, \mathfrak{t}'\right] \\ \text{where } i = 1, 2 \end{array} \right)$$

Now

$$\left(\hat{S}_1 \otimes \hat{S}_2\right)^c = \left(\begin{array}{c} \left\langle \dot{S}_{\left[\Psi_1 + \Psi_2 - \frac{\Psi_1\Psi_2}{\mathfrak{t}}, \Psi_1' + \Psi_2' - \frac{\Psi_1'\Psi_2'}{\mathfrak{t}}\right]}, \dot{S}_{\left[\frac{\breve{\xi}_1\breve{\xi}_2}{\mathfrak{t}}, \frac{\breve{\xi}_1'\breve{\xi}_2'}{\mathfrak{t}'}\right]} \right\rangle, \\ \left\langle \ddot{S}_{\left[\breve{\nu}_1 + \breve{\nu}_2 - \frac{\breve{\nu}_1\breve{\nu}_2}{\mathfrak{t}'}, \breve{\nu}_1' + \breve{\nu}_2' - \frac{\breve{\nu}_1'\breve{\nu}_2'}{\mathfrak{t}'}\right]}, \ddot{S}_{\left[\frac{\breve{\mu}_1\breve{\mu}_2}{\mathfrak{t}'}, \frac{\breve{\mu}_1'\breve{\mu}_2'}{\mathfrak{t}'}\right]} \right\rangle; \\ \left[\breve{\xi}_i, \breve{\xi}_i'\right], [\Psi_i, \Psi_i'] \subseteq [0, \mathfrak{t}]; [\breve{\mu}_i, \breve{\mu}_i'], [\breve{\nu}_i, \breve{\nu}_i'] \subseteq \left[0, \mathfrak{t}'\right] \\ \text{where } i = 1, 2 \end{array} \right) \quad (4.14)$$

Consider R.H.S. $\hat{S}{}^c_1 \oplus \hat{S}{}^c_2$

$$\hat{S}{}^c_1 = \left(\begin{array}{c} \left\langle \dot{S}_{[\Psi_1,\Psi_1']}, \dot{S}_{\left[\breve{\xi}_1,\breve{\xi}_1'\right]} \right\rangle, \left\langle \ddot{S}_{[\breve{\nu}_1,\breve{\nu}_1']}, \ddot{S}_{[\breve{\mu}_1,\breve{\mu}_1']} \right\rangle; \\ \left[\breve{\xi}_i, \breve{\xi}_i'\right], [\Psi_i, \Psi_i'] \subseteq [0, \mathfrak{t}]; [\breve{\mu}_1, \breve{\mu}_1'], [\breve{\nu}_1, \breve{\nu}_1'] \subseteq \left[0, \mathfrak{t}'\right] \end{array} \right)$$

$$\hat{S}{}^c_2 = \left(\begin{array}{c} \left\langle \dot{S}_{[\Psi_2,\Psi_2']}, \dot{S}_{\left[\breve{\xi}_2,\breve{\xi}_2'\right]} \right\rangle, \left\langle \ddot{S}_{[\breve{\nu}_2,\breve{\nu}_2']}, \ddot{S}_{[\breve{\mu}_2,\breve{\mu}_2']} \right\rangle; \\ \left[\breve{\xi}_2, \breve{\xi}_2'\right], [\Psi_2, \Psi_2'] \subseteq [0, \mathfrak{t}]; [\breve{\mu}_2, \breve{\mu}_2'], [\breve{\nu}_2, \breve{\nu}_2'] \subseteq \left[0, \mathfrak{t}'\right] \end{array} \right)$$

By using Definition (27) of Complement

$$\hat{S}{}^c_1 \oplus \hat{S}{}^c_2 = \left(\begin{array}{c} \left\langle \dot{S}_{\left[\Psi_1+\Psi_2-\frac{\Psi_1\Psi_2}{\mathfrak{t}},\ \Psi_1'+\Psi_2'-\frac{\Psi_1'\Psi_2'}{\mathfrak{t}}\right]}, \dot{S}_{\left[\frac{\breve{\xi}_1\breve{\xi}_2}{\mathfrak{t}}, \frac{\breve{\xi}_1'\breve{\xi}_2'}{\mathfrak{t}'}\right]} \right\rangle, \\ \left\langle \ddot{S}_{\left[\breve{\nu}_1 + \breve{\nu}_2-\frac{\breve{\nu}_1\breve{\nu}_2}{\mathfrak{t}'}, \breve{\nu}_1' + \breve{\nu}_2'-\frac{\breve{\nu}_1'\breve{\nu}'_2}{\mathfrak{t}'}\right]}, \ddot{S}_{\left[\frac{\breve{\mu}_1\breve{\mu}_2}{\mathfrak{t}'}, \frac{\breve{\mu}_1'\breve{\mu}_2'}{\mathfrak{t}'}\right]} \right\rangle; \\ \left[\breve{\xi}_i, \breve{\xi}_i'\right], [\Psi_i, \Psi_i'] \subseteq [0, \mathfrak{t}]; [\breve{\mu}_i, \breve{\mu}_i'], [\breve{\nu}_i, \breve{\nu}_i'] \subseteq \left[0, \mathfrak{t}'\right] \\ \text{where } i = 1, 2 \end{array} \right) \quad (4.15)$$

By 4.14 and 4.15,we get the required result

$$\left(\hat{S}_1 \otimes \hat{S}_2\right)^c = \hat{S}{}^c_1 \oplus \hat{S}{}^c_2$$

Hence Proved. ∎

Theorem 37 *Let* $\hat{S}, \hat{S}_1, \hat{S}_2$ *be any three General linguistic interval valued intuitionistc fuzzy soft expert sets GLIVIFSES Sets and* $\lambda, \lambda_1, \lambda_2$ *be any positive real numbers such as:*

$$\hat{S} = \left(\begin{array}{c} \left\langle \dot{S}_{\left[\breve{\xi},\breve{\xi}'\right]}, \dot{S}_{[\Psi,\Psi']} \right\rangle, \left\langle \ddot{S}_{[\breve{\mu},\breve{\mu}']}, \ddot{S}_{[\nu,\nu']} \right\rangle; \\ \left[\breve{\xi}, \breve{\xi}'\right], [\Psi, \Psi'] \subseteq [0, \mathfrak{t}]; [\breve{\mu}, \breve{\mu}'], [\breve{\nu}, \breve{\nu}'] \subseteq \left[0, \mathfrak{t}'\right] \end{array} \right),$$

$$\hat{S}_1 = \left(\begin{array}{c} \left\langle \dot{S}_{\left[\breve{\xi}_1,\ \breve{\xi}_1'\right]}, \dot{S}_{[\Psi_1,\ \Psi_1']} \right\rangle, \left\langle \ddot{S}_{[\breve{\mu}_1,\ \breve{\mu}_1']}, \ddot{S}_{\left[\breve{\nu}_1,\ \breve{\nu}_1'\right]} \right\rangle; \\ \left[\breve{\xi}_1, \breve{\xi}_1'\right], [\Psi_1, \Psi_1'] \subseteq [0, \mathfrak{t}]; [\breve{\mu}_1, \breve{\mu}_1'], [\breve{\nu}_1, \breve{\nu}_1'] \subseteq \left[0, \mathfrak{t}'\right] \end{array} \right)$$

$$\hat{S}_2 = \left(\begin{array}{c} \left\langle \dot{S}_{\left[\breve{\xi}_2,\ \breve{\xi}_2'\right]}, \dot{S}_{[\Psi_2,\ \Psi_2']} \right\rangle, \left\langle \ddot{S}_{[\breve{\mu}_2,\ \breve{\mu}_2']}, \ddot{S}_{\left[\breve{\nu}_2,\ \breve{\nu}_2'\right]} \right\rangle; \\ \left[\breve{\xi}_2, \breve{\xi}_2'\right], [\Psi_2, \Psi_2'] \subseteq [0, t]; [\breve{\mu}_2, \breve{\mu}_2'], [\breve{\nu}_2, \breve{\nu}_2'] \subseteq \left[0, t'\right] \end{array} \right)$$

where

$$(\left\langle \dot{S}_{\left[\breve{\xi}_i,\breve{\xi}_i'\right]}, \dot{S}_{[\Psi_i,\Psi_i']} \right\rangle; \dot{S}_{[0,0]} \subseteq \dot{S}_{\left[\breve{\xi}_i,\breve{\xi}_i'\right]} + \dot{S}_{[\Psi_i,\Psi_i']} \subseteq \dot{S}_{[t,t]}) \in \hat{S}^{[1]}$$

$$(\left\langle \ddot{S}_{[\breve{\mu}_i,\breve{\mu}_i']}, \ddot{S}_{[\breve{\nu}_i,\breve{\nu}_i']} \right\rangle; \ddot{S}_{[0.0,\ 0.0]} \subseteq \ddot{S}_{[\breve{\mu}_i,\breve{\mu}_i']} + \ddot{S}_{[\breve{\nu}_i,\breve{\nu}_i']} \subseteq \ddot{S}_{[t',t']}) \in \hat{S}^{[2]}$$

then:

i) $\lambda(\hat{S}_1 \oplus \hat{S}_2) = \lambda(\hat{S}_1) \oplus \lambda(\hat{S}_2)$

ii) $\lambda_1(\hat{S}) \oplus \lambda_2(\hat{S}) = (\lambda_1 + \lambda_2)\hat{S}$

iii) $\lambda_1(\lambda_2\hat{S}) = (\lambda_1\lambda_2)\hat{S}$

iv) $(\hat{S}_1 \otimes \hat{S}_2)^{\lambda} = \hat{S}_1^{\lambda} \otimes \hat{S}_2^{\lambda}$

v) $\hat{S}^{\lambda_1} \otimes \hat{S}^{\lambda_2} = \hat{S}^{(\lambda_1+\lambda_2)}$

vi) $(\hat{S}^{\lambda_1})^{\lambda_2} = \hat{S}^{(\lambda_1\lambda_2)}$

vii) $\lambda(\hat{S}_1 \oplus \hat{S}_2) = \lambda(\hat{S}_1) \oplus \lambda(\hat{S}_2)$

where $\dot{S}_{[\breve{\xi},\breve{\xi}']}, \dot{S}_{[\breve{\xi}_1,\breve{\xi}_1']}, \dot{S}_{[\breve{\xi}_2,\breve{\xi}_2']}$defines the degree of belongingness and $\dot{S}_{[\Psi,\Psi']}, \dot{S}_{[\Psi_1,\Psi_1']}, \dot{S}_{[\Psi_2,\Psi_2']}$describes MD in $\hat{S}_i$ and $\left\langle \dot{S}_{[\breve{\xi}_i,\breve{\xi}_i']}, \dot{S}_{[\Psi_i,\Psi_i']} \right\rangle$ are representing experts opinion about evaluated objects (attributes), while $\ddot{S}_{[\breve{\mu},\breve{\mu}']}, \ddot{S}_{[\breve{\mu}_1,\breve{\mu}_1']}$and $\ddot{S}_{[\breve{\mu}_2,\breve{\mu}_2']}$defines the degree of belongingness $\ddot{S}_{[\nu,\nu']}, \ddot{S}_{[\nu_1,\nu_1']}$and $\ddot{S}_{[\nu_2,\nu_2']}$are describing theNMD in $\hat{S}_i$.

Proof. Consider L.H.S $\lambda(\overset{\wedge}{S_1} \oplus \overset{\wedge}{S_2})$

$$= \lambda \left(\begin{array}{c} \left\langle \dot{S}_{\left[\breve{\xi}_1+\breve{\xi}_2-\frac{\breve{\xi}_1\breve{\xi}_2}{\text{ţ}},\breve{\xi}'_1+\breve{\xi}'_2-\frac{\breve{\xi}'_1\breve{\xi}'_2}{\text{ţ}}\right]}, \dot{S}_{\left[\frac{\Psi_1\Psi_2}{\text{ţ}},\frac{\Psi'_1\Psi'_2}{\text{ţ}}\right]} \right\rangle, \\ \left\langle \ddot{S}_{\left[\breve{\mu}_1+\breve{\mu}_2-\frac{\breve{\mu}_1\breve{\mu}_2}{\text{ţ}'},\breve{\mu}'_1+\breve{\mu}'_2-\frac{\breve{\mu}'_1\breve{\mu}'_2}{\text{ţ}'}\right]}, \ddot{S}_{\left[\frac{\breve{\nu}_1\breve{\nu}_2}{\text{ţ}'},\frac{\breve{\nu}'_1\breve{\nu}'_2}{\text{ţ}'}\right]} \right\rangle; \\ \left[\breve{\xi}_i,\breve{\xi}_i^{'}\right], [\Psi_i,\Psi'_i] \subseteq [0,\text{ţ}]; [\breve{\mu}_i,\breve{\mu}'_i], [\breve{\nu}_i,\breve{\nu}'_i] \subseteq \left[0,\text{ţ}^{'}\right], \text{ where} i=1,2 \end{array} \right)$$

By using Definition of Sum (18) of any two General Linguistic Interval Valued Intuitionistc Fuzzy Soft Expert Sets.

$$= \left(\begin{array}{c} \left\langle \dot{S}_{\text{ţ}\left[1-\left(1-\frac{\left(\breve{\xi}_{1+}\breve{\xi}_2-\frac{\breve{\xi}_1\breve{\xi}_2}{\text{ţ}}\right)}{\text{ţ}}\right)^{\lambda},1-\left(1-\frac{\left(\breve{\xi}'_{1+}\breve{\xi}'_2-\frac{\breve{\xi}'_1\breve{\xi}'_2}{\text{ţ}}\right)}{\text{ţ}}\right)^{\lambda}\right]}, \dot{S}_{\text{ţ}\left[\left(\frac{\Psi_1\Psi_2}{\text{ţ}^2}\right)^{\lambda},\left(\frac{\Psi'_1\Psi'_2}{\text{ţ}^2}\right)^{\lambda}\right]} \right\rangle, \\ \left\langle \ddot{S}_{\text{ţ}\left[\left(1-\left(1-\frac{\breve{\mu}_1+\breve{\mu}_2}{\text{ţ}}-\frac{\breve{\mu}_1+\breve{\mu}_2}{\text{ţ}^2}\right)^{\lambda}\right),\left(1-\left(1-\frac{\breve{\mu}'_1+\breve{\mu}'_2}{\text{ţ}}-\frac{\breve{\mu}'_1+\breve{\mu}'_2}{\text{ţ}^2}\right)^{\lambda}\right)\right]}, \dot{S}_{\text{ţ}\left[\left(\frac{\breve{\nu}_2\breve{\nu}_1}{\text{ţ}^2}\right)^{\lambda},\left(\frac{\breve{\nu}'_2\breve{\nu}'_1}{\text{ţ}^2}\right)^{\lambda}\right]} \right\rangle; \\ \left[\breve{\xi}_i,\breve{\xi}_i^{'}\right], [\Psi_i,\Psi'_i] \subseteq [0,\text{ţ}]; [\breve{\mu}_i,\breve{\mu}'_i], [\breve{\nu}_i,\breve{\nu}'_i] \subseteq \left[0,\text{ţ}^{'}\right] \text{ where } i=1,2 \end{array} \right) \quad (4.16)$$

Consider R.H.S.

$$\lambda\overset{\wedge}{S_1} \oplus \lambda\overset{\wedge}{S_2}$$

Now using the Definition of scalar product (22)of any two General linguistic interval valued intuitionistc fuzzy soft expert sets ,

$$\lambda\overset{\wedge}{S_1} = \left(\begin{array}{c} \left\langle \dot{S}_{\text{ţ}\left[\left(1-\left(1-\frac{(\breve{\xi}_1)}{\text{ţ}}\right)^{\lambda}\right),\left(1-\left(1-\frac{(\breve{\xi}'_1)}{\text{ţ}}\right)^{\lambda}\right)\right]}, \dot{S}_{\text{ţ}\left[\left(\frac{\Psi_1}{\text{ţ}}\right)^{\lambda},\left(\frac{\Psi'_1}{\text{ţ}}\right)^{\lambda}\right]} \right\rangle, \\ \left\langle \ddot{S}_{\text{ţ}'\left[\left(1-\left(1-\frac{(\breve{\mu}_1)}{\text{ţ}'}\right)^{\lambda}\right),\left(1-\left(1-\frac{(\breve{\mu}'_1)}{\text{ţ}'}\right)^{\lambda}\right)\right]}, \ddot{S}_{\text{ţ}'\left[\left(\frac{\breve{\nu}_1}{\text{ţ}'}\right)^{\lambda},\left(\frac{\breve{\nu}'_1}{\text{ţ}'}\right)^{\lambda}\right]} \right\rangle; \\ \left[\breve{\xi}_1,\breve{\xi}_1^{'}\right], [\Psi_1,\Psi'_1] \subseteq [0,\text{ţ}]; [\breve{\mu}_1,\breve{\mu}'_1], [\breve{\nu}_1,\breve{\nu}'_1] \subseteq \left[0,\text{ţ}^{'}\right] \end{array} \right)$$

$$\lambda\overset{\wedge}{S_2}=\left(\left\langle \dot{S}_{\mathfrak{t}\left[\left(1-\left(1-\frac{\breve{\xi}_2}{\mathfrak{t}}\right)^{\lambda}\right),\left(1-\left(1-\frac{\breve{\xi}'_2}{\mathfrak{t}}\right)^{\lambda}\right)\right]},\dot{S}_{\mathfrak{t}\left[\left(\frac{\Psi_2}{\mathfrak{t}}\right)^{\lambda},\left(\frac{\Psi'_2}{\mathfrak{t}}\right)^{\lambda}\right]}\right\rangle,\right.$$

$$\left.\left\langle \ddot{S}_{\mathfrak{t}'\left[\left(1-\left(1-\frac{\breve{\mu}_2}{\mathfrak{t}'}\right)^{\lambda}\right),\left(1-\left(1-\frac{\breve{\mu}'_2}{\mathfrak{t}'}\right)^{\lambda}\right)\right]},\ddot{S}_{\mathfrak{t}'\left[\left(\frac{\breve{\nu}_2}{\mathfrak{t}'}\right)^{\lambda},\left(\frac{\breve{\nu}'_2}{\mathfrak{t}'}\right)^{\lambda}\right]}\right\rangle\right)$$

$$\lambda\overset{\wedge}{S_1}\oplus\lambda\overset{\wedge}{S_2}$$

$$=\left(\begin{array}{c}\left\langle \dot{S}_{\left[\begin{array}{c}\mathfrak{t}\left(1-\left(1-\frac{\breve{\xi}_1}{\mathfrak{t}}\right)^{\lambda}\right)+\mathfrak{t}\left(1-\left(1-\frac{\breve{\xi}_2}{\mathfrak{t}}\right)^{\lambda}\right)-\mathfrak{t}^2\frac{\left(1-\left(1-\frac{\breve{\xi}_1}{\mathfrak{t}}\right)^{\lambda}\right)\left(1-\left(1-\frac{\breve{\xi}_2}{\mathfrak{t}}\right)^{\lambda}\right)}{\mathfrak{t}},\\ \mathfrak{t}\left(1-\left(1-\frac{\breve{\xi}'_1}{\mathfrak{t}}\right)^{\lambda}\right)+\mathfrak{t}\left(1-\left(1-\frac{\breve{\xi}'_2}{\mathfrak{t}}\right)^{\lambda}\right)-\frac{\mathfrak{t}^2\left(1-\left(1-\frac{\breve{\xi}'_1}{\mathfrak{t}}\right)^{\lambda}\right)\left(1-\left(1-\frac{\breve{\xi}'_2}{\mathfrak{t}}\right)^{\lambda}\right)}{\mathfrak{t}}\end{array}\right]},\right.\\ \left.\dot{S}_{\left[\mathfrak{t}^2\frac{\left(\frac{\Psi_1\Psi_2}{\mathfrak{t}^2}\right)^{\lambda}}{\mathfrak{t}},\mathfrak{t}^2\frac{\left(\frac{\Psi'_1\Psi'_2}{\mathfrak{t}^2}\right)^{\lambda}}{\mathfrak{t}}\right]}\right\rangle\\ \left\langle \ddot{S}_{\left[\begin{array}{c}\mathfrak{t}'\left(1-\left(1-\frac{(\breve{\mu}_1)}{\mathfrak{t}'}\right)^{\lambda}\right)+\mathfrak{t}'\left(1-\left(1-\frac{(\breve{\mu}_2)}{\mathfrak{t}'}\right)^{\lambda}\right)-\mathfrak{t}'^2\frac{\left(1-\left(1-\frac{\breve{\mu}_1}{\mathfrak{t}'}\right)^{\lambda}\right)\left(1-\left(1-\frac{\breve{\mu}_2}{\mathfrak{t}'}\right)^{\lambda}\right)}{\mathfrak{t}'},\\ \mathfrak{t}'\left(1-\left(1-\frac{\breve{\mu}'_1}{\mathfrak{t}'}\right)^{\lambda}\right)+\mathfrak{t}'\left(1-\left(1-\frac{\breve{\mu}'_2}{\mathfrak{t}'}\right)^{\lambda}\right)-\frac{\mathfrak{t}'^2\left(1-\left(1-\frac{\breve{\mu}'_1}{\mathfrak{t}'}\right)^{\lambda}\right)\left(1-\left(1-\frac{\breve{\mu}'_2}{\mathfrak{t}'}\right)^{\lambda}\right)}{\mathfrak{t}'}\end{array}\right]},\ddot{S}_{\left[\frac{\mathfrak{t}'^2}{\mathfrak{t}'}\left(\frac{\breve{\nu}_1\breve{\nu}_2}{\mathfrak{t}'^2}\right)^{\lambda},\frac{\mathfrak{t}'^2}{\mathfrak{t}'}\left(\frac{\breve{\nu}'_1\breve{\nu}'_2}{\mathfrak{t}'^2}\right)^{\lambda}\right]}\right\rangle;\\ \left[\breve{\xi}_i,\breve{\xi}'_i\right],[\Psi_i,\Psi'_i]\subseteq[0,\mathfrak{t}];[\breve{\mu}_i,\breve{\mu}'_i],[\breve{\nu}_i,\breve{\nu}'_i]\subseteq\left[0,\mathfrak{t}'\right]\quad\text{where } i=1,2.\end{array}\right)$$

$$= \left(\begin{array}{c} \left\langle \dot{S}_{\mathfrak{t}\left[\begin{array}{c} \left(1-\left(1-\frac{\breve{\xi}_1}{\mathfrak{t}}\right)^{\lambda}-\left(1-\frac{\breve{\xi}_2}{\mathfrak{t}}\right)^{\lambda}+\left(1-\frac{\breve{\xi}_1}{\mathfrak{t}}\right)^{\lambda}+\left(1-\frac{\breve{\xi}_2}{\mathfrak{t}}\right)^{\lambda}-\left(1-\frac{\breve{\xi}_1}{\mathfrak{t}}\right)^{\lambda}\left(1-\frac{\breve{\xi}_2}{\mathfrak{t}}\right)^{\lambda}\right) \\ ,\mathfrak{t}\left(1-\left(1-\frac{\breve{\xi}_1'}{\mathfrak{t}}\right)^{\lambda}\left(1-\frac{\breve{\xi}_2'}{\mathfrak{t}}\right)^{\lambda}\right) \end{array} \right]}, \right\rangle, \\ \dot{S}_{\mathfrak{t}\left[\left(\frac{\Psi_1\Psi_2}{\mathfrak{t}^2}\right)^{\lambda},\left(\frac{\Psi_1'\Psi_2'}{\mathfrak{t}^2}\right)^{\lambda}\right]} \\ \left\langle \ddot{S}_{\left[\mathfrak{t}\left(1-\left(1-\frac{\breve{\mu}_1}{\mathfrak{t}}\right)^{\lambda}\left(1-\frac{\breve{\mu}_2}{\mathfrak{t}}\right)^{\lambda}\right),\mathfrak{t}\left(1-\left(1-\frac{\breve{\mu}_1'}{\mathfrak{t}}\right)^{\lambda}\left(1-\frac{\breve{\mu}_2'}{\mathfrak{t}}\right)^{\lambda}\right)\right]}, \ddot{S}_{\left[\mathfrak{t}\left(\frac{\breve{\nu}_1\breve{\nu}_2}{\mathfrak{t}^2}\right)^{\lambda},\mathfrak{t}\left(\frac{\breve{\nu}_1'\breve{\nu}_2'}{\mathfrak{t}^2}\right)^{\lambda}\right]} \right\rangle; \\ \left[\breve{\xi}_i,\breve{\xi}_i'\right],\left[\Psi_i,\Psi_i'\right]\subseteq[0,\mathfrak{t}];\left[\breve{\mu}_i,\breve{\mu}_i'\right],\left[\breve{\nu}_i,\breve{\nu}_i'\right]\subseteq\left[0,\mathfrak{t}'\right] \text{ where } i=1,2 \end{array} \right)$$

$$= \left(\begin{array}{c} \left\langle \dot{S}_{\mathfrak{t}\left[\left(1-\left(1-\frac{\breve{\xi}_1}{\mathfrak{t}}-\frac{\breve{\xi}_2}{\mathfrak{t}}+\frac{\breve{\xi}_1\breve{\xi}_2}{\mathfrak{t}^2}\right)^{\lambda}\right),\left(1-\left(1-\frac{\breve{\xi}_1'}{\mathfrak{t}}-\frac{\breve{\xi}_2'}{\mathfrak{t}}+\frac{\breve{\xi}_1'\breve{\xi}_2'}{\mathfrak{t}^2}\right)^{\lambda}\right)\right]}, \dot{S}_{\mathfrak{t}\left[\left(\frac{\Psi_1\Psi_2}{\mathfrak{t}^2}\right)^{\lambda},\left(\frac{\Psi_1'\Psi_2'}{\mathfrak{t}^2}\right)^{\lambda}\right]} \right\rangle, \\ \left\langle \ddot{S}_{\mathfrak{t}'\left[\left(1-\left(1-\frac{\breve{\mu}_1+\breve{\mu}_2}{\mathfrak{t}'}+\frac{\breve{\mu}_1\breve{\mu}_2}{\mathfrak{t}'^2}\right)^{\lambda},\left(1-\left(1-\frac{\breve{\mu}_1'}{\mathfrak{t}'}-\frac{\breve{\mu}_2'}{\mathfrak{t}'}+\frac{\breve{\mu}_1'\breve{\mu}_2'}{\mathfrak{t}'^2}\right)^{\lambda}\right)\right)\right]}, \ddot{S}_{\mathfrak{t}\left[\left(\left(\frac{\breve{\nu}_1\breve{\nu}_2}{\mathfrak{t}^2}\right)^{\lambda},\left(\frac{\breve{\nu}_1'\breve{\nu}_2'}{\mathfrak{t}^2}\right)^{\lambda}\right)\right]} \right\rangle \\ \text{Where } \left[\breve{\xi}_i,\breve{\xi}_i'\right],\left[\Psi_i,\Psi_i'\right]\subseteq[0,\mathfrak{t}];\left[\breve{\mu}_i,\breve{\mu}_i'\right],\left[\breve{\nu}_i,\breve{\nu}_i'\right]\subseteq\left[0,\mathfrak{t}'\right] \text{ where } i=1,2 \end{array} \right) \quad (4.17)$$

By 4.16 and 4.17 we have

$$\lambda(\hat{S}_1 \oplus \hat{S}_2) = \lambda\hat{S}_1 \oplus \lambda\hat{S}_2$$

Hence proved. ■

(ii) $\lambda_1\hat{S} \oplus \lambda_2\hat{S} = (\lambda_1+\lambda_2)\hat{S}$

Proof. Consider L.H.S. $\lambda_1\hat{S} \oplus \lambda_2\hat{S}$

$$\lambda_1\hat{S} = \left(\begin{array}{c} \left\langle \dot{S}_{\mathfrak{t}\left[\left(1-\left(1-\frac{\breve{\xi}}{\mathfrak{t}}\right)^{\lambda_1}\right),\left(1-\left(1-\frac{\breve{\xi}'}{\mathfrak{t}}\right)^{\lambda_1}\right)\right]}, \dot{S}_{\mathfrak{t}\left[\left(\frac{\Psi}{\mathfrak{t}}\right)^{\lambda_1},\left(\frac{\Psi'}{\mathfrak{t}}\right)^{\lambda_1}\right]} \right\rangle, \\ \left\langle \ddot{S}_{\mathfrak{t}'\left[\left(1-\left(1-\frac{\breve{\mu}}{\mathfrak{t}'}\right)^{\lambda_1}\right),\left(1-\left(1-\frac{\breve{\mu}'}{\mathfrak{t}'}\right)^{\lambda_1}\right)\right]}, \ddot{S}_{\mathfrak{t}'\left[\left(\frac{\breve{\nu}}{\mathfrak{t}'}\right)^{\lambda},\left(\frac{\breve{\nu}'}{\mathfrak{t}'}\right)^{\lambda_1}\right]} \right\rangle; \\ \left[\breve{\xi}_i,\breve{\xi}_i'\right],\left[\Psi_i,\Psi_i'\right]\subseteq[0,\mathfrak{t}];\left[\breve{\mu}_i,\breve{\mu}_i'\right],\left[\breve{\nu}_i,\breve{\nu}_i'\right]\subseteq\left[0,\mathfrak{t}'\right] \text{ where } i=1,2 \end{array} \right)$$

$$\lambda_2\hat{S} = \left(\left\langle \dot{S}_{\mathfrak{t}\left[\left(1-\left(1-\frac{\breve{\xi}}{\mathfrak{t}}\right)^{\lambda_2}\right),\left(1-\left(1-\frac{\breve{\xi}_1'}{\mathfrak{t}}\right)^{\lambda_2}\right)\right]}, \dot{S}_{\left[\mathfrak{t}\left[\left(\frac{\Psi}{\mathfrak{t}}\right)^{\lambda_2},\left(\frac{\Psi'}{\mathfrak{t}}\right)^{\lambda_2}\right]\right]} \right\rangle, \left\langle \ddot{S}_{\mathfrak{t}'\left[\left(1-\left(1-\frac{\breve{\mu}}{\mathfrak{t}'}\right)^{\lambda_2}\right),\left(1-\left(1-\frac{\breve{\mu}_2'}{\mathfrak{t}'}\right)^{\lambda_2}\right)\right]}, \ddot{S}_{\left[\mathfrak{t}'\left[\left(\frac{\breve{\nu}}{\mathfrak{t}'}\right)^{\lambda_2},\left(\frac{\breve{\nu}'}{\mathfrak{t}'}\right)^{\lambda_2}\right]\right]} \right\rangle\right)$$

Now $\lambda_1\hat{S} \oplus \lambda_2\hat{S}$

$$= \left(\begin{array}{c} \left\langle \dot{S}_{\mathfrak{t}\left[\begin{array}{c} 2-\left(1-\frac{\breve{\xi}}{\mathfrak{t}}\right)^{\lambda_1}-\left(1-\frac{\breve{\xi}'}{\mathfrak{t}}\right)^{\lambda_2}-\left(1-\left(1-\frac{\breve{\xi}'}{\mathfrak{t}}\right)^{\lambda_1}\right)\left(1-\left(1-\frac{\breve{\xi}'}{\mathfrak{t}}\right)^{\lambda_2}\right), \\ 2-\left(1-\frac{\breve{\xi}'}{\mathfrak{t}}\right)^{\lambda_1}-\left(1-\frac{\breve{\xi}'}{\mathfrak{t}}\right)^{\lambda_2}-\left(1-\left(1-\frac{\breve{\xi}'}{\mathfrak{t}}\right)^{\lambda_1}\right)\left(1-\left(1-\frac{\breve{\xi}'}{\mathfrak{t}}\right)^{\lambda_2}\right) \end{array}\right]}, \right\rangle, \\ \dot{S}_{\mathfrak{t}\left[\left(\frac{\Psi}{\mathfrak{t}}\right)^{\lambda_1}\left(\frac{\Psi}{\mathfrak{t}}\right)^{\lambda_2},\left(\frac{\Psi'}{\mathfrak{t}}\right)^{\lambda_1}\left(\frac{\Psi'}{\mathfrak{t}}\right)^{\lambda_2}\right]} \\ \left\langle \begin{array}{c} \ddot{S}_{\mathfrak{t}'\left[\left(1-\left(1-\frac{\breve{\mu}}{\mathfrak{t}'}\right)^{\lambda_1}\left(1-\frac{\breve{\mu}}{\mathfrak{t}'}\right)^{\lambda_2}\right),1-\left(1-\frac{\breve{\mu}'}{\mathfrak{t}'}\right)^{\lambda_1}\left(1-\frac{\breve{\mu}'}{\mathfrak{t}'}\right)^{\lambda_2}\right]}, \\ \ddot{S}_{\mathfrak{t}'\left[\left(\frac{\breve{\nu}}{\mathfrak{t}'}\right)^{\lambda_1}\left(\frac{\breve{\nu}}{\mathfrak{t}'}\right)^{\lambda_2},\left(\frac{\breve{\nu}'}{\mathfrak{t}'}\right)^{\lambda_1}\left(\frac{\breve{\nu}'}{\mathfrak{t}'}\right)^{\lambda_2}\right]} \end{array} \right\rangle \end{array}\right)$$

Where $\left[\breve{\xi},\breve{\xi}^{'}\right],\left[\Psi,\Psi'\right] \subseteq [0,\mathfrak{t}];\left[\breve{\mu},\breve{\mu}'\right],\left[\breve{\nu},\breve{\nu}'\right] \subseteq \left[0,\mathfrak{t}^{'}\right]$

$$\lambda_1\hat{S}\oplus\lambda_2\hat{S} = \left(\begin{array}{c} \left\langle \dot{S}_{\mathfrak{t}\left[1-\left(1-\frac{\breve{\xi}}{\mathfrak{t}}\right)^{\lambda_1+\lambda_2},1-\left(1-\frac{\breve{\xi}'}{\mathfrak{t}}\right)^{\lambda_1+\lambda_2}\right]}, \dot{S}_{\mathfrak{t}\left[\left(\frac{\Psi}{\mathfrak{t}}\right)^{\lambda_1+\lambda_2},\left(\frac{\Psi'}{\mathfrak{t}}\right)^{\lambda_1+\lambda_2}\right]} \right\rangle \\ \left\langle \ddot{S}_{\mathfrak{t}'\left[1-\left(1-\frac{\breve{\mu}}{\mathfrak{t}'}\right)^{\lambda_1+\lambda_2},1-\left(1-\frac{\breve{\mu}'}{\mathfrak{t}'}\right)^{\lambda_1+\lambda_2}\right]}, \ddot{S}_{\mathfrak{t}'\left[\left(\left(\frac{\breve{\nu}}{\mathfrak{t}'}\right)^{\lambda_1}\right)\left(\left(\frac{\breve{\nu}'}{\mathfrak{t}}\right)^{\lambda_2}\right),\left(\left(\frac{\breve{\nu}}{\mathfrak{t}'}\right)^{\lambda_1}\right),\left(\left(\frac{\breve{\nu}'}{\mathfrak{t}'}\right)^{\lambda_2}\right)\right]} \right\rangle; \\ \left[\breve{\xi},\breve{\xi}^{'}\right],\left[\Psi,\Psi'\right] \subseteq [0,\mathfrak{t}];\left[\breve{\mu},\breve{\mu}'\right],\left[\breve{\nu},\breve{\nu}'\right] \subseteq \left[0,\mathfrak{t}^{'}\right] \end{array}\right). \tag{4.18}$$

Consider R.H.S.

$$(\lambda_1 \oplus \lambda_2)\hat{S} = \left(\begin{array}{c} \left\langle \dot{S}_{\mathfrak{t}}\left[\left(1-\left(1-\frac{\breve{\xi}}{\mathfrak{t}}\right)^{\lambda_1+\lambda_2}\right),\left(1-\left(1-\frac{\breve{\xi}'}{\mathfrak{t}}\right)^{\lambda_1+\lambda_2}\right)\right], \dot{S}_{\mathfrak{t}}\left[\left(\left(\frac{\Psi}{\mathfrak{t}}\right)^{\lambda_1+\lambda_2}\right),\left(\left(\frac{\Psi'}{\mathfrak{t}}\right)^{\lambda_1+\lambda_2}\right)\right], \right\rangle \\ \left\langle \ddot{S}_{\mathfrak{t}'}\left[\left(1-\left(1-\frac{\breve{\mu}}{\mathfrak{t}'}\right)^{\lambda_1+\lambda_2}\right),\left(1-\left(1-\frac{\breve{\mu}'}{\mathfrak{t}'}\right)^{\lambda_1+\lambda_2}\right)\right], \ddot{S}_{\mathfrak{t}'}\left[\left(\left(\frac{\breve{\nu}}{\mathfrak{t}'}\right)^{\lambda_1+\lambda_2}\right)\left(\left(\frac{\breve{\nu}'}{\mathfrak{t}}\right)^{\lambda_1+\lambda_2}\right)\right] \right\rangle; \\ \left[\breve{\xi},\breve{\xi}^{'}\right], [\Psi,\Psi'] \subseteq [0,\mathfrak{t}]; [\breve{\mu},\breve{\mu}'], [\breve{\nu},\breve{\nu}'] \subseteq \left[0,\mathfrak{t}^{'}\right] \end{array} \right). \tag{4.19}$$

By considering 4.18 and 4.19 we get the desired result such as:

$$\lambda_1\hat{S} + \lambda_2\hat{S} = (\lambda_1+\lambda_2)\hat{S}$$

Hence proved. ■

(iii) $\lambda_1\left(\lambda_2\hat{S}\right) = (\lambda_1\lambda_2)\hat{S}$

Proof. Consider R.H.S

$$(\lambda_1\lambda_2)\hat{S} = \left(\begin{array}{c} \left\langle \dot{S}_{\mathfrak{t}}\left[\left(1-\left(1-\frac{\breve{\xi}}{\mathfrak{t}}\right)^{\lambda_1\lambda_2}\right),\left(1-\left(1-\frac{\breve{\xi}'}{\mathfrak{t}'}\right)^{\lambda_1\lambda_2}\right)\right], \dot{S}\left[\mathfrak{t}\left(\frac{\Psi}{\mathfrak{t}}\right)^{\lambda_1\lambda_2},\mathfrak{t}\left(\frac{\Psi'}{\mathfrak{t}}\right)^{\lambda_1\lambda_2}\right] \right\rangle, \\ \left\langle \ddot{S}_{\mathfrak{t}'}\left[\left(1-\left(1-\frac{\breve{\mu}}{\mathfrak{t}'}\right)^{\lambda_1\lambda_2}\right),\left(1-\left(1-\frac{\breve{\mu}'}{\mathfrak{t}'}\right)^{\lambda_1\lambda_2}\right)\right], \ddot{S}\,\mathfrak{t}'\left[\left(\frac{\breve{\nu}}{\mathfrak{t}'}\right)^{\lambda_1\lambda_2},\mathfrak{t}'\left(\frac{\breve{\nu}'}{\mathfrak{t}'}\right)^{\lambda_1\lambda_2}\right] \right\rangle; \\ \left[\breve{\xi},\breve{\xi}^{'}\right], [\Psi,\Psi'] \subseteq [0,\mathfrak{t}]; [\breve{\mu},\breve{\mu}'], [\breve{\nu},\breve{\nu}'] \subseteq \left[0,\mathfrak{t}^{'}\right] \end{array} \right) \tag{4.20}$$

Consider L.H.S. $\lambda_1\left(\lambda_2\hat{S}\right)$

$$\lambda_2\hat{S} = \left(\begin{array}{c} \left\langle \dot{S}_{\mathfrak{t}}\left[\left(1-\left(1-\frac{\breve{\xi}}{\mathfrak{t}}\right)^{\lambda_2}\right),\left(1-\left(1-\frac{\breve{\xi}'}{\mathfrak{t}}\right)^{\lambda_2}\right)\right], \dot{S}\left[\mathfrak{t}\left(\frac{\Psi}{\mathfrak{t}}\right)^{\lambda_2},\mathfrak{t}\left(\frac{\Psi'}{\mathfrak{t}}\right)^{\lambda_2}\right] \right\rangle, \\ \left\langle \ddot{S}_{\mathfrak{t}'}\left[\left(1-\left(1-\frac{\breve{\mu}}{\mathfrak{t}'}\right)^{\lambda_2}\right),\left(1-\left(1-\frac{\breve{\mu}'}{\mathfrak{t}'}\right)^{\lambda_2}\right)\right], \ddot{S}_{\mathfrak{t}'}\left[\left(\frac{\breve{\nu}}{\mathfrak{t}'}\right)^{\lambda_2},\mathfrak{t}'\left(\frac{\breve{\nu}'}{\mathfrak{t}'}\right)^{\lambda_2}\right] \right\rangle; \\ \left[\breve{\xi},\breve{\xi}^{'}\right], [\Psi,\Psi'] \subseteq [0,\mathfrak{t}]; [\breve{\mu},\breve{\mu}'], [\breve{\nu},\breve{\nu}'] \subseteq \left[0,\mathfrak{t}^{'}\right] \end{array} \right)$$

Now

$$\lambda_1\left(\lambda_2\hat{S}\right)=\left(\begin{array}{c}\left\langle \dot{S}_{\text{ţ}\left[1-\left(1-\text{ţ}\frac{\left(1-\left(1-\frac{\breve{\xi}}{\text{ţ}}\right)^{\lambda_2}\right)^{\lambda_1}}{\text{ţ}}\right),1-\left(1-\text{ţ}\frac{\left(1-\left(1-\frac{\breve{\xi}'}{\text{ţ}}\right)^{\lambda_2}\right)^{\lambda_1}}{\text{ţ}}\right)\right],} \right\rangle, \\ \dot{S}_{\text{ţ}\left[\left(\frac{\text{ţ}\left(\frac{\Psi}{\text{ţ}}\right)^{\lambda_2}}{\text{ţ}}\right)^{\lambda_1},\left(\frac{\text{ţ}\left(\frac{\Psi'}{\text{ţ}}\right)^{\lambda_2}}{\text{ţ}}\right)^{\lambda_1}\right]} \\ \left\langle \ddot{S}_{\text{ţ}'\left[1-\left(1-\text{ţ}\left(\frac{1-\left(1-\frac{\breve{\mu}}{\text{ţ}}\right)^{\lambda_2}}{\text{ţ}}\right)^{\lambda_1}\right),1-\left(1-\text{ţ}'\left(\frac{1-\left(1-\frac{\breve{\mu}'}{\text{ţ}}\right)^{\lambda_2}}{\text{ţ}}\right)^{\lambda_1}\right)\right],} \right\rangle; \\ \ddot{S}_{\text{ţ}'\left[\left(\frac{\text{ţ}}{\text{ţ}}\left(\frac{\breve{\nu}}{\text{ţ}}\right)^{\lambda_1}\right)^{\lambda_2},\left(\frac{\text{ţ}}{\text{ţ}}\left(\frac{\breve{\nu}'}{\text{ţ}}\right)^{\lambda_2}\right)^{\lambda_1}\right]} \\ \left[\breve{\xi},\breve{\xi}'\right],[\Psi,\Psi']\subseteq[0,\text{ţ}];[\breve{\mu},\breve{\mu}'],[\breve{\nu},\breve{\nu}']\subseteq\left[0,\text{ţ}'\right]\end{array}\right)$$

$$=\left(\begin{array}{c}\left\langle \dot{S}_{\text{ţ}\left[1-\left(1-\frac{\breve{\xi}}{\text{ţ}}\right)^{\lambda_2\lambda_1},\left(1-\left(1-\frac{\breve{\xi}'}{\text{ţ}}\right)^{\lambda_2\lambda_1}\right)\right]},\dot{S}_{\left[\text{ţ}\left(\frac{\Psi}{\text{ţ}}\right)^{\lambda_2\lambda_1},\text{ţ}\left(\frac{\Psi'}{\text{ţ}}\right)^{\lambda_2\lambda_1}\right]}\right\rangle, \\ \left\langle \ddot{S}_{\text{ţ}'\left[1-\left(1-\frac{\breve{\mu}}{\text{ţ}'}\right)^{\lambda_2\lambda_1},\left(1-\left(1-\frac{\breve{\mu}'}{\text{ţ}'}\right)^{\lambda_2\lambda_1}\right)\right]},\ddot{S}_{\left[\text{ţ}'\left(\frac{\breve{\nu}}{\text{ţ}'}\right)^{\lambda_2\lambda_1},\text{ţ}'\left(\frac{\breve{\nu}'}{\text{ţ}'}\right)^{\lambda_2\lambda_1}\right]}\right\rangle; \\ \left[\breve{\xi},\breve{\xi}'\right],[\Psi,\Psi']\subseteq[0,\text{ţ}];[\breve{\mu},\breve{\mu}'],[\breve{\nu},\breve{\nu}']\subseteq\left[0,\text{ţ}'\right]\end{array}\right) \tag{4.21}$$

$$\lambda_2(\lambda_1\hat{S})=(\lambda_1\lambda_2)\hat{S}\ (\lambda_1,\ \lambda_2 \text{ are Positive Real Numbers})$$

Hence proved the desired result. ■

(iv) $\left(\overset{\wedge}{S_1}\otimes\overset{\wedge}{S_2}\right)^{\lambda}=\overset{\wedge}{S_1}^{\lambda}\otimes\overset{\wedge}{S_2}^{\lambda}$

Proof. Consider L.H.S. $\left(\overset{\wedge}{S_1}\otimes\overset{\wedge}{S_2}\right)^{\lambda}$

$$\left(\overset{\wedge}{S_1}\otimes\overset{\wedge}{S_2}\right)=\left(\begin{array}{c}\left\langle \dot{S}_{\left[\frac{\breve{\xi}_1\breve{\xi}_2}{\text{ţ}},\frac{\breve{\xi}_1'\breve{\xi}_2'}{\text{ţ}}\right]},\dot{S}_{\left[\Psi_1+\Psi_2-\frac{\Psi_1\Psi_2}{\text{ţ}},\Psi_1'\Psi_2'-\frac{\Psi_1'\Psi_2'}{\text{ţ}}\right]}\right\rangle, \\ \left\langle \ddot{S}_{\text{ţ}'\left[\frac{\breve{\mu}_1\breve{\mu}_2}{\text{ţ}'},\frac{\breve{\mu}_1'\breve{\mu}_2'}{\text{ţ}'}\right]},\ddot{S}_{\left[\breve{\nu}_1\breve{\nu}_2-\frac{\breve{\nu}_1\breve{\nu}_2}{\text{ţ}'},\breve{\nu}_1'\breve{\nu}_2'-\frac{\breve{\nu}_1'\breve{\nu}_2'}{\text{ţ}'}\right]}\right\rangle; \\ \left[\breve{\xi}_i,\breve{\xi}_i'\right],[\Psi_i,\Psi_i']\subseteq[0,\text{ţ}];[\breve{\mu}_i,\breve{\mu}_i'],[\breve{\nu}_i,\breve{\nu}_i']\subseteq\left[0,\text{ţ}'\right] \text{ where } i=1,2\end{array}\right)$$

Now

$$\left(\overset{\wedge}{S_1}\otimes\overset{\wedge}{S_2}\right)^{\lambda}=\left(\begin{array}{c}\left\langle \dot{S}_{\mathfrak{t}}\left[\left(\frac{\breve{\xi}_1\breve{\xi}_2}{\mathfrak{t}'}\right)^{\lambda},\left(\frac{\breve{\xi}_1'\breve{\xi}_2'}{\mathfrak{t}'}\right)^{\lambda}\right],\dot{S}_{\mathfrak{t}}\left[\left(1-\left(1-\left(\frac{\Psi_1+\Psi_2}{\mathfrak{t}}-\frac{\Psi_1\Psi_2}{\mathfrak{t}}\right)\right)^{\lambda}\right),\left(1-\left(1-\left(\frac{\Psi_1'+\Psi_2'-}{\mathfrak{t}}-\frac{\Psi_1'\Psi_2'}{\mathfrak{t}'}\right)\right)^{\lambda}\right)\right]\right\rangle, \\ \left\langle \ddot{S}_{\mathfrak{t}'}\left[\left(\frac{\breve{\mu}_1\breve{\mu}_2}{\mathfrak{t}'}\right)^{\lambda},\left(\frac{\breve{\mu}_1'\breve{\mu}_2'}{\mathfrak{t}'}\right)^{\lambda}\right],\ddot{S}_{\mathfrak{t}'}\left[\left(1-\left(1-\left(\frac{\breve{\nu}_1+\breve{\nu}_2}{\mathfrak{t}'}-\frac{\breve{\nu}_1\breve{\nu}_2}{\mathfrak{t}'}\right)\right)^{\lambda}\right),\left(1-\left(1-\left(\frac{\breve{\nu}_1'+\breve{\nu}_2'-}{\mathfrak{t}'}-\frac{\breve{\nu}_1'\breve{\nu}_2'}{\mathfrak{t}'}\right)\right)^{\lambda}\right)\right]\right\rangle; \\ \left[\breve{\xi}_i,\breve{\xi}_i^{'}\right],[\Psi_i,\Psi_i']\subseteq[0,\mathfrak{t}];[\breve{\mu}_i,\breve{\mu}_i'],[\breve{\nu}_i,\breve{\nu}_i']\subseteq\left[0,\mathfrak{t}^{'}\right]\text{ where } i=1,2\end{array}\right) \tag{4.22}$$

Consider R.H.S.

$$\overset{\wedge}{S_1}^{\lambda}=\left(\begin{array}{c}\left\langle \dot{S}_{\mathfrak{t}}\left[\left(\frac{\breve{\xi}_1}{\mathfrak{t}}\right)^{\lambda},\left(\frac{\breve{\xi}_1'}{\mathfrak{t}}\right)^{\lambda}\right],\dot{S}_{\mathfrak{t}}\left[\left(1-\left(1-\frac{\Psi_1}{\mathfrak{t}}\right)^{\lambda}\right),\left(1-\left(1-\frac{\Psi_1'}{\mathfrak{t}}\right)^{\lambda}\right)\right]\right\rangle, \\ \left\langle \ddot{S}_{\mathfrak{t}'}\left[\left(\frac{\breve{\mu}_1}{\mathfrak{t}'}\right)^{\lambda},\left(\frac{\breve{\mu}_1'}{\mathfrak{t}'}\right)^{\lambda}\right],\ddot{S}_{\mathfrak{t}'}\left[\left(1-\left(1-\frac{\breve{\nu}_1}{\mathfrak{t}}\right)^{\lambda}\right),\left(1-\left(1-\frac{\breve{\nu}_1'}{\mathfrak{t}}\right)^{\lambda}\right)\right]\right\rangle; \\ \left[\breve{\xi}_1,\breve{\xi}_1^{'}\right],[\Psi_1,\Psi_1']\subseteq[0,\mathfrak{t}];[\breve{\mu}_1,\breve{\mu}_1'],[\breve{\nu}_1,\breve{\nu}_1']\subseteq\left[0,\mathfrak{t}^{'}\right]\end{array}\right)$$

$$\overset{\wedge}{S_2}^{\lambda}=\left(\begin{array}{c}\left\langle \dot{S}_{\mathfrak{t}}\left[\left(\frac{\breve{\xi}_2}{\mathfrak{t}}\right)^{\lambda},\left(\frac{\breve{\xi}_2'}{\mathfrak{t}}\right)^{\lambda}\right],\dot{S}_{\mathfrak{t}}\left[\left(1-\left(1-\frac{\Psi_2}{\mathfrak{t}}\right)^{\lambda}\right),\left(1-\left(1-\frac{\Psi_2'}{\mathfrak{t}}\right)^{\lambda}\right)\right]\right\rangle, \\ \left\langle \ddot{S}_{\mathfrak{t}'}\left[\left(\frac{\breve{\mu}_2}{\mathfrak{t}'}\right)^{\lambda},\left(\frac{\breve{\mu}_2'}{\mathfrak{t}'}\right)^{\lambda}\right],\ddot{S}_{\mathfrak{t}'}\left[\left(1-\left(1-\frac{\breve{\nu}_2}{\mathfrak{t}'}\right)^{\lambda}\right),\left(1-\left(1-\frac{\breve{\nu}_2'}{\mathfrak{t}'}\right)^{\lambda}\right)\right]\right\rangle; \\ \left[\breve{\xi}_2,\breve{\xi}_2^{'}\right],[\Psi_2,\Psi_2']\subseteq[0,\mathfrak{t}];[\breve{\mu}_2,\breve{\mu}_2'],[\breve{\nu}_2,\breve{\nu}_2']\subseteq\left[0,\mathfrak{t}^{'}\right]\end{array}\right)$$

Now $\hat{S}_1^{\lambda} \otimes \hat{S}_2^{\lambda}$ by using the Definition (20) of product

$$= \left(\begin{array}{c} \dot{S}_{\mathfrak{t}\left[\frac{\mathfrak{t}^2}{\mathfrak{t}}\left(\frac{\breve{\xi}_1}{\mathfrak{t}}\right)^{\lambda}\left(\frac{\breve{\xi}_2}{\mathfrak{t}}\right)^{\lambda},\frac{\mathfrak{t}^2}{\mathfrak{t}}\left(\frac{\breve{\xi}_1'}{\mathfrak{t}}\right)^{\lambda}\left(\frac{\breve{\xi}_2'}{\mathfrak{t}}\right)^{\lambda}\right]}, \\ \left\langle \dot{S}\left[\begin{array}{c} 2\mathfrak{t}-\mathfrak{t}\left(1-\frac{\Psi_1}{\mathfrak{t}}\right)^{\lambda}-\mathfrak{t}\left(1-\left(\frac{\Psi_2}{\mathfrak{t}}\right)\right)^{\lambda}-\frac{\mathfrak{t}^2\left(1-\left(1-\frac{\Psi_1}{\mathfrak{t}}\right)^{\lambda}\right)\left(1-\left(1-\frac{\Psi_2}{\mathfrak{t}}\right)^{\lambda}\right)}{\mathfrak{t}}, \\ 2\mathfrak{t}-\mathfrak{t}\left(1-\frac{\Psi_1'}{\mathfrak{t}'}\right)^{\lambda}-\mathfrak{t}\left(1-\frac{\Psi_2'}{\mathfrak{t}'}\right)^{\lambda}-\frac{\mathfrak{t}^2\left(1-\left(1-\frac{\Psi_1'}{\mathfrak{t}}\right)^{\lambda}\right)\left(1-\left(1-\frac{\Psi_2'}{\mathfrak{t}}\right)^{\lambda}\right)}{\mathfrak{t}} \end{array}\right] \right\rangle, \\ \ddot{S}_{\left[\frac{\mathfrak{t}'^2}{\mathfrak{t}'}\left(\frac{\breve{\mu}_1}{\mathfrak{t}'}\right)^{\lambda}\left(\frac{\breve{\mu}_2}{\mathfrak{t}'}\right)^{\lambda},\frac{\mathfrak{t}^2}{\mathfrak{t}}\left(\frac{\breve{\mu}_1'\breve{\mu}_2'}{\mathfrak{t}}\right)\right]}, \\ \left\langle \ddot{S}\left[\begin{array}{c} 2\mathfrak{t}'-\mathfrak{t}'\left(1-\frac{\breve{\nu}_1}{\mathfrak{t}'}\right)^{\lambda}-\mathfrak{t}'\left(1-\frac{\breve{\nu}_2}{\mathfrak{t}}\right)^{\lambda}-\frac{\mathfrak{t}'^2}{\mathfrak{t}'}\left(1-\left(1-\frac{\breve{\nu}_1}{\mathfrak{t}'}\right)^{\lambda}\right)\left(1-\left(1-\frac{\breve{\nu}_2}{\mathfrak{t}}\right)^{\lambda}\right), \\ \mathfrak{t}'\left(2-\left(1-\frac{\breve{\nu}_1'}{\mathfrak{t}'}\right)^{\lambda}-\left(1-\frac{\breve{\nu}_2'}{\mathfrak{t}'}\right)^{\lambda}+\left(1-\frac{\breve{\nu}_1'}{\mathfrak{t}'}\right)^{\lambda}+\left(1-\frac{\breve{\nu}_2'}{\mathfrak{t}'}\right)^{\lambda}-\left(1-\frac{\breve{\nu}_1}{\mathfrak{t}}\right)^{\lambda}\left(1-\frac{\breve{\nu}_2}{\mathfrak{t}}\right)^{\lambda}\right) \end{array}\right] \right\rangle \\ \left[\breve{\xi}_i,\breve{\xi}_i'\right],[\Psi_i,\Psi_i']\subseteq[0,\mathfrak{t}];[\breve{\mu}_i,\breve{\mu}_i'],[\breve{\nu}_i,\breve{\nu}_i']\subseteq\left[0,\mathfrak{t}'\right] \text{ where } i=1,2 \end{array} \right)$$

$$= \left(\begin{array}{c} \left\langle \begin{array}{c} \dot{S}_{\mathfrak{t}'}\left[\left(\frac{\breve{\xi}_1\breve{\xi}_2}{\mathfrak{t}^2}\right)^{\lambda},\left(\frac{\breve{\xi}_1'\breve{\xi}_2'}{\mathfrak{t}^2}\right)^{\lambda}\right], \\ \dot{S}_{\mathfrak{t}}\left[1-\left(1-\frac{\Psi_1}{\mathfrak{t}}\right)^{\lambda}-\left(1-\frac{\Psi_2}{\mathfrak{t}}\right)^{\lambda}-1+\left(1-\frac{\Psi_1}{\mathfrak{t}}\right)^{\lambda}+\left(1-\frac{\Psi_2}{\mathfrak{t}}\right)^{\lambda}-\left(1-\frac{\Psi_1}{\mathfrak{t}}\right)^{\lambda}\left(1-\frac{\Psi_2}{\mathfrak{t}}\right)^{\lambda},\left(2-1-\left(1-\frac{\Psi_1'}{\mathfrak{t}}\right)^{\lambda}\left(1-\frac{\Psi_2'}{\mathfrak{t}}\right)^{\lambda}\right)\right] \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \ddot{S}_{\mathfrak{t}'}\left[\left(\frac{\breve{\mu}_1\breve{\mu}_2}{\mathfrak{t}^2}\right)^{\lambda},\left(\frac{\breve{\mu}_1'\breve{\mu}_2'}{\mathfrak{t}^2}\right)^{\lambda}\right], \\ \ddot{S}_{\mathfrak{t}'}\left[1-\left(\left(1-\frac{\breve{\nu}_1}{\mathfrak{t}'}\right)\left(1-\frac{\breve{\nu}_2}{\mathfrak{t}'}\right)\right)^{\lambda},\left(1-\left(1-\frac{\breve{\nu}_1'}{\mathfrak{t}'}\right)\left(1-\frac{\breve{\nu}_2'}{\mathfrak{t}'}\right)^{\lambda}\right)\right] \end{array} \right\rangle; \\ \left[\breve{\xi}_i,\breve{\xi}_i'\right],[\Psi_i,\Psi_i']\subseteq[0,\mathfrak{t}];[\breve{\mu}_i,\breve{\mu}_i'],[\breve{\nu}_i,\breve{\nu}_i']\subseteq\left[0,\mathfrak{t}'\right] \text{ where } i=1,2 \end{array} \right)$$

$$= \left(\begin{array}{c} \left\langle \dot{S}_{\mathfrak{t}}\left[\left(\frac{\breve{\xi}_1\breve{\xi}_2}{\mathfrak{t}^2}\right)^{\lambda},\left(\frac{\breve{\xi}_1'\breve{\xi}_2'}{\mathfrak{t}^2}\right)^{\lambda}\right],\dot{S}_{\mathfrak{t}}\left[1-\left(\left(1-\left(\frac{\Psi_1}{\mathfrak{t}}+\frac{\Psi_2}{\mathfrak{t}}\right)-\frac{\Psi_1\Psi_2}{\mathfrak{t}^2}\right)^{\lambda}-\left(1-\frac{\Psi_2}{\mathfrak{t}}\right)^{\lambda}\right),\left(1-\left(\frac{\Psi_1'+\Psi_2'}{\mathfrak{t}}-\frac{\Psi_1'\Psi_2'}{\mathfrak{t}^2}\right)\right)^{\lambda}-\right] \right\rangle, \\ \left\langle \ddot{S}_{\mathfrak{t}'}\left[\left(\frac{\breve{\mu}_1\breve{\mu}_2}{\mathfrak{t}'^2}\right)^{\lambda},\left(\frac{\breve{\mu}_1'\breve{\mu}_2'}{\mathfrak{t}'^2}\right)^{\lambda}\right],\ddot{S}_{\mathfrak{t}'}\left[\left(1-\left(1-\left(\frac{\breve{\nu}_1+\breve{\nu}_2}{\mathfrak{t}'}-\frac{\breve{\nu}_1\breve{\nu}_2}{\mathfrak{t}'^2}\right)^{\lambda}\right)\right),\left(1-\left(\frac{\breve{\nu}_1'+\breve{\nu}_2'}{\mathfrak{t}'}-\frac{\breve{\nu}_1'\breve{\nu}_2'}{\mathfrak{t}'^2}\right)^{\lambda}\right)\right] \right\rangle; \\ \left[\breve{\xi}_i,\breve{\xi}_i'\right],[\Psi_i,\Psi_i']\subseteq[0,\mathfrak{t}];[\breve{\mu}_i,\breve{\mu}_i'],[\breve{\nu}_i,\breve{\nu}_i']\subseteq\left[0,\mathfrak{t}'\right] \text{ where } i=1,2 \end{array} \right) \tag{4.23}$$

By 4.22 and 4.23 we have

$$\left(\overset{\wedge}{S_1} \otimes \overset{\wedge}{S_2}\right)^{\lambda} = \overset{\wedge}{S_1}^{\lambda} \otimes \overset{\wedge}{S_2}^{\lambda}$$

Hence proved ■

(v) $\hat{S}^{\lambda_1} \otimes \hat{S}^{\lambda_2} = \hat{S}^{(\lambda_1+\lambda_2)}$

Proof. Consider R.H.S

$$\hat{S}^{(\lambda_1+\lambda_2)} = \left(\begin{array}{c} \left\langle \begin{array}{c} \dot{S}_{\mathfrak{t}}\left[\left(\frac{\breve{\xi}}{\mathfrak{t}}\right)^{\lambda_1+\lambda_2}, \left(\frac{\breve{\xi}'}{\mathfrak{t}}\right)^{\lambda_1+\lambda_2}\right], \\ \dot{S}_{\mathfrak{t}}\left[\left(1-\left(1-\frac{\Psi}{\mathfrak{t}}\right)^{\lambda_1+\lambda_2}\right), \left(1-\left(1-\frac{\Psi'}{\mathfrak{t}}\right)^{\lambda_1+\lambda_2}\right)\right] \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \ddot{S}_{\mathfrak{t}'}\left[\left(\frac{\breve{\mu}}{\mathfrak{t}'}\right)^{\lambda_1+\lambda_2}, \left(\frac{\breve{\mu}'}{\mathfrak{t}'}\right)^{\lambda_1+\lambda_2}\right], \\ \ddot{S}_{\mathfrak{t}'}\left[\left(1-\left(1-\frac{\breve{\nu}}{\mathfrak{t}'}\right)^{\lambda_1+\lambda_2}\right), \left(1-\left(1-\frac{\breve{\nu}'}{\mathfrak{t}'}\right)^{\lambda_1+\lambda_2}\right)\right] \end{array} \right\rangle; \\ \left[\breve{\xi}, \breve{\xi}'\right], [\Psi, \Psi'] \subseteq [0, \mathfrak{t}]; [\breve{\mu}, \breve{\mu}'], [\breve{\nu}, \breve{\nu}'] \subseteq \left[0, \mathfrak{t}'\right] \end{array} \right) \tag{4.24}$$

Consider L.H.S

$$\hat{S}^{\lambda_1} = \left(\begin{array}{c} \left\langle \dot{S}_{\mathfrak{t}}\left[\left(\frac{\breve{\xi}}{\mathfrak{t}}\right)^{\lambda_1}, \left(\frac{\breve{\xi}'}{\mathfrak{t}}\right)^{\lambda_1}\right], \dot{S}_{\mathfrak{t}}\left[\left(1-\left(1-\frac{\Psi}{\mathfrak{t}}\right)^{\lambda_1}\right), \left(1-\left(1-\frac{\Psi'}{\mathfrak{t}}\right)^{\lambda_1}\right)\right] \right\rangle, \\ \left\langle \ddot{S}_{\mathfrak{t}'}\left[\left(\frac{\breve{\mu}}{\mathfrak{t}'}\right)^{\lambda_1}, \left(\frac{\breve{\mu}'}{\mathfrak{t}'}\right)^{\lambda_1}\right], \ddot{S}_{\mathfrak{t}'}\left[\left(1-\left(1-\frac{\breve{\nu}}{\mathfrak{t}'}\right)^{\lambda_1}\right), \left(1-\left(1-\frac{\breve{\nu}'}{\mathfrak{t}'}\right)^{\lambda_1}\right)\right] \right\rangle \end{array} \right)$$

$$\hat{S}^{\lambda_2} = \left(\begin{array}{c} \left\langle \dot{S}_{\mathfrak{t}}\left[\left(\frac{\breve{\xi}}{\mathfrak{t}}\right)^{\lambda_2}, \left(\frac{\breve{\xi}'}{\mathfrak{t}}\right)^{\lambda_2}\right], \dot{S}_{\mathfrak{t}}\left[\left(1-\left(1-\frac{\Psi}{\mathfrak{t}}\right)^{\lambda_2}\right), \left(1-\left(1-\frac{\Psi'}{\mathfrak{t}}\right)^{\lambda_2}\right)\right] \right\rangle, \\ \left\langle \ddot{S}_{\mathfrak{t}'}\left[\left(\frac{\breve{\mu}}{\mathfrak{t}'}\right)^{\lambda_2}, \left(\frac{\breve{\mu}'}{\mathfrak{t}'}\right)^{\lambda_2}\right], \ddot{S}_{\mathfrak{t}'}\left[\left(1-\left(1-\frac{\breve{\nu}}{\mathfrak{t}'}\right)^{\lambda_2}\right), \left(1-\left(1-\frac{\breve{\nu}'}{\mathfrak{t}'}\right)^{\lambda_2}\right)\right] \right\rangle; \\ \left[\breve{\xi}, \breve{\xi}'\right], [\Psi, \Psi'] \subseteq [0, \mathfrak{t}]; [\breve{\mu}, \breve{\mu}'], [\breve{\nu}, \breve{\nu}'] \subseteq \left[0, \mathfrak{t}'\right] \end{array} \right)$$

Now $\hat{S}^{\lambda_1} \otimes \hat{S}^{\lambda_2}$

$$= \left(\begin{array}{c} \dot{S}_{\frac{\text{ţ}^2}{\text{ţ}}\left[\left(\frac{\breve{\xi}}{\text{ţ}}\right)^{\lambda_1}\left(\frac{\breve{\xi}'}{\text{ţ}}\right)^{\lambda_2}, \frac{\text{ţ}^2}{\text{ţ}}\left(\frac{\breve{\xi}'}{\text{ţ}}\right)^{\lambda_1+\lambda_2}\right]}, \\ \left\langle \dot{S}_{\text{ţ}} \left[\begin{array}{c} \left(1-\left(1-\frac{\Psi}{\text{ţ}}\right)^{\lambda_1}+\left(1-\left(1-\frac{\Psi}{\text{ţ}}\right)^{\lambda_2}\right)\right)-\frac{\text{ţ}\left(1-\left(1-\frac{\Psi}{\text{ţ}}\right)^{\lambda_1}\right)}{\text{ţ}}, \\ 1-\left(1-\frac{\Psi'}{\text{ţ}}\right)^{\lambda_1}+\left(\begin{array}{c} 1-\left(1-\frac{\Psi'}{\text{ţ}}\right)^{\lambda_2} \\ +1-1\frac{\text{ţ}}{\text{ţ}}\left(1-\left(1-\frac{\Psi}{\text{ţ}}\right)^{\lambda_1}\right)\left(1-\left(1-\frac{\Psi}{\text{ţ}}\right)^{\lambda_2}\right) \end{array} \right) \end{array} \right] \right\rangle, \\ \ddot{S}_{\left[\frac{\text{ţ}'^2}{\text{ţ}}\left(\frac{\breve{\mu}}{\text{ţ}'}\right)^{\lambda_1\lambda_2}, \frac{\text{ţ}'}{\text{ţ}'}\left(\frac{\breve{\mu}'}{\text{ţ}'}\right)^{\lambda_1\lambda_2}\right]}, \\ \left\langle \ddot{S}_{\text{ţ}'} \left[\begin{array}{c} \left(1-\left(1-\frac{\breve{\nu}}{\text{ţ}'}\right)^{\lambda_1}+1-\left(1-\frac{\breve{\nu}}{\text{ţ}'}\right)^{\lambda_2}-\frac{\text{ţ}'}{\text{ţ}'}\left(1-\left(1-\frac{\breve{\nu}}{\text{ţ}'}\right)^{\lambda_1}\left(1-\frac{\breve{\nu}}{\text{ţ}'}\right)^{\lambda_2}\right)\right), \\ \left(2-\left(1-\frac{\breve{\nu}'}{\text{ţ}'}\right)^{\lambda_1}-\left(1-\frac{\breve{\nu}}{\text{ţ}'}\right)^{\lambda_2}-\left(1-\left(1-\frac{\breve{\nu}}{\text{ţ}'}\right)^{\lambda_1}\left(1-\left(1-\frac{\breve{\nu}}{\text{ţ}'}\right)^{\lambda_2}\right)\right)\right) \end{array} \right] \right\rangle; \\ \left[\breve{\xi}, \breve{\xi}'\right], \left[\Psi, \Psi'\right] \subseteq [0, \text{ţ}]; \left[\breve{\mu}, \breve{\mu}'\right], \left[\breve{\nu}, \breve{\nu}'\right] \subseteq \left[0, \text{ţ}'\right] \end{array} \right)$$

$$= \left(\begin{array}{c} \left\langle \begin{array}{c} \dot{S}_{\text{ţ}\left[\left(\frac{\breve{\xi}}{\text{ţ}}\right)^{\lambda_1+\lambda_2}\left(\frac{\breve{\xi}'}{\text{ţ}}\right)^{\lambda_1+\lambda_2}\right]}, \\ \dot{S}_{\text{ţ}\left[1-\left(1-\frac{\Psi}{\text{ţ}}\right)^{\lambda_1}\left(1-\frac{\Psi}{\text{ţ}}\right)^{\lambda_2}, \left(1-\left(1-\frac{\Psi'}{\text{ţ}}\right)^{\lambda_1}\left(1-\frac{\Psi'}{\text{ţ}}\right)^{\lambda_2}\right)\right]} \end{array} \right\rangle, \\ \left\langle \left\langle \begin{array}{c} \ddot{S}_{\text{ţ}'\left[\left(\frac{\breve{\mu}}{\text{ţ}'}\right)^{\lambda_1+\lambda_2}\left(\frac{\breve{\mu}'}{\text{ţ}'}\right)^{\lambda_1+\lambda_2}\right]}, \\ \ddot{S}_{\text{ţ}'\left[1-\left(1-\frac{\breve{\nu}}{\text{ţ}'}\right)^{\lambda_1}\left(1-\frac{\breve{\nu}}{\text{ţ}'}\right)^{\lambda_2}, \left(1-\left(1-\frac{\breve{\nu}'}{\text{ţ}'}\right)^{\lambda_1}\left(1-\frac{\breve{\nu}'}{\text{ţ}'}\right)^{\lambda_2}\right)\right]} \end{array} \right\rangle \right\rangle; \\ \left[\breve{\xi}, \breve{\xi}'\right], \left[\Psi, \Psi'\right] \subseteq [0, \text{ţ}]; \left[\breve{\mu}, \breve{\mu}'\right], \left[\breve{\nu}, \breve{\nu}'\right] \subseteq \left[0, \text{ţ}'\right] \end{array} \right)$$

$$= \left(\begin{array}{c} \left\langle \dot{S}_{\text{ţ}\left[\left(\frac{\breve{\xi}}{\text{ţ}}\right)^{\lambda_1+\lambda_2}\left(\frac{\breve{\xi}'}{\text{ţ}}\right)^{\lambda_1+\lambda_2}\right]}, \dot{S}_{\text{ţ}\left[1-\left(1-\frac{\Psi}{\text{ţ}}\right)^{\lambda_1+\lambda_2}, \left(1-\left(1-\frac{\Psi'}{\text{ţ}}\right)^{\lambda_1+\lambda_2}\right)\right]} \right\rangle, \\ \left\langle \ddot{S}_{\text{ţ}'\left[\left(\frac{\breve{\mu}}{\text{ţ}'}\right)^{\lambda_1+\lambda_2}\left(\frac{\breve{\mu}'}{\text{ţ}'}\right)^{\lambda_1+\lambda_2}\right]}, \ddot{S}_{\text{ţ}'\left[1-\left(1-\frac{\breve{\nu}}{\text{ţ}'}\right)^{\lambda_1+\lambda_2}, \left(1-\left(1-\frac{\breve{\nu}'}{\text{ţ}'}\right)^{\lambda_1+\lambda_2}\right)\right]} \right\rangle; \\ \left[\breve{\xi}, \breve{\xi}'\right], \left[\Psi, \Psi'\right] \subseteq [0, \text{ţ}]; \left[\breve{\mu}, \breve{\mu}'\right], \left[\breve{\nu}, \breve{\nu}'\right] \subseteq \left[0, \text{ţ}'\right] \end{array} \right) \tag{4.25}$$

By equation no. 4.24 and equation no.4.25 we have Proved the desired result, that is:

$$\hat{S}^{\lambda_1} \otimes \hat{S}^{\lambda_2} = \hat{S}^{(\lambda_1+\lambda_2)}$$

■

(vi) $$\left(\hat{S}^{\lambda_1}\right)^{\lambda_2} = \hat{S}^{(\lambda_1\lambda_2)}$$

Proof. Consider R.H.S. $\hat{S}^{(\lambda_1\lambda_2)}$

$$\hat{S}^{(\lambda_1\lambda_2)} = \left(\begin{array}{c} \left\langle \dot{S}_{\mathfrak{t}}\left[\left(\frac{\breve{\xi}}{\mathfrak{t}}\right)^{\lambda_1\lambda_2}, \left(\frac{\breve{\xi}'}{\mathfrak{t}}\right)^{\lambda_1\lambda_2}\right], \dot{S}_{\mathfrak{t}}\left[\left(1-\left(1-\frac{\Psi}{\mathfrak{t}}\right)^{\lambda_1\lambda_2}\right), \left(1-\left(1-\frac{\Psi'}{\mathfrak{t}}\right)^{\lambda_1\lambda_2}\right)\right] \right\rangle, \\ \left\langle \ddot{S}_{\mathfrak{t}'}\left[\left(\frac{\breve{\mu}}{\mathfrak{t}'}\right)^{\lambda_1\lambda_2}, \left(\frac{\breve{\mu}'}{\mathfrak{t}'}\right)^{\lambda_1\lambda_2}\right], \ddot{S}_{\mathfrak{t}'}\left[\left(1-\left(1-\frac{\breve{\nu}}{\mathfrak{t}'}\right)^{\lambda_1\lambda_2}\right), \left(1-\left(1-\frac{\breve{\nu}'}{\mathfrak{t}'}\right)^{\lambda_1\lambda_2}\right)\right] \right\rangle; \\ \left[\breve{\xi}, \breve{\xi}'\right], [\Psi, \Psi'] \subseteq [0, \mathfrak{t}]; [\breve{\mu}, \breve{\mu}'], [\breve{\nu}, \breve{\nu}'] \subseteq \left[0, \mathfrak{t}'\right] \end{array} \right) \tag{4.26}$$

Now consider L.H.S. $\left(\hat{S}^{\lambda_1}\right)^{\lambda_2}$

$$\hat{S}^{\lambda_1} = \left(\begin{array}{c} \left\langle \dot{S}_{\mathfrak{t}}\left[\left(\frac{\breve{\xi}}{\mathfrak{t}}\right)^{\lambda_1}, \left(\frac{\breve{\xi}'}{\mathfrak{t}}\right)^{\lambda_1}\right], \dot{S}_{\mathfrak{t}}\left[\left(1-\left(1-\frac{\Psi}{\mathfrak{t}}\right)^{\lambda_1}\right), \left(1-\left(1-\frac{\Psi'}{\mathfrak{t}}\right)^{\lambda_1}\right)\right] \right\rangle, \\ \left\langle \ddot{S}_{\mathfrak{t}'}\left[\left(\frac{\breve{\mu}}{\mathfrak{t}'}\right)^{\lambda_1}, \left(\frac{\breve{\mu}'}{\mathfrak{t}'}\right)^{\lambda_1}\right], \ddot{S}_{\mathfrak{t}'}\left[\left(1-\left(1-\frac{\breve{\nu}}{\mathfrak{t}'}\right)^{\lambda_1}\right), \left(1-\left(1-\frac{\breve{\nu}'}{\mathfrak{t}'}\right)^{\lambda_1}\right)\right] \right\rangle; \\ \left[\breve{\xi}, \breve{\xi}'\right], [\Psi, \Psi'] \subseteq [0, \mathfrak{t}]; [\breve{\mu}, \breve{\mu}'], [\breve{\nu}, \breve{\nu}'] \subseteq \left[0, \mathfrak{t}'\right] \end{array} \right)$$

Now

$$\left(\hat{S}^{\lambda_1}\right)^{\lambda_2} = \begin{pmatrix} \left\langle \begin{array}{c} \dot{S}_{\mathfrak{t}}\left[\left(\frac{\mathfrak{t}\left(\frac{\breve{\xi}}{\mathfrak{t}}\right)^{\lambda_1}}{\mathfrak{t}}\right)^{\lambda_2}, \left(\frac{\mathfrak{t}\left(\frac{\breve{\xi}'}{\mathfrak{t}}\right)^{\lambda_1}}{\mathfrak{t}}\right)^{\lambda_2}\right], \\ \dot{S}_{\mathfrak{t}}\left[\left(1-\left(1-\mathfrak{t}\frac{\left(1-\left(\frac{1-\Psi}{\mathfrak{t}}\right)^{\lambda_1}\right)}{\mathfrak{t}}\right)^{\lambda_2}\right), \left(1-\left(1-\mathfrak{t}\frac{\left(1-\left(\frac{1-\Psi'}{\mathfrak{t}}\right)^{\lambda_1}\right)}{\mathfrak{t}}\right)^{\lambda_2}\right)\right] \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \ddot{S}_{\mathfrak{t}'}\left[\left(\frac{\mathfrak{t}'\left(\frac{\breve{\mu}}{\mathfrak{t}'}\right)^{\lambda_1}}{\mathfrak{t}'}\right)^{\lambda_2}, \left(\frac{\mathfrak{t}'\left(\frac{\breve{\mu}'}{\mathfrak{t}'}\right)^{\lambda_1}}{\mathfrak{t}'}\right)^{\lambda_2}\right], \\ \ddot{S}_{\mathfrak{t}'}\left[\left(1-\left(1-\mathfrak{t}'\frac{\left(1-\left(\frac{1-\breve{\nu}'}{\mathfrak{t}'}\right)^{\lambda_1}\right)}{\mathfrak{t}'}\right)^{\lambda_2}\right), \left(1-\left(1-\mathfrak{t}'\frac{\left(1-\left(\frac{1-\breve{\nu}'}{\mathfrak{t}}\right)^{\lambda_1}\right)}{\mathfrak{t}'}\right)^{\lambda_2}\right)\right] \end{array} \right\rangle; \\ \left[\breve{\xi}, \breve{\xi}'\right], [\Psi, \Psi'] \subseteq [0, \mathfrak{t}]; [\breve{\mu}, \breve{\mu}'], [\breve{\nu}, \breve{\nu}'] \subseteq \left[0, \mathfrak{t}'\right] \end{pmatrix}$$

$$= \begin{pmatrix} \left\langle \dot{S}_{\mathfrak{t}}\left[\left(\frac{\breve{\xi}}{\mathfrak{t}}\right)^{\lambda_1\lambda_2} \left(\frac{\breve{\xi}'}{\mathfrak{t}}\right)^{\lambda_1\lambda_2}\right], \dot{S}_{\mathfrak{t}}\left[1-\left(1-\frac{\Psi}{\mathfrak{t}}\right)^{\lambda_1\lambda_2}, \left(1-\left(1-\frac{\Psi'}{\mathfrak{t}}\right)^{\lambda_1\lambda_2}\right)\right] \right\rangle, \\ \left\langle \ddot{S}_{\mathfrak{t}'}\left[\left(\frac{\breve{\mu}}{\mathfrak{t}'}\right)^{\lambda_1\lambda_2} \left(\frac{\breve{\mu}'}{\mathfrak{t}'}\right)^{\lambda_1\lambda_2}\right], \ddot{S}_{\mathfrak{t}'}\left[1-\left(1-\frac{\breve{\nu}}{\mathfrak{t}'}\right)^{\lambda_1\lambda_2}, \left(1-\left(1-\frac{\breve{\nu}'}{\mathfrak{t}'}\right)^{\lambda_1\lambda_2}\right)\right] \right\rangle; \\ \left[\breve{\xi}, \breve{\xi}'\right], [\Psi, \Psi'] \subseteq [0, \mathfrak{t}]; [\breve{\mu}, \breve{\mu}'], [\breve{\nu}, \breve{\nu}'] \subseteq \left[0, \mathfrak{t}'\right] \end{pmatrix} \quad (4.27)$$

By 4.26 and 4.27 we have Proved the desired result such that:

$$\left(\hat{S}^{\lambda_1}\right)^{\lambda_2} = \hat{S}^{\lambda_1\lambda_2}$$

■

Definition 38 *For some particular cases of* λ *and* $\hat{S}$, *Where* λ *be any scalar number and* $\hat{S}$ *be GLIVIFSES then* $\lambda\hat{S}$ *and* $\hat{S}^{\lambda}$ *will be defined as:*

Case (a) If

$$\hat{S} = \left(\begin{array}{c} \left\langle \dot{S}_{\left[\breve{\xi},\breve{\xi}'\right]}, \dot{S}_{[\Psi,\Psi']} \right\rangle, \left\langle \ddot{S}_{[\breve{\mu},\breve{\mu}']}, \ddot{S}_{[\breve{\nu},\breve{\nu}']} \right\rangle; \\ [\Psi,\Psi'] \subseteq [0,\mathfrak{t}]; [\breve{\mu},\breve{\mu}'], [\breve{\nu},\breve{\nu}'] \subseteq \left[0,\mathfrak{t}'\right] \end{array} \right)$$
$$= \left(\left\langle \dot{S}_{[\mathfrak{t},\mathfrak{t}]}, \dot{S}_{[0,0]} \right\rangle, \left\langle \ddot{S}_{[\mathfrak{t}',\mathfrak{t}']}, \ddot{S}_{[0,0]} \right\rangle \right)$$

Then

$$\lambda\hat{S} = \left(\begin{array}{c} \left\langle \dot{S}_{\mathfrak{t}\left[\left(1-\left(1-\frac{\mathfrak{t}}{\mathfrak{t}}\right)^{\lambda}\right),1-\left(1-\frac{\mathfrak{t}}{\mathfrak{t}}\right)^{\lambda}\right]}, \dot{S}_{\mathfrak{t}\left[\left(\frac{0}{\mathfrak{t}}\right)^{\lambda},\left(\frac{0}{\mathfrak{t}}\right)^{\lambda}\right]} \right\rangle, \\ \left\langle \ddot{S}_{\mathfrak{t}'\left[\left(1-\left(1-\frac{\mathfrak{t}'}{\mathfrak{t}'}\right)^{\lambda}\right),1-\left(1-\frac{\mathfrak{t}'}{\mathfrak{t}'}\right)^{\lambda}\right]}, \ddot{S}_{\mathfrak{t}'\left[\left(\frac{0}{\mathfrak{t}'}\right)^{\lambda},\left(\frac{0}{\mathfrak{t}'}\right)^{\lambda}\right]} \right\rangle \end{array} \right)$$
$$= \left(\left\langle \dot{S}_{\mathfrak{t}\left[\left(1-(1-1)^{\lambda}\right),1-(1-1)^{\lambda}\right]}, \dot{S}_{\mathfrak{t}[0,0]} \right\rangle, \left\langle \ddot{S}_{\mathfrak{t}'\left[1-(1-1)^{\lambda},1-(1-1)^{\lambda}\right]}, \ddot{S}_{\mathfrak{t}'[0,0]} \right\rangle \right)$$
$$= \left(\left\langle \dot{S}_{\mathfrak{t}[\mathfrak{t},\mathfrak{t}]}, \dot{S}_{\mathfrak{t}[0,0]} \right\rangle, \left\langle \ddot{S}_{\mathfrak{t}'[\mathfrak{t}',\mathfrak{t}']}, \ddot{S}_{\mathfrak{t}'[0,0]} \right\rangle \right)$$
$$\hat{S}^{\lambda} = \left(\begin{array}{c} \left\langle \dot{S}_{\mathfrak{t}\left[\left(\frac{\mathfrak{t}}{\mathfrak{t}}\right)^{\lambda},\left(\frac{\mathfrak{t}}{\mathfrak{t}}\right)^{\lambda}\right]}, \dot{S}_{\mathfrak{t}\left[1-\left(1-\frac{0}{\mathfrak{t}}\right)^{\lambda},1-\left(1-\frac{0}{\mathfrak{t}}\right)^{\lambda}\right]} \right\rangle, \\ \left\langle \ddot{S}_{\mathfrak{t}'\left[\left(\frac{\mathfrak{t}'}{\mathfrak{t}'}\right)^{\lambda},\left(\frac{\mathfrak{t}'}{\mathfrak{t}'}\right)^{\lambda}\right]}, \ddot{S}_{\mathfrak{t}'\left[1-\left(1-\frac{0}{\mathfrak{t}'}\right)^{\lambda},1-\left(1-\frac{0}{\mathfrak{t}'}\right)^{\lambda}\right]} \right\rangle \end{array} \right)$$
$$= \left(\left\langle \dot{S}_{\mathfrak{t}[\mathfrak{t},\mathfrak{t}]}, \dot{S}_{\mathfrak{t}[0,0]} \right\rangle, \left\langle \ddot{S}_{\mathfrak{t}'[\mathfrak{t}',\mathfrak{t}']}, \ddot{S}_{\mathfrak{t}'[0,0]} \right\rangle \right)$$

Case (b) If

$$\hat{S} = \left(\begin{array}{c} \left\langle \dot{S}_{\left[\breve{\xi},\breve{\xi}'\right]}, \dot{S}_{[\Psi,\Psi']} \right\rangle, \left\langle \ddot{S}_{[\breve{\mu},\breve{\mu}']}, \ddot{S}_{[\breve{\nu},\breve{\nu}']} \right\rangle; \\ \left[\breve{\xi},\breve{\xi}'\right], [\Psi,\Psi'] \subseteq [0,\mathfrak{t}]; [\breve{\mu},\breve{\mu}'], [\breve{\nu},\breve{\nu}'] \subseteq \left[0,\mathfrak{t}'\right] \end{array} \right)$$
$$= \left(\left\langle \dot{S}_{[\mathfrak{t},\mathfrak{t}]}, \dot{S}_{[0,0]} \right\rangle, \left\langle \ddot{S}_{[0,0]}, \ddot{S}_{[\mathfrak{t}',\mathfrak{t}']} \right\rangle \right)$$

Then

$$\lambda\hat{S} = \left(\begin{array}{c} \left\langle \dot{S}_{\mathfrak{t}\left[\left(1-\left(1-\frac{\mathfrak{t}}{\mathfrak{t}}\right)^{\lambda}\right),1-\left(1-\frac{\mathfrak{t}}{\mathfrak{t}}\right)^{\lambda}\right]}, \dot{S}_{\mathfrak{t}\left[\left(\frac{0}{\mathfrak{t}}\right)^{\lambda},\left(\frac{0}{\mathfrak{t}}\right)^{\lambda}\right]} \right\rangle, \\ \left\langle \ddot{S}_{\mathfrak{t}'\left[\left(1-\left(1-\frac{0}{\mathfrak{t}}\right)^{\lambda}\right),1-\left(1-\frac{0}{\mathfrak{t}}\right)^{\lambda}\right]}, \ddot{S}_{\mathfrak{t}'\left[\left(\frac{\mathfrak{t}'}{\mathfrak{t}'}\right)^{\lambda},\left(\frac{\mathfrak{t}'}{\mathfrak{t}'}\right)^{\lambda}\right]} \right\rangle \end{array} \right)$$

$$= \left(\left\langle \dot{S}_{[\text{ţ}\ ,\ \text{ţ}]}, \dot{S}_{[0\ ,\ 0]} \right\rangle, \left\langle \ddot{S}_{[0\ ,\ 0]}, \ddot{S}_{\left[\text{ţ}'\ ,\ \text{ţ}'\right]} \right\rangle \right)$$

$$\hat{S}^{\lambda} = \left(\begin{array}{c} \left\langle \dot{S}_{\text{ţ}}\left[\left(\frac{\text{ţ}}{\text{ţ}}\right)^{\lambda},\left(\frac{\text{ţ}}{\text{ţ}}\right)^{\lambda}\right], \dot{S}_{\text{ţ}}\left[1-\left(1-\frac{0}{\text{ţ}}\right)^{\lambda},1-\left(1-\frac{0}{\text{ţ}}\right)^{\lambda}\right] \right\rangle, \\ \left\langle \ddot{S}_{\text{ţ}'}\left[\left(\frac{0}{\text{ţ}'}\right)^{\lambda},\left(\frac{0}{\text{ţ}'}\right)^{\lambda}\right], \ddot{S}_{\text{ţ}'}\left[1-\left(1-\frac{\text{ţ}'}{\text{ţ}'}\right)^{\lambda},1-\left(1-\frac{\text{ţ}'}{\text{ţ}'}\right)^{\lambda}\right] \right\rangle \end{array} \right)$$

$$= \left(\left\langle \dot{S}_{[\text{ţ}\ ,\ \text{ţ}]}, \dot{S}_{[0\ ,\ 0]} \right\rangle, \left\langle \ddot{S}_{\text{ţ}'\,[0\ ,\ 0]}, \ddot{S}_{\text{ţ}'\,[\text{ţ}',\ \text{ţ}']} \right\rangle \right)$$

Case (c) If

$$\hat{S} = \left(\begin{array}{c} \left\langle \dot{S}_{\left[\ \breve{\xi}\ ,\ \breve{\xi}'\right]}, \dot{S}_{[\Psi,\Psi']} \right\rangle, \left\langle \ddot{S}_{[\breve{\mu},\breve{\mu}']}, \ddot{S}_{[\breve{\nu},\breve{\nu}']} \right\rangle; \\ [\Psi,\Psi'] \subseteq [0,\text{ţ}]\,;[\breve{\mu},\breve{\mu}']\,,[\breve{\nu},\breve{\nu}'] \subseteq \left[0,\text{ţ}'\right] \end{array} \right)$$

$$= \left(\left\langle \dot{S}_{[0\ \ ,\ \ 0]}, \dot{S}_{[\text{ţ}\ ,\ \text{ţ}]} \right\rangle, \left\langle \ddot{S}_{\left[\text{ţ}'\ \ ,\ \ \text{ţ}'\right]}, \ddot{S}_{[0\ \ ,\ \ 0]} \right\rangle \right)$$

Then

$$\lambda\hat{S} = \left(\begin{array}{c} \left\langle \dot{S}_{\text{ţ}}\left[\left(1-\left(1-\frac{0}{\text{ţ}}\right)^{\lambda}\right),1-\left(1-\frac{0}{\text{ţ}}\right)^{\lambda}\right], \dot{S}_{\text{ţ}}\left[\left(\frac{\text{ţ}}{\text{ţ}}\right)^{\lambda},\left(\frac{\text{ţ}}{\text{ţ}}\right)^{\lambda}\right] \right\rangle, \\ \left\langle \ddot{S}_{\text{ţ}'}\left[\left(1-\left(1-\frac{\text{ţ}'}{\text{ţ}'}\right)^{\lambda}\right),1-\left(1-\frac{\text{ţ}'}{\text{ţ}'}\right)^{\lambda}\right], \ddot{S}_{\text{ţ}'}\left[\left(\frac{0}{\text{ţ}'}\right)^{\lambda},\left(\frac{0}{\text{ţ}'}\right)^{\lambda}\right] \right\rangle \end{array} \right)$$

$$= \left(\left\langle \dot{S}_{[0\ ,\ 0]}, \dot{S}_{[\text{ţ}\ ,\ \text{ţ}]} \right\rangle, \left\langle \ddot{S}_{\left[\text{ţ}'\ ,\ \text{ţ}'\right]}, \ddot{S}_{[0\ ,\ 0]} \right\rangle \right)$$

$$\hat{S}^{\lambda} = \left(\begin{array}{c} \left\langle \dot{S}_{\text{ţ}}\left[\left(\frac{0}{\text{ţ}}\right)^{\lambda},\left(\frac{0}{\text{ţ}}\right)^{\lambda}\right], \dot{S}_{\text{ţ}}\left[1-\left(1-\frac{\text{ţ}}{\text{ţ}}\right)^{\lambda},1-\left(1-\frac{\text{ţ}}{\text{ţ}}\right)^{\lambda}\right] \right\rangle, \\ \left\langle \ddot{S}_{\text{ţ}'}\left[\left(\frac{\text{ţ}'}{\text{ţ}'}\right)^{\lambda},\left(\frac{\text{ţ}'}{\text{ţ}'}\right)^{\lambda}\right], \ddot{S}_{\text{ţ}'}\left[1-\left(1-\frac{0}{\text{ţ}'}\right)^{\lambda},1-\left(1-\frac{0}{\text{ţ}'}\right)^{\lambda}\right] \right\rangle \end{array} \right)$$

$$= \left(\left\langle \dot{S}_{[0\ ,\ 0]}, \dot{S}_{[\text{ţ}\ ,\ \text{ţ}]} \right\rangle, \left\langle \ddot{S}_{\text{ţ}'\,[\text{ţ}',\ \text{ţ}']}, \ddot{S}_{\text{ţ}'\,[0\ ,\ 0]} \right\rangle \right)$$

Case (d) If

$$\hat{S} = \left(\begin{array}{c} \left\langle \dot{S}_{\left[\ \breve{\xi}\ ,\ \breve{\xi}'\right]}, \dot{S}_{[\Psi,\Psi']} \right\rangle, \left\langle \ddot{S}_{[\breve{\mu},\breve{\mu}']}, \ddot{S}_{[\breve{\nu},\breve{\nu}']} \right\rangle; \\ [\Psi,\Psi'] \subseteq [0,\text{ţ}]\,;[\breve{\mu},\breve{\mu}']\,,[\breve{\nu},\breve{\nu}'] \subseteq \left[0,\text{ţ}'\right] \end{array} \right)$$

$$= \left(\left\langle \dot{S}_{[0\ \ ,\ \ 0]}, \dot{S}_{[\text{ţ}\ ,\ \text{ţ}]} \right\rangle, \left\langle \ddot{S}_{[0\ \ ,\ \ 0]}, \ddot{S}_{\left[\text{ţ}'\ \ ,\ \ \text{ţ}'\right]} \right\rangle \right)$$

Then

$$\lambda\hat{S} = \begin{pmatrix} \left\langle \dot{S}_{\mathfrak{t}}\left[\left(1-\left(1-\frac{0}{\mathfrak{t}}\right)^{\lambda}\right),1-\left(1-\frac{0}{\mathfrak{t}}\right)^{\lambda}\right], \dot{S}_{\mathfrak{t}}\left[\left(\frac{\mathfrak{t}}{\mathfrak{t}}\right)^{\lambda},\left(\frac{\mathfrak{t}}{\mathfrak{t}}\right)^{\lambda}\right] \right\rangle, \\ \left\langle \ddot{S}_{\mathfrak{t}'}\left[\left(1-\left(1-\frac{0}{\mathfrak{t}'}\right)^{\lambda}\right),1-\left(1-\frac{0}{\mathfrak{t}'}\right)^{\lambda}\right], \ddot{S}_{\mathfrak{t}'}\left[\left(\frac{\mathfrak{t}'}{\mathfrak{t}'}\right)^{\lambda},\left(\frac{\mathfrak{t}'}{\mathfrak{t}'}\right)^{\lambda}\right] \right\rangle \end{pmatrix}$$

$$= \left(\left\langle \dot{S}_{\mathfrak{t}[0\ ,\ 0]}, \dot{S}_{\mathfrak{t}[1\ ,\ 1]} \right\rangle, \left\langle \ddot{S}_{\mathfrak{t}[0\ ,\ 0]}, \ddot{S}_{\mathfrak{t}[1\ ,\ 1]} \right\rangle\right)$$

$$= \left(\left\langle \dot{S}_{[0\ ,\ 0]}, \dot{S}_{[\mathfrak{t}\ ,\ \mathfrak{t}]} \right\rangle, \left\langle S_{[0\ ,\ 0]}, \ddot{S}_{[\mathfrak{t}'\ ,\ \mathfrak{t}']} \right\rangle\right)$$

$$\hat{S}^{\lambda} = \begin{pmatrix} \left\langle \dot{S}_{\mathfrak{t}}\left[\left(\frac{0}{\mathfrak{t}}\right)^{\lambda},\left(\frac{0}{\mathfrak{t}}\right)^{\lambda}\right], \dot{S}_{\mathfrak{t}}\left[1-\left(1-\frac{\mathfrak{t}}{\mathfrak{t}}\right)^{\lambda},1-\left(1-\frac{\mathfrak{t}}{\mathfrak{t}}\right)^{\lambda}\right] \right\rangle, \\ \left\langle \ddot{S}_{\mathfrak{t}'}\left[\left(\frac{0}{\mathfrak{t}'}\right)^{\lambda},\left(\frac{0}{\mathfrak{t}'}\right)^{\lambda}\right], \ddot{S}_{\mathfrak{t}'}\left[1-\left(1-\frac{\mathfrak{t}'}{\mathfrak{t}'}\right)^{\lambda},1-\left(1-\frac{\mathfrak{t}'}{\mathfrak{t}'}\right)^{\lambda}\right] \right\rangle \end{pmatrix}$$

$$= \left(\left\langle \dot{S}_{\mathfrak{t}[0\ ,\ 0]}, \dot{S}_{\mathfrak{t}[1\ ,\ 1]} \right\rangle, \left\langle \ddot{S}_{\mathfrak{t}'[0\ ,\ 0]}, \ddot{S}_{\mathfrak{t}'[1\ ,\ 1]} \right\rangle\right)$$

$$= \left(\left\langle \dot{S}_{[0\ ,\ 0]}, \dot{S}_{[\mathfrak{t}\ ,\ \mathfrak{t}]} \right\rangle, \left\langle \ddot{S}_{\mathfrak{t}'[0\ ,\ 0]}, \ddot{S}_{\mathfrak{t}'[\mathfrak{t}',\ \mathfrak{t}']} \right\rangle\right)$$

Case (e) If

$$\alpha = \left(\left\langle \dot{S}_{\left[\ \breve{\xi}\ ,\ \breve{\xi}'\right]}, \dot{S}_{[\Psi,\Psi']} \right\rangle, \left\langle \ddot{S}_{[\breve{\mu},\breve{\mu}']}, \ddot{S}_{[\breve{\nu},\breve{\nu}']} \right\rangle\right)$$

$$= \left(\left\langle \dot{S}_{[0\ \ ,\ \ 0]}, \dot{S}_{[0\ \ ,\ \ 0]} \right\rangle, \left\langle \ddot{S}_{[0\ \ ,\ \ 0]}, \ddot{S}_{[\mathfrak{t}'\ \ ,\ \ \mathfrak{t}']} \right\rangle\right)$$

Then

$$\lambda\hat{S} = \begin{pmatrix} \left\langle \dot{S}_{\mathfrak{t}}\left[\left(1-\left(1-\frac{0}{\mathfrak{t}}\right)^{\lambda}\right),1-\left(1-\frac{0}{\mathfrak{t}}\right)^{\lambda}\right], \dot{S}_{\mathfrak{t}}\left[\left(\frac{0}{\mathfrak{t}}\right)^{\lambda},\left(\frac{0}{\mathfrak{t}}\right)^{\lambda}\right] \right\rangle, \\ \left\langle \ddot{S}_{\mathfrak{t}'}\left[\left(1-\left(1-\frac{0}{\mathfrak{t}'}\right)^{\lambda}\right),1-\left(1-\frac{0}{\mathfrak{t}'}\right)^{\lambda}\right], \ddot{S}_{\mathfrak{t}'}\left[\left(\frac{\mathfrak{t}'}{\mathfrak{t}'}\right)^{\lambda},\left(\frac{\mathfrak{t}'}{\mathfrak{t}'}\right)^{\lambda}\right] \right\rangle \end{pmatrix}$$

$$= \left(\left\langle \dot{S}_{\mathfrak{t}[0\ ,\ 0]}, \dot{S}_{\mathfrak{t}[0\ ,\ 0]} \right\rangle, \left\langle \ddot{S}_{\mathfrak{t}[0\ ,\ 0]}, \ddot{S}_{\mathfrak{t}[1\ ,\ 1]} \right\rangle\right)$$

$$= \left(\left\langle \dot{S}_{[0\ ,\ 0]}, \dot{S}_{[0\ ,\ 0]} \right\rangle, \left\langle S_{[0\ ,\ 0]}, \ddot{S}_{[\mathfrak{t}'\ ,\ \mathfrak{t}']} \right\rangle\right)$$

$$\hat{S}^{\lambda} = \begin{pmatrix} \left\langle \dot{S}_{\mathfrak{t}}\left[\left(\frac{0}{\mathfrak{t}}\right)^{\lambda},\left(\frac{0}{\mathfrak{t}}\right)^{\lambda}\right], \dot{S}_{\mathfrak{t}}\left[1-\left(1-\frac{0}{\mathfrak{t}}\right)^{\lambda},1-\left(1-\frac{0}{\mathfrak{t}}\right)^{\lambda}\right] \right\rangle, \\ \left\langle \ddot{S}_{\mathfrak{t}'}\left[\left(\frac{0}{\mathfrak{t}'}\right)^{\lambda},\left(\frac{0}{\mathfrak{t}'}\right)^{\lambda}\right], \ddot{S}_{\mathfrak{t}'}\left[1-\left(1-\frac{\mathfrak{t}'}{\mathfrak{t}'}\right)^{\lambda},1-\left(1-\frac{\mathfrak{t}'}{\mathfrak{t}'}\right)^{\lambda}\right] \right\rangle \end{pmatrix}$$

$$= \left(\left\langle \dot{S}_{\mathfrak{t}[0\ ,\ 0]}, \dot{S}_{\mathfrak{t}[0\ ,\ 0]} \right\rangle, \left\langle \ddot{S}_{\mathfrak{t}'[0\ ,\ 0]}, \ddot{S}_{\mathfrak{t}'[1\ ,\ 1]} \right\rangle \right)$$

$$= \left(\left\langle \dot{S}_{[0\ ,\ 0]}, \dot{S}_{[0\ ,\ 0]} \right\rangle, \left\langle \ddot{S}_{[0\ ,\ 0]}, \ddot{S}_{[\mathfrak{t}',\ \mathfrak{t}']} \right\rangle \right)$$

Case (f) If

$$\hat{S} = \left(\left\langle \dot{S}_{[\ \check{\xi}\ ,\ \check{\xi}']}, \dot{S}_{[\Psi,\Psi']} \right\rangle, \left\langle \ddot{S}_{[\check{\mu},\check{\mu}']}, \ddot{S}_{[\check{\nu},\check{\nu}']} \right\rangle \right)$$

$$= \left(\left\langle \dot{S}_{[0\ ,\ 0]}, \dot{S}_{[0\ ,\ 0]} \right\rangle, \left\langle \ddot{S}_{[\mathfrak{t}'\ ,\ \mathfrak{t}']}, \ddot{S}_{[0\ ,\ 0]} \right\rangle \right)$$

Then

$$\lambda\hat{S} = \begin{pmatrix} \left\langle \dot{S}_{\mathfrak{t}\left[\left(1-\left(1-\frac{0}{\mathfrak{t}}\right)^{\lambda}\right), 1-\left(1-\frac{0}{\mathfrak{t}}\right)^{\lambda}\right]}, \dot{S}_{\mathfrak{t}\left[\left(\frac{0}{\mathfrak{t}}\right)^{\lambda}, \left(\frac{0}{\mathfrak{t}}\right)^{\lambda}\right]} \right\rangle, \\ \left\langle \ddot{S}_{\mathfrak{t}'\left[\left(1-\left(1-\frac{\mathfrak{t}'}{\mathfrak{t}'}\right)^{\lambda}\right), 1-\left(1-\frac{\mathfrak{t}'}{\mathfrak{t}'}\right)^{\lambda}\right]}, \ddot{S}_{\mathfrak{t}'\left[\left(\frac{0}{\mathfrak{t}'}\right)^{\lambda}, \left(\frac{0}{\mathfrak{t}'}\right)^{\lambda}\right]} \right\rangle \end{pmatrix}$$

$$= \left(\left\langle \dot{S}_{\mathfrak{t}[0\ ,\ 0]}, \dot{S}_{\mathfrak{t}[0\ ,\ 0]} \right\rangle, \left\langle \ddot{S}_{\mathfrak{t}[1\ ,\ 1]}, \ddot{S}_{\mathfrak{t}[0\ ,\ 0]} \right\rangle \right)$$

$$= \left(\left\langle \dot{S}_{[0\ ,\ 0]}, \dot{S}_{[0\ ,\ 0]} \right\rangle, \left\langle S_{[\mathfrak{t}'\ ,\ \mathfrak{t}']}, \ddot{S}_{[0\ ,\ 0]} \right\rangle \right)$$

$$\hat{S}^{\lambda} = \begin{pmatrix} \left\langle \dot{S}_{\mathfrak{t}\left[\left(\frac{0}{\mathfrak{t}}\right)^{\lambda}, \left(\frac{0}{\mathfrak{t}}\right)^{\lambda}\right]}, \dot{S}_{\mathfrak{t}\left[1-\left(1-\frac{0}{\mathfrak{t}}\right)^{\lambda}, 1-\left(1-\frac{0}{\mathfrak{t}}\right)^{\lambda}\right]} \right\rangle, \\ \left\langle \ddot{S}_{\mathfrak{t}'\left[\left(\frac{\mathfrak{t}'}{\mathfrak{t}'}\right)^{\lambda}, \left(\frac{\mathfrak{t}'}{\mathfrak{t}'}\right)^{\lambda}\right]}, \ddot{S}_{\mathfrak{t}'\left[1-\left(1-\frac{0}{\mathfrak{t}'}\right)^{\lambda}, 1-\left(1-\frac{0}{\mathfrak{t}'}\right)^{\lambda}\right]} \right\rangle \end{pmatrix}$$

$$= \left(\left\langle \dot{S}_{\mathfrak{t}[0\ ,\ 0]}, \dot{S}_{\mathfrak{t}[0\ ,\ 0]} \right\rangle, \left\langle \ddot{S}_{\mathfrak{t}'[1\ ,\ 1]}, \ddot{S}_{\mathfrak{t}'[0\ ,\ 0]} \right\rangle \right)$$

$$= \left(\left\langle \dot{S}_{[0\ ,\ 0]}, \dot{S}_{[0\ ,\ 0]} \right\rangle, \left\langle \ddot{S}_{[0\ ,\ 0]}, \ddot{S}_{[\mathfrak{t}',\ \mathfrak{t}']} \right\rangle \right)$$

Case (g) If

$$\hat{S} = \left(\left\langle \dot{S}_{[\ \check{\xi}\ ,\ \check{\xi}']}, \dot{S}_{[\Psi,\Psi']} \right\rangle, \left\langle \ddot{S}_{[\check{\mu},\check{\mu}']}, \ddot{S}_{[\check{\nu},\check{\nu}']} \right\rangle \right)$$

$$= \left(\left\langle \dot{S}_{[\mathfrak{t}\ ,\ \mathfrak{t}]}, \dot{S}_{[0\ ,\ 0]} \right\rangle, \left\langle \ddot{S}_{[0\ ,\ 0]}, \ddot{S}_{[0\ ,\ 0]} \right\rangle \right)$$

Then

$$\lambda\hat{S} = \begin{pmatrix} \left\langle \dot{S}_{\mathfrak{t}\left[\left(1-\left(1-\frac{\mathfrak{t}}{\mathfrak{t}}\right)^{\lambda}\right), 1-\left(1-\frac{\mathfrak{t}}{\mathfrak{t}}\right)^{\lambda}\right]}, \dot{S}_{\mathfrak{t}\left[\left(\frac{0}{\mathfrak{t}}\right)^{\lambda}, \left(\frac{0}{\mathfrak{t}}\right)^{\lambda}\right]} \right\rangle, \\ \left\langle \ddot{S}_{\mathfrak{t}'\left[\left(1-\left(1-\frac{0}{\mathfrak{t}'}\right)^{\lambda}\right), 1-\left(1-\frac{0}{\mathfrak{t}'}\right)^{\lambda}\right]}, \ddot{S}_{\mathfrak{t}'\left[\left(\frac{0}{\mathfrak{t}'}\right)^{\lambda}, \left(\frac{0}{\mathfrak{t}'}\right)^{\lambda}\right]} \right\rangle \end{pmatrix}$$

$$= \left(\left\langle \dot{S}_{ţ[1\ ,\ 1]}, \dot{S}_{ţ[0\ ,\ 0]} \right\rangle, \left\langle \ddot{S}_{ţ[0\ ,\ 0]}, \ddot{S}_{ţ[0\ ,\ 0]} \right\rangle\right)$$

$$= \left(\left\langle \dot{S}_{[ţ\ ,\ ţ]}, \dot{S}_{[0\ ,\ 0]} \right\rangle, \left\langle S_{[0\ ,\ 0]}, \ddot{S}_{[0\ ,\ 0]} \right\rangle\right)$$

$$\hat{S}^{\lambda} = \begin{pmatrix} \left\langle \dot{S}_{ţ\left[\left(\frac{ţ}{ţ}\right)^{\lambda},\left(\frac{ţ}{ţ}\right)^{\lambda}\right]}, \dot{S}_{ţ\left[1-\left(1-\frac{0}{ţ}\right)^{\lambda},1-\left(1-\frac{0}{ţ}\right)^{\lambda}\right]} \right\rangle, \\ \left\langle \ddot{S}_{ţ'\left[\left(\frac{0}{ţ'}\right)^{\lambda},\left(\frac{0}{ţ'}\right)^{\lambda}\right]}, \ddot{S}_{ţ'\left[1-\left(1-\frac{0}{ţ'}\right)^{\lambda},1-\left(1-\frac{0}{ţ'}\right)^{\lambda}\right]} \right\rangle \end{pmatrix}$$

$$= \left(\left\langle \dot{S}_{ţ[1\ ,\ 1]}, \dot{S}_{ţ[0\ ,\ 0]} \right\rangle, \left\langle \ddot{S}_{ţ'[0\ ,\ 0]}, \ddot{S}_{ţ'[0\ ,\ 0]} \right\rangle\right)$$

$$= \left(\left\langle \dot{S}_{[ţ\ ,\ ţ]}, \dot{S}_{[0\ ,\ 0]} \right\rangle, \left\langle \ddot{S}_{[0\ ,\ 0]}, \ddot{S}_{[0\ ,\ 0]} \right\rangle\right)$$

Case (h) if

$$\hat{S} = \left(\left\langle \dot{S}_{\left[\breve{\xi}\ ,\ \breve{\xi}'\right]}, \dot{S}_{[\Psi,\Psi']} \right\rangle, \left\langle \ddot{S}_{[\breve{\mu},\breve{\mu}']}, \ddot{S}_{[\breve{\nu},\breve{\nu}']} \right\rangle\right)$$

$$= \left(\left\langle \dot{S}_{[0\ ,\ 0]}, \dot{S}_{[ţ\ ,\ ţ]} \right\rangle, \left\langle \ddot{S}_{[0\ ,\ 0]}, \ddot{S}_{[0\ ,\ 0]} \right\rangle\right)$$

Then

$$\lambda\hat{S} = \begin{pmatrix} \left\langle \dot{S}_{ţ\left[\left(1-\left(1-\frac{0}{ţ}\right)^{\lambda}\right),1-\left(1-\frac{0}{ţ}\right)^{\lambda}\right]}, \dot{S}_{ţ\left[\left(\frac{ţ}{ţ}\right)^{\lambda},\left(\frac{ţ}{ţ}\right)^{\lambda}\right]} \right\rangle, \\ \left\langle \ddot{S}_{ţ'\left[\left(1-\left(1-\frac{0}{ţ'}\right)^{\lambda}\right),1-\left(1-\frac{0}{ţ'}\right)^{\lambda}\right]}, \ddot{S}_{ţ'\left[\left(\frac{0}{ţ'}\right)^{\lambda},\left(\frac{0}{ţ'}\right)^{\lambda}\right]} \right\rangle \end{pmatrix}$$

$$= \left(\left\langle \dot{S}_{ţ[0\ ,\ 0]}, \dot{S}_{ţ[1\ ,\ 1]} \right\rangle, \left\langle \ddot{S}_{ţ[0\ ,\ 0]}, \ddot{S}_{ţ[0\ ,\ 0]} \right\rangle\right)$$

$$= \left(\left\langle \dot{S}_{[0\ ,\ 0]}, \dot{S}_{[ţ\ ,\ ţ]} \right\rangle, \left\langle S_{[0\ ,\ 0]}, \ddot{S}_{[0\ ,\ 0]} \right\rangle\right)$$

$$\hat{S}^{\lambda} = \begin{pmatrix} \left\langle \dot{S}_{ţ\left[\left(\frac{0}{ţ}\right)^{\lambda},\left(\frac{0}{ţ}\right)^{\lambda}\right]}, \dot{S}_{ţ\left[1-\left(1-\frac{ţ}{ţ}\right)^{\lambda},1-\left(1-\frac{ţ}{ţ}\right)^{\lambda}\right]} \right\rangle, \\ \left\langle \ddot{S}_{ţ'\left[\left(\frac{0}{ţ'}\right)^{\lambda},\left(\frac{0}{ţ'}\right)^{\lambda}\right]}, \ddot{S}_{ţ'\left[1-\left(1-\frac{0}{ţ'}\right)^{\lambda},1-\left(1-\frac{0}{ţ'}\right)^{\lambda}\right]} \right\rangle \end{pmatrix}$$

$$= \left(\left\langle \dot{S}_{ţ[0\ ,\ 0]}, \dot{S}_{ţ[1\ ,\ 1]} \right\rangle, \left\langle \ddot{S}_{ţ'[0\ ,\ 0]}, \ddot{S}_{ţ'[0\ ,\ 0]} \right\rangle\right)$$

Case (i) If

$$\hat{S} = \left(\left\langle \dot{S}_{\left[\ \breve{\xi}\ ,\ \breve{\xi}'\right]}, \dot{S}_{[\Psi,\Psi']} \right\rangle, \left\langle \ddot{S}_{[\breve{\mu},\breve{\mu}']}, \ddot{S}_{[\breve{\nu},\breve{\nu}']} \right\rangle\right)$$
$$= \left(\left\langle \dot{S}_{[\mathfrak{t}\ ,\ \mathfrak{t}]}, \dot{S}_{[\mathfrak{t}\ ,\ \mathfrak{t}]} \right\rangle, \left\langle \ddot{S}_{[0\ ,\ 0]}, \ddot{S}_{[0\ \ ,\ \ \ 0]} \right\rangle\right)$$
$$\lambda \to 0,$$

Then

$$\lambda\hat{S} = \begin{pmatrix} \left\langle \dot{S}_{\mathfrak{t}}\left[\left(1-\left(1-\frac{\breve{\xi}}{\mathfrak{t}}\right)^{\lambda}\right),1-\left(1-\frac{\breve{\xi}'}{\mathfrak{t}}\right)^{\lambda}\right], \dot{S}_{\mathfrak{t}}\left[\left(\frac{\Psi}{\mathfrak{t}}\right)^{\lambda},\left(\frac{\Psi'}{\mathfrak{t}}\right)^{\lambda}\right] \right\rangle, \\ \left\langle \ddot{S}_{\mathfrak{t}'}\left[\left(1-\left(1-\frac{\breve{\mu}}{\mathfrak{t}'}\right)^{\lambda}\right),1-\left(1-\frac{\breve{\mu}'}{\mathfrak{t}'}\right)^{\lambda}\right], \ddot{S}_{\mathfrak{t}'}\left[\left(\frac{\breve{\nu}}{\mathfrak{t}'}\right)^{\lambda},\left(\frac{\breve{\nu}'}{\mathfrak{t}'}\right)^{\lambda}\right] \right\rangle \end{pmatrix}$$

$$\lambda \to 0 = \begin{pmatrix} \left\langle \dot{S}_{\mathfrak{t}}\left[\left(1-\left(1-\frac{\breve{\xi}}{\mathfrak{t}}\right)^{0}\right),1-\left(1-\frac{\breve{\xi}'}{\mathfrak{t}}\right)^{0}\right], \dot{S}_{\mathfrak{t}}\left[\left(\frac{\Psi}{\mathfrak{t}}\right)^{0},\left(\frac{\Psi'}{\mathfrak{t}}\right)^{0}\right] \right\rangle, \\ \left\langle \ddot{S}_{\mathfrak{t}'}\left[\left(1-\left(1-\frac{\breve{\mu}}{\mathfrak{t}'}\right)^{0}\right),1-\left(1-\frac{\breve{\mu}'}{\mathfrak{t}'}\right)^{0}\right], \ddot{S}_{\mathfrak{t}'}\left[\left(\frac{\breve{\nu}}{\mathfrak{t}'}\right)^{0},\left(\frac{\breve{\nu}'}{\mathfrak{t}'}\right)^{0}\right] \right\rangle \end{pmatrix}$$

$$\hat{S}^{\lambda} = \begin{pmatrix} \left\langle \dot{S}_{\mathfrak{t}}\left[\left(\frac{0}{\mathfrak{t}}\right)^{\lambda},\left(\frac{0}{\mathfrak{t}}\right)^{\lambda}\right], \dot{S}_{\mathfrak{t}}\left[1-\left(1-\frac{\mathfrak{t}}{\mathfrak{t}}\right)^{\lambda},1-\left(1-\frac{\mathfrak{t}}{\mathfrak{t}}\right)^{\lambda}\right] \right\rangle, \\ \left\langle \ddot{S}_{\mathfrak{t}'}\left[\left(\frac{0}{\mathfrak{t}'}\right)^{\lambda},\left(\frac{0}{\mathfrak{t}'}\right)^{\lambda}\right], \ddot{S}_{\mathfrak{t}'}\left[1-\left(1-\frac{0}{\mathfrak{t}'}\right)^{\lambda},1-\left(1-\frac{0}{\mathfrak{t}'}\right)^{\lambda}\right] \right\rangle \end{pmatrix}$$

$$= \left(\left\langle \dot{S}_{\mathfrak{t}[1-1\ ,\ 1-1]}, \dot{S}_{\mathfrak{t}[1\ ,\ 1]} \right\rangle, \left\langle \ddot{S}_{\mathfrak{t}'\ [1-1\ ,\ 1-1]}, \ddot{S}_{\mathfrak{t}'\ [1\ ,\ 1]} \right\rangle\right)$$

$$= \left(\left\langle \dot{S}_{[0\ \ \ \ 0]}, \dot{S}_{[\mathfrak{t}\ ,\ \mathfrak{t}]} \right\rangle, \left\langle \ddot{S}_{[0\ \ \ \ 0]}, \ddot{S}_{[\mathfrak{t}\ '\ ,\ \ \ \mathfrak{t}']} \right\rangle\right)$$

$$\hat{S}^{\lambda} = \begin{pmatrix} \left\langle \dot{S}_{\mathfrak{t}}\left[\left(\frac{\breve{\xi}}{\mathfrak{t}}\right)^{\lambda},\left(\frac{\breve{\xi}'}{\mathfrak{t}}\right)^{\lambda}\right], \dot{S}_{\mathfrak{t}}\left[1-\left(1-\frac{\Psi}{\mathfrak{t}}\right)^{\lambda},1-\left(1-\frac{\Psi'}{\mathfrak{t}}\right)^{\lambda}\right] \right\rangle, \\ \left\langle \ddot{S}_{\mathfrak{t}'}\left[\left(\frac{\breve{\mu}}{\mathfrak{t}'}\right)^{\lambda},\left(\frac{\breve{\mu}'}{\mathfrak{t}'}\right)^{\lambda}\right], \ddot{S}_{\mathfrak{t}'}\left[1-\left(1-\frac{\breve{\nu}}{\mathfrak{t}'}\right)^{\lambda},1-\left(1-\frac{\breve{\nu}'}{\mathfrak{t}'}\right)^{\lambda}\right] \right\rangle \end{pmatrix}$$

$$\hat{S}^{\lambda} = \begin{pmatrix} \left\langle \dot{S}_{\mathfrak{t}}\left[\left(\frac{\breve{\xi}}{\mathfrak{t}}\right)^{0},\left(\frac{\breve{\xi}'}{\mathfrak{t}}\right)^{0}\right], \dot{S}_{\mathfrak{t}}\left[1-\left(1-\frac{\Psi}{\mathfrak{t}}\right)^{0},1-\left(1-\frac{\Psi'}{\mathfrak{t}}\right)^{0}\right] \right\rangle, \\ \left\langle \ddot{S}_{\mathfrak{t}'}\left[\left(\left(\frac{\breve{\mu}}{\mathfrak{t}'}\right)^{0}\right),\left(\frac{\breve{\mu}'}{\mathfrak{t}'}\right)^{0}\right], \ddot{S}_{\mathfrak{t}'}\left[1-\left(1-\frac{\breve{\nu}}{\mathfrak{t}'}\right)^{0},1-\left(1-\frac{\breve{\nu}'}{\mathfrak{t}'}\right)^{0}\right] \right\rangle \end{pmatrix}$$

$$= \left(\left\langle \dot{S}_{\mathfrak{t}[1\ \ 1]}, \dot{S}_{\mathfrak{t}[1-1\ ,\ 1-1]} \right\rangle, \left\langle \ddot{S}_{\mathfrak{t}'\ [1\ \ \ 1]}, \ddot{S}_{\mathfrak{t}'\ [1\ -1,\ 1-1]} \right\rangle\right)$$

$$= \left(\left\langle \dot{S}_{[\mathfrak{t} \quad \mathfrak{t}]}, \dot{S}_{[0 \quad 0]} \right\rangle, \left\langle \ddot{S}_{[\mathfrak{t}\,', \quad \mathfrak{t}']}, \ddot{S}_{[0 \quad 0]} \right\rangle \right)$$

Case (j) If

$$\hat{S} = \left(\left\langle \dot{S}_{\left[\breve{\xi}, \breve{\xi}'\right]}, \dot{S}_{[\Psi,\Psi']} \right\rangle, \left\langle \ddot{S}_{[\breve{\mu},\breve{\mu}']}, \ddot{S}_{[\breve{\nu},\breve{\nu}']} \right\rangle \right)$$

and

$$\lambda \to \infty$$

Then

$$\lambda \hat{S} = \begin{pmatrix} \left\langle \dot{S}_{\mathfrak{t}\left[\left(1-\left(1-\frac{\breve{\xi}}{\mathfrak{t}}\right)^{\lambda}\right), 1-\left(1-\frac{\breve{\xi}'}{\mathfrak{t}}\right)^{\lambda}\right]}, \dot{S}_{\mathfrak{t}\left[\left(\frac{\Psi}{\mathfrak{t}}\right)^{\lambda}, \left(\frac{\Psi'}{\mathfrak{t}}\right)^{\lambda}\right]} \right\rangle, \\ \left\langle \ddot{S}_{\mathfrak{t}'\left[\left(1-\left(1-\frac{\breve{\mu}}{\mathfrak{t}'}\right)^{\lambda}\right), 1-\left(1-\frac{\breve{\mu}'}{\mathfrak{t}'}\right)^{\lambda}\right]}, \ddot{S}_{\mathfrak{t}'\left[\left(\frac{\breve{\nu}}{\mathfrak{t}'}\right)^{\lambda}, \left(\frac{\breve{\nu}'}{\mathfrak{t}'}\right)^{\lambda}\right]} \right\rangle \end{pmatrix}$$

$\lambda \to \infty$ conider Term

$$\dot{S}_{\mathfrak{t}\left[\left(1-\left(1-\frac{\breve{\xi}}{\mathfrak{t}}\right)^{\lambda}\right), 1-\left(1-\frac{\breve{\xi}'}{\mathfrak{t}}\right)^{\lambda}\right]}$$

$$\left(1 - \frac{\breve{\xi}}{\mathfrak{t}}\right)^{\lambda}$$

when λ approches to positive infinity that is $\lambda \to +\infty$ then,

$$\begin{aligned} &\because o \leq \breve{\xi} < 1 \qquad \therefore o < \left|\frac{\breve{\xi}}{\mathfrak{t}}\right| \breve{\xi} << 1 \\ &\left(1 - \frac{\breve{\xi}}{\mathfrak{t}}\right)^{\lambda} = 1 - \frac{\lambda\breve{\xi}}{\mathfrak{t}} + \frac{\lambda(\lambda-1)}{21}\left(\frac{\breve{\xi}}{\mathfrak{t}}\right)^{2} - \frac{\lambda(\lambda-1)(\lambda-2)}{31}\left(\frac{\breve{\xi}}{\mathfrak{t}}\right)^{3} + \ldots\ldots\ldots.., +\infty \\ &\because o < \left(\frac{\breve{\xi}}{\mathfrak{t}}\right) << 1 \\ &= 1 - 1 = 0 \end{aligned}$$

As the higher tends to Zero,

so

$$\left(1 - \frac{\breve{\xi}}{\mathfrak{t}}\right)^{\lambda} \to 0 \qquad as \qquad \lambda \to \infty$$

Now

$$\begin{array}{l}\left(\frac{\breve{\xi}}{\text{ţ}}\right)^{\lambda} \qquad as \qquad \lambda \to \infty \\ \Psi < 1 \qquad , \text{ţ} > 0 \text{ positive even integers} \\ 0 < \frac{\Psi}{\text{ţ}} << 1 \\ \left(\frac{\Psi}{\text{ţ}}\right)^{1}\left(\frac{\Psi}{\text{ţ}}\right)^{2}...\left(\frac{\Psi}{\text{ţ}}\right)^{n} \text{approaches to zero} \\ \left(\frac{\Psi}{\text{ţ}}\right)^{\lambda} \to 0\end{array}$$

when λ tends to infinity $i.e.\,(\lambda \to \infty)$

$$\lambda \hat{S} = \left(\begin{array}{c} \left\langle \dot{S}_{\text{ţ}\left[1-\left(1-\frac{\breve{\xi}}{\text{ţ}}\right)^{\lambda},1-\left(1-\frac{\breve{\xi}'}{\text{ţ}}\right)^{\lambda}\right]}, \dot{S}_{\text{ţ}\left[\left(\frac{\Psi}{\text{ţ}}\right)^{\lambda},\left(\frac{\breve{\xi}'}{\text{ţ}}\right)^{\lambda}\right]} \right\rangle, \\ \left\langle \ddot{S}_{\text{ţ}\left[1-\left(1-\frac{\breve{\mu}}{\text{ţ}}\right)^{\lambda},1-\left(1-\frac{\breve{\mu}'}{\text{ţ}}\right)^{\lambda}\right]}, \ddot{S}_{\text{ţ}\left[\left(\frac{\breve{\nu}}{\text{ţ}}\right)^{\lambda},\left(\frac{\breve{\nu}'}{\text{ţ}}\right)^{\lambda}\right]} \right\rangle \end{array} \right)$$

$$\lambda \hat{S} = \left(\left\langle \dot{S}_{\text{ţ}[1-0,1-0]}, \dot{S}_{\text{ţ}[0\ ,0]}\right\rangle, \left\langle \ddot{S}_{\text{ţ}[1-0\ ,1-0]}, \ddot{S}_{\text{ţ}[0\ ,0]}\right\rangle\right)$$

$$= \left(\left\langle \dot{S}_{[\text{ţ}\ ,\ \text{ţ}]}, \dot{S}_{[0\ ,0]}\right\rangle, \left\langle \ddot{S}_{[\text{ţ}'\ ,\ \text{ţ}']}, \ddot{S}_{[0\ ,0]}\right\rangle\right)$$

$$\lambda \hat{S} = \left(\begin{array}{c} \left\langle \dot{S}_{\text{ţ}\left[\left(\frac{\breve{\xi}}{\text{ţ}}\right)^{\lambda},\left(\frac{\breve{\xi}'}{\text{ţ}}\right)^{\lambda}\right]}, \dot{S}_{\text{ţ}\left[1-\left(1-\frac{\Psi}{\text{ţ}}\right)^{\lambda},1-\left(1-\frac{\Psi'}{\text{ţ}}\right)^{\lambda}\right]} \right\rangle, \\ \left\langle \ddot{S}_{\text{ţ}'\left[\left(\frac{\breve{\mu}}{\text{ţ}}\right)^{\lambda},\left(\frac{\breve{\mu}'}{\text{ţ}}\right)^{\lambda}\right]}, \ddot{S}_{\text{ţ}'\left[1-\left(1-\frac{\breve{\nu}}{\text{ţ}'}\right)^{\lambda},1-\left(1-\frac{\breve{\nu}'}{\text{ţ}'}\right)^{\lambda}\right]} \right\rangle \end{array} \right)$$

$$As \quad \lambda \to \infty \qquad so \qquad \left(\frac{\breve{\xi}}{\text{ţ}}\right) \to 0$$

$$\& \ \left(1 - \frac{\Psi}{\text{ţ}}\right) \to 0$$

$$= \left(\left\langle \dot{S}_{\text{ţ}[0\ ,\ 0]}, \dot{S}_{\text{ţ}[1-0\ ,1-0]}\right\rangle, \left\langle \ddot{S}_{\text{ţ}'[0\ ,\ 0]}, \ddot{S}_{\text{ţ}'[1-0\ ,1-0]}\right\rangle\right)$$

$$= \left(\left\langle \dot{S}_{[0\ ,\ 0]}, \dot{S}_{[\text{ţ}\ ,\text{ţ}]}\right\rangle, \left\langle \ddot{S}_{[0\ ,\ 0]}, \ddot{S}_{[\text{ţ}'\ ,\text{ţ}']}\right\rangle\right)$$

Theorem 39 *For GLIVIFSESs.* $\hat{S}$, $\hat{S}_1, \hat{S}_2$ *are the sets obtain by definition 12 are still GLIVIFSESs.*

Proof. For this we shall prove only $\hat{s}_1 \oplus \ \hat{s}_2 and \ \ \lambda \hat{s}$ are GLIVIFSESs, while others can be proved similarly.

Since

$$\overset{\wedge}{S_i} = \left(\left\langle \dot{S}_{\left[\breve{\xi}_i,\ \breve{\xi}'_i\right]}, \dot{S}_{\left[\tilde{\Psi}_i,\ \tilde{\Psi}'_i\right]} \right\rangle, \left\langle \ddot{S}_{\left[\breve{\mu}_i,\ \breve{\mu}'_i\right]}, \ddot{S}_{\left[\breve{\nu}_i,\ \breve{\nu}'_i\right]} \right\rangle \right) \qquad (i = 1, 2)$$

Where

$$\dot{S}_{\left[\breve{\xi}_i,\ \breve{\xi}'_i\right]}, \dot{S}_{\left[\tilde{\Psi}_i,\ \tilde{\Psi}'_i\right]}, \in \dot{S}^{[1]} \left[o, \mathfrak{t}\right] \ and \ \ddot{S}_{\left[\breve{\mu}_i,\ \breve{\mu}'_i\right]}, \ddot{S}_{\left[\breve{\nu}_i,\ \breve{\nu}'_i\right]} \ \in \ddot{S}^{[2]} \left[0, \mathfrak{t}'\right] \ ,$$

So

$$[0.0,\ 0.0] \leq \left[\breve{\xi}_i,\ \breve{\xi}'_i\right], \left[\tilde{\Psi}_i,\ \tilde{\Psi}'_i\right] \leq [\mathfrak{t}, \mathfrak{t}], [0.0,\ 0.0] \leq \left[\breve{\mu}_i,\ \breve{\mu}'_i\right], \left[\breve{\nu}_i,\ \breve{\nu}'_i\right] \leq \left[\mathfrak{t}', \mathfrak{t}'\right]$$

and

$$\breve{\xi}_i + \ \tilde{\Psi}_i \leq \mathfrak{t}, \breve{\xi}'_i + \ \tilde{\Psi}'_i \leq \mathfrak{t} \ , \ \breve{\mu}_i + \breve{\nu}_i \leq \mathfrak{t}', \quad \breve{\mu}'_i + \breve{\nu}'_i \leq \mathfrak{t}' \ for \ i \ = \ 1, 2$$

Now for

$$[0.0,\ 0.0] \leq \left[\breve{\xi}_1,\ \breve{\xi}'_1\right], \left[\breve{\xi}_2,\ \breve{\xi}'_2\right] \leq [\mathfrak{t}, \mathfrak{t}]$$

$$\Longrightarrow [0.0,\ 0.0] \leq \left[\frac{\breve{\xi}_1}{\mathfrak{t}},\ \frac{\breve{\xi}'_1}{\mathfrak{t}}\right], \left[\frac{\breve{\xi}_2}{\mathfrak{t}},\ \frac{\breve{\xi}'_2}{\mathfrak{t}}\right] \leq [1, 1]$$

$$\Longrightarrow [0.0,\ 0.0] \leq \left[1 - \frac{\breve{\xi}_1}{\mathfrak{t}},\ 1 - \frac{\breve{\xi}'_1}{\mathfrak{t}}\right], \left[1 - \frac{\breve{\xi}_2}{\mathfrak{t}}, 1 - \ \frac{\breve{\xi}'_2}{\mathfrak{t}}\right] \leq [1, 1]$$

$$\Longleftrightarrow [0.0,\ 0.0] \leq \left[\left(1 - \frac{\breve{\xi}_1}{\mathfrak{t}}\right),\ \left(1 - \frac{\breve{\xi}_2}{\mathfrak{t}}\right)\right], \left[\left(1 - \frac{\breve{\xi}'_1}{\mathfrak{t}}\right), \left(1 - \ \frac{\breve{\xi}'_2}{\mathfrak{t}}\right)\right] \leq [1, 1]$$

$$\Longleftrightarrow 0 \leq \left(1 - \frac{\breve{\xi}_1}{\mathfrak{t}}\right),\ \left(1 - \frac{\breve{\xi}_2}{\mathfrak{t}}\right) \leq 1, 0 \leq \left(1 - \frac{\breve{\xi}'_1}{\mathfrak{t}}\right), \left(1 - \ \frac{\breve{\xi}'_2}{\mathfrak{t}}\right) \leq 1$$

$$\Longleftrightarrow 0 \leq 1 - \left(1 - \frac{\breve{\xi}_1}{\mathfrak{t}}\right)\ \left(1 - \frac{\breve{\xi}_2}{\mathfrak{t}}\right) \leq 1, 0 \leq 1 - \left(1 - \frac{\breve{\xi}'_1}{\mathfrak{t}}\right), \left(1 - \ \frac{\breve{\xi}'_2}{\mathfrak{t}}\right) \leq 1$$

Thus

$$\breve{\xi}_1 + \breve{\xi}_2 - \frac{\breve{\xi}_1\breve{\xi}_2}{\c{t}} + \frac{\tilde{\Psi}_1\tilde{\Psi}_2}{\c{t}} \leq \breve{\xi}_1 + \breve{\xi}_2 - \frac{\breve{\xi}_1\breve{\xi}_2}{\c{t}} + \frac{\left(\c{t}-\breve{\xi}_1\right)}{\c{t}}\frac{\left(\c{t}-\breve{\xi}_2\right)}{\c{t}} = \c{t}$$
$$\Longrightarrow \breve{\xi}_1 + \breve{\xi}_2 - \frac{\breve{\xi}_1\breve{\xi}_2}{\c{t}} + \frac{\tilde{\Psi}_1\tilde{\Psi}_2}{\c{t}} \leq \c{t}$$

Similarly we can prove that

$$\breve{\mu}_1 + \breve{\mu}_2 - \frac{\breve{\mu}_1\breve{\mu}_2}{\c{t}} \leq \c{t}', \breve{\mu}'_1 + \breve{\mu}'_2 - \frac{\breve{\mu}'_1\breve{\mu}'_2}{\c{t}} \leq \c{t}',$$

$$\breve{\nu}_1 + \breve{\nu}_2 - \frac{\breve{\nu}_1\breve{\nu}_2}{\c{t}} \leq \c{t}', \breve{\nu}'_1 + \breve{\nu}'_2 - \frac{\breve{\nu}'_1\breve{\nu}'_2}{\c{t}} \leq \c{t}',$$

and

$$\breve{\mu}'_1 + \breve{\mu}'_2 - \frac{\breve{\mu}'_1\breve{\mu}'_2}{\c{t}'} + \frac{\breve{\nu}'_1\breve{\nu}'_2}{\c{t}'} \leq \c{t}'$$

Thus $\hat{S}_1 \oplus \hat{S}_2$ is GLIVIFSESs.

Also

$$0 \leq \left(1 - \frac{\breve{\xi}}{\c{t}}\right)^{\lambda} \leq 1 \Longrightarrow 0 \leq \left(1 - \left(1 - \frac{\breve{\xi}}{\c{t}}\right)^{\lambda}\right) \leq 1$$
$$\Longleftrightarrow 0 \leq \c{t}\left(1 - \left(1 - \frac{\breve{\xi}}{\c{t}}\right)^{\lambda}\right) \leq 1 \textit{ and } 0 \leq \c{t}\left(1 - \left(1 - \frac{\breve{\xi}'}{\c{t}}\right)^{\lambda}\right) \leq \c{t}$$

Also

$$0 \leq \c{t}\left(\frac{\tilde{\Psi}}{\c{t}}\right)^{\lambda} \leq \c{t} \textit{ and } 0 \leq \c{t}\left(\frac{\tilde{\Psi}'}{\c{t}}\right)^{\lambda} \leq \c{t}$$

Further

$$\mathfrak{t}\left(1-\left(1-\frac{\breve{\xi}}{\mathfrak{t}}\right)^{\lambda}\right)+\mathfrak{t}\left(\frac{\tilde{\Psi}}{\mathfrak{t}}\right)^{\lambda}\leq\mathfrak{t}\left(1-\left(1-\frac{\breve{\xi}}{\mathfrak{t}}\right)^{\lambda}+\left(1-\frac{\breve{\xi}}{\mathfrak{t}}\right)^{\lambda}\right)=\mathfrak{t}$$
$$\Longrightarrow\mathfrak{t}\left(1-\left(1-\frac{\breve{\xi}}{\mathfrak{t}}\right)^{\lambda}\right)+\mathfrak{t}\left(\frac{\tilde{\Psi}}{\mathfrak{t}}\right)^{\lambda}\leq\mathfrak{t}$$

Similarly, we can prove that

$$0\leq\mathfrak{t}'\left(1-\left(1-\frac{\breve{\mu}}{\mathfrak{t}'}\right)^{\lambda}\right)\leq\mathfrak{t},$$

$$0\leq\mathfrak{t}'\left(1-\left(1-\frac{\breve{\mu}'}{\mathfrak{t}'}\right)^{\lambda}\right)\leq\mathfrak{t}',$$

Also

$$0\leq\mathfrak{t}'\left(\frac{\breve{\nu}}{\mathfrak{t}'}\right)^{\lambda}\leq\mathfrak{t}',\ 0\leq\mathfrak{t}'\left(\frac{\breve{\nu}'}{\mathfrak{t}'}\right)^{\lambda}\leq\mathfrak{t}',$$

$$\mathfrak{t}'\left(1-\left(1-\frac{\breve{\mu}}{\mathfrak{t}'}\right)^{\lambda}\right)+\mathfrak{t}'\left(\frac{\breve{\nu}}{\mathfrak{t}'}\right)^{\lambda}\leq\mathfrak{t}'$$

and

$$\mathfrak{t}'\left(1-\left(1-\frac{\breve{\mu}'}{\mathfrak{t}'}\right)^{\lambda}\right)+\mathfrak{t}'\left(\frac{\breve{\nu}'}{\mathfrak{t}'}\right)^{\lambda}\leq\mathfrak{t}'$$

Hence $\lambda\hat{S}$ is a GLIVIFSESs.

$$\Longleftrightarrow 0\leq 1-\left(1-\frac{\breve{\xi}_1}{\mathfrak{t}}-\frac{\breve{\xi}_2}{\mathfrak{t}}+\frac{\breve{\xi}_1\breve{\xi}_2}{\mathfrak{t}^2}\right)\leq 1,$$

$$0\leq 1-\left(1-\frac{\breve{\xi}_1'}{\mathfrak{t}}-\frac{\breve{\xi}_2'}{\mathfrak{t}}+\frac{\breve{\xi}_1'\breve{\xi}_2'}{\mathfrak{t}^2}\right)\leq 1$$

$$\Longleftrightarrow 0\leq\frac{\breve{\xi}_1}{\mathfrak{t}}+\frac{\breve{\xi}_2}{\mathfrak{t}}+\frac{\breve{\xi}_1\breve{\xi}_2}{\mathfrak{t}^2}\leq 1,\ 0\leq\frac{\breve{\xi}_1'}{\mathfrak{t}}-\frac{\breve{\xi}_2'}{\mathfrak{t}}+\frac{\breve{\xi}_1'\breve{\xi}_2'}{\mathfrak{t}^2}\leq 1$$

$$\Longleftrightarrow 0\leq\breve{\xi}_1+\breve{\xi}_2+\frac{\breve{\xi}_1\breve{\xi}_2}{\mathfrak{t}}\leq\mathfrak{t},\ 0\leq\breve{\xi}_1+\breve{\xi}_2-\frac{\breve{\xi}_1\breve{\xi}_2}{\mathfrak{t}^2}\leq\mathfrak{t}$$

Similarly for

$$[0.0,\ 0.0] \leq \left[\tilde{\Psi}_1,\ \tilde{\Psi}_1'\right], \left[\tilde{\Psi}_2,\ \tilde{\Psi}_2'\right] \leq [\text{ţ}, \text{ţ}]$$

$$\Longleftrightarrow [0.0,\ 0.0] \leq \left[\frac{\tilde{\Psi}_1}{\text{ţ}},\ \frac{\tilde{\Psi}_1'}{\text{ţ}}\right], \left[\frac{\tilde{\Psi}_2}{\text{ţ}},\ \frac{\tilde{\Psi}_2'}{\text{ţ}}\right] \leq [1, 1]$$

$$\Longrightarrow [0.0,\ 0.0] \leq \left[\frac{\tilde{\Psi}_1\tilde{\Psi}_2}{\text{ţ}},\ \frac{\tilde{\Psi}_1'\tilde{\Psi}_2'}{\text{ţ}}\right] \leq [1, 1]$$

$$\Longrightarrow [0.0,\ 0.0] \leq \left[\frac{\tilde{\Psi}_1\tilde{\Psi}_2}{\text{ţ}},\ \frac{\tilde{\Psi}_1'\tilde{\Psi}_2'}{\text{ţ}}\right] \leq [\text{ţ}, \text{ţ}]$$

$$\Longleftrightarrow 0 \leq \frac{\tilde{\Psi}_1\tilde{\Psi}_2}{\text{ţ}} \leq \text{ţ}\ ,\ 0 \leq \frac{\tilde{\Psi}_1'\tilde{\Psi}_2'}{\text{ţ}} \leq \text{ţ}$$

Further

$$\breve{\xi}_1 + \breve{\xi}_2 - \frac{\breve{\xi}_1\breve{\xi}_2}{\text{ţ}} + \frac{\tilde{\Psi}_1\tilde{\Psi}_2}{\text{ţ}} \leq \breve{\xi}_1 + \breve{\xi}_2 - \frac{\breve{\xi}_1\breve{\xi}_2}{\text{ţ}} + \frac{\left(\text{ţ} - \breve{\xi}_1\right)\left(\text{ţ} - \breve{\xi}_2\right)}{\text{ţ}} = \text{ţ}$$

and

$$\breve{\xi}_1' + \breve{\xi}_2' - \frac{\breve{\xi}_1'\breve{\xi}_2'}{\text{ţ}} + \frac{\tilde{\Psi}_1'\tilde{\Psi}_2'}{\text{ţ}} \leq \breve{\xi}_1' + \breve{\xi}_2' - \frac{\breve{\xi}_1'\breve{\xi}_2'}{\text{ţ}} + \frac{\left(\text{ţ} - \tilde{\Psi}_1'\right)\left(\text{ţ} - \tilde{\Psi}_2'\right)}{\text{ţ}} = \text{ţ}$$

Thus General Linguistic Interval Valued Intuitionistic Fuzzy Soft Expert Sets GLIVIFSESs $\hat{S},\ \hat{S}_1, \hat{S}_2$ are the sets obtain by definition 12 are still General Linguistic Interval Valued Intuitionistic Fuzzy Soft Expert Sets GLIVIFSESs. ■

Chapter 5

General Linguistic Interval Valued Intuitionistic Fuzzy Soft Expert Aggregation Operators

In this chapter, we will define some arithemetic and geometric aggregational operators which based on proposed operational laws for GLIVIFSESs, for putting together the collection of GLIVIFSESs as

$$\hat{S}_i = \left(\left\langle \dot{S}_{\left[\breve{\xi}_i, \breve{\xi}'_i\right]}, \dot{S}_{[\tilde{\Psi}_i, \tilde{\Psi}'_i]} \right\rangle, \left\langle \ddot{S}_{[\breve{\mu}_i, \breve{\mu}'_i]}, \ddot{S}_{[\breve{\nu}_i, \breve{\nu}'_i]} \right\rangle;\ \text{where } i = 1,\ 2,\ 3..., n\right)$$

Let Υ be the collection of all GLIVIFSESs.

5.1 General Linguistic Interval Valued Intuitionistic Fuzzy Soft Expert Weighted Averaging (GLIVIFSEWA) Operator

GLIVIFSESs . $\hat{S}_i = \left(\left\langle \dot{S}_{\left[\breve{\xi}_i,\breve{\xi}'_i\right]}, \dot{S}_{[\tilde{\Psi}_i,\tilde{\Psi}'_i]}\right\rangle, \left\langle \ddot{S}_{[\breve{\mu}_i,\breve{\mu}'_i]}, \ddot{S}_{[\breve{\nu}_i,\breve{\nu}'_i]}\right\rangle;\ \text{where } (i = 1,\ 2,\ 3...,n)\right)$ be the collection of n General Lingusitic Interval Valued Intuitionistic Fuzzy Soft Expert sets, then the GLIVIFSEWAOs i.e. General Lingusitic Interval Valued Intuitionistic Fuzzy Soft Expert Weighted Averaging Operator of n dimensions is defined as:

A mapping or a function from $\Upsilon^n \to \Upsilon$ such as; GLIVIFSEWA:$\Upsilon^n \to \Upsilon$; that is associated with the weight vector $\tilde{\varpi} = (\tilde{\varpi}_1, \tilde{\varpi}_2, ...\tilde{\varpi}_n)^{\mathfrak{t}}$such that $\tilde{\varpi}_i \in [0,1]$ and $\sum \tilde{\varpi}_i = 1$ then

$$\text{GLIVIFSEWA}(\hat{S}_1, \hat{S}_2.....\hat{S}_n) = \oplus^n \tilde{\varpi}_i S_i$$

Then GLIVIFSEWA is called the Gernalized Lingusitic Interval Valued Intuitionistic Fuzzy Soft Expert WAO, $\varpi = (\varpi_1, \varpi_2, ...\varpi_n)^{\mathfrak{t}}$ is then weighting vector of $\hat{S}_i$ with $\varpi_i \in [0, \mathfrak{t}]$, $\sum \varpi_i = 1$. Particularly if $\varpi = \left(\frac{1}{n}, \frac{1}{n}....\frac{1}{n}\right)^{\mathfrak{t}}$, then the GLIVIFSEWA operator diminished into GLIVIFSEAO Presented as:

$$GLIVIFSEWA\left(\hat{S}_1,\ \hat{S}_2, \hat{S}_3...\hat{S}_n\right) = \frac{1}{n}\left(\overset{n}{\underset{i=1}{\oplus}} \hat{S}_i\right)$$

Theorem 40 *Let*

$$\hat{S}_i = \left(\left\langle \dot{S}_{\left[\breve{\xi}_i\ ,\ \breve{\xi}'_i\right]},\ \dot{S}_{[\tilde{\Psi}_i\ ,\ \tilde{\Psi}'_i]}\right\rangle, \left\langle \ddot{S}_{[\breve{\mu}_i\ ,\ \breve{\mu}'_i]},\ \ddot{S}_{[\breve{\nu}_i\ ,\ \breve{\nu}'_i]}\right\rangle\right), i = 1, 2, 3 - n$$

be a collection of GLIVIFSESs, then GLIVIFWA $\left(\hat{S}_1,\ \hat{S}_2, ...\hat{S}_n\right) =$

$$= \left(\begin{array}{c} \left\langle \begin{array}{c} \dot{S}_{\mathfrak{t}} \left[1-\prod_{i=1}^{n}\left(1-\frac{\check{\xi}_i}{\mathfrak{t}}\right)^{\varpi i}, 1-\prod_{i=1}^{n}\left(1-\frac{\check{\xi}'_i}{\mathfrak{t}}\right)^{\varpi i} \right], \\ \dot{S}_{\mathfrak{t}} \left[\prod_{i=1}^{n}\left(\frac{\tilde{\Psi}_i}{\mathfrak{t}}\right)^{\varpi i}, \prod_{i=1}^{n}\left(\frac{\tilde{\Psi}'_i}{\mathfrak{t}}\right)^{\varpi i} \right] \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \ddot{S}_{\mathfrak{t}'} \left[1-\prod_{i=1}^{n}\left(1-\frac{\breve{\mu}_i}{\mathfrak{t}}\right)^{\varpi i}, 1-\prod_{i=1}^{n}\left(1-\frac{\breve{\mu}'_i}{\mathfrak{t}}\right) \right], \\ \ddot{S}_{\mathfrak{t}'} \left[\prod_{i=1}^{n}\left(\frac{\breve{\nu}_i}{\mathfrak{t}'}\right)^{\varpi i}, \prod_{i=1}^{n}\left(\frac{\breve{\nu}'_i}{\mathfrak{t}'}\right)^{\varpi i} \right] \end{array} \right\rangle \end{array} \right)$$

Proof. By using mathematical induction, we prove the above result, for n

Let n=2, for GLIVIFSESs. from Operational Laws given in definition number 9, we obtain,

$$\varpi_1 \hat{S}_1 = \left(\begin{array}{c} \left\langle \begin{array}{c} \dot{S}_{\mathfrak{t}} \left[1-\left(1-\frac{\check{\xi}_1}{\mathfrak{t}}\right)^{\varpi_1}, 1-\left(1-\frac{\check{\xi}'_1}{\mathfrak{t}}\right)^{\varpi_1} \right], \\ \dot{S}_{\mathfrak{t}} \left[\left(\frac{\tilde{\Psi}_1}{\mathfrak{t}}\right)^{\varpi_1}, \left(\frac{\tilde{\Psi}'_1}{\mathfrak{t}}\right)^{\varpi_1} \right] \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \ddot{S}_{\mathfrak{t}'} \left[1-\left(1-\frac{\breve{\mu}_1}{\mathfrak{t}'}\right)^{\varpi_1}, \left(1-\frac{\breve{\mu}'_1}{\mathfrak{t}'}\right) \right], \\ \ddot{S}_{\mathfrak{t}'} \left[\left(\frac{\breve{\nu}_1}{\mathfrak{t}'}\right)^{\varpi_1}, \left(\frac{\breve{\nu}'_1}{\mathfrak{t}'}\right)^{\varpi_1} \right] \end{array} \right\rangle \end{array} \right)$$

$$\varpi_2 \hat{S}_2 = \left(\begin{array}{c} \left\langle \begin{array}{c} \dot{S}_{\mathfrak{t}} \left[1-\left(1-\frac{\check{\xi}_2}{\mathfrak{t}}\right)^{\varpi_2}, 1-\left(1-\frac{\check{\xi}'_2}{\mathfrak{t}}\right)^{\varpi_2} \right], \\ \dot{S}_{\mathfrak{t}} \left[\left(\frac{\tilde{\Psi}_2}{\mathfrak{t}}\right)^{\varpi_2}, \left(\frac{\tilde{\Psi}'_2}{\mathfrak{t}}\right)^{\varpi_2} \right] \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \ddot{S}_{\mathfrak{t}'} \left[1-\left(1-\frac{\breve{\mu}_2}{\mathfrak{t}'}\right)^{\varpi_2}, \left(1-\frac{\breve{\mu}'_2}{\mathfrak{t}'}\right) \right], \\ \ddot{S}_{\mathfrak{t}'} \left[\left(\frac{\breve{\nu}_2}{\mathfrak{t}'}\right)^{\varpi_2}, \left(\frac{\breve{\nu}'_2}{\mathfrak{t}'}\right)^{\varpi_2} \right] \end{array} \right\rangle \end{array} \right)$$

and Hence GLIVIFSEWA $\left(\hat{S}_1, \hat{S}_2\right) = \varpi_1 \hat{S}_1 \oplus \varpi_2 \hat{S}_2$

$$
= \left(\begin{array}{c}
\left\langle \dot{S}\left[\begin{array}{c}
\text{ţ}\left(1-\left(1-\frac{\breve{\xi}_1}{\text{ţ}}\right)^{\varpi_1}\right)+\text{ţ}\left(1-\left(1-\frac{\breve{\xi}_2}{\text{ţ}}\right)^{\varpi_2}\right) \\
-\frac{\text{ţ}\left(1-\left(1-\frac{\breve{\xi}_1}{\text{ţ}}\right)^{\varpi_1}\right)\text{ţ}\left(1-\left(1-\frac{\breve{\xi}_2}{\text{ţ}}\right)^{\varpi_2}\right)}{\text{ţ}}, \\
\text{ţ}\left(1-\left(1-\frac{\breve{\xi}'_1}{\text{ţ}}\right)^{\varpi_1}\right)+\text{ţ}\left(1-\left(1-\frac{\breve{\xi}'_2}{\text{ţ}}\right)^{\varpi_2}\right) \\
\frac{-\text{ţ}\left(1-\left(1-\frac{\breve{\xi}'_1}{\text{ţ}}\right)^{\varpi_1}\right)+\text{ţ}\left(1-\left(1-\frac{\breve{\xi}'_2}{\text{ţ}}\right)^{\varpi_2}\right)}{\text{ţ}}
\end{array} \right], \right\rangle, \\
\dot{S}_{\text{ţ}}\left[\frac{\text{ţ}\left(\frac{\tilde{\Psi}_1}{\text{ţ}}\right)^{\varpi_1}\text{ţ}\left(\frac{\tilde{\Psi}_2}{\text{ţ}}\right)^{\varpi_2}}{\text{ţ}},\frac{\text{ţ}^2\left(\frac{\tilde{\Psi}'_1}{\text{ţ}}\right)\left(\frac{\tilde{\Psi}'_2}{\text{ţ}}\right)}{\text{ţ}}\right] \\
\left\langle \ddot{S}\left[\begin{array}{c}
\text{ţ}'\left(1-\left(1-\frac{\breve{\mu}_1}{\text{ţ}'}\right)^{\varpi_1}\right)+\text{ţ}'\left(1-\left(1-\frac{\breve{\mu}_2}{\text{ţ}'}\right)^{\varpi_2}\right) \\
-\frac{\text{ţ}'^2\left(1-\left(1-\frac{\breve{\mu}_1}{\text{ţ}'}\right)^{\varpi_1}\right)\text{ţ}\left(1-\left(1-\frac{\breve{\mu}_2}{\text{ţ}'}\right)^{\varpi_2}\right)}{\text{ţ}'}, \\
\text{ţ}'\left(1-\left(1-\frac{\breve{\mu}'_1}{\text{ţ}'}\right)^{\varpi_1}\right)+\text{ţ}'\left(1-\left(1-\frac{\breve{\mu}'_2}{\text{ţ}'}\right)^{\varpi_2}\right) \\
-\frac{-\text{ţ}'\left(1-\left(1-\frac{\breve{\mu}'_1}{\text{ţ}'}\right)^{\varpi_1}\right)+\text{ţ}\left(1-\left(1-\frac{\breve{\mu}'_2}{\text{ţ}'}\right)^{\varpi_2}\right)}{\text{ţ}'}
\end{array} \right], \right\rangle \\
\ddot{S}\left[\frac{\text{ţ}'\left(\frac{\breve{\nu}_1}{\text{ţ}'}\right)^{\varpi_1}\text{ţ}\left(\frac{\breve{\nu}_2}{\text{ţ}'}\right)^{\varpi_2}}{\text{ţ}'},\frac{\text{ţ}'^2\left(\frac{\breve{\nu}'_1}{\text{ţ}'}\right)\left(\frac{\breve{\nu}'_2}{\text{ţ}'}\right)}{\text{ţ}'}\right]
\end{array} \right)
$$

$$= \begin{pmatrix} \left\langle \dot{S}_{\mathfrak{t}}\begin{bmatrix} 1-\left(1-\frac{\breve{\xi}_1}{\mathfrak{t}}\right)^{\varpi_1}+1-\left(1-\frac{\breve{\xi}_2}{\mathfrak{t}}\right)^{\varpi_2} \\ -\left(1-\left(1-\frac{\breve{\xi}_1}{\mathfrak{t}}\right)^{\varpi_1}\right)\left(1-\left(1-\frac{\breve{\xi}_2}{\mathfrak{t}}\right)^{\varpi_2}\right), \\ 1-\left(1-\frac{\breve{\xi}'_1}{\mathfrak{t}}\right)^{\varpi_1}+1-\left(1-\frac{\breve{\xi}'_2}{\mathfrak{t}}\right)^{\varpi_2} \\ -\left(1-\left(1-\frac{\breve{\xi}'_1}{\mathfrak{t}}\right)^{\varpi_1}\right)\left(1-\left(1-\frac{\breve{\xi}'_2}{\mathfrak{t}}\right)^{\varpi_2}\right) \end{bmatrix}, \right\rangle, \\ \dot{S}_{\mathfrak{t}\left[\left(\frac{\tilde{\Psi}_1}{\mathfrak{t}}\right)^{\varpi_1}\cdot\left(\frac{\tilde{\Psi}_2}{\mathfrak{t}}\right)^{\varpi_2},\left(\frac{\tilde{\Psi}'_1}{\mathfrak{t}}\right)\left(\frac{\tilde{\Psi}'_2}{\mathfrak{t}}\right)\right]} \\ \left\langle \ddot{S}_{\mathfrak{t}'}\begin{bmatrix} 1-\left(1-\frac{\breve{\mu}_1}{\mathfrak{t}'}\right)^{\varpi_1}+1-\left(1-\frac{\breve{\mu}_2}{\mathfrak{t}'}\right)^{\varpi_2} \\ -\left(1-\left(1-\frac{\breve{\mu}_1}{\mathfrak{t}'}\right)^{\varpi_1}\right)\left(1-\left(1-\frac{\breve{\mu}_2}{\mathfrak{t}'}\right)^{\varpi_2}\right), \\ 1-\left(1-\frac{\breve{\mu}'_1}{\mathfrak{t}'}\right)^{\varpi_1}+1-\left(1-\frac{\breve{\mu}'_2}{\mathfrak{t}'}\right)^{\varpi_2} \\ -\left(1-\left(1-\frac{\breve{\mu}'_1}{\mathfrak{t}'}\right)^{\varpi_1}\right)\left(1-\left(1-\frac{\breve{\mu}'_2}{\mathfrak{t}'}\right)^{\varpi_2}\right) \end{bmatrix}, \right\rangle \\ \ddot{S}_{\mathfrak{t}\left[\left(\frac{\breve{\nu}_1}{\mathfrak{t}'}\right)^{\varpi_1}\cdot\left(\frac{\breve{\nu}_2}{\mathfrak{t}'}\right)^{\varpi_2},\left(\frac{\breve{\nu}'_1}{\mathfrak{t}'}\right)\left(\frac{\breve{\nu}'_2}{\mathfrak{t}'}\right)\right]} \end{pmatrix}$$

$$
= \left(\begin{array}{c}
\left\langle \begin{array}{c}
\dot{S}_{\text{ţ}} \left[\begin{array}{c}
2 - \left(1 - \frac{\breve{\xi}_1}{\text{ţ}}\right)^{\varpi_1} - \left(1 - \frac{\breve{\xi}_2}{\text{ţ}}\right)^{\varpi_2} - \\
\left(1 - \left(1 - \frac{\breve{\xi}_1}{\text{ţ}}\right)^{\varpi_1} - \left(1 - \frac{\breve{\xi}_2}{\text{ţ}}\right)^{\varpi_2} + \left(1 - \frac{\breve{\xi}_1}{\text{ţ}}\right)^{\varpi_1} \left(1 - \frac{\breve{\xi}_2}{\text{ţ}}\right)^{\varpi_2}\right), \\
2 - \left(1 - \frac{\breve{\xi}'_1}{\text{ţ}}\right)^{\varpi_1} - \left(1 - \frac{\breve{\xi}'_2}{\text{ţ}}\right)^{\varpi_2} - \\
\left(1 - \left(1 - \frac{\breve{\xi}'_1}{\text{ţ}}\right)^{\varpi_1} - \left(1 - \frac{\breve{\xi}'_2}{\text{ţ}}\right)^{\varpi_2} + \left(1 - \frac{\breve{\xi}_1}{\text{ţ}}\right)^{\varpi_1} \left(1 - \frac{\breve{\xi}_2}{\text{ţ}}\right)^{\varpi_2}\right)
\end{array} \right] \\
\dot{S}_{\text{ţ}\left[\overset{2}{\underset{i=1}{\pi}} \left(\frac{\tilde{\Psi}_i}{\text{ţ}}\right)^{\varpi_i}, \overset{2}{\underset{i=1}{\pi}} \left(\frac{\tilde{\Psi}'_i}{\text{ţ}}\right)^{\varpi_i} \right]}
\end{array} \right\rangle, \\
\left\langle \begin{array}{c}
\ddot{S}_{\text{ţ}'} \left[\begin{array}{c}
2 - \left(1 - \frac{\breve{\mu}_1}{\text{ţ}}\right)^{\varpi_1} - \left(1 - \frac{\breve{\mu}_2}{\text{ţ}}\right)^{\varpi_2} - \\
\left(1 - \left(1 - \frac{\breve{\mu}_1}{\text{ţ}}\right)^{\varpi_1} - \left(1 - \frac{\breve{\mu}_2}{\text{ţ}}\right)^{\varpi_2} + \left(1 - \frac{\breve{\mu}_1}{\text{ţ}}\right)^{\varpi_1} \left(1 - \frac{\breve{\mu}_2}{\text{ţ}}\right)^{\varpi_2}\right), \\
2 - \left(1 - \frac{\breve{\mu}'_1}{\text{ţ}}\right)^{\varpi_1} - \left(1 - \frac{\breve{\mu}'_2}{\text{ţ}}\right)^{\varpi_2} - \\
\left(1 - \left(1 - \frac{\breve{\mu}'_1}{\text{ţ}}\right)^{\varpi_1} - \left(1 - \frac{\breve{\mu}'_2}{\text{ţ}}\right)^{\varpi_2} + \left(1 - \frac{\breve{\mu}_1}{\text{ţ}}\right)^{\varpi_1} \left(1 - \frac{\breve{\mu}_2}{\text{ţ}}\right)^{\varpi_2}\right)
\end{array} \right] \\
\ddot{S}_{\text{ţ}'\left[\overset{2}{\underset{i=1}{\pi}} \left(\frac{\breve{\nu}_i}{\text{ţ}}\right)^{\varpi_i}, \overset{2}{\underset{i=1}{\pi}} \left(\frac{\breve{\nu}'_i}{\text{ţ}}\right)^{\varpi_i} \right]}
\end{array} \right\rangle
\end{array} \right)
$$

$$= \left(\begin{array}{c} \left\langle \begin{array}{c} \dot{S}_{\mathfrak{t}} \left[\begin{array}{c} 2-1-\left(1-\frac{\breve{\xi}_1}{\mathfrak{t}}\right)^{\varpi_1}+\left(1-\frac{\breve{\xi}_1}{\mathfrak{t}}\right)^{\varpi_1} \\ -\left(1-\frac{\breve{\xi}_2}{\mathfrak{t}}\right)^{\varpi_2}+\left(1-\frac{\breve{\xi}_2}{\mathfrak{t}}\right)^{\varpi_2} \\ -\left(1-\frac{\breve{\xi}_1}{\mathfrak{t}}\right)^{\varpi_1}\left(1-\frac{\breve{\xi}_2}{\mathfrak{t}}\right)^{\varpi_2}, \\ 2-1-\left(1-\frac{\breve{\xi}'_1}{\mathfrak{t}}\right)^{\varpi_1}+\left(1-\frac{\breve{\xi}'_1}{\mathfrak{t}}\right) \\ -\left(1-\frac{\breve{\xi}'_2}{\mathfrak{t}}\right)^{\varpi_2}+\left(1-\frac{\breve{\xi}'_2}{\mathfrak{t}}\right)^{\varpi_2}- \\ \left(1-\frac{\breve{\xi}'_1}{\mathfrak{t}}\right)^{\varpi_1}\left(1-\frac{\breve{\xi}'_2}{\mathfrak{t}}\right)^{\varpi_2} \end{array} \right], \\ \dot{S}_{\mathfrak{t}\left[\prod_{i=1}^{2}\left(\frac{\tilde{\Psi}_i}{\mathfrak{t}}\right)^{\varpi_i},\prod_{i=1}^{2}\left(\frac{\tilde{\Psi}_i}{\mathfrak{t}}\right)^{\varpi_i}\right]} \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \ddot{S}_{\mathfrak{t}'} \left[\begin{array}{c} 2-1-\left(1-\frac{\breve{\mu}_1}{\mathfrak{t}}\right)^{\varpi_1}+\left(1-\frac{\breve{\mu}_1}{\mathfrak{t}}\right)^{\varpi_1} \\ -\left(1-\frac{\breve{\mu}_2}{\mathfrak{t}}\right)^{\varpi_2}+\left(1-\frac{\breve{\mu}_2}{\mathfrak{t}}\right)^{\varpi_2} \\ -\left(1-\frac{\breve{\mu}_1}{\mathfrak{t}}\right)^{\varpi_1}\left(1-\frac{\breve{\mu}_2}{\mathfrak{t}}\right)^{\varpi_2}, \\ 2-1-\left(1-\frac{\breve{\mu}'_1}{\mathfrak{t}}\right)^{\varpi_1}+\left(1-\frac{\breve{\mu}'_1}{\mathfrak{t}}\right)- \\ \left(1-\frac{\breve{\mu}'_2}{\mathfrak{t}}\right)^{\varpi_2}+\left(1-\frac{\breve{\mu}'_2}{\mathfrak{t}}\right)^{\varpi_2} \\ -\left(1-\frac{\breve{\mu}'_1}{\mathfrak{t}}\right)^{\varpi_1}\left(1-\frac{\breve{\mu}'_2}{\mathfrak{t}}\right)^{\varpi_2} \end{array} \right], \\ \ddot{S}_{\mathfrak{t}'\left[\prod_{i=1}^{2}\left(\frac{\breve{\nu}_i}{\mathfrak{t}}\right)^{\varpi_i},\prod_{i=1}^{2}\left(\frac{\breve{\nu}_i}{\mathfrak{t}}\right)^{\varpi_i}\right]} \end{array} \right\rangle \end{array} \right)$$

$$= \left(\begin{array}{c} \left\langle \dot{S}_{\mathfrak{t}\left[1-\prod_{i=1}^{2}\left(1-\frac{\breve{\xi}_i}{\mathfrak{t}}\right)^{\varpi i},1-\prod_{i=1}^{2}\left(1-\frac{\breve{\xi}'_i}{\mathfrak{t}}\right)^{\varpi i}\right]}, \dot{S}_{\mathfrak{t}\left[\prod_{i=1}^{2}\left(\frac{\tilde{\Psi}_i}{\mathfrak{t}}\right)^{\varpi i},\prod_{i=1}^{2}\left(\frac{\tilde{\Psi}'_i}{\mathfrak{t}}\right)^{\varpi i}\right]} \right\rangle, \\ \left\langle \ddot{S}_{\mathfrak{t}'\left[1-\prod_{i=1}^{2}\left(1-\frac{\breve{\mu}_i}{\mathfrak{t}}\right)^{\varpi i},1-\prod_{i=1}^{2}\left(1-\frac{\breve{\mu}'_i}{\mathfrak{t}}\right)\right]}, \ddot{S}_{\mathfrak{t}'\left[\prod_{i=1}^{2}\left(\frac{\breve{\nu}_i}{\mathfrak{t}'}\right)^{\varpi i},\prod_{i=1}^{2}\left(\frac{\breve{\nu}'_i}{\mathfrak{t}'}\right)^{\varpi i}\right]} \right\rangle \end{array} \right)$$

Thus

$$GLIVIFSEWA\left(\hat{S}_1,\hat{S}_2\right)=\left(\begin{array}{c}\left\langle \dot{S}_{\mathfrak{t}}\left[1-\underset{i=1}{\overset{2}{\pi}}\left(1-\frac{\check{\xi}'_i}{\mathfrak{t}}\right)^{\varpi i},1-\underset{i=1}{\overset{2}{\pi}}\left(1-\frac{\check{\xi}'_i}{\mathfrak{t}}\right)^{\varpi i}\right], \dot{S}_{\mathfrak{t}}\left[\underset{i=1}{\overset{2}{\pi}}\left(\frac{\tilde{\Psi}'_i}{\mathfrak{t}}\right)^{\varpi i},\underset{i=1}{\overset{2}{\pi}}\left(\frac{\tilde{\Psi}'_i}{\mathfrak{t}}\right)^{\varpi i}\right]\right\rangle, \\ \left\langle \ddot{S}_{\mathfrak{t}'}\left[1-\underset{i=1}{\overset{2}{\pi}}\left(1-\frac{\breve{\mu}'_i}{\mathfrak{t}'}\right)^{\varpi i},1-\underset{i=1}{\overset{2}{\pi}}\left(1-\frac{\breve{\mu}'_i}{\mathfrak{t}'}\right)^{\varpi i}\right], \ddot{S}_{\mathfrak{t}}\left[\underset{i=1}{\overset{2}{\pi}}\left(\frac{\breve{\nu}'_i}{\mathfrak{t}'}\right)^{\varpi i},\underset{i=1}{\overset{2}{\pi}}\left(\frac{\breve{\nu}'_i}{\mathfrak{t}'}\right)^{\varpi i}\right]\right\rangle\end{array}\right)$$

It is clearly seen that result exist for n = 2

Further, we assume that equation # is valid for n=k, i.e

GLIVIFSEWA $(\hat{S}_1,\hat{S}_2,...\hat{S}_k)$

$$=\left(\begin{array}{c}\left\langle\begin{array}{c}\dot{S}_{\mathfrak{t}}\left[1-\prod_{i=1}^{k}\left(1-\frac{\check{\xi}_i}{\mathfrak{t}}\right)^{\varpi i},1-\prod_{i=1}^{k}\left(1-\frac{\check{\xi}'_i}{\mathfrak{t}}\right)^{\varpi i}\right], \\ \dot{S}_{\mathfrak{t}}\left[\prod_{i=1}^{k}\left(\frac{\tilde{\Psi}_i}{\mathfrak{t}}\right)^{\varpi i},\prod_{i=1}^{k}\left(\frac{\tilde{\Psi}'_i}{\mathfrak{t}}\right)^{\varpi i}\right]\end{array}\right\rangle, \\ \left\langle\begin{array}{c}\ddot{S}_{\mathfrak{t}'}\left[1-\prod_{i=1}^{k}\left(1-\frac{\breve{\mu}_i}{\mathfrak{t}'}\right)^{\varpi i},\left(1-\prod_{i=1}^{k}\left(1-\frac{\breve{\mu}'_i}{\mathfrak{t}'}\right)^{\varpi i}\right)\right], \\ \ddot{S}_{\mathfrak{t}'}\left[\prod_{i=1}^{k}\left(\frac{\breve{\nu}_i}{\mathfrak{t}'}\right)^{\varpi i},\prod_{i=1}^{k}\left(\frac{\breve{\nu}'_i}{\mathfrak{t}'}\right)^{\varpi i}\right]\end{array}\right\rangle\end{array}\right)$$

when n = k +1, according to the 12 and further operations, we get

GLIFIFSEWA$\left(\hat{S}_1,\ \hat{S}_2,....\ ,\hat{S}_k,\hat{S}_{k+1}\right)=\ GLIVIFSEWA\ \left(\hat{S}_1,\hat{S}_2,....\hat{S}_k\right)\oplus\varpi_{k+1}\hat{S}_{k+1}$

$$=\left(\begin{array}{c}\left\langle\begin{array}{c}\dot{S}_{\mathfrak{t}}\left[1-\prod_{i=1}^{k}\left(1-\frac{\check{\xi}_i}{\mathfrak{t}}\right)^{\varpi i},1-\prod_{i=1}^{k}\left(1-\frac{\check{\xi}'_i}{\mathfrak{t}}\right)^{\varpi i}\right], \\ \dot{S}_{\mathfrak{t}}\left[\prod_{i=1}^{k}\left(\frac{\tilde{\Psi}_i}{\mathfrak{t}}\right)^{\varpi i},\prod_{i=1}^{k}\left(\frac{\tilde{\Psi}'_i}{\mathfrak{t}}\right)^{\varpi i}\right]\end{array}\right\rangle, \\ \left\langle\begin{array}{c}\ddot{S}_{\mathfrak{t}'}\left[1-\prod_{i=1}^{k}\left(1-\frac{\breve{\mu}_i}{\mathfrak{t}'}\right)^{\varpi i},1-\prod_{i=1}^{k}\left(1-\frac{\breve{\mu}'_i}{\mathfrak{t}'}\right)\right], \\ \ddot{S}_{\mathfrak{t}'}\left[\prod_{i=1}^{k}\left(\frac{\breve{\nu}_i}{\mathfrak{t}'}\right)^{\varpi i},\prod_{i=1}^{k}\left(\frac{\breve{\nu}'_i}{\mathfrak{t}'}\right)^{\varpi i}\right]\end{array}\right\rangle\end{array}\right)$$

$$\oplus\left(\begin{array}{c}\left\langle\begin{array}{c}\dot{S}_{\mathfrak{t}}\left[1-\left(1-\frac{\breve{\xi}_{k+1}}{\mathfrak{t}}\right)^{\varpi k+1},1-\left(1-\frac{\breve{\xi}'_{k+1}}{\mathfrak{t}}\right)^{\varpi k+1}\right],\\ \dot{S}_{\mathfrak{t}}\left[\left(\frac{\tilde{\Psi}_{k+1}}{\mathfrak{t}}\right)^{\varpi k+1},\prod_{i=1}^{k}\left(\frac{\tilde{\Psi}'_{k+1}}{\mathfrak{t}}\right)^{\varpi k+1}\right]\end{array}\right\rangle,\\ \left\langle\begin{array}{c}\ddot{S}_{\mathfrak{t}'}\left[1-\left(1-\frac{\breve{\mu}_{k+1}}{\mathfrak{t}'}\right)^{\varpi k+1},1-\left(1-\frac{\breve{\mu}'_{k+1}}{\mathfrak{t}'}\right)^{\varpi k+1}\right],\\ \ddot{S}_{\mathfrak{t}'}\left[\left(\frac{\breve{\nu}_{k+1}}{\mathfrak{t}'}\right)^{\varpi k+1},\left(\frac{\breve{\nu}'_{k+1}}{\mathfrak{t}'}\right)^{\varpi k+1}\right]\end{array}\right\rangle\end{array}\right)$$

$$=\left(\begin{array}{c}\left\langle\begin{array}{c}\dot{S}\left[\begin{array}{c}\mathfrak{t}\left(1-\prod_{i=1}^{k}\left(1-\frac{\breve{\xi}_i}{\mathfrak{t}}\right)^{\varpi i}\right)+\mathfrak{t}\left(1-\left(1-\frac{\breve{\xi}_{k+1}}{\mathfrak{t}}\right)^{\varpi k+1}\right)\\ -\frac{\mathfrak{t}^2\left(1-\prod_{i=1}^{k}\left(1-\frac{\breve{\xi}_i}{\mathfrak{t}}\right)^{\varpi i}\right)\left(1-\left(1-\frac{\breve{\xi}_{k+1}}{\mathfrak{t}}\right)^{\varpi k+1}\right)}{\mathfrak{t}},\\ \mathfrak{t}\left(1-\prod_{i=1}^{k}\left(1-\frac{\breve{\xi}'_i}{\mathfrak{t}}\right)^{\varpi i}\right)+\mathfrak{t}\left(1-\left(1-\frac{\breve{\xi}'_{k+1}}{\mathfrak{t}}\right)^{\varpi k+1}\right)\\ -\mathfrak{t}^2\left(1-\prod_{i=1}^{k}\left(1-\frac{\breve{\xi}'_i}{\mathfrak{t}}\right)^{\varpi i}\right)1-\left(1-\frac{\breve{\xi}_{k+1}}{\mathfrak{t}}\right)^{\varpi k+1}\end{array}\right],\\ \dot{S}\left[\mathfrak{t}\left(-\prod_{i=1}^{k}\left(\frac{\tilde{\Psi}_i}{\mathfrak{t}}\right)^{\varpi i}\left(\frac{\tilde{\Psi}_{k+1}}{\mathfrak{t}}\right)^{\varpi k+1},\prod_{i=1}^{k}\left(\frac{\tilde{\Psi}'_i}{\mathfrak{t}}\right)^{\varpi i}\left(\frac{\tilde{\Psi}'_{k+1}}{\mathfrak{t}}\right)^{\varpi k+1}\right)\right]\end{array}\right\rangle\\ \left\langle\begin{array}{c}\ddot{S}\left[\begin{array}{c}\mathfrak{t}'\left(1-\prod_{i=1}^{k}\left(1-\frac{\breve{\mu}_i}{\mathfrak{t}}\right)^{\varpi i}\right)+\mathfrak{t}\left(1-\left(1-\frac{\breve{\mu}_{k+1}}{\mathfrak{t}}\right)^{\varpi k+1}\right)\\ -\frac{\mathfrak{t}^2\left(1-\prod_{i=1}^{k}\left(1-\frac{\breve{\mu}_i}{\mathfrak{t}}\right)^{\varpi i}\right)\left(1-\left(1-\frac{\breve{\mu}_{k+1}}{\mathfrak{t}}\right)^{\varpi k+1}\right)}{\mathfrak{t}},\\ \mathfrak{t}\left(1-\prod_{i=1}^{k}\left(1-\frac{\breve{\mu}'_i}{\mathfrak{t}}\right)^{\varpi i}\right)+\mathfrak{t}\left(1-\left(1-\frac{\breve{\mu}'_{k+1}}{\mathfrak{t}}\right)^{\varpi k+1}\right)\\ -\mathfrak{t}^2\left(1-\prod_{i=1}^{k}\left(1-\frac{\breve{\mu}'_i}{\mathfrak{t}}\right)^{\varpi i}\right)1-\left(1-\frac{\breve{\mu}_{k+1}}{\mathfrak{t}}\right)^{\varpi k+1}\end{array}\right],\\ \ddot{S}\left[\mathfrak{t}'\left(-\prod_{i=1}^{k}\left(\frac{\breve{\nu}_i}{\mathfrak{t}}\right)^{\varpi i}\left(\frac{\breve{\nu}_{k+1}}{\mathfrak{t}}\right)^{\varpi k+1},\prod_{i=1}^{k}\left(\frac{\breve{\nu}'_i}{\mathfrak{t}}\right)^{\varpi i}\left(\frac{\breve{\nu}'_{k+1}}{\mathfrak{t}}\right)^{\varpi k+1}\right)\right]\end{array}\right\rangle\end{array}\right)$$

$$
= \left(\begin{array}{c}
\left\langle \dot{S}_{\mathfrak{t}} \left[\begin{array}{c}
1 - \prod_{i=1}^{k} \left(1 - \frac{\breve{\xi}_i}{\mathfrak{t}}\right)^{\varpi i} + 1 - \left(1 - \frac{\breve{\xi}_{k+1}}{\mathfrak{t}}\right)^{\varpi_{k+1}} - 1 \\
+ \prod_{i=1}^{k} \left(1 - \frac{\breve{\xi}_i}{\mathfrak{t}}\right)^{\varpi i} + \left(1 - \frac{\breve{\xi}_{k+1}}{\mathfrak{t}}\right)^{\varpi_{k+1}} \\
- \prod_{i=1}^{k} \left(1 - \frac{\breve{\xi}_i}{\mathfrak{t}}\right)^{\varpi i} \left(1 - \frac{\breve{\xi}_{k+1}}{\mathfrak{t}}\right)^{\varpi_{k+1}}, \\
1 - \prod_{i=1}^{k} \left(1 - \frac{\breve{\xi}'_i}{\mathfrak{t}}\right)^{\varpi i} + 1 - \left(1 - \frac{\breve{\xi}'_{k+1}}{\mathfrak{t}}\right)^{\varpi_{k+1}} \\
-1 + \prod_{i=1}^{k} \left(1 - \frac{\breve{\xi}'_i}{\mathfrak{t}}\right)^{\varpi i} + \left(1 - \frac{\breve{\xi}'_{k+1}}{\mathfrak{t}}\right)^{\varpi_{k+1}} \\
- \prod_{i=1}^{k} \left(1 - \frac{\breve{\xi}'_i}{\mathfrak{t}}\right) \left(1 - \frac{\breve{\xi}_{k+1}}{\mathfrak{t}}\right)^{\varpi_{k+1}}
\end{array} \right] \right\rangle, \\
\dot{S}_{\mathfrak{t}} \left[\prod_{i=1}^{k} \left(\frac{\tilde{\Psi}_i}{\mathfrak{t}}\right), \prod_{i=1}^{k} \left(\frac{\tilde{\Psi}'_i}{\mathfrak{t}}\right)^{\varpi i} \right]^{\tilde{\Psi}} \\
\ddot{S}_{\mathfrak{t}} \left[\begin{array}{c}
1 - \prod_{i=1}^{k} \left(1 - \frac{\breve{\mu}_i}{\mathfrak{t}}\right)^{\varpi i} + 1 - \left(1 - \frac{\breve{\mu}_{k+1}}{\mathfrak{t}}\right)^{\varpi_{k+1}} - 1 + \\
\prod_{i=1}^{k} \left(1 - \frac{\breve{\mu}_i}{\mathfrak{t}}\right)^{\varpi i} + \left(1 - \frac{\breve{\mu}_{k+1}}{\mathfrak{t}}\right)^{\varpi_{k+1}} - \prod_{i=1}^{k} \left(1 - \frac{\breve{\mu}_i}{\mathfrak{t}}\right)^{\varpi i} \left(1 - \frac{\breve{\mu}_{k+1}}{\mathfrak{t}}\right)^{\varpi_{k+1}}, \\
1 - \prod_{i=1}^{k} \left(1 - \frac{\breve{\mu}'_i}{\mathfrak{t}}\right)^{\varpi i} + 1 - \left(1 - \frac{\breve{\mu}'_{k+1}}{\mathfrak{t}}\right)^{\varpi_{k+1}} - 1 \\
+ \prod_{i=1}^{k} \left(1 - \frac{\breve{\mu}'_i}{\mathfrak{t}}\right)^{\varpi i} + \left(1 - \frac{\breve{\mu}'_{k+1}}{\mathfrak{t}}\right)^{\varpi_{k+1}} - \prod_{i=1}^{k} \left(1 - \frac{\breve{\mu}'_i}{\mathfrak{t}}\right) \left(1 - \frac{\breve{\mu}_{k+1}}{\mathfrak{t}}\right)^{\varpi_{k+1}}
\end{array} \right], \\
\dot{S}_{\mathfrak{t}} \left[\prod_{i=1}^{k} \left(\frac{\breve{\nu}_i}{\mathfrak{t}}\right), \prod_{i=1}^{k} \left(\frac{\breve{\nu}'_i}{\mathfrak{t}}\right)^{\varpi i} \right]^{\tilde{\Psi}}
\end{array} \right)
$$

$$
= \left(\begin{array}{c}
\left\langle \dot{S}_{\mathfrak{t}} \left[1 - \prod_{i=1}^{k} \left(1 - \frac{\breve{\xi}_i}{\mathfrak{t}}\right)^{\varpi i} \left(1 - \frac{\breve{\xi}_{k+1}}{\mathfrak{t}}\right)^{\varpi_{k+1}}, 1 - \prod_{i=1}^{k} \left(1 - \frac{\breve{\xi}'_i}{\mathfrak{t}}\right)^{\varpi i} \left(1 - \frac{\breve{\xi}_{k+1}}{\mathfrak{t}}\right)^{\varpi_{k+1}} \right], \dot{S}_{\mathfrak{t}} \, 1 - \prod_{i=1}^{k} \left(\frac{\tilde{\Psi}_i}{\mathfrak{t}}\right)^{\varpi i}, \ \prod_{i=1}^{k} \left(\frac{\tilde{\Psi}'_i}{\mathfrak{t}}\right)^{\varpi i} \right\rangle, \\
\left\langle \ddot{S}_{\mathfrak{t}} \left[1 - \prod_{i=1}^{k} \left(1 - \frac{\breve{\mu}_i}{\mathfrak{t}}\right)^{\varpi i} \left(1 - \frac{\breve{\mu}_{k+1}}{\mathfrak{t}}\right)^{\varpi_{k+1}}, 1 - \prod_{i=1}^{k} \left(1 - \frac{\breve{\mu}'_i}{\mathfrak{t}}\right)^{\varpi i} \left(1 - \frac{\breve{\mu}_{k+1}}{\mathfrak{t}}\right)^{\varpi_{k+1}} \right], \ddot{S}_{\mathfrak{t}} \, 1 - \prod_{i=1}^{k} \left(\frac{\breve{\nu}_i}{\mathfrak{t}}\right)^{\varpi i}, \ \prod_{i=1}^{k} \left(\frac{\breve{\nu}'_i}{\mathfrak{t}}\right)^{\varpi i} \right\rangle
\end{array} \right)
$$

$$
= \left(\begin{array}{c}
\left\langle \dot{S}_{\mathfrak{t}} \left[1 - \prod_{i=1}^{k} \left(1 - \frac{\breve{\xi}_i}{\mathfrak{t}}\right)^{\varpi i}, \ 1 - \prod_{i=1}^{k} \left(1 - \frac{\breve{\xi}'_i}{\mathfrak{t}}\right)^{\varpi i} \right], \dot{S}_{\mathfrak{t}} \left[\prod_{i=1}^{k} \left(\frac{\tilde{\Psi}_i}{\mathfrak{t}}\right)^{\varpi i}, \ \prod_{i=1}^{k} \left(\frac{\tilde{\Psi}'_i}{\mathfrak{t}}\right)^{\varpi i} \right] \right\rangle, \\
\left\langle \ddot{S}_{\mathfrak{t}} \left[1 - \prod_{i=1}^{k} \left(1 - \frac{\breve{\mu}_i}{\mathfrak{t}}\right)^{\varpi i}, \ 1 - \prod_{i=1}^{k} \left(1 - \frac{\breve{\mu}'_i}{\mathfrak{t}}\right)^{\varpi i} \right], \ddot{S}_{\mathfrak{t}} \left[\prod_{i=1}^{k} \left(\frac{\breve{\nu}_i}{\mathfrak{t}}\right)^{\varpi i}, \ \prod_{i=1}^{k} \left(\frac{\breve{\nu}'_i}{\mathfrak{t}}\right)^{\varpi i} \right] \right\rangle
\end{array} \right) \tag{5.1}
$$

So above equation holds for n=k+1 Thus holds for all n Hence Proved. ■

Theorem 41 *Let*

$$\hat{S}_i = \left(\left\langle \dot{S}_{\left[\breve{\xi}_i\ ,\ \breve{\xi}'_i\right]} \dot{S}_{\mathfrak{t}\left[\tilde{\Psi}_i\ ,\ \tilde{\Psi}'_i\right]} \right\rangle, \left\langle \ddot{S}_{\left[\breve{\mu}_i\ ,\ \breve{\mu}'_i\right]} \ddot{S}_{\mathfrak{t}\left[\breve{\nu}_i\ ,\ \breve{\nu}'_i\right]} \right\rangle\right)$$

be a collection of GLIVIFSESs, then the aggregated value according to the GLIVIFSEWA operator is also a GLIVIFSES

Proof. The proof of this Theorem is directly follows from Theorem # 1 ■

The Proposed GLIVIFSEWA operator satisfies the following properties.

Property No. 1 [*Idempotency*] :

$$GLIFIFWA(\hat{S}_1, \hat{S}_2, \ ....\hat{S}_n) = \ \hat{S}.$$

Proof. As

$$\hat{S}_i \ = \ \hat{S} \ = \left(\begin{array}{c} \left\langle \dot{S}_{\left[\breve{\xi}\ ,\ \breve{\xi}'\right]} \dot{S}_{\mathfrak{t}\left[\tilde{\Psi}\ ,\ \tilde{\Psi}'\right]} \right\rangle, \left\langle \ddot{S}_{\left[\breve{\mu}\ ,\ \breve{\mu}'\right]} \ddot{S}_{\mathfrak{t}\left[\breve{\nu}\ ,\ \breve{\nu}'\right]} \right\rangle; \\ \left[\breve{\xi}_i, \breve{\xi}_i'\right], \left[\tilde{\Psi}_i, \tilde{\Psi}'_i\right] \subseteq [0, \mathfrak{t}]; [\breve{\mu}_i, \breve{\mu}'_i], [\breve{\nu}_i, \breve{\nu}'_i] \subseteq \left[0, \mathfrak{t}'\right] \ , \text{ for all } i \end{array}\right)$$

so by using equation 5.1, we get

$$GLIVIFSEWA(\hat{S}_1, \hat{S}_2, \hat{S}_3....\hat{S}_n) = \left(\begin{array}{c} \left\langle \begin{array}{c} \dot{S}_{\mathfrak{t}\left[1-\prod_{i=1}^{n}\left(1-\frac{\breve{\xi}}{\mathfrak{t}}\right)^{\varpi_i},\ 1-\prod_{i=1}^{n}\left(1-\frac{\breve{\xi}'_i}{\mathfrak{t}}\right)^{\varpi_i}\right],} \\ \dot{S}_{\mathfrak{t}\left[\prod_{i=1}^{n}\left(\frac{\tilde{\Psi}}{\mathfrak{t}}\right)^{\varpi_i},\ \prod_{i=1}^{n}\left(\frac{\tilde{\Psi}'}{\mathfrak{t}}\right)^{\varpi_i}\right]} \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \ddot{S}_{\mathfrak{t}'\left[1-\prod_{i=1}^{n}\left(1-\frac{\breve{\mu}}{\mathfrak{t}}\right)^{\varpi_i},\ 1-\prod_{i=1}^{n}\left(1-\frac{\breve{\mu}'_i}{\mathfrak{t}}\right)^{\varpi_i}\right],} \\ \ddot{S}_{\mathfrak{t}'\left[\prod_{i=1}^{n}\left(\frac{\breve{\nu}}{\mathfrak{t}}\right)^{\varpi_i},\ \prod_{i=1}^{n}\left(\frac{\breve{\nu}'}{\mathfrak{t}}\right)^{\varpi_i}\right]} \end{array} \right\rangle; \end{array}\right)$$

$$= \left(\begin{array}{c} \left\langle \begin{array}{c} \dot{S}_{\mathfrak{t}} \left[\begin{array}{c} 1-\left(1-\frac{\breve{\xi}}{\mathfrak{t}}\right)^{\varpi_1} \quad \left(1-\frac{\breve{\xi}}{\mathfrak{t}}\right)^{\varpi_2} .... \left(1-\frac{\breve{\xi}}{\mathfrak{t}}\right)^{\varpi n}, \\ 1-\left(1-\frac{\breve{\xi}'}{\mathfrak{t}}\right)^{\varpi_1} \left(1-\frac{\breve{\xi}'}{\mathfrak{t}}\right)^{\varpi_2} \left(1-\frac{\breve{\xi}'}{\mathfrak{t}}\right)^{\varpi_3} ... \left(1-\frac{\breve{\xi}'}{\mathfrak{t}}\right)^{\varpi n} \end{array} \right], \\ \dot{S}_{\mathfrak{t}} \left[\begin{array}{c} \left(\frac{\tilde{\Psi}}{\mathfrak{t}}\right)^{\varpi_1} \quad \left(\frac{\tilde{\Psi}}{\mathfrak{t}}\right)^{\varpi_2} \left(\frac{\tilde{\Psi}}{\mathfrak{t}}\right)^{\varpi_3} .... \left(\frac{\tilde{\Psi}}{\mathfrak{t}}\right)^{\varpi n}, \\ \left(\frac{\tilde{\Psi}'}{\mathfrak{t}}\right)^{\varpi_1} \quad \left(\frac{\tilde{\Psi}'}{\mathfrak{t}}\right)^{\varpi_2} \left(\frac{\tilde{\Psi}'}{\mathfrak{t}}\right)^{\varpi_3} .... \left(\frac{\tilde{\Psi}'}{\mathfrak{t}}\right)^{\varpi_n} \end{array} \right] \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \ddot{S}_{\mathfrak{t}} \left[\begin{array}{c} 1-\left(1-\frac{\breve{\mu}}{\mathfrak{t}}\right)^{\varpi_1}, \left(1-\frac{\breve{\mu}}{\mathfrak{t}}\right)^{\varpi_2} .... \left(1-\frac{\breve{\mu}}{\mathfrak{t}}\right)^{\varpi n}, \\ 1-\left(1-\frac{\breve{\mu}'}{\mathfrak{t}}\right)^{\varpi_1} \left(1-\frac{\breve{\mu}'}{\mathfrak{t}}\right)^{\varpi_2} \left(1-\frac{\breve{\mu}'}{\mathfrak{t}}\right)^{\varpi_3} ... \left(1-\frac{\breve{\mu}'}{\mathfrak{t}}\right)^{\varpi n} \end{array} \right], \\ \ddot{S}_{\mathfrak{t}} \left[\begin{array}{c} \left(\frac{\breve{\nu}}{\mathfrak{t}}\right)^{\varpi_1}, \left(\frac{\breve{\nu}}{\mathfrak{t}}\right)^{\varpi_2}, \left(\frac{\breve{\nu}}{\mathfrak{t}}\right)^{\varpi_3} .... \left(\frac{\breve{\nu}}{\mathfrak{t}}\right)^{\varpi n}, \\ \left(\frac{\breve{\nu}}{\mathfrak{t}}\right)^{\varpi_1}, \left(\frac{\breve{\nu}'}{\mathfrak{t}}\right)^{\varpi_1}, \left(\frac{\breve{\nu}'}{\mathfrak{t}}\right)^{\varpi_1} .... \left(\frac{\breve{\nu}'}{\mathfrak{t}}\right)^{\varpi_n} \end{array} \right] \end{array} \right\rangle \end{array} \right)$$

Since $\sum_{i=1}^{n} \varpi_i = 1$. *i.e.* $\varpi_1 + \varpi_2 \; ... + \; \varpi_n = 1$. $\forall \varpi_i \in [0 \;\; 1]$

$$= \left(\begin{array}{c} \left\langle \begin{array}{c} \dot{S}_{\mathfrak{t}} \left[1-\left(1-\frac{\breve{\xi}}{\mathfrak{t}}\right)^{\varpi_1+\varpi_2+\varpi_3 \cdots \varpi_n}, \quad \left(1-\frac{\breve{\xi}'}{\mathfrak{t}}\right)^{\varpi_1+\varpi_2+\varpi_3 \cdots \varpi_n} \right], \\ \dot{S}_{\mathfrak{t}} \left[\left(\frac{\tilde{\Psi}}{\mathfrak{t}}\right)^{\varpi_1+\varpi_2+\varpi_3 \cdots \varpi_n}, \quad \left(\frac{\tilde{\Psi}'}{\mathfrak{t}}\right)^{\tilde{\varpi}_1+\tilde{\varpi}_2+\tilde{\varpi}_3 \cdots \tilde{\varpi}_n} \right] \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \ddot{S}_{\mathfrak{t}'} \left[1-\left(1-\frac{\breve{\mu}}{\mathfrak{t}}\right)^{\tilde{\varpi}_1+\tilde{\varpi}_2+\tilde{\varpi}_3 \cdots \tilde{\varpi}_n}, \quad \left(1-\frac{\breve{\mu}'}{\mathfrak{t}}\right)^{\tilde{\varpi}_1+\tilde{\varpi}_2+\tilde{\varpi}_3 \cdots \tilde{\varpi}_n} \right], \\ \ddot{S}_{\mathfrak{t}'} \left[\left(\frac{\breve{\nu}}{\mathfrak{t}}\right)^{\tilde{\varpi}_1+\tilde{\varpi}_2+\tilde{\varpi}_3 \cdots \tilde{\varpi}_n}, \quad \left(\frac{\breve{\nu}'}{\mathfrak{t}}\right)^{\tilde{\varpi}_1+\tilde{\varpi}_2+\tilde{\varpi}_3 \cdots \tilde{\varpi}_n} \right] \end{array} \right\rangle \end{array} \right)$$

$$= \left(\begin{array}{c} \left\langle \dot{S}_{\mathfrak{t}} \left[1-\left(1-\frac{\breve{\xi}}{\mathfrak{t}}\right), \quad 1-\left(1-\frac{\breve{\xi}'}{\mathfrak{t}}\right) \right], \dot{S}_{\mathfrak{t}} \left[\left(\frac{\tilde{\Psi}}{\mathfrak{t}}\right), \quad \left(\frac{\tilde{\Psi}'}{\mathfrak{t}}\right) \right] \right\rangle, \\ \left\langle \ddot{S}_{\mathfrak{t}'} \left[1-\left(1-\frac{\breve{\mu}}{\mathfrak{t}}\right), \quad 1-\left(1-\frac{\breve{\mu}'}{\mathfrak{t}}\right) \right], \ddot{S}_{\mathfrak{t}'} \left[\left(\frac{\breve{\nu}}{\mathfrak{t}'}\right), \quad \left(\frac{\breve{\nu}'}{\mathfrak{t}'}\right) \right] \right\rangle \end{array} \right)$$

$$= \left(\left\langle \dot{S}_{\mathfrak{t}} \left[\frac{\breve{\xi}}{\mathfrak{t}}, \frac{\breve{\xi}'}{\mathfrak{t}} \right], \dot{S}_{\mathfrak{t}} \left[\tilde{\Psi}, \tilde{\Psi}' \right] \right\rangle, \left\langle \dot{S}_{\mathfrak{t}'} \left[\frac{\breve{\mu}}{\mathfrak{t}'}, \frac{\breve{\mu}'}{\mathfrak{t}'} \right], \ddot{S}_{\mathfrak{t}'} \left[\breve{\nu}, \breve{\nu}' \right] \right\rangle \right)$$

$$= \left(\left\langle \dot{S}_{\c{t}\left[\breve{\xi}\ ,\ \breve{\xi}'\right]}, \dot{S}_{\c{t}\left[\tilde{\Psi}\ ,\ \tilde{\Psi}'\right]} \right\rangle, \left\langle \dot{S}_{\c{t}'\left[\breve{\mu}\ ,\ \breve{\mu}'\right]}, \ddot{S}_{\c{t}'\left[\breve{\nu}\ ,\ \breve{\nu}'\right]} \right\rangle\right)$$

$$= \hat{S}$$

■

Property # 2 $[Monotonicity]$:

Let
$$\hat{S}_i = \left(\begin{array}{c} \left\langle \dot{S}_{\left[\breve{\xi}_i\ ,\ \breve{\xi}'_i\right]}, \dot{S}_{\left[\tilde{\Psi}_i\ ,\ \tilde{\Psi}'_i\right]} \right\rangle, \left\langle \ddot{S}_{\left[\breve{\mu}_i\ ,\ \breve{\mu}'_i\right]}, \ddot{S}_{\left[\breve{\nu}_i\ ,\ \breve{\nu}'_i\right]} \right\rangle; \\ \left[\breve{\xi}_i, \breve{\xi}_i^{'}\right], \left[\tilde{\Psi}_i, \tilde{\Psi}'_i\right] \subseteq [0, \c{t}] ; [\breve{\mu}_i, \breve{\mu}'_i], [\breve{\nu}_i, \breve{\nu}'_i] \subseteq \left[0, \c{t}^{'}\right] \text{ for all i} \end{array} \right)$$

and

$$\overset{\wedge}{\c{S}}_i = \left(\left\langle \dot{S}_{[\zeta_i\ ,\ \zeta'_i]}, \dot{S}_{\left[\varphi_i\ ,\ \varphi'_i\right]} \right\rangle, \left\langle \ddot{S}_{[\eta\ ,\ \eta']}, \ddot{S}_{[\gamma\ ,\ \gamma']} \right\rangle\right)$$

such that $\hat{S}_i \leq \overset{\wedge}{\c{S}}_i \forall_i$, then $GLIVIFSEWA\,(\hat{S}_1, \hat{S}_2,\ ....\hat{S}_n) \leq GLIFIFSEWA\left(\overset{\wedge}{\c{S}}_1,\ \overset{\wedge}{\c{S}}_2,\ \overset{\wedge}{\c{S}}_3 ...\ \overset{\wedge}{\c{S}}_n\right)$

Proof. Since

$$\hat{S}_i \leq \overset{\wedge}{\c{S}}_i \forall i,$$

then

$$\oplus \tilde{\varpi}_i \leq \overset{n}{\underset{i=1}{\oplus}} \tilde{\varpi}_i\ \overset{\wedge}{\c{S}}_i$$

OR

$$GLIVIFSEWA(\hat{S}_1, \hat{S}_2,\ ....\hat{S}_n) \leq GLIFIFSEWA\left(\overset{\wedge}{\c{S}}_1,\ \overset{\wedge}{\c{S}}_2,\ \overset{\wedge}{\c{S}}_3 ...\ \overset{\wedge}{\c{S}}_n\right)$$

Hence Proved ■

Property # 3 $[Boundedness]$:

Let $\hat{S}^{-}, \hat{S}_1, \hat{S}_2,\ ....\hat{S}_n$ be the collection of General linguistic interval valued fuzzy soft expert sets and

$$\hat{S}^{-} = \min\,(\hat{S}_1, \hat{S}_2,\ ....\hat{S}_n)\ and\ \hat{S}^{+} = \max\,(\hat{S}_1, \hat{S}_2,\ ....\hat{S}_n),$$

then

$$\hat{S}^{-} \leq GLIVIFSEWA(\hat{S}_1, \hat{S}_2, \ ....\hat{S}_n) \leq \ \hat{S}^{+}$$

Proof. It can be proved by using *Property # 2* ■

Property # 4

$$\text{If} \qquad \hat{\text{Ş}} = \left(\left\langle \dot{S}_{[\zeta_i \ , \ \zeta_i'],} \ \dot{S}_{\left[\varphi_i \ , \ \varphi_i'\right]} \right\rangle, \left\langle \ddot{S}_{\left[\eta_i \ , \ \eta_i'\right],} \ \ddot{S}_{\left[\gamma_i \ , \ \gamma_i'\right]} \right\rangle\right)$$

be on other GLIVIFSESs, then

$$GLIVIFSEWA\left(\hat{S}_1 \oplus \ \hat{\text{Ş}}, \ \ \hat{S}_2 \oplus \ \ \hat{\text{Ş}}, \ \hat{S}_3 \oplus \ \ \hat{\text{Ş}}, ... \ \hat{S}_n \oplus \ \ \hat{\text{Ş}}, \right)$$
$$= GLIVIFSEWA(\hat{S}_1 \ominus, \hat{S}_2, \ ....\hat{S}_n) \oplus \ \hat{\text{Ş}}$$

Proof. Since $\hat{S}_i, \ \hat{\text{Ş}} \in GLIVIFSESs : So$

$$\hat{S}_i \oplus \ \hat{\text{Ş}} = \left(\begin{array}{c} \left\langle \dot{S}_{\left[\breve{\xi}_i \ + \ \zeta - \frac{\breve{\xi}_i \ \ \zeta}{\text{ţ}}, \breve{\xi}_i' \ + \ \zeta' - \frac{\breve{\xi}_i' \ \ \zeta'}{\text{ţ}}\right],} \ \dot{S}_{\left[\frac{\breve{\xi}_i \ \ \zeta}{\text{ţ}}, \frac{\breve{\xi}_i' \ \ \zeta'}{\text{ţ}}\right]} \right\rangle, \\ \left\langle \ddot{S}_{\left[\breve{\mu}_i \ + \ \varphi - \frac{\breve{\mu}_i \ \ \varphi}{\text{ţ}'}, \ \breve{\mu}_i' \ + \ \varphi' - \frac{\breve{\mu}_i' \ \ \varphi'}{\text{ţ}}\right],} \ \ddot{S}_{\left[\frac{\breve{\nu}_i \ \ \gamma}{\text{ţ}}, \frac{\breve{\nu}_i' \ \ \gamma'}{\text{ţ}}\right]} \right\rangle; \\ \left[\breve{\xi}_i, \breve{\xi}_i'\right], \left[\tilde{\Psi}_i, \tilde{\Psi}_i'\right] \subseteq [0, \text{ţ}]; [\breve{\mu}_i, \breve{\mu}_i'], [\breve{\nu}_i, \breve{\nu}_i'] \subseteq \left[0, \text{ţ}'\right] \ \text{for all i} \end{array} \right)$$

therefore GLIVIFSEWA$\left(\hat{S}_1 \oplus \ \hat{\text{Ş}}, \ \ \hat{S}_2 \oplus \ \ \hat{\text{Ş}}, \ \hat{S}_3 \oplus \ \ \hat{\text{Ş}}, ... \ \hat{S}_n \oplus \ \ \hat{\text{Ş}} \right)$.

$$= \left(\begin{array}{c} \left\langle \begin{array}{c} \dot{S}_{\text{ţ}} \left[1 - \left(\prod_{i=1}^{n}\left(1 - \frac{\breve{\xi}_i}{\text{ţ}}\right)^{\tilde{\varpi}_i}\right)\left(1 - \frac{\zeta}{\text{ţ}}\right), \ 1 - \prod_{i=1}^{n}\left(1 - \frac{\breve{\xi}_i'}{\text{ţ}}\right)^{\tilde{\varpi}_i}\left(1 - \frac{\zeta'}{\text{ţ}}\right)\right], \\ \dot{S}_{\text{ţ}} \left[\prod_{i=1}^{n}\left(\frac{\tilde{\Psi}_i}{\text{ţ}}\right)^{\tilde{\varpi}_i}\left(\frac{\varphi}{\text{ţ}}\right), \ \prod_{i=1}^{n}\left(\frac{\tilde{\Psi}_i'}{\text{ţ}}\right)^{\tilde{\varpi}_i}\left(\frac{\varphi'}{\text{ţ}}\right)\right] \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \ddot{S}_{\text{ţ}'} \left[1 - \left(\prod_{i=1}^{n}\left(1 - \frac{\breve{\mu}_i}{\text{ţ}'}\right)^{\tilde{\varpi}_i}\right)\left(1 - \frac{\eta}{\text{ţ}'}\right), \ 1 - \prod_{i=1}^{n}\left(1 - \frac{\breve{\mu}_i'}{\text{ţ}'}\right)^{\tilde{\varpi}_i}\left(1 - \frac{\eta'}{\text{ţ}'}\right)\right], \\ \ddot{S}_{\text{ţ}} \left[\prod_{i=1}^{n}\left(\frac{\breve{\nu}_i}{\text{ţ}'}\right)^{\tilde{\varpi}_i}\left(\frac{\gamma}{\text{ţ}'}\right), \ \prod_{i=1}^{n}\left(\frac{\breve{\nu}_i'}{\text{ţ}'}\right)^{\tilde{\varpi}_i}\left(\frac{\gamma'}{\text{ţ}'}\right)\right] \end{array} \right\rangle \end{array} \right)$$

$$= \left(\begin{array}{c} \left\langle \begin{array}{c} \dot{S}_{\mathfrak{t}} \left[1 - \prod_{i=1}^{n} \left(1 - \frac{\check{\xi}_i}{\mathfrak{t}}\right)^{\tilde{\varpi}_i}, 1 - \prod_{i=1}^{n} \left(1 - \frac{\check{\xi}'_i}{\mathfrak{t}}\right)^{\tilde{\varpi}_i} \right], \\ \dot{S}_{\mathfrak{t}} \left[\prod_{i=1}^{n} \left(\frac{\tilde{\Psi}_i}{\mathfrak{t}}\right)^{\tilde{\varpi}_i}, \prod_{i=1}^{n} \left(\frac{\tilde{\Psi}'_i}{\mathfrak{t}}\right)^{\tilde{\varpi}_i} \right] \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \ddot{S}_{\mathfrak{t}'} \left[1 - \left(\prod_{i=1}^{n} \left(1 - \frac{\check{\mu}_i}{\mathfrak{t}'}\right)^{\tilde{\varpi}_i} \right), 1 - \prod_{i=1}^{n} \left(1 - \frac{\check{\mu}'_i}{\mathfrak{t}'}\right)^{\tilde{\varpi}_i} \right], \\ \ddot{S}_{\mathfrak{t}} \left[\prod_{i=1}^{n} \left(\frac{\check{\nu}_i}{\mathfrak{t}'}\right)^{\tilde{\varpi}_i}, \prod_{i=1}^{n} \left(\frac{\check{\nu}'_i}{\mathfrak{t}'}\right)^{\tilde{\varpi}_i} \right] \end{array} \right\rangle \end{array} \right)$$

$$= GLIVIFSEWA\left(\hat{S}_1, \hat{S}_2, ...\hat{S}_n\right) \oplus \hat{\c{S}}$$

Hence proved ■

Property # 5

If $\beta \geq 0$ be a real number, then

$$GLIVIFWA\left(\beta\hat{S}_1, \beta\hat{S}_2, ..., \beta\hat{S}_n\right) = \beta\left(GLIVIFSEWA\left(\hat{S}_1, \hat{S}_2, ...\hat{S}_n\right)\right).$$

Proof. Since $\hat{S}_i \in GLIVIFSES \ \ \forall_i$, then for any $\beta > 0$ we get

$$\beta\hat{S}_i = \left(\begin{array}{c} \left\langle \dot{S}_{\mathfrak{t}} \left[1 - \left(1 - \frac{\check{\xi}_i}{\mathfrak{t}}\right)^{\beta}, 1 - \left(1 - \frac{\check{\xi}'_i}{\mathfrak{t}}\right)^{\beta} \right], \dot{S}_{\mathfrak{t}} \left[\left(\frac{\tilde{\Psi}_i}{\mathfrak{t}}\right)^{\beta}, \left(\frac{\tilde{\Psi}'_i}{\mathfrak{t}}\right)^{\beta} \right] \right\rangle, \\ \left\langle \ddot{S}_{\mathfrak{t}'} \left[1 - \left(1 - \frac{\check{\mu}_i}{\mathfrak{t}'}\right)^{\beta}, 1 - \left(1 - \frac{\check{\mu}'_i}{\mathfrak{t}'}\right)^{\beta} \right], \ddot{S}_{\mathfrak{t}'} \left[\left(\frac{\check{\nu}_i}{\mathfrak{t}'}\right)^{\beta}, \left(\frac{\check{\nu}'_i}{\mathfrak{t}'}\right)^{\beta} \right] \right\rangle \end{array} \right)$$

$$\text{GLIVISEWA}\left(\beta\hat{S}_1,\beta\hat{S}_2,....,\beta\hat{S}_n\right)$$

$$=\left(\begin{array}{c}\left\langle\begin{array}{c}\dot{S}_{\text{ţ}}\left[1-\prod_{i=1}^{n}\left(\left(1-\frac{\breve{\xi}_i}{\text{ţ}}\right)^{\beta}\right)^{\tilde{\varpi}_i},1-\prod_{i=1}^{n}\left(\left(1-\frac{\breve{\xi}'_i}{\text{ţ}}\right)^{\beta}\right)^{\tilde{\varpi}_i}\right],\\ \dot{S}_{\text{ţ}}\left[\prod_{i=1}^{n}\left(\left(\frac{\tilde{\Psi}_i}{\text{ţ}}\right)^{\beta}\right)^{\tilde{\varpi}_i},\prod_{i=1}^{n}\left(\left(\frac{\tilde{\Psi}'_i}{\text{ţ}}\right)^{\beta}\right)^{\tilde{\varpi}_i}\right]\end{array}\right\rangle,\\ \left\langle\begin{array}{c}\ddot{S}_{\text{ţ}'}\left[1-\prod_{i=1}^{n}\left(\left(1-\frac{\breve{\mu}_i}{\text{ţ}'}\right)^{\beta}\right)^{\tilde{\varpi}_i},1-\prod_{i=1}^{n}\left(\left(1-\frac{\breve{\mu}'_i}{\text{ţ}'}\right)^{\beta}\right)^{\tilde{\varpi}_i}\right],\\ \ddot{S}_{\text{ţ}}\left[\prod_{i=1}^{n}\left(\left(\frac{\breve{\nu}_i}{\text{ţ}'}\right)^{\beta}\right)^{\tilde{\varpi}_i},\prod_{i=1}^{n}\left(\left(\frac{\breve{\nu}'_i}{\text{ţ}'}\right)^{\beta}\right)^{\tilde{\varpi}_i}\right]\end{array}\right\rangle\end{array}\right)$$

$$=\left(\begin{array}{c}\left\langle\begin{array}{c}\dot{S}_{\text{ţ}}\left[1-\prod_{i=1}^{n}\left(\left(1-\frac{\breve{\xi}_i}{\text{ţ}}\right)^{\tilde{\varpi}_i}\right)^{\beta},1-\prod_{i=1}^{n}\left(\left(1-\frac{\breve{\xi}'_i}{\text{ţ}}\right)^{\tilde{\varpi}_i}\right)^{\beta}\right],\\ \dot{S}_{\text{ţ}}\left[\prod_{i=1}^{n}\left(\left(\frac{\tilde{\Psi}_i}{\text{ţ}}\right)^{\tilde{\varpi}_i}\right)^{\beta},\prod_{i=1}^{n}\left(\left(\frac{\tilde{\Psi}'_i}{\text{ţ}}\right)^{\tilde{\varpi}_i}\right)^{\beta}\right]\end{array}\right\rangle,\\ \left\langle\begin{array}{c}\ddot{S}_{\text{ţ}'}\left[1-\prod_{i=1}^{n}\left(\left(1-\frac{\breve{\mu}_i}{\text{ţ}'}\right)^{\tilde{\varpi}_i}\right)^{\beta},1-\prod_{i=1}^{n}\left(\left(1-\frac{\breve{\mu}'_i}{\text{ţ}'}\right)^{\tilde{\varpi}_i}\right)^{\beta}\right],\\ \ddot{S}_{\text{ţ}}\left[\prod_{i=1}^{n}\left(\left(\frac{\breve{\nu}_i}{\text{ţ}'}\right)^{\tilde{\varpi}_i}\right)^{\beta},\prod_{i=1}^{n}\left(\left(\frac{\breve{\nu}'_i}{\text{ţ}'}\right)^{\tilde{\varpi}_i}\right)^{\beta}\right]\end{array}\right\rangle\end{array}\right)$$

$$=\beta\left(GLIVIFSEWA\left(\hat{S}_1,\hat{S}_2,....\hat{S}_n\right)\right).$$

Hence Proved the desired result. ■

Property # 6

$$\text{If}\qquad \overset{\wedge}{\text{Ş}}=\left(\left\langle\dot{S}_{\left[\zeta\ ,\ \breve{\xi}'\right]},\ \dot{S}_{[\varphi\ ,\ \varphi']}\right\rangle,\left\langle\ddot{S}_{[\eta\ ,\ \eta'],}\ \ddot{S}_{[\gamma\ ,\ \gamma']}\right\rangle\right)$$

and $\beta>0$ be a real number, Then,

$$\text{GLIVIFSEWA}\left(\beta \hat{S}_1 \oplus \overset{\wedge}{Ş}, \beta \hat{S}_2 \oplus \overset{\wedge}{Ş}, \beta \hat{S}_3 \oplus \overset{\wedge}{Ş}, ..., \beta \hat{S}_n \oplus \overset{\wedge}{Ş},\right)$$
$$=\beta\left(GLIVIFSEWA\left(\hat{S}_1, \hat{S}_2, ...\hat{S}_n\right)\right) \oplus \overset{\wedge}{Ş}$$

Proof. By property 4, we have

$$\begin{aligned}&\text{GLIVIFSEWA}\left(\beta \hat{S}_1 \oplus \overset{\wedge}{Ş}, \beta \hat{S}_2 \oplus \overset{\wedge}{Ş}, \beta \hat{S}_3 \oplus \overset{\wedge}{Ş}, ..., \beta \hat{S}_n \oplus \overset{\wedge}{Ş},\right)\\ &= GLIVIFSEWA\left(\beta \hat{S}_1, \beta \hat{S}_2, ..., \beta \hat{S}_n\right) \oplus \overset{\wedge}{Ş}\end{aligned} \tag{5.2}$$

and by using property # 5, we get

$$\begin{aligned}&GLIVIFSEWA\left(\beta \hat{S}_1, \beta \hat{S}_2, ..., \beta \hat{S}_n\right)\\ &= \beta\left(GLIVIFSEWA\left(\hat{S}_1, \hat{S}_2, ...\hat{S}_n\right)\right)\end{aligned} \tag{5.3}$$

Combaining 5.1 and 5.1 we have

$$\text{GLIVIFSEWA}\left(\beta \hat{S}_1 \oplus \overset{\wedge}{Ş}, \beta \hat{S}_2 \oplus \overset{\wedge}{Ş}, \beta \hat{S}_3 \oplus \overset{\wedge}{Ş}, ..., \beta \hat{S}_n \oplus \overset{\wedge}{Ş},\right)$$
$$=\beta\left(GLIVIFSEWA\left(\hat{S}_1, \hat{S}_2, ...\hat{S}_n\right)\right) \oplus \overset{\wedge}{Ş}$$

Hence proved the desired result. ■

Property # 7

Let $\hat{S}_i = \left(\left\langle \dot{S}_{\left[\breve{\xi}_i\ ,\ \breve{\xi}'_i\right]}, \dot{S}_{\left[\tilde{\Psi}_i\ ,\ \tilde{\Psi}'_i\right]}\right\rangle, \left\langle \ddot{S}_{\left[\breve{\mu}_i\ ,\ \breve{\mu}'_i\right]}, \ddot{S}_{\left[\breve{\nu}_i\ ,\ \breve{\nu}'_i\right]}\right\rangle\right)$

and

$$\overset{\wedge}{\c{S}} = \left(\left\langle \dot{S}_{\left[\zeta_i\ ,\ \breve{\xi}_i'\right]},\ \dot{S}_{\left[\varphi_i\ ,\ \varphi_i'\right]} \right\rangle, \left\langle \ddot{S}_{\left[\eta_i\ ,\ \eta_i'\right]},\ \ddot{S}_{\left[\gamma_i\ ,\ \gamma_i'\right]} \right\rangle\right)$$

by any two collections of GLIVIFSESs, Then

$$\text{GLIVIFWA}\left(\hat{S}_1 \oplus \overset{\wedge}{\c{S}}_1, S_2 \overset{\wedge}{\oplus} \overset{\wedge}{\c{S}}_2, ...\hat{S}_n \oplus \overset{\wedge}{\c{S}}_n\right)$$
$$=\text{GLIVIFWA}\left(\hat{S}_1, \hat{S}_2, \hat{S}_3, ...\hat{S}_n\right) \oplus \left(\overset{\wedge}{\c{S}}_1, \overset{\wedge}{\c{S}}_2, \overset{\wedge}{\c{S}}_3, ...\overset{\wedge}{\c{S}}_n\right).$$

Proof. Since $\hat{S}_i, \overset{\wedge}{\c{S}}_i \in GLIVIFSES$,So

$$\hat{S}_i \oplus \overset{\wedge}{\c{S}}_i = \left(\begin{array}{c} \left\langle \dot{S}_{\left[\breve{\xi}_i\ +\ \zeta_i - \frac{\breve{\xi}_i\ \zeta_i}{\mathfrak{t}}, \breve{\xi}_i'\ +\ \zeta_i' - \frac{\breve{\xi}_i'\ \zeta_i'}{\mathfrak{t}}\right]},\ \dot{S}_{\left[\frac{\tilde{\Psi}_i\ \varphi}{\mathfrak{t}},\ \frac{\tilde{\Psi}_i'\varphi'}{\mathfrak{t}}\right]} \right\rangle, \\ \left\langle \ddot{S}_{\left[\breve{\mu}_i\ +\ \eta_i - \frac{\breve{\mu}_i\ \eta_i}{\mathfrak{t}'}, \breve{\mu}_i'\ +\ \eta_i' - \frac{\breve{\mu}_i'\ \eta_i'}{\mathfrak{t}'}\right]},\ \ddot{S}_{\left[\frac{\breve{\nu}_i\ \gamma}{\mathfrak{t}'},\ \frac{\breve{\nu}_i'\gamma'}{\mathfrak{t}'}\right]} \right\rangle \end{array}\right)$$

therefore

$$GLIVIFWA\left(\hat{S}_1 \oplus \overset{\wedge}{\c{S}}_1, S_2 \overset{\wedge}{\oplus} \overset{\wedge}{\c{S}}_2, ...\hat{S}_n \oplus \overset{\wedge}{\c{S}}_n\right)$$

$$= \left(\begin{array}{c} \left\langle \begin{array}{c} \dot{S}_{\mathfrak{t}}\left[1-\prod_{i=1}^{n}\left(\left(1-\frac{\breve{\xi}_i}{\mathfrak{t}}\right)^{\varpi_i}\left(1-\frac{\zeta_i}{\mathfrak{t}}\right)\right)^{\varpi_i},\ 1-\prod_{i=1}^{n}\left(\left(1-\frac{\breve{\xi}_i'}{\mathfrak{t}}\right)\left(1-\frac{\zeta_i'}{\mathfrak{t}}\right)\right)^{\varpi_i}\right], \\ \dot{S}_{\mathfrak{t}}\left[\prod_{i=1}^{n}\left(\left(\frac{\tilde{\Psi}_i}{\mathfrak{t}}\right)\left(\frac{\varphi_i}{\mathfrak{t}}\right)\right)^{\varpi_i},\ \prod_{i=1}^{n}\left(\left(\frac{\tilde{\Psi}_i'}{\mathfrak{t}}\right)\left(\frac{\varphi_i'}{\mathfrak{t}}\right)\right)^{\varpi_i}\right] \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \ddot{S}_{\mathfrak{t}'}\left[1-\prod_{i=1}^{n}\left(\left(1-\frac{\breve{\mu}_i}{\mathfrak{t}'}\right)\left(1-\frac{\eta_i}{\mathfrak{t}'}\right)\right)^{\varpi_i},\ 1-\prod_{i=1}^{n}\left(\left(1-\frac{\breve{\mu}_i'}{\mathfrak{t}'}\right)\left(1-\frac{\eta_i'}{\mathfrak{t}'}\right)\right)^{\varpi_i}\right], \\ \ddot{S}_{\mathfrak{t}}\left[\prod_{i=1}^{n}\left(\left(\frac{\breve{\nu}_i}{\mathfrak{t}'}\right)\left(\frac{\gamma_i}{\mathfrak{t}'}\right)\right)^{\varpi_i},\ \prod_{i=1}^{n}\left(\left(\frac{\breve{\nu}_i'}{\mathfrak{t}'}\right)\left(\frac{\gamma_i'}{\mathfrak{t}'}\right)\right)^{\varpi_i}\right] \end{array} \right\rangle \end{array}\right)$$

$$= \left(\begin{array}{c} \left\langle \begin{array}{c} \dot{S}_{\mathfrak{t}} \left[1-\prod_{i=1}^{n}\left(1-\frac{\check{\xi}_i}{\mathfrak{t}}\right)^{\tilde{\varpi}_i}, 1-\prod_{i=1}^{n}\left(1-\frac{\check{\xi}'_i}{\mathfrak{t}}\right)^{\tilde{\varpi}_i} \right], \\ \dot{S}_{\mathfrak{t}} \left[\prod_{i=1}^{n}\left(\frac{\tilde{\Psi}_i}{\mathfrak{t}}\right)^{\tilde{\varpi}_i}, \prod_{i=1}^{n}\left(\frac{\tilde{\Psi}'_i}{\mathfrak{t}}\right)^{\tilde{\varpi}_i} \right] \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \ddot{S}_{\mathfrak{t}'} \left[1-\prod_{i=1}^{n}\left(1-\frac{\check{\mu}_i}{\mathfrak{t}'}\right)^{\tilde{\varpi}_i}, 1-\prod_{i=1}^{n}\left(1-\frac{\check{\mu}'_i}{\mathfrak{t}'}\right)^{\tilde{\varpi}_i} \right], \\ \ddot{S}_{\mathfrak{t}} \left[\prod_{i=1}^{n}\left(\frac{\check{\nu}_i}{\mathfrak{t}'}\right)^{\tilde{\varpi}_i}, \prod_{i=1}^{n}\left(\frac{\check{\nu}'_i}{\mathfrak{t}'}\right)^{\tilde{\varpi}_i} \right] \end{array} \right\rangle \end{array} \right)$$

$$= \left(\begin{array}{c} \left\langle \begin{array}{c} \dot{S}_{\mathfrak{t}} \left[1-\prod_{i=1}^{n}\left(1-\frac{\zeta_i}{\mathfrak{t}}\right)^{\tilde{\varpi}_i}, 1-\prod_{i=1}^{n}\left(1-\frac{\zeta'_i}{\mathfrak{t}}\right)^{\tilde{\varpi}_i} \right], \\ \dot{S}_{\mathfrak{t}} \left[\prod_{i=1}^{n}\left(\frac{\varphi_i}{\mathfrak{t}}\right)^{\tilde{\varpi}_i}, \prod_{i=1}^{n}\left(\frac{\varphi'_i}{\mathfrak{t}}\right)^{\tilde{\varpi}_i} \right] \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \ddot{S}_{\mathfrak{t}'} \left[1-\prod_{i=1}^{n}\left(1-\frac{\eta_i}{\mathfrak{t}'}\right)^{\tilde{\varpi}_i}, 1-\prod_{i=1}^{n}\left(1-\frac{\eta'_i}{\mathfrak{t}'}\right)^{\tilde{\varpi}_i} \right], \\ \ddot{S}_{\mathfrak{t}} \left[\prod_{i=1}^{n}\left(\frac{\gamma_i}{\mathfrak{t}'}\right)^{\tilde{\varpi}_i}, \prod_{i=1}^{n}\left(\frac{\gamma'_i}{\mathfrak{t}'}\right)^{\tilde{\varpi}_i} \right] \end{array} \right\rangle \end{array} \right)$$

$$\begin{aligned} = \quad & GLIVIFSEWA\left(\hat{S}_1, \hat{S}_2, \hat{S}_3, ...\hat{S}_n\right) \\ & \oplus GLIVIFSEWA\left(\overset{\wedge}{\mathfrak{S}}_1, \overset{\wedge}{\mathfrak{S}}_2, \overset{\wedge}{\mathfrak{S}}_3, ...\overset{\wedge}{\mathfrak{S}}_n\right) \end{aligned}$$

Hence proved the desired result. ■

In 1988 Yager give the notion of ordered WA operator,we define General linguistic interval valued intuitionistic fuzzy soft weighted averaging (GLIVIFSEOWA) operator.

5.2 General Linguistic Interval Valued Intuitionistic Fuzzy Soft Ordered Weighted Averaging (GLIVIFSEOWA) Operator

GLIVIFSESs . $\hat{S}_i = \left(\left\langle \dot{S}_{\left[\breve{\xi}_i,\breve{\xi}'_i\right]}, \dot{S}_{[\tilde{\Psi}_i,\tilde{\Psi}'_i]} \right\rangle, \left\langle \ddot{S}_{[\breve{\mu}_i,\breve{\mu}'_i]}, \ddot{S}_{[\breve{\nu}_i,\breve{\nu}'_i]} \right\rangle ;\ \text{where } (i = 1,\ 2,\ 3...,n) \right)$ be the collection of n General Lingusitic Interval Valued Intuitionistic Fuzzy Soft Expert sets, then the GLIVIFSEOWAOs i.e. General Lingusitic Interval Valued Intuitionistic Fuzzy Soft Expert Ordered Weighted Averaging Operator of n dimensions is defined as:

A mapping or a function from $\Upsilon^n \rightarrow \Upsilon$ such as; GLIVIFSEOWA:$\Upsilon^n \rightarrow \Upsilon$; that is associated with the weight vector $\tilde{\varpi} = (\tilde{\varpi}_1, \tilde{\varpi}_2, ...\tilde{\varpi}_n)^{\mathfrak{t}}$such that $\tilde{\varpi}_i \in [0,1]$ and $\sum \tilde{\varpi}_i = 1$ then

$$\text{GLIVIFSEOWA}(\hat{S}_1, \hat{S}_2.....\hat{S}_n) = \oplus^n \tilde{\varpi}_i S_{m(i)}$$

Then GLIVIFSEOWA is called the Gernalized Lingusitic Interval Valued Intuitionistic Fuzzy Soft Expert Ordered Weighted Averaging operator, associated with weight vector $\varpi = (\varpi_1, \varpi_2, ...\varpi_n)^{\mathfrak{t}}$ is then weighting vector of $\hat{S}_i$ with $\varpi_i \in [0, \mathfrak{t}]$, $\sum \varpi_i = 1$.

Theorem 42 *Let*

$$\hat{S}_i = \left(\left\langle \dot{S}_{\left[\breve{\xi}_i\ ,\ \breve{\xi}'_i\right]} \dot{S}_{\mathfrak{t}[\tilde{\Psi}_i\ ,\ \tilde{\Psi}'_i]} \right\rangle, \left\langle \ddot{S}_{[\breve{\mu}_i\ ,\ \breve{\mu}'_i]} \ddot{S}_{\mathfrak{t}[\breve{\nu}_i\ ,\ \breve{\nu}'_i]} \right\rangle, i = 1, 2, 3, ..., n \right)$$

be a collection of n GLIVIFSESs, then the aggregated value according to the GLIVIFSEOWA operator is also a GLIVIFSES

Proof. The proof of this Theorem is directly follows from Theorem 40 ■

The Proposed GLIVIFSEOWA operator obey the properties given bellow.

Property # 1 $[Idempotency]$:

Proof. GLIVIFSEOWA $(\hat{S}_1, \hat{S}_2, \ ....\hat{S}_n) = \ \hat{S}$. As

$$\overset{\wedge}{S_i} \ = \hat{S} \ = \left(\begin{array}{c} \left\langle \dot{S}_{\left[\breve{\xi}\ ,\ \breve{\xi}'\right]} \dot{S}_{\mathfrak{t}\left[\tilde{\Psi}\ ,\ \tilde{\Psi}'\right]} \right\rangle, \left\langle \ddot{S}_{[\breve{\mu}\ ,\ \breve{\mu}']} \ddot{S}_{\mathfrak{t}[\breve{\nu}\ ,\ \breve{\nu}']} \right\rangle, \\ \left[\breve{\xi}_i, \breve{\xi}_i^{'}\right], \left[\tilde{\Psi}_i, \tilde{\Psi}'_i\right] \subseteq [0, \mathfrak{t}]; [\breve{\mu}_i, \breve{\mu}'_i], [\breve{\nu}_i, \breve{\nu}'_i] \subseteq \left[0, \mathfrak{t}^{'}\right] \text{For all } i \end{array} \right)$$

so by using related Theorem 5.1 which is previouly proved , we get

$$GLIVIFSEOWA(\hat{S}_1, \hat{S}_2, \hat{S}_3....\hat{S}_n)$$

$$= \left(\begin{array}{c} \left\langle \begin{array}{c} \dot{S}_{\mathfrak{t}} \left[1 - \prod_{i=1}^{n} \left(1 - \frac{\breve{\xi}_m}{\mathfrak{t}}\right)^{\tilde{\varpi}_i}, \ 1 - \prod_{i=1}^{n} \left(1 - \frac{\breve{\xi}'_m}{\mathfrak{t}}\right)^{\tilde{\varpi}_i} \right], \\ \dot{S}_{\mathfrak{t}} \left[\prod_{i=1}^{n} \left(\frac{\tilde{\Psi}_m}{\mathfrak{t}}\right)^{\tilde{\varpi}_i}, \ \prod_{i=1}^{n} \left(\frac{\tilde{\Psi}'_m}{\mathfrak{t}}\right)^{\tilde{\varpi}_i} \right] \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \ddot{S}_{\mathfrak{t}'} \left[1 - \prod_{i=1}^{n} \left(1 - \frac{\breve{\mu}_m}{\mathfrak{t}}\right)^{\tilde{\varpi}_i}, \ 1 - \prod_{i=1}^{n} \left(1 - \frac{\breve{\mu}'_m}{\mathfrak{t}}\right)^{\tilde{\varpi}_i} \right], \\ \ddot{S}_{\mathfrak{t}'} \left[\prod_{i=1}^{n} \left(\frac{\breve{\nu}_m}{\mathfrak{t}}\right)^{\tilde{\varpi}_i}, \ \prod_{i=1}^{n} \left(\frac{\breve{\nu}'_m}{\mathfrak{t}}\right)^{\tilde{\varpi}_i} \right] \end{array} \right\rangle; \\ \left[\breve{\xi}_m, \breve{\xi}_m^{'}\right], \left[\tilde{\Psi}_m, \tilde{\Psi}'_m\right] \subseteq [0, \mathfrak{t}]; [\breve{\mu}_m, \breve{\mu}'_m], [\breve{\nu}_m, \breve{\nu}'_m] \subseteq \left[0, \mathfrak{t}^{'}\right] \end{array} \right)$$

$$= \left(\begin{array}{c} \left\langle \begin{array}{c} \dot{S}_{\text{ţ}} \left[\begin{array}{c} 1-\left(1-\frac{\breve{\xi}_m}{\text{ţ}}\right)^{\tilde{\varpi}_1} \quad \left(1-\frac{\breve{\xi}_m}{\text{ţ}}\right)^{\tilde{\varpi}_2} .... \left(1-\frac{\breve{\xi}_m}{\text{ţ}}\right)^{\tilde{\varpi}n}, \\ 1-\left(1-\frac{\breve{\xi}'_m}{\text{ţ}}\right)^{\tilde{\varpi}_1} \left(1-\frac{\breve{\xi}'_m}{\text{ţ}}\right)^{\tilde{\varpi}_2} \left(1-\frac{\breve{\xi}'_m}{\text{ţ}}\right)^{\tilde{\varpi}_3} ... \left(1-\frac{\breve{\xi}'_m}{\text{ţ}}\right)^{\tilde{\varpi}n} \end{array} \right], \\ \dot{S}_{\text{ţ}} \left[\begin{array}{c} \left(\frac{\tilde{\Psi}_m}{\text{ţ}}\right)^{\tilde{\varpi}_1} \quad \left(\frac{\tilde{\Psi}_m}{\text{ţ}}\right)^{\tilde{\varpi}_2} \left(\frac{\tilde{\Psi}_m}{\text{ţ}}\right)^{\tilde{\varpi}_3} .... \left(\frac{\tilde{\Psi}_m}{\text{ţ}}\right)^{\tilde{\varpi}n}, \\ \left(\frac{\tilde{\Psi}'_m}{\text{ţ}}\right)^{\tilde{\varpi}_1} \quad \left(\frac{\tilde{\Psi}'_m}{\text{ţ}}\right)^{\tilde{\varpi}_2} \left(\frac{\tilde{\Psi}'_m}{\text{ţ}}\right)^{\tilde{\varpi}_3} .... \left(\frac{\tilde{\Psi}'_m}{\text{ţ}}\right)^{\tilde{\varpi}_n} \end{array} \right] \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \ddot{S}_{\text{ţ}} \left[\begin{array}{c} 1-\left(1-\frac{\breve{\mu}_m}{\text{ţ}}\right)^{\tilde{\varpi}_1}, \left(1-\frac{\breve{\mu}_m}{\text{ţ}}\right)^{\tilde{\varpi}_2} .... \left(1-\frac{\breve{\mu}_m}{\text{ţ}}\right)^{\tilde{\varpi}n}, \\ 1-\left(1-\frac{\breve{\mu}'_m}{\text{ţ}}\right)^{\tilde{\varpi}_1} \left(1-\frac{\breve{\mu}'_m}{\text{ţ}}\right)^{\tilde{\varpi}_2} \left(1-\frac{\breve{\mu}'_m}{\text{ţ}}\right)^{\tilde{\varpi}_3} ..... \left(1-\frac{\breve{\mu}'_m}{\text{ţ}}\right)^{\tilde{\varpi}n} \end{array} \right], \\ \ddot{S}_{\text{ţ}} \left[\begin{array}{c} \left(\frac{\breve{\nu}_m}{\text{ţ}}\right)^{\tilde{\varpi}_1}, \left(\frac{\breve{\nu}_m}{\text{ţ}}\right)^{\tilde{\varpi}_2}, \left(\frac{\breve{\nu}_m}{\text{ţ}}\right)^{\tilde{\varpi}_3} .... \left(\frac{\breve{\nu}_m}{\text{ţ}}\right)^{wn}, \\ \left(\frac{\breve{\nu}_m}{\text{ţ}}\right)^{\tilde{\varpi}_1}, \left(\frac{\breve{\nu}'_m}{\text{ţ}}\right)^{\tilde{\varpi}_1}, \left(\frac{\breve{\nu}'_m}{\text{ţ}}\right)^{\tilde{\varpi}_1} .... \left(\frac{\breve{\nu}'_m}{\text{ţ}}\right)^{\tilde{\varpi}_n} \end{array} \right] \end{array} \right\rangle; \\ \left[\breve{\xi}_m, \breve{\xi}_m^{'}\right], \left[\tilde{\Psi}_m, \tilde{\Psi}'_m\right] \subseteq [0, \text{ţ}]; [\breve{\mu}_m, \breve{\mu}'_m], [\breve{\nu}_m, \breve{\nu}'_m] \subseteq \left[0, \text{ţ}^{'}\right] \end{array} \right)$$

Since $\sum\limits_{i=1}^{n} \tilde{\varpi}_i = 1. \quad i.e. \quad \tilde{\varpi}_1 + \tilde{\varpi}_2 \; ... \; \tilde{\varpi}_n = 1. \qquad \forall \tilde{\varpi}_i \in [0 \quad 1]$

$$= \left(\begin{array}{c} \left\langle \begin{array}{c} \dot{S}_{\text{ţ}} \left[1-\left(1-\frac{\breve{\xi}_m}{\text{ţ}}\right)^{\tilde{\varpi}_1+\tilde{\varpi}_2+\tilde{\varpi}_3 ... \tilde{\varpi}_n}, \quad \left(1-\frac{\breve{\xi}'_m}{\text{ţ}}\right)^{\tilde{\varpi}_1+\tilde{\varpi}_2+\tilde{\varpi}_3 ... \tilde{\varpi}_n} \right], \\ \dot{S}_{\text{ţ}} \left[\left(\frac{\tilde{\Psi}_m}{\text{ţ}}\right)^{\tilde{\varpi}_1+\tilde{\varpi}_2+\tilde{\varpi}_3 ... \tilde{\varpi}_n}, \quad \left(\frac{\tilde{\Psi}'_m}{\text{ţ}}\right)^{\tilde{\varpi}_1+\tilde{\varpi}_2+\tilde{\varpi}_3 ... \tilde{\varpi}_n} \right] \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \ddot{S}_{\text{ţ}'} \left[1-\left(1-\frac{\breve{\mu}_m}{\text{ţ}}\right)^{\tilde{\varpi}_1+\tilde{\varpi}_2+\tilde{\varpi}_3 ... \tilde{\varpi}_n}, \quad \left(1-\frac{\breve{\mu}'_m}{\text{ţ}}\right)^{\tilde{\varpi}_1+\tilde{\varpi}_2+\tilde{\varpi}_3 ... \tilde{\varpi}_n} \right], \\ \ddot{S}_{\text{ţ}'} \left[\left(\frac{\breve{\nu}_m}{\text{ţ}}\right)^{\tilde{\varpi}_1+\tilde{\varpi}_2+\tilde{\varpi}_3 ... \tilde{\varpi}_n}, \quad \left(\frac{\breve{\nu}'_m}{\text{ţ}}\right)^{\tilde{\varpi}_1+\tilde{\varpi}_2+\tilde{\varpi}_3 ... \tilde{\varpi}_n} \right] \end{array} \right\rangle; \\ \left[\breve{\xi}_m, \breve{\xi}_m^{'}\right], \left[\tilde{\Psi}_m, \tilde{\Psi}'_m\right] \subseteq [0, \text{ţ}]; [\breve{\mu}_m, \breve{\mu}'_m], [\breve{\nu}_m, \breve{\nu}'_m] \subseteq \left[0, \text{ţ}^{'}\right] \end{array} \right)$$

$$= \left(\begin{array}{c} \left\langle \dot{S}_{\text{ţ}} \left[1-\left(1-\frac{\breve{\xi}_m}{\text{ţ}}\right), \quad 1-\left(1-\frac{\breve{\xi}'_m}{\text{ţ}}\right) \right], \dot{S}_{\text{ţ}} \left[\left(\frac{\tilde{\Psi}_m}{\text{ţ}}\right), \quad \left(\frac{\tilde{\Psi}'_m}{\text{ţ}}\right) \right] \right\rangle, \\ \left\langle \ddot{S}_{\text{ţ}'} \left[1-\left(1-\frac{\breve{\mu}_m}{\text{ţ}}\right), \quad 1-\left(1-\frac{\breve{\mu}'_m}{\text{ţ}}\right) \right], \ddot{S}_{\text{ţ}'} \left[\left(\frac{\breve{\nu}_m}{\text{ţ}'}\right), \quad \left(\frac{\breve{\nu}'_m}{\text{ţ}'}\right) \right] \right\rangle; \\ \left[\breve{\xi}_m, \breve{\xi}_m^{'}\right], \left[\tilde{\Psi}_m, \tilde{\Psi}'_m\right] \subseteq [0, \text{ţ}]; [\breve{\mu}_m, \breve{\mu}'_m], [\breve{\nu}_m, \breve{\nu}'_m] \subseteq \left[0, \text{ţ}^{'}\right] \end{array} \right)$$

$$= \left(\begin{array}{c} \left\langle \dot{S}_{\mathfrak{t}}\left[\frac{\breve{\xi}_m}{\mathfrak{t}}, \frac{\breve{\xi}'_m}{\mathfrak{t}}\right], \dot{S}_{\left[\tilde{\Psi}_m , \tilde{\Psi}'_m\right]} \right\rangle, \left\langle \dot{S}_{\mathfrak{t}'}\left[\frac{\breve{\mu}_m}{\mathfrak{t}'}, \frac{\breve{\mu}'_m}{\mathfrak{t}'}\right], \ddot{S}_{\left[\breve{\nu}_m , \breve{\nu}'_m\right]} \right\rangle; \\ \left[\breve{\xi}_m, \breve{\xi}'_m\right], \left[\tilde{\Psi}_m, \tilde{\Psi}'_m\right] \subseteq [0, \mathfrak{t}]; \left[\breve{\mu}_m, \breve{\mu}'_m\right], \left[\breve{\nu}_m, \breve{\nu}'_m\right] \subseteq \left[0, \mathfrak{t}'\right] \end{array} \right)$$

$$= \left(\begin{array}{c} \left\langle \dot{S}_{\left[\breve{\xi}_m , \breve{\xi}'_m\right]}, \dot{S}_{\left[\tilde{\Psi}_m , \tilde{\Psi}'_m\right]} \right\rangle, \left\langle \dot{S}_{\left[\breve{\mu}_m , \breve{\mu}'\right]}, \ddot{S}_{\left[\breve{\nu}_m , \breve{\nu}'_m\right]} \right\rangle; \\ \left[\breve{\xi}_m, \breve{\xi}'_m\right], \left[\tilde{\Psi}_m, \tilde{\Psi}'_m\right] \subseteq [0, \mathfrak{t}]; \left[\breve{\mu}_m, \breve{\mu}'_m\right], \left[\breve{\nu}_m, \breve{\nu}'_m\right] \subseteq \left[0, \mathfrak{t}'\right] \end{array} \right)$$

$$= \hat{S}_m \in \hat{S}$$

where $\hat{S}_m$ is the mth largest value of $\hat{S}_i$. Thus we can clearly see that GLIVIFSEOWA operator is also a GLIVIFSESs. ■

Property # 2 [*Monotonicity*] :

Let $\hat{S}_i$and $\overset{\wedge}{\mathfrak{S}}_i$be two collections of General Linguistic Interval Valued Intuitionistic Fuzzy Soft Expert Sets GLIVIFSESs.

$$\hat{S}_i = \left(\begin{array}{c} \left\langle \dot{S}_{\left[\breve{\xi}_i , \breve{\xi}'_i\right]}, \dot{S}_{\left[\tilde{\Psi}_i , \tilde{\Psi}'_i\right]} \right\rangle, \left\langle \ddot{S}_{\left[\breve{\mu}_i , \breve{\mu}'_i\right]}, \ddot{S}_{\left[\breve{\nu}_i , \breve{\nu}'_i\right]} \right\rangle; \\ \left[\breve{\xi}_i, \breve{\xi}'_i\right], \left[\tilde{\Psi}_i, \tilde{\Psi}'_i\right] \subseteq [0, \mathfrak{t}]; \left[\breve{\mu}_i, \breve{\mu}'_i\right], \left[\breve{\nu}_i, \breve{\nu}'_i\right] \subseteq \left[0, \mathfrak{t}'\right] \text{ For all } i \end{array} \right)$$

and

$$\overset{\wedge}{\mathfrak{S}}_i = \left(\begin{array}{c} \left\langle \dot{S}_{\left[\zeta_i , \zeta'_i\right]}, \dot{S}_{\left[\varphi_i , \varphi'_i\right]} \right\rangle, \left\langle \ddot{S}_{\left[\eta , \eta'\right]}, \ddot{S}_{\left[\gamma , \gamma'\right]} \right\rangle; \\ \left[\zeta_i , \zeta'_i\right], \left[\varphi_i , \varphi'_i\right] \subseteq [0, \mathfrak{t}]; \left[\eta , \eta'\right], \left[\gamma , \gamma'\right] \subseteq \left[0, \mathfrak{t}'\right] \text{ For all } i \end{array} \right)$$

such that $\hat{S}_i \leq \overset{\wedge}{\mathfrak{S}}_i \forall_i$, then

$$GLIVIFSEOWA\ (\hat{S}_1, \hat{S}_2, \ ....\hat{S}_n) \leq GLIVIFSEOWA \left(\overset{\wedge}{\mathfrak{S}}_1, \overset{\wedge}{\mathfrak{S}}_2, \overset{\wedge}{\mathfrak{S}}_3 ... \overset{\wedge}{\mathfrak{S}}_n \right)$$

Proof. Let $\hat{S}_i$and $\overset{\wedge}{\mathfrak{S}}_i$be two collections of General Linguistic Interval Valued Fuzzy Soft Expert Sets and

$$\hat{S}_i \leq \overset{\wedge}{\mathfrak{S}}_i \quad \text{for all i,}$$

then,

$$\oplus \tilde{\varpi}_i \hat{S}_i \leq \overset{n}{\underset{i=1}{\oplus}} \tilde{\varpi}_i \overset{\wedge}{Ş}_i$$

OR

$$GLIVIFSEOWA(\hat{S}_1, \hat{S}_2, \; ....\hat{S}_n) \leq GLIVIFSEOWA\left(\overset{\wedge}{Ş}_1, \; \overset{\wedge}{Ş}_2, \; \overset{\wedge}{Ş}_3 ... \; \overset{\wedge}{Ş}_n \right)$$

Hence Proved the required result. ■

Property # 3 [*Boundedness*] :

Let $\hat{S}, \hat{S}_1, \hat{S}_2, \; ....\hat{S}_n$ be the collections of General Linguistic Interval Valued Intuitionistic Fuzzy Soft Expert Sets such that:

$$\hat{S}^{-} = \min\,(\hat{S}_1, \hat{S}_2, \; ....\hat{S}_n) \; and \; \hat{S}^{+} = \max\,(\hat{S}_1, \hat{S}_2, \; ....\hat{S}_n),$$

then

$$\hat{S}^{-} \leq GLIVIFSEOWA(\hat{S}_1, \hat{S}_2, \; ....\hat{S}_n) \leq \; \hat{S}^{+}$$

Proof. It can be proved by using *Property 5.2* ■

Property # 4 [*Commutativity*] :

Let $\hat{S}_i$ and $\overset{\wedge}{Ş}_i$be any two collections of General Linguistic Interval Valued Intuitionistic Fuzzy Soft Expert Sets such that

$$\hat{S}_i = \left(\begin{array}{c} \left\langle \dot{S}_{\left[\breve{\xi}_i \; , \; \breve{\xi}'_i\right]} \dot{S}_{ţ\left[\tilde{\Psi}_i \; , \; \tilde{\Psi}'_i\right]} \right\rangle, \left\langle \ddot{S}_{\left[\breve{\mu}_i \; , \; \breve{\mu}'_i\right]} \ddot{S}_{ţ\left[\breve{\nu}_i \; , \; \breve{\nu}'_i\right]} \right\rangle; \\ \left[\breve{\xi}_i, \breve{\xi}_i^{'}\right], \left[\tilde{\Psi}_i, \tilde{\Psi}'_i\right] \subseteq [0, ţ]\,; [\breve{\mu}_i, \breve{\mu}'_i]\,, [\breve{\nu}_i, \breve{\nu}'_i] \subseteq \left[0, ţ^{'}\right] \text{,where } i = 1, 2, 3, ..., n \end{array} \right)$$

then,

$$GLIVIFSEOWA\left(\hat{S}_1, \hat{S}_2, \hat{S}_3, ...\hat{S}_n\right) = GLIVIFSEOWA\left(\overset{\wedge}{Ş}_1, \overset{\wedge}{Ş}_2, \overset{\wedge}{Ş}_3, ...\overset{\wedge}{Ş}_n\right)$$

where $\overset{\wedge}{Ş}_i (i = 1, 2, 3, ..., n)$ be any permutation of $\hat{S}_i$.

Remark 43 *If*

$$\tilde{\varpi} = (1, 0, 0, 0, ..., 0)^t,$$

then

$$GLIVIFSEOWA\left(\hat{S}_1, \hat{S}_2, \hat{S}_3, ...\hat{S}_n\right) = \overset{\wedge}{S_{m(i)}} = \hat{S}_1 = \max\left(\hat{S}_1, \hat{S}_2, \hat{S}_3, ..., \hat{S}_n\right)$$

Remark 44 *If*

$$\tilde{\varpi} = (0, 0, 0, ..., 1)^T,$$

then

$$GLIVIFSEOWA\left(\hat{S}_1, \hat{S}_2, \hat{S}_3, ...\hat{S}_n\right) = \overset{\wedge}{S_{m(i)}} = \hat{S}_n = \min\left(\hat{S}_1, \hat{S}_2, \hat{S}_3, ..., \hat{S}_n\right)$$

where $\overset{\wedge}{S_{m(i)}}$*is the ith greatest / largest value of* $\hat{S}_i$.

Remark 45 *if*

$$\tilde{\varpi}_i = 1, \tilde{\varpi}_j = 0 and i \neq j,$$

then

$$GLIVIFSEOWA\left(\hat{S}_1, \hat{S}_2, \hat{S}_3, ...\hat{S}_n\right) = S_i$$

5.3 General Linguistic Interval Valued Intuitionistic Fuzzy Soft Expert Weighted Geometric (GLIVIFSEWG) Operator

In this section, we will define General Linguistic Interval Valued Intuitionistic Fuzzy Soft Expert Weighted Geometric (GLIVIFSEWG) Operator on GLIVIFSESs. Then some properties related to described operator.

Let

$$\overset{\wedge}{S_i} = \left(\left\langle \dot{S}_{\left[\breve{\xi}_i\ ,\ \breve{\xi}'_i\right]},\ \dot{S}_{\left[\tilde{\Psi}_i\ ,\ \tilde{\Psi}'_i\right]}\right\rangle, \left\langle \ddot{S}_{\left[\breve{\mu}_i\ ,\ \breve{\mu}'_i\right]},\ \ddot{S}_{\left[\breve{\nu}_i\ ,\ \breve{\nu}'_i\right]}\right\rangle; \left[\breve{\xi}_i, \breve{\xi}_i^{'}\right],\ \left[\tilde{\Psi}_i, \tilde{\Psi}'_i\right] \subseteq [0, \mathfrak{t}];\ \left[\breve{\mu}_i, \breve{\mu}'_i\right], \left[\breve{\nu}_i, \breve{\nu}'_i\right] \subseteq \left[0, \mathfrak{t}^{'}\right]\right)$$

where $i = 1, 2, 3, ..., n,$ be n GLIVIFSESs.

$$\text{if} \quad GLIVIFSEWG\left(\overset{\wedge}{S_1},\ \overset{\wedge}{S_2},\ \overset{\wedge}{S_3},\ ....\overset{\wedge}{S_n}\right) = \underset{i=1}{\overset{n}{\otimes}} S_i^{\overset{\wedge}{\tilde{\varpi}}_i}$$

then $GLIVIFSEWG$ is called General Linguistic Interval Valued Intuitionistic Fuzzy Soft Expert Weighted Geometric (GLIVIFSEWG) Operator,where $\tilde{\varpi} = (\tilde{\varpi}_1, \tilde{\varpi}_2, ...\tilde{\varpi}_n)^T$ is then weighting vector of $\overset{\wedge}{S_i}$ with $\tilde{\varpi}_i \in [0, 1], \sum \tilde{\varpi}_i = 1$. Especially,if $\tilde{\varpi} = \left(\frac{1}{n}, \frac{1}{n}....\frac{1}{n}\right)^T$, Then GLIVIFSEWG operator reduced into GLIVIFSEG operator presented as

$$GLIVIFSEWG\left(\overset{\wedge}{S_1}, \overset{\wedge}{S_2}.....\overset{\wedge}{S_n}\right) = \underset{i=1}{\overset{n}{\otimes}} \overset{\wedge}{S_i}^{\frac{1}{n}}$$

Theorem 46 *Let*

$$\overset{\wedge}{S_i} = \left(\begin{array}{c} \left\langle \dot{S}_{\left[\breve{\xi}_i\ ,\ \breve{\xi}'_i\right]},\ \dot{S}_{\left[\tilde{\Psi}_i\ ,\ \tilde{\Psi}'_i\right]}\right\rangle, \left\langle \ddot{S}_{\left[\breve{\mu}_i\ ,\ \breve{\mu}'_i\right]},\ \ddot{S}_{\left[\breve{\nu}_i\ ,\ \breve{\nu}'_i\right]}\right\rangle; \\ \left[\breve{\xi}_i, \breve{\xi}_i^{'}\right], \left[\tilde{\Psi}_i, \tilde{\Psi}'_i\right] \subseteq [0, \mathfrak{t}]; \left[\breve{\mu}_i, \breve{\mu}'_i\right], \left[\breve{\nu}_i, \breve{\nu}'_i\right] \subseteq \left[0, \mathfrak{t}^{'}\right] \end{array}\right)$$

where $i = 1, 2, 3, ..., n$, be a collection of n $GLIVIFSESs$,then aggregated value by using *the GLIVIFSEWG operator is also a* $GLIVIFSESs$ *and*

$$GLIVIFSEWG\left(\hat{S}_1,\hat{S}_2.....\hat{S}_n\right)$$

$$=\left(\begin{array}{c}\left\langle\begin{array}{c}\dot{S}_{\mathfrak{t}\left[\prod_{i=1}^{n}\left(\frac{\breve{\xi}_i}{\mathfrak{t}}\right)^{\tilde{\varpi}_i},\ \prod_{i=1}^{n}\left(\frac{\breve{\xi}'_i}{\mathfrak{t}}\right)^{\tilde{\varpi}_i}\right]},\\ \dot{S}_{\mathfrak{t}\left[1-\prod_{i=1}^{n}\left(1-\frac{\tilde{\psi}_i}{\mathfrak{t}}\right)^{\tilde{\varpi}_i},\ 1-\prod_{i=1}^{n}\left(1-\frac{\tilde{\psi}'_i}{\mathfrak{t}}\right)^{\tilde{\varpi}_i}\right]}\end{array}\right\rangle,\\ \left\langle\begin{array}{c}\ddot{S}_{\mathfrak{t}\left[\prod_{i=1}^{n}\left(\frac{\breve{\mu}_i}{\mathfrak{t}'}\right)^{\tilde{\varpi}_i},\ \prod_{i=1}^{n}\left(\frac{\breve{\mu}'_i}{\mathfrak{t}'}\right)^{\tilde{\varpi}_i}\right]},\\ \ddot{S}_{\mathfrak{t}'\left[1-\prod_{i=1}^{n}\left(1-\frac{\breve{\nu}_i}{\mathfrak{t}'}\right)^{\tilde{\varpi}_i},\ 1-\prod_{i=1}^{n}\left(1-\frac{\breve{\nu}'_i}{\mathfrak{t}'}\right)^{\tilde{\varpi}_i}\right]}\end{array}\right\rangle;\\ \left[\breve{\xi}_i,\breve{\xi}_i'\right],\left[\tilde{\Psi}_i,\tilde{\Psi}'_i\right]\subseteq[0,\mathfrak{t}];\left[\breve{\mu}_i,\breve{\mu}'_i\right],\left[\breve{\nu}_i,\breve{\nu}'_i\right]\subseteq\left[0,\mathfrak{t}'\right]\ For\ all\ i\end{array}\right).$$

Proof. On the basis of operational laws for $GLIVIFSESs$ described in definition 12, related theorem can be clearly proved ■

In a same way to the $GLIVIFSEWA$ operator, the proposed $GLIVIFSEWG$ opertor also obey the properties given below:

Property # 1 [*Idempotency*] :if

$$\hat{S}_i=\hat{S}=\left(\begin{array}{c}\left\langle\dot{S}_{\left[\breve{\xi}\ ,\ \breve{\xi}'\right]}\dot{S}_{\mathfrak{t}\left[\tilde{\Psi}\ ,\ \tilde{\Psi}'\right]}\right\rangle\ ,\ \left\langle\ddot{S}_{\left[\breve{\mu}\ ,\ \breve{\mu}'\right]}\ddot{S}_{\mathfrak{t}\left[\breve{\nu}\ ,\ \breve{\nu}'\right]}\right\rangle;\\ \left[\breve{\xi}_i,\breve{\xi}_i'\right]\ ,\ \left[\tilde{\Psi}_i,\tilde{\Psi}'_i\right]\subseteq[0,\mathfrak{t}];\\ \left[\breve{\mu}_i,\breve{\mu}'_i\right]\ ,\ \left[\breve{\nu}_i,\breve{\nu}'_i\right]\ \subseteq\ \left[0,\mathfrak{t}'\right]\ \text{For all } i\end{array}\right)$$

then

$$GLIVIFSEWG(\hat{S}_1,\hat{S}_2,\ ....\hat{S}_n)=\hat{S}.$$

Proof. As

$$\hat{S}_i = \hat{S} = \begin{pmatrix} \left\langle \dot{S}_{\left[\breve{\xi}\ ,\ \breve{\xi}'\right]}, S_{[\tilde{\Psi}\ ,\ \tilde{\Psi}']} \right\rangle, \left\langle \ddot{S}_{[\breve{\mu}\ ,\ \breve{\mu}'],} S_{[\breve{\nu}\ ,\ \breve{\nu}']} \right\rangle; \\ \left[\breve{\xi}_i, \breve{\xi}_i^{'}\right], \left[\tilde{\Psi}_i, \tilde{\Psi}'_i\right] \subseteq [0, \mathfrak{t}]; \\ [\breve{\mu}_i, \breve{\mu}'_i], \ \ [\breve{\nu}_i, \breve{\nu}'_i] \subseteq \left[0, \mathfrak{t}^{'}\right] \text{ For all } i \end{pmatrix}$$

$$GLIVIFSEWG(\hat{S}_1, \hat{S}_2, \hat{S}_3....\hat{S}_n)$$

$$= \begin{pmatrix} \left\langle \dot{S}_{\mathfrak{t}}\left[\prod_{i=1}^{n}\left(\frac{\breve{\xi}}{\mathfrak{t}}\right)^{\tilde{\varpi}_i}, \ \prod_{i=1}^{n}\left(\frac{\breve{\xi}'}{\mathfrak{t}}\right)^{\tilde{\varpi}_i}\right], \dot{S}_{\mathfrak{t}}\left[1-\prod_{i=1}^{n}\left(1-\frac{\tilde{\psi}}{\mathfrak{t}}\right)^{\tilde{\varpi}_i}, \ 1-\prod_{i=1}^{n}\left(1-\frac{\tilde{\psi}'}{\mathfrak{t}}\right)^{\tilde{\varpi}_i}\right] \right\rangle, \\ \left\langle \ddot{S}_{\mathfrak{t}}\left[\prod_{i=1}^{n}\left(\frac{\breve{\mu}}{\mathfrak{t}'}\right)^{\tilde{\varpi}_i}, \ \prod_{i=1}^{n}\left(\frac{\breve{\mu}'}{\mathfrak{t}'}\right)^{\tilde{\varpi}_i}\right], \ddot{S}_{\mathfrak{t}'}\left[1-\prod_{i=1}^{n}\left(1-\frac{\breve{\nu}}{\mathfrak{t}'}\right)^{\tilde{\varpi}_i}, \ 1-\prod_{i=1}^{n}\left(1-\frac{\breve{\nu}'}{\mathfrak{t}'}\right)^{\tilde{\varpi}_i}\right] \right\rangle; \\ \left[\breve{\xi}, \breve{\xi}^{'}\right], \left[\tilde{\Psi}, \tilde{\Psi}'\right] \subseteq [0, \mathfrak{t}]; [\breve{\mu}, \breve{\mu}'], [\breve{\nu}, \breve{\nu}'] \subseteq \left[0, \mathfrak{t}^{'}\right] \ \text{ For all } i \end{pmatrix}$$

$$= \begin{pmatrix} \left\langle \begin{matrix} \dot{S}_{\mathfrak{t}}\begin{bmatrix} \left(\frac{\breve{\xi}}{\mathfrak{t}}\right)^{\tilde{\varpi}_1} \ \left(\frac{\breve{\xi}}{\mathfrak{t}}\right)^{\tilde{\varpi}_2} \left(\frac{\breve{\xi}}{\mathfrak{t}}\right)^{\tilde{\varpi}_3} .... \left(\frac{\breve{\xi}}{\mathfrak{t}}\right)^{wn}, \\ \left(\frac{\breve{\xi}'}{\mathfrak{t}}\right)^{\tilde{\varpi}_1} \ \left(\frac{\breve{\xi}'}{\mathfrak{t}}\right)^{\tilde{\varpi}_2} \left(\frac{\breve{\xi}'}{\mathfrak{t}}\right)^{\tilde{\varpi}_3} .... \left(\frac{\breve{\xi}'}{\mathfrak{t}}\right)^{\tilde{\varpi}n} \end{bmatrix}, \\ \dot{S}_{\mathfrak{t}}\begin{bmatrix} 1-\left(1-\frac{\tilde{\Psi}}{\mathfrak{t}}\right)^{\tilde{\varpi}_1} \ \left(1-\frac{\tilde{\Psi}}{\mathfrak{t}}\right)^{\tilde{\varpi}_2} .... \left(1-\frac{\tilde{\Psi}}{\mathfrak{t}}\right)^{wn}, \\ 1-\left(1-\frac{\tilde{\Psi}'}{\mathfrak{t}}\right)^{\tilde{\varpi}_1} \left(1-\frac{\tilde{\Psi}'}{\mathfrak{t}}\right)^{\tilde{\varpi}_2} \left(1-\frac{\tilde{\Psi}'}{\mathfrak{t}}\right)^{\tilde{\varpi}_3} ... \left(1-\frac{\tilde{\Psi}'}{\mathfrak{t}}\right)^{\tilde{\varpi}n} \end{bmatrix}, \end{matrix} \right\rangle, \\ \left\langle \begin{matrix} \ddot{S}_{\mathfrak{t}}\begin{bmatrix} \left(\frac{\breve{\mu}}{\mathfrak{t}}\right)^{\tilde{\varpi}_1} \ \left(\frac{\breve{\mu}}{\mathfrak{t}}\right)^{\tilde{\varpi}_2} \left(\frac{\breve{\mu}}{\mathfrak{t}}\right)^{\tilde{\varpi}_3} .... \left(\frac{\breve{\mu}}{\mathfrak{t}}\right)^{\tilde{\varpi}n}, \\ \left(\frac{\breve{\mu}'}{\mathfrak{t}}\right)^{\tilde{\varpi}_1} \ \left(\frac{\breve{\mu}'}{\mathfrak{t}}\right)^{\tilde{\varpi}_1} \left(\frac{\breve{\mu}'}{\mathfrak{t}}\right)^{\tilde{\varpi}_1}, ..., \left(\frac{\breve{\mu}'}{\mathfrak{t}}\right)^{\tilde{\varpi}_n} \end{bmatrix}, \\ \ddot{S}_{\mathfrak{t}}\begin{bmatrix} 1-\left(1-\frac{\breve{\nu}}{\mathfrak{t}}\right)^{\tilde{\varpi}_1} \ \left(1-\frac{\breve{\nu}}{\mathfrak{t}}\right)^{\tilde{\varpi}_2} .... \left(1-\frac{\breve{\nu}}{\mathfrak{t}}\right)^{\varpi n}, \\ 1-\left(1-\frac{\breve{\nu}'}{\mathfrak{t}}\right)^{\tilde{\varpi}_1} \left(1-\frac{\breve{\nu}'}{\mathfrak{t}}\right)^{\tilde{\varpi}_2} \left(1-\frac{\breve{\nu}'}{\mathfrak{t}}\right)^{\tilde{\varpi}_3} ... \left(1-\frac{\breve{\nu}'}{\mathfrak{t}}\right)^{\tilde{\varpi}n} \end{bmatrix}, \end{matrix} \right\rangle; \\ \left[\breve{\xi}, \breve{\xi}^{'}\right], \left[\tilde{\Psi}, \tilde{\Psi}'\right] \subseteq [0, \mathfrak{t}]; [\breve{\mu}, \breve{\mu}'], [\breve{\nu}, \breve{\nu}'] \subseteq \left[0, \mathfrak{t}^{'}\right] \end{pmatrix}$$

Since $\sum_{i=1}^{n}\tilde{\varpi}_i = 1.$ $i.e.$ $\tilde{\varpi}_1 + \tilde{\varpi}_2 ... + \tilde{\varpi}_n = 1.$ for all $\tilde{\varpi}_i \in [0 \;\; 1]$

$$= \begin{pmatrix} \left\langle \begin{matrix} \dot{S}_{\mathfrak{t}}\left[1-\left(1-\frac{\breve{\xi}}{\mathfrak{t}}\right)^{\tilde{\varpi}_1+\tilde{\varpi}_2+\tilde{\varpi}_3...\tilde{\varpi}_n}, \left(1-\frac{\breve{\xi}'}{\mathfrak{t}}\right)^{\tilde{\varpi}_1+\tilde{\varpi}_2+\tilde{\varpi}_3...\tilde{\varpi}_n}\right], \\ \dot{S}_{\mathfrak{t}}\left[\left(\frac{\tilde{\Psi}}{\mathfrak{t}}\right)^{\tilde{\varpi}_1+\tilde{\varpi}_2+\tilde{\varpi}_3...\tilde{\varpi}_n}, \left(\frac{\tilde{\Psi}'}{\mathfrak{t}}\right)^{\tilde{\varpi}_1+\tilde{\varpi}_2+\tilde{\varpi}_3...\tilde{\varpi}_n}\right] \end{matrix} \right\rangle, \\ \left\langle \begin{matrix} \ddot{S}_{\mathfrak{t}'}\left[1-\left(1-\frac{\breve{\mu}}{\mathfrak{t}}\right)^{\tilde{\varpi}_1+\tilde{\varpi}_2+\tilde{\varpi}_3...\tilde{\varpi}_n}, \left(1-\frac{\breve{\mu}'}{\mathfrak{t}}\right)^{\tilde{\varpi}_1+\tilde{\varpi}_2+\tilde{\varpi}_3...\tilde{\varpi}_n}\right], \\ \ddot{S}_{\mathfrak{t}'}\left[\left(\frac{\breve{\nu}}{\mathfrak{t}}\right)^{\tilde{\varpi}_1+\tilde{\varpi}_2+\tilde{\varpi}_3...\tilde{\varpi}_n}, \left(\frac{\breve{\nu}'}{\mathfrak{t}}\right)^{\tilde{\varpi}_1+\tilde{\varpi}_2+\tilde{\varpi}_3...\tilde{\varpi}_n}\right] \end{matrix} \right\rangle \end{pmatrix}$$

$$= \begin{pmatrix} \left\langle \dot{S}_{\mathfrak{t}}\left[\frac{\breve{\xi}}{\mathfrak{t}},\frac{\breve{\xi}'}{\mathfrak{t}}\right], \dot{S}_{\mathfrak{t}}\left[1-\left(1-\frac{\tilde{\Psi}}{\mathfrak{t}}\right), 1-\left(1-\frac{\tilde{\Psi}'}{\mathfrak{t}}\right)\right] \right\rangle, \\ \left\langle \ddot{S}_{\mathfrak{t}'}\left[\frac{\breve{\mu}}{\mathfrak{t}'}, \frac{\breve{\mu}'}{\mathfrak{t}'}\right], \ddot{S}_{\mathfrak{t}'}\left[1-\left(1-\frac{\breve{\nu}}{\mathfrak{t}'}\right), 1-\left(1-\frac{\breve{\nu}'}{\mathfrak{t}'}\right)\right] \right\rangle; \end{pmatrix}$$

$$= \left(\left\langle \dot{S}_{\mathfrak{t}}\left[\frac{\breve{\xi}}{\mathfrak{t}}, \frac{\breve{\xi}'}{\mathfrak{t}}\right], \dot{S}_{[\tilde{\Psi}, \tilde{\Psi}']} \right\rangle, \left\langle \dot{S}_{\mathfrak{t}'}\left[\frac{\breve{\mu}}{\mathfrak{t}'}, \frac{\breve{\mu}'}{\mathfrak{t}'}\right], \ddot{S}_{[\breve{\nu}, \breve{\nu}']} \right\rangle\right),$$

$$= \left(\left\langle \dot{S}_{\left[\breve{\xi}, \breve{\xi}'\right]}, \dot{S}_{[\tilde{\Psi}, \tilde{\Psi}']} \right\rangle, \left\langle \dot{S}_{[\breve{\mu}, \breve{\mu}']}, \ddot{S}_{[\breve{\nu}, \breve{\nu}']} \right\rangle\right).$$

$$= \hat{S}$$

■

Property # 2 $[Monotonicity]$:

Let $\overset{\wedge}{S_i}$ and $\overset{\wedge}{\c{S}}_i$ be any two collections of GLIVIFSE Sets .

$$\overset{\wedge}{S_i} = \begin{pmatrix} \left\langle \dot{S}_{\left[\breve{\xi}_i, \breve{\xi}_i'\right]}, \dot{S}_{[\tilde{\Psi}_i, \tilde{\Psi}_i']} \right\rangle, \left\langle \ddot{S}_{[\breve{\mu}_i, \breve{\mu}_i']}, \ddot{S}_{[\breve{\nu}_i, \breve{\nu}_i']} \right\rangle, \\ \left[\breve{\xi}_i, \breve{\xi}_i{}'\right], \left[\tilde{\Psi}_i, \tilde{\Psi}_i'\right] \subseteq [0, \mathfrak{t}]; \\ [\breve{\mu}_i, \breve{\mu}_i'], [\breve{\nu}_i, \breve{\nu}_i'] \subseteq \left[0, \mathfrak{t}'\right] \text{,where } i = 1, 2, 3, ..., n \end{pmatrix}$$

and

$$\overset{\wedge}{\c{S}}_i = \left(\left\langle \dot{S}_{[\zeta_i, \zeta_i']}, \dot{S}_{\left[\varphi_i, \varphi_i'\right]} \right\rangle, \left\langle \ddot{S}_{[\eta, \eta']}, \ddot{S}_{[\gamma, \gamma']} \right\rangle, i = 1, 2, 3, ..., n\right)$$

such that

$$\hat{S}_i \leq \overset{\wedge}{Ş}_i \forall_i,$$

then

$$GLIVIFSEWG\ (\hat{S}_1, \hat{S}_2,\ ....\hat{S}_n) \leq GLIFIFSEWG \left(\overset{\wedge}{Ş}_1,\ \overset{\wedge}{Ş}_2,\ \overset{\wedge}{Ş}_3 ...\ \overset{\wedge}{Ş}_n \right)$$

Proof. As $\hat{S}_i$ and $\overset{\wedge}{Ş}_i$ be any two collections of GLIVIFSE Sets

Also

$$\hat{S}_i \leq \overset{\wedge}{Ş}_i \forall i$$

then

$$\underset{i=1}{\overset{n}{\oplus}} \hat{S}_i^{\tilde{\varpi}_i} \leq \underset{i=1}{\overset{n}{\oplus}} \overset{\wedge}{Ş}_i^{\tilde{\varpi}_i}$$

OR

$$GLIVIFSEWG(\hat{S}_1, \hat{S}_2,\ ....\hat{S}_n) \leq GLIFIFSEWG \left(\overset{\wedge}{Ş}_1,\ \overset{\wedge}{Ş}_2,\ \overset{\wedge}{Ş}_3 ...\ \overset{\wedge}{Ş}_n \right).$$

Hence Proved ■

Property # 3 [*Boundedness*] :

Let $\hat{S}_i$ be the collections of General Linguistic Interval Valued Intuitionistic Fuzzy Soft Expert Sets such that:

$$\hat{S}^{-} = \min\ (\hat{S}_1, \hat{S}_2,\ ....\hat{S}_n)\ and\ \hat{S}^{+} = \max\ (\hat{S}_1, \hat{S}_2,\ ....\hat{S}_n),$$

then

$$\hat{S}^{-} \leq GLIVIFSEWG(\hat{S}_1, \hat{S}_2,\ ....\hat{S}_n) \leq \hat{S}^{+}$$

Proof. It can be proved by using *Property 5.3* ■

Property # 4

If

$$\hat{\text{Ş}} = \begin{pmatrix} \left\langle \dot{S}_{[\zeta_i\ ,\ \zeta_i'],}\ \dot{S}_{[\varphi_i\ ,\ \varphi_i']} \right\rangle, \\ \left\langle \ddot{S}_{[\eta_i\ ,\ \eta_i'],}\ \ddot{S}_{[\gamma_i\ ,\ \gamma_i']} \right\rangle; \\ [\zeta\ ,\ \zeta'], [\varphi\ ,\ \varphi'] \subseteq [0, \text{ţ}]; \\ [\eta\ ,\ \eta'], [\gamma\ ,\ \gamma'] \subseteq \left[0, \text{ţ}'\right] \end{pmatrix}$$

be on other GLIVIFSESs, then

$$GLIVIFSEWG\left(\hat{S}_1 \otimes\ \hat{\text{Ş}},\ \hat{S}_2 \otimes\ \hat{\text{Ş}},\ \hat{S}_3 \otimes\ \hat{\text{Ş}}, \ldots \hat{S}_n \otimes\ \hat{\text{Ş}},\right)$$

$$= \ GLIVIFSEWA(\hat{S}_1, \hat{S}_2,\ \ldots.\hat{S}_n) \otimes\ \hat{\text{Ş}}$$

Proof. Since $\hat{S}_i,\ \hat{\text{Ş}} \in GLIVIFSESs$, So

$$\hat{S}_i \otimes\ \hat{\text{Ş}} = \begin{pmatrix} \left\langle \dot{S}_{\left[\frac{\breve{\xi}_i\ \ \zeta}{\text{ţ}}, \frac{\breve{\xi}_i'\ \ \zeta'}{\text{ţ}}\right]}, \dot{S}_{\left[\tilde{\psi}_i\ +\ \varphi - \frac{\tilde{\psi}_i\ \varphi}{\text{ţ}'},\ \tilde{\psi}_i'\ +\ \varphi' - \frac{\tilde{\psi}_i'\ \ \varphi'}{\text{ţ}}\right]} \right\rangle, \\ \left\langle \ddot{S}_{\left[\frac{\breve{\mu}_i\ \ \eta}{\text{ţ}}, \frac{\breve{\mu}_i'\ \ \eta'}{\text{ţ}}\right]}, \ddot{S}_{\left[\breve{\nu}_i +\ \ \gamma - \frac{\breve{\nu}_i\ \ \ \gamma}{\text{ţ}}, \breve{\nu}_i' +\ \gamma' - \frac{\breve{\nu}_i'\ \ \gamma'}{\text{ţ}}\right]} \right\rangle \end{pmatrix}$$

where

$$\left[\breve{\xi}_i, \breve{\xi}_i'\right], \left[\tilde{\Psi}_i, \tilde{\Psi}_i'\right] \subseteq [0, \text{ţ}]; \left[\breve{\mu}_i, \breve{\mu}_i'\right], \left[\breve{\nu}_i, \breve{\nu}_i'\right] \subseteq \left[0, \text{ţ}'\right] \text{ for all } i$$

and

$$[\zeta_i\ ,\ \zeta_i'], [\varphi_i\ ,\ \varphi_i'] \subseteq [0, \text{ţ}]; [\eta_i\ ,\ \eta_i'], [\gamma_i\ ,\ \gamma_i'] \subseteq \left[0, \text{ţ}'\right] \text{ for all } i$$

therefore

$$GLIVIFSEWG\left(\hat{S}_1 \otimes \overset{\wedge}{\c{S}}, \hat{S}_2 \otimes \overset{\wedge}{\c{S}}, \hat{S}_3 \otimes \overset{\wedge}{\c{S}}, ... \hat{S}_n \otimes \overset{\wedge}{\c{S}}\right)$$

$$= \left(\begin{array}{c} \left\langle \begin{array}{c} \dot{S}_{\c{t}}\left[\left(\prod_{i=1}^{n}\left(\frac{\breve{\xi}_i}{\c{t}}\right)^{\tilde{\varpi}_i}\right)\left(\frac{\zeta}{\c{t}}\right), 1-\prod_{i=1}^{n}\left(\frac{\breve{\xi}'_i}{\c{t}}\right)^{\tilde{\varpi}_i}\left(\frac{\zeta'}{\c{t}}\right)\right], \\ \dot{S}_{\c{t}}\left[\prod_{i=1}^{n}\left(1-\frac{\tilde{\Psi}_i}{\c{t}}\right)^{\tilde{\varpi}_i}\left(1-\frac{\varphi}{\c{t}}\right), 1-\prod_{i=1}^{n}\left(1-\frac{\tilde{\Psi}'_i}{\c{t}}\right)^{\tilde{\varpi}_i}\left(1-\frac{\varphi'}{\c{t}}\right)\right] \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \ddot{S}_{\c{t}'}\left[\left(\prod_{i=1}^{n}\left(\frac{\breve{\mu}_i}{\c{t}'}\right)^{\tilde{\varpi}_i}\right)\left(\frac{\eta}{\c{t}'}\right), \prod_{i=1}^{n}\left(\frac{\breve{\mu}'_i}{\c{t}'}\right)^{\tilde{\varpi}_i}\left(\frac{\eta'}{\c{t}'}\right)\right], \\ \ddot{S}_{\c{t}'}\left[1-\prod_{i=1}^{n}\left(1-\frac{\breve{\nu}_i}{\c{t}'}\right)^{\tilde{\varpi}_i}\left(1-\frac{\gamma}{\c{t}'}\right), 1-\prod_{i=1}^{n}\left(1-\frac{\breve{\nu}'_i}{\c{t}'}\right)^{\tilde{\varpi}_i}\left(1-\frac{\gamma'}{\c{t}'}\right)\right] \end{array} \right\rangle \end{array}\right)$$

$$= \left(\begin{array}{c} \left\langle \begin{array}{c} \dot{S}_{\c{t}}\left[\prod_{i=1}^{n}\left(\frac{\breve{\xi}_i}{\c{t}}\right)^{\tilde{\varpi}_i}, \prod_{i=1}^{n}\left(\frac{\breve{\xi}'_i}{\c{t}}\right)^{\tilde{\varpi}_i}\right], \\ \dot{S}_{\c{t}}\left[1-\prod_{i=1}^{n}\left(1-\frac{\tilde{\Psi}_i}{\c{t}}\right)^{\tilde{\varpi}_i}, 1-\prod_{i=1}^{n}\left(1-\frac{\tilde{\Psi}'_i}{\c{t}}\right)^{\tilde{\varpi}_i}\right] \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \ddot{S}_{\c{t}'}\left[\prod_{i=1}^{n}\left(\frac{\breve{\mu}_i}{\c{t}'}\right)^{\tilde{\varpi}_i}, \prod_{i=1}^{n}\frac{\breve{\mu}'_i}{\c{t}'}^{\tilde{\varpi}_i}\right], \\ \ddot{S}_{\c{t}'}\left[1-\prod_{i=1}^{n}\left(1-\frac{\breve{\nu}_i}{\c{t}'}\right)^{\tilde{\varpi}_i}, \prod_{i=1}^{n}\left(1-\frac{\breve{\nu}'_i}{\c{t}'}\right)^{\tilde{\varpi}_i}\right] \end{array} \right\rangle \end{array}\right) \otimes$$

$$\left(\begin{array}{c} \left\langle \dot{S}_{\c{t}}\left[\frac{\zeta_i}{\c{t}}, \frac{\zeta'_i}{\c{t}}\right], \dot{S}_{\c{t}}\left[1-(1-\frac{\varphi_i}{\c{t}}), 1-(1-\frac{\varphi'_i}{\c{t}})\right]\right\rangle, \\ \left\langle \ddot{S}_{\c{t}'}\left[\frac{\eta_i}{\c{t}'}, \frac{\eta'_i}{\c{t}'}\right], \ddot{S}_{\c{t}'}\left[1-(1-\frac{\gamma_i}{\c{t}'}), 1-(1-\frac{\gamma'_i}{\c{t}'})\right]\right\rangle \end{array}\right)$$

where

$$\left[\breve{\xi}_i, \breve{\xi}_i^{'}\right], \left[\tilde{\Psi}_i, \tilde{\Psi}'_i\right] \subseteq [0, \c{t}]; \left[\breve{\mu}_i, \breve{\mu}'_i\right], \left[\breve{\nu}_i, \breve{\nu}'_i\right] \subseteq \left[0, \c{t}^{'}\right] \text{ for all } i$$

and

$$\left[\zeta_i , \zeta'_i\right], \left[\varphi_i , \varphi'_i\right] \subseteq [0, \c{t}]; \left[\eta_i , \eta'_i\right], \left[\gamma_i , \gamma'_i\right] \subseteq \left[0, \c{t}^{'}\right] \text{ for all } i$$

$$\left(\begin{array}{c}\left\langle\begin{array}{c}\dot{S}_{\text{ţ}}\left[\prod_{i=1}^{n}\left(\frac{\breve{\xi}_i}{\text{ţ}}\right)^{\tilde{\varpi}_i},\ \prod_{i=1}^{n}\left(\frac{\breve{\xi}'_i}{\text{ţ}}\right)^{\tilde{\varpi}_i}\right],\\ \dot{S}_{\text{ţ}}\left[1-\prod_{i=1}^{n}\left(1-\frac{\tilde{\Psi}_i}{\text{ţ}}\right)^{\tilde{\varpi}_i},1-\prod_{i=1}^{n}\left(1-\frac{\tilde{\Psi}'_i}{\text{ţ}}\right)^{\tilde{\varpi}_i}\right]\end{array}\right\rangle,\\ \left\langle\begin{array}{c}\ddot{S}_{\text{ţ}'}\left[\prod_{i=1}^{n}\left(\frac{\breve{\mu}_i}{\text{ţ}'}\right)^{\tilde{\varpi}_i},\ \prod_{i=1}^{n}\frac{\breve{\mu}'_i}{\text{ţ}'}^{\tilde{\varpi}_i}\right],\\ \ddot{S}_{\text{ţ}'}\left[1-\prod_{i=1}^{n}\left(1-\frac{\breve{\nu}_i}{\text{ţ}'}\right)^{\tilde{\varpi}_i},1-\prod_{i=1}^{n}\left(1-\frac{\breve{\nu}'_i}{\text{ţ}'}\right)^{\tilde{\varpi}_i}\right]\end{array}\right\rangle\end{array}\right)\otimes$$

$$\left(\left\langle\dot{S}_{[\zeta_i\ ,\ \zeta'_i]},\ \dot{S}_{[\varphi_i\ ,\ \varphi'_i]}\right\rangle,\left\langle\ddot{S}_{[\eta_i\ ,\ \eta'_i]},\ \ddot{S}_{[\gamma_i\ ,\ \gamma'_i]}\right\rangle\right)$$

where

$$\left[\breve{\xi}_i,\breve{\xi}_i^{'}\right],\left[\tilde{\Psi}_i,\tilde{\Psi}'_i\right]\subseteq[0,\text{ţ}];\left[\breve{\mu}_i,\breve{\mu}'_i\right],\left[\breve{\nu}_i,\breve{\nu}'_i\right]\subseteq\left[0,\text{ţ}^{'}\right]\ \text{for all } i$$

and

$$\left[\zeta_i,\zeta'_i\right],\left[\varphi_i\ ,\ \varphi'_i\right]\subseteq[0,\text{ţ}];\left[\eta_i\ ,\ \eta'_i\right],\left[\gamma_i\ ,\ \gamma'_i\right]\subseteq\left[0,\text{ţ}^{'}\right]\ \text{for all } i$$

$$=GLIVIFSEWG\left(\hat{S}_1,\ \hat{S}_2,\ ...\ \hat{S}_n\right)\otimes\overset{\wedge}{\text{Ş}}$$

Hence proved ■

Property # 5

If $\beta\geq0$be a read number, then

$$GLIVIFEWG\left(\ \beta\hat{S}_1,\ \beta\hat{S}_2,\ ....,\beta\hat{S}_n\ \right)=\beta\left(GLIVIFSEWG\ \left(\hat{S}_1,\ \hat{S}_2,\ ....,\ \hat{S}_n\right)\right).$$

Proof. Since $\overset{\wedge}{S_i} \in GLIVIFSES$ for all i, then for any $\beta > 0$ we get

$$\beta \overset{\wedge}{S_i} = \left(\begin{array}{c} \left\langle \begin{array}{c} \dot{S}_{\text{ţ}} \left[1-\left(1-\frac{\breve{\xi}_i}{\text{ţ}}\right)^{\beta}, 1-\left(1-\frac{\breve{\xi}'_i}{\text{ţ}}\right)^{\beta} \right], \\ \dot{S}_{\text{ţ}} \left[\left(\frac{\tilde{\Psi}_i}{\text{ţ}}\right)^{\beta}, \left(\frac{\tilde{\Psi}'_i}{\text{ţ}}\right)^{\beta} \right] \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \ddot{S}_{\text{ţ}'} \left[1-\left(1-\frac{\breve{\mu}_i}{\text{ţ}'}\right)^{\beta}, 1-\left(1-\frac{\breve{\mu}'_i}{\text{ţ}'}\right)^{\beta} \right], \\ \ddot{S}_{\text{ţ}'} \left[\left(\frac{\breve{\nu}_i}{\text{ţ}'}\right)^{\beta}, \left(\frac{\breve{\nu}'_i}{\text{ţ}'}\right)^{\beta} \right] \end{array} \right\rangle; \\ \left[\breve{\xi}_i, \breve{\xi}_i^{'}\right], \left[\tilde{\Psi}_i, \tilde{\Psi}'_i\right] \subseteq [0, \text{ţ}]; \\ [\breve{\mu}_i, \breve{\mu}'_i], [\breve{\nu}_i, \breve{\nu}'_i] \subseteq \left[0, \text{ţ}^{'}\right] \text{ For all } i \end{array} \right)$$

$$GLIVIFSEWG\left(\beta \overset{\wedge}{S_1}, \beta \overset{\wedge}{S_2}, ..., \beta \overset{\wedge}{S_n}\right)$$

$$= \begin{array}{c} \left(\left\langle \begin{array}{c} \dot{S}_{\text{ţ}} \left[1-\prod_{i=1}^{n}\left(\left(1-\frac{\breve{\xi}_i}{\text{ţ}}\right)^{\beta}\right)^{\tilde{\varpi}_i}, 1-\prod_{i=1}^{n}\left(\left(1-\frac{\breve{\xi}'_i}{\text{ţ}}\right)^{\beta}\right)^{\tilde{\varpi}_i} \right], \\ \dot{S}_{\text{ţ}} \left[\prod_{i=1}^{n}\left(\left(\frac{\tilde{\Psi}_i}{\text{ţ}}\right)^{\beta}\right)^{\tilde{\varpi}_i}, \prod_{i=1}^{n}\left(\left(\frac{\tilde{\Psi}'_i}{\text{ţ}}\right)^{\beta}\right)^{\tilde{\varpi}_i} \right] \end{array} \right\rangle, \right. \\ \left\langle \begin{array}{c} \ddot{S}_{\text{ţ}'} \left[1-\prod_{i=1}^{n}\left(\left(1-\frac{\breve{\mu}_i}{\text{ţ}'}\right)^{\beta}\right)^{\tilde{\varpi}_i}, 1-\prod_{i=1}^{n}\left(\left(1-\frac{\breve{\mu}'_i}{\text{ţ}'}\right)^{\beta}\right)^{\tilde{\varpi}_i} \right], \\ \ddot{S}_{\text{ţ}} \left[\prod_{i=1}^{n}\left(\left(\frac{\breve{\nu}_i}{\text{ţ}'}\right)^{\beta}\right)^{\tilde{\varpi}_i}, \prod_{i=1}^{n}\left(\left(\frac{\breve{\nu}'_i}{\text{ţ}'}\right)^{\beta}\right)^{\tilde{\varpi}_i} \right] \end{array} \right\rangle; \\ \left[\breve{\xi}_i, \breve{\xi}_i^{'}\right], \left[\tilde{\Psi}_i, \tilde{\Psi}'_i\right] \subseteq [0, \text{ţ}]; [\breve{\mu}_i, \breve{\mu}'_i], [\breve{\nu}_i, \breve{\nu}'_i] \subseteq \left[0, \text{ţ}^{'}\right] \text{ For all } i \Big) \end{array}$$

$$= \left(\begin{array}{c} \left\langle \begin{array}{c} \dot{S}_{\mathfrak{t}} \left[1-\prod_{i=1}^{n}\left(\left(1-\frac{\breve{\xi}_i}{\mathfrak{t}}\right)^{\tilde{\varpi}_i}\right)^{\beta}, 1-\prod_{i=1}^{n}\left(\left(1-\frac{\breve{\xi}'_i}{\mathfrak{t}}\right)^{\tilde{\varpi}_i}\right)^{\beta} \right], \\ \dot{S}_{\mathfrak{t}} \left[\prod_{i=1}^{n}\left(\left(\frac{\tilde{\Psi}_i}{\mathfrak{t}}\right)^{\tilde{\varpi}_i}\right)^{\beta}, \prod_{i=1}^{n}\left(\left(\frac{\tilde{\Psi}'_i}{\mathfrak{t}}\right)^{\tilde{\varpi}_i}\right)^{\beta} \right] \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \ddot{S}_{\mathfrak{t}'} \left[1-\prod_{i=1}^{n}\left(\left(1-\frac{\breve{\mu}_i}{\mathfrak{t}'}\right)^{\tilde{\varpi}_i}\right)^{\beta}, 1-\prod_{i=1}^{n}\left(\left(1-\frac{\breve{\mu}'_i}{\mathfrak{t}'}\right)^{\tilde{\varpi}_i}\right)^{\beta} \right], \\ \ddot{S}_{\mathfrak{t}} \left[\prod_{i=1}^{n}\left(\left(\frac{\breve{\nu}_i}{\mathfrak{t}'}\right)^{\tilde{\varpi}_i}\right)^{\beta}, \prod_{i=1}^{n}\left(\left(\frac{\breve{\nu}'_i}{\mathfrak{t}'}\right)^{\tilde{\varpi}_i}\right)^{\beta} \right] \end{array} \right\rangle; \\ \left[\breve{\xi}_i, \breve{\xi}_i^{'}\right], \left[\tilde{\Psi}_i, \tilde{\Psi}'_i\right] \subseteq [0, \mathfrak{t}]; [\breve{\mu}_i, \breve{\mu}'_i], [\breve{\nu}_i, \breve{\nu}'_i] \subseteq \left[0, \mathfrak{t}^{'}\right] \text{ For all } i \end{array} \right)$$

$$= \beta\left(GLIVIFSEWG\left(\hat{S}_1, \hat{S}_2, ...\hat{S}_n\right)\right).$$

Hence Proved the desired result. ■

Property # 6

If

$$\overset{\wedge}{\c{S}} = \left(\begin{array}{c} \left\langle \dot{S}_{[\zeta\ ,\ \zeta'],}\ \dot{S}_{[\varphi\ ,\ \varphi']} \right\rangle, \left\langle \ddot{S}_{[\eta\ ,\ \eta'],}\ \ddot{S}_{[\gamma\ ,\ \gamma']} \right\rangle; \\ [\zeta\ ,\ \zeta'], [\varphi\ ,\ \varphi'] \subseteq [0, \mathfrak{t}]; [\eta\ ,\ \eta'], [\gamma\ ,\ \gamma'] \subseteq \left[0, \mathfrak{t}^{'}\right] \end{array} \right)$$

and $\beta > 0$ be a read number, Then,

$$\text{GLIVIFSEWA}\left(\beta\hat{S}_1 \oplus \overset{\wedge}{\c{S}}, \beta\hat{S}_2 \oplus \overset{\wedge}{\c{S}}, \beta\hat{S}_3 \oplus \overset{\wedge}{\c{S}}, ..., \beta\hat{S}_n \oplus \overset{\wedge}{\c{S}},\right)$$
$$= \beta\left(GLIVIFSEWA\ \left(\hat{S}_1, \hat{S}_2, ...\hat{S}_n\right)\right) \oplus \overset{\wedge}{\c{S}}$$

Proof. By using property 5.3, we have GLIVIFSEWA $\left(\beta\hat{S}_1 \oplus \overset{\wedge}{\c{S}}, \beta\hat{S}_2 \oplus \overset{\wedge}{\c{S}}, \beta\hat{S}_3 \oplus \overset{\wedge}{\c{S}}, ..., \beta\hat{S}_n \oplus \overset{\wedge}{\c{S}},\right)$

$$= GLIVIFSEWG\left(\beta\hat{S}_1, \beta\hat{S}_2, ..., \beta\hat{S}_n\right) \oplus \overset{\wedge}{\c{S}} \tag{5.4}$$

and by using property 5.2, we get

$$= GLIVIFSEWG\left(\beta\hat{S}_1, \beta\hat{S}_2, ..., \beta\hat{S}_n\right) \quad (5.5)$$

$$= \beta \quad \left(GLIVIFSEWG\left(\hat{S}_1, \hat{S}_2, ...\hat{S}_n\right)\right) \quad (5.6)$$

Combining equation 5.4 and equation 5.5 we have

$$\beta\left(GLIVIFSEWA\left(\hat{S}_1, \hat{S}_2, ...\hat{S}_n\right)\right) \oplus \overset{\wedge}{\c{S}}$$

Hence proved. ■

Property # 7

Let

$$\hat{S}_i = \left(\begin{array}{c} \left\langle \dot{S}_{\left[\breve{\xi}_i\ ,\ \breve{\xi}_i'\right]}, \dot{S}_{\left[\tilde{\Psi}_i\ ,\ \tilde{\Psi}_i'\right]} \right\rangle, \\ \left\langle \ddot{S}_{\left[\breve{\mu}_i\ ,\ \breve{\mu}_i'\right]}, \ddot{S}_{\left[\breve{\nu}_i\ ,\ \breve{\nu}_i'\right]} \right\rangle; \\ \left[\breve{\xi}_i, \breve{\xi}_i^{'}\right], \left[\tilde{\Psi}_i, \tilde{\Psi}_i'\right] \subseteq [0, \c{t}]; \left[\breve{\mu}_i, \breve{\mu}_i'\right], \left[\breve{\nu}_i, \breve{\nu}_i'\right] \subseteq \left[0, \c{t}^{'}\right] \text{ for all i} \end{array}\right)$$

and

$$\overset{\wedge}{\c{S}} = \left(\begin{array}{c} \left\langle \dot{S}_{\left[\zeta_i\ ,\ \zeta_i'\right]}, \dot{S}_{\left[\varphi_i\ ,\ \varphi_i'\right]} \right\rangle, \\ \left\langle \ddot{S}_{\left[\eta_i\ ,\ \eta_i'\right]}, \ddot{S}_{\left[\gamma_i\ ,\ \gamma_i'\right]} \right\rangle; \\ \left[\zeta_i\ ,\ \zeta_i'\right], \left[\varphi_i\ ,\ \varphi_i'\right] \subseteq [0, \c{t}]; \left[\eta_i\ ,\ \eta_i'\right], \left[\gamma_i\ ,\ \gamma_i'\right] \subseteq \left[0, \c{t}^{'}\right] \end{array}\right)$$

By any two collections of GLIFIFSESs, Then

$$GLIVIFWG\left(\begin{array}{c} \hat{S}_1 \oplus \overset{\wedge}{\c{S}}_1, S_2 \overset{\wedge}{\oplus} \overset{\wedge}{\c{S}}_2, ...\hat{S}_n \oplus \overset{\wedge}{\c{S}}_n \\ \end{array}\right)$$

$$= GLIVIFWG\left(\hat{S}_1, \hat{S}_2, \hat{S}_3, ...\hat{S}_n\right) \oplus \left(\overset{\wedge}{\c{S}}_1, \overset{\wedge}{\c{S}}_2, \overset{\wedge}{\c{S}}_3, ...\overset{\wedge}{\c{S}}_n\right).$$

5.4 General Linguistic Interval Valued Fuzzy Soft Expert Ordered Weighted Geometric (GLIVIFSEOWG) Operator

In this section, we will define General Linguistic Interval Valued Intuitionistic Fuzzy Soft Expert Ordered Weighted Geometric (GLIVIFSEOWG) Operator on GLIVIFSESs. Then some properties related to described operator.

Definition 47 *Let*

$$\hat{S}_i = \begin{pmatrix} \left\langle \dot{S}_{\left[\breve{\xi}_i\ ,\ \breve{\xi}_i'\right]},\ \dot{S}_{\left[\tilde{\Psi}_i\ ,\ \tilde{\Psi}_i'\right]} \right\rangle, \left\langle \ddot{S}_{\left[\breve{\mu}_i\ ,\ \breve{\mu}_i'\right]},\ \ddot{S}_{\left[\breve{\nu}_i\ ,\ \breve{\nu}_i'\right]} \right\rangle; \\ \left[\breve{\xi}_i, \breve{\xi}_i'\right],\ \left[\tilde{\Psi}_i, \tilde{\Psi}_i'\right] \subseteq [0, t];\ [\breve{\mu}_i, \breve{\mu}_i'],\ [\breve{\nu}_i, \breve{\nu}_i'] \subseteq \left[0, t'\right], \\ where\ i = 1, 2, 3, ..., n \end{pmatrix}$$

be n $GLIVIFSESs.if$

$$GLIVIFSEOWG\ \left(\hat{S}_1,\ \hat{S}_2,\ \hat{S}_3,\ ....\hat{S}_n\right) = \overset{n}{\underset{i=1}{\otimes}} S_{m(i)}^{\overset{\wedge}{\tilde{\varpi}}_i}$$

then $GLIVIFSEOWG$ *is called General linguistic interval valued intuitionistic fuzzy soft expert ordered weighted geometric (GLIVIFSEOWG) operator,where* $\tilde{\varpi} = (\tilde{\varpi}_1, \tilde{\varpi}_2, ...\tilde{\varpi}_n)^T$ *is then weighting vector of* $\hat{S}_i$ *with* $\tilde{\varpi}_i \in [0,1], \sum \tilde{\varpi}_i = 1$. *Especially,if* $\tilde{\varpi} = \left(\frac{1}{n}, \frac{1}{n}....\frac{1}{n}\right)^T$, *Then GLIVIFSEWG operator reduced into GLIVIFSEG operator presented as*

$$GLIVIFSEOWG\left(\hat{S}_1, \hat{S}_2.....\hat{S}_n\right) = \overset{n}{\underset{i=1}{\otimes}} \hat{S}_{m(i)}^{\frac{1}{n}}$$

Theorem 48 *Let*

$$\hat{S}_i = \begin{pmatrix} \left\langle \dot{S}_{\left[\breve{\xi}_i\ ,\ \breve{\xi}_i'\right]},\ \dot{S}_{\left[\tilde{\Psi}_i\ ,\ \tilde{\Psi}_i'\right]} \right\rangle, \left\langle \ddot{S}_{\left[\breve{\mu}_i\ ,\ \breve{\mu}_i'\right]},\ \ddot{S}_{\left[\breve{\nu}_i\ ,\ \breve{\nu}_i'\right]} \right\rangle; \\ \left[\breve{\xi}_i, \breve{\xi}_i'\right], \left[\tilde{\Psi}_i, \tilde{\Psi}_i'\right] \subseteq [0, t]; [\breve{\mu}_i, \breve{\mu}_i'], [\breve{\nu}_i, \breve{\nu}_i'] \subseteq \left[0, t'\right], \\ where\ i = 1, 2, 3, ..., n \end{pmatrix}$$

be a collection of n GLIVIFSESs,then aggregated value by using the GLIVIFSEOWG operator is also a GLIVIFSESs and

$$GLIVIFSEOWG\left(\hat{S}_1, \hat{S}_2.....\hat{S}_n\right)$$

$$= \left(\begin{array}{c} \left\langle \begin{array}{c} \dot{S}_{\mathfrak{t}}\left[\prod_{i=1}^{n}\left(\frac{\breve{\xi}_{m(i)}}{\mathfrak{t}}\right)^{\tilde{\varpi}_i}, \prod_{i=1}^{n}\left(\frac{\breve{\xi}'_{m(i)}}{\mathfrak{t}}\right)^{\tilde{\varpi}_i}\right], \\ \dot{S}_{\mathfrak{t}}\left[1-\prod_{i=1}^{n}\left(1-\frac{\tilde{\psi}_{m(i)}}{\mathfrak{t}}\right)^{\tilde{\varpi}_i}, 1-\prod_{i=1}^{n}\left(1-\frac{\tilde{\psi}'_{m(i)}}{\mathfrak{t}}\right)^{\tilde{\varpi}_i}\right] \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \ddot{S}_{\mathfrak{t}}\left[\prod_{i=1}^{n}\left(\frac{\breve{\mu}_{m(i)}}{\mathfrak{t}'}\right)^{\tilde{\varpi}_i}, \prod_{i=1}^{n}\left(\frac{\breve{\mu}'_{m(i)}}{\mathfrak{t}'}\right)^{\tilde{\varpi}_i}\right], \\ \ddot{S}_{\mathfrak{t}'}\left[1-\prod_{i=1}^{n}\left(1-\frac{\breve{\nu}_{m(i)}}{\mathfrak{t}'}\right)^{\tilde{\varpi}_i}, 1-\prod_{i=1}^{n}\left(1-\frac{\breve{\nu}'_{m(i)}}{\mathfrak{t}'}\right)^{\tilde{\varpi}_i}\right] \end{array} \right\rangle; \\ \left[\breve{\xi}_{m(i)}, \breve{\xi}'_{m(i)}\right], \left[\tilde{\Psi}_{m(i)}, \tilde{\Psi}'_{m(i)}\right] \subseteq [0, \mathfrak{t}]; \left[\breve{\mu}_{m(i)}, \breve{\mu}'_{m(i)}\right], \left[\breve{\nu}_{m(i)}, \breve{\nu}'_{m(i)}\right] \subseteq \left[0, \mathfrak{t}'\right], \\ \textit{where } m(i) \textit{ is largest value of } \hat{S}_i \end{array} \right).$$

Proof. On the basis of operational laws for *GLIVIFSESs* described in definition,theorem 7 can be clearly proved by using previous related theorems. ■

All the other properties namely monotonicty, idempotency, boundedness, and commutattivity holds in GLIVIFSEOWG.

Remark 49 *If* $\tilde{\varpi} = (1, 0, 0, 0, ..., 0)^{\mathfrak{t}}$ *,then*

$$GLIVIFSEOWG\left(\hat{S}_1, \hat{S}_2, \hat{S}_3, ...\hat{S}_n\right)$$
$$= \hat{S}_{m(i)} = \hat{S}_1 = \max\left(\hat{S}_1, \hat{S}_2, \hat{S}_3, ..., \hat{S}_n\right)$$

Remark 50 *If* $\tilde{\varpi} = (0, 0, 0, ..., 1)^T$ *,then*

$$
\begin{aligned}
& GLIVIFSEOWG\left(\hat{S}_1, \hat{S}_2, \hat{S}_3, ...\hat{S}_n\right) \\
= \quad & \hat{S}_{m(i)} = \hat{S}_n = \min\left(\hat{S}_1, \hat{S}_2, \hat{S}_3, ..., \hat{S}_n\right)
\end{aligned}
$$

where $\hat{S}_{m(i)}$ *is the ith greatest / largest value of* $\hat{S}_i$.

Remark 51 *if* $\tilde{\varpi}_i = 1$, $\tilde{\varpi}_j = 0$ *and* $i \neq j$, *then*

$$
GLIVIFSEOWG\left(\hat{S}_1, \hat{S}_2, \hat{S}_3, ...\hat{S}_n\right) = \hat{S}_{m(i)} = \hat{S}_i
$$

5.5 Decision Analysis on GLIVIFSESs.

In this section, a real-world problem is discussed in which we aim to search for the best team for cricket. Since Cricket is a bat-ball game, played on a big ground (oval shaped) among two teams, each team having 11 players. In the middle of the ground, 22 yards rectangular area, called a pitch also plays an important role during the match. Cricket is one of the most favorite and entertaining games for most of the people in different countries. Because of its reputation, glamour, and the excitement involved in it, more and more people from all over the world are taking interest in this game. The actual purpose of any of these teams is to win the match, but the success and failure of any team depend upon the abilities and skills of the team's players. Thus, the team's performance depends upon the performance of the players, and the player's performance is assessed on the basis of some parameters like strike rate, and averages, total runs scored by batsman, number of wickets taken by bowler, bowling averages, and economy rate. Team's strength or weakness depends on all of these parameters which are evaluating the players performances.

So, for estimating the team's performance, which are made of different eleven players of different caliber, so we may estimate or evaluate by using General Linguistic Interval Valued Intuitionistic Fuzzy Soft Expert Sets, the more stimulating or cchallenging condition can be faced when the teams are made up of different worth of players, having the different ranges of attributes, and we have to judge or give our opinions collectively about the teams, this is possible now by using recently construted structure, that is General Linguistic Interval Valued Intuitionistic Fuzzy Soft Expert Sets.

At the international level different types of cricket played, like test match / cricket which consists of 5 days and each day has 90 overs, One Day match, which is 50 overs per innings match, and Twenty–20 in which 20 Overs per innings were played.

Recently in Pakistan, the ''Pakistan Super League 2023 (PSL 2023)" is going to be held, which will be a source of entertainment and interest. In our decision analysis problem, we wish to select the best team among T_{LQ} (Team of Lahore Qalandars) T_{IU} (Team of Islamabad United) and T_{PZ} (Team of Peshawar Zalmi) evaluation of teams is done by comparing their attributes described as;

A_1. The team strength,

A_2. The team against strength,

A_3 The most balanced team.

Where Team strength is defined and considered as the internal bounding between players, that is, how they played like a time, or we may say to check their team-work altogether.

Team against strength is defined and considered as strategies and bounding against the opposite team that is how the players perform like a team to deal with the pressure built by opposite team on them.

How much the team is balanced we may say on standards the balanced team is defined and considered as the team, is balanced if it has the five best batsmen, five bestand strong bowler according to the pitch of ground, one of them must be all rounder player.

We assign the weight vector to each attribute of its importance

$\hat{\mathrm{W}}$=(0.30,0.30,0.4)^T

Which is given on the bases of expert's preferences.

Three experts E $_1$, E $_2$, E $_3$ are evaluating the teams performance, where E_1 is medical/Physician E_2 is the Instructor/Selector of players of team, E_3 is coach who is experienced player and having the strong previous record also included in top ranking players previously. The weight vector $\hat{\mathrm{W}}$ = (0.25, 0.5, 0.25)^T Associated with Experts familiarities.

Here our aim is to select the best team among the T_{IU}, T_{PZ}, T_{LQ}. The experts give their preferences on each characteristic or attribute in term of GLIVIFSESs according to following General linguistic fuzzy soft expert terms

$$\overset{\wedge[1]}{S} = \left\{ \begin{array}{c} \dot{S}_{Z_0} = \text{very srong}, \dot{S}_{Z_1} = \text{ strong }, \dot{S}_{z_2} = \text{ slighlty strong}, \dot{S}_{z_3} = \text{ neutral}, \\ \dot{S}_{z_4} = \text{ slightly weak}, \dot{S}_{z_6} = \text{ weak}, \dot{S}_7 = \text{ very weak}, \end{array} \right\}$$

where

$$\left(\left\langle \dot{S}_{d_0} , \dot{S}_{d_0'} \right\rangle\right) = \dot{S}_{Z_0} = \left(\left\langle \dot{S}_{[0.8 , 0.95]}, \dot{S}_{[0 , 0.05]} \right\rangle\right)$$
$$\left(\left\langle \dot{S}_{d_1} , \dot{S}_{d_1'} \right\rangle\right) = \dot{S}_{Z_1} = \left(\left\langle \dot{S}_{[0.7 , 0.8]}, \dot{S}_{[0.1 , 0.2]} \right\rangle\right)$$
$$\left(\left\langle \dot{S}_{d_2} , \dot{S}_{d_2'} \right\rangle\right) = \dot{S}_{Z_2} = \left(\left\langle \dot{S}_{[0.6 , 0.7]}, \dot{S}_{[0.1 , 0.3]} \right\rangle\right)$$
$$\left(\left\langle \dot{S}_{d_3} , \dot{S}_{d_3'} \right\rangle\right) = \dot{S}_{Z_3} = \left(\left\langle \dot{S}_{[0.5 , 0.5]}, \dot{S}_{[0.5 , 0.5]} \right\rangle\right)$$

$$\left(\left\langle \dot{S}_{d_4}\ ,\ \dot{S}_{d'_4}\right\rangle\right)=\dot{S}_{Z_4}=\ \left(\left\langle \dot{S}_{[0.4\ ,\ 0.5]},\ \dot{S}_{[0.3\ ,\ 0.4]}\right\rangle\right)$$
$$\left(\left\langle \dot{S}_{d_5}\ ,\ \dot{S}_{d'_5}\right\rangle\right)=\dot{S}_{Z_5}=\ \left(\left\langle \dot{S}_{[0.3\ ,\ 0.4]},\ \dot{S}_{[0.2\ ,\ 0.4]}\right\rangle\right)$$
$$\left(\left\langle \dot{S}_{d_6}\ ,\ \dot{S}_{d'_6}\right\rangle\right)=\dot{S}_{Z_6}=\ \left(\left\langle \dot{S}_{[0.1\ ,\ 0.2]},\ \dot{S}_{[0.5\ ,\ 0.8]}\right\rangle\right)$$

be the first General linguistic interval valued intuitionstic fuzzy soft expert term and

Let

$$\hat{S}^{[2]}=\left\{\begin{array}{c}\ddot{S}_{Z'_0}=\text{ externly familiar, }\ddot{S}_{Z_1}=\text{ moderately ,}\\ \ddot{S}_{z_2}=\text{ some what familiar,}\ddot{S}_{z_3}=\text{ slightly familiar, }\ddot{S}_{z_4}=\text{ not at all familiar}\end{array}\right\}$$

$$\ddot{S}_{Z'_0}=\left(\left\langle \ddot{S}_{d_0}\ ,\ \ddot{S}_{d'_0}\right\rangle\right)=\ \left(\left\langle \ddot{S}_{[0.9\ ,\ 0.95]},\ \ddot{S}_{[0\ ,\ 0.05]}\right\rangle\right)$$
$$\ddot{S}_{Z'_1}=\left(\left\langle \ddot{S}_{d_1}\ ,\ \ddot{S}_{d'_1}\right\rangle\right)=\ \left(\left\langle \ddot{S}_{[0.7\ ,\ 0.9]},\ \ddot{S}_{[0.05\ ,\ 0.1]}\right\rangle\right)$$
$$\ddot{S}_{Z'_2}=\left(\left\langle \ddot{S}_{d_2}\ ,\ \ddot{S}_{d'_2}\right\rangle\right)=\ \left(\left\langle \ddot{S}_{[0.5\ ,\ 0.7]},\ \ddot{S}_{[0.1\ ,\ 0.3]}\right\rangle\right)$$
$$\ddot{S}_{Z'_3}=\left(\left\langle \ddot{S}_{d_3}\ ,\ \ddot{S}_{d'_3}\right\rangle\right)=\ \left(\left\langle \ddot{S}_{[0.4\ ,\ 0.5]},\ \ddot{S}_{[0.3\ ,\ 0.5]}\right\rangle\right)$$
$$\ddot{S}_{Z'_4}=\left(\left\langle \ddot{S}_{d_4}\ ,\ \ddot{S}_{d'_4}\right\rangle\right)=\ \left(\left\langle \ddot{S}_{[0\ ,\ 0]},\ \ddot{S}_{[1\ ,\ 1]}\right\rangle\right)$$

be the second General linguistic interval valued intuitionistic fuzzy soft expert term. After assessement of available Teams or alternatives, The experts constructs the following GLIVIFSE matrixes $\tilde{\mathrm{M}}^{(q)}=\left[S_{ij}^{(q)}\right],(q=1,2,3)$ as given in following Tables. Table # 1, Table # 2, and Table # 3.

Step 1: There is no need to do the normalization process on atributes, as all of these attributes are of same kind.

Step 2: Since the weight vector is associated with expert (decision makers) is already normalized so there is no need to put any operation on it.

Step 3: General linguistic interval valued intuitionistic fuzzy soft matrix $\tilde{\mathrm{M}}^{[q]}$ provided or given by experts E_1, E_2 , E_3. Here we have t = 6 , and $t' = 4$

Table 1

General linguistic Interval valued intuitionistic fuzzy soft expert Analysis Matrix $\tilde{M}^{(1)}$provided by Expert or decision makes E_1, E_2, E_3 about the attributes A_1 , A_2 , A_3, of all the teams T_{IU} , T_{PZ} , T_{LQ}respectively.

	A_1	A_2	A_3
T_{IU}	$\left(\begin{matrix}\left\langle \dot{S}_{[0.6\ \ 0.7]}, \dot{S}_{[0.1\ \ 0.3]}\right\rangle, \\ \left\langle \ddot{S}_{[0.4\ \ 0.5]}, \ddot{S}_{[0.3\ \ 0.5]}\right\rangle\end{matrix}\right)$	$\left(\begin{matrix}\left\langle \dot{S}_{[0.3\ \ 0.4]}, \dot{S}_{[0.2\ \ 0.4]}\right\rangle, \\ \left\langle \ddot{S}_{[0.4\ \ 0.5]}, \ddot{S}_{[0.3\ \ 0.5]}\right\rangle\end{matrix}\right)$	$\left(\begin{matrix}\left\langle \dot{S}_{[0.7\ \ 0.8]}, \dot{S}_{[0.1\ \ 0.2]}\right\rangle, \\ \left\langle \ddot{S}_{[0.5\ \ 0.7]}, \ddot{S}_{[0.5\ \ 0.1]}\right\rangle\end{matrix}\right)$
T_{PZ}	$\left(\begin{matrix}\left\langle \dot{S}_{[0.7\ \ 0.8]}, \dot{S}_{[0.1\ \ 0.2]}\right\rangle, \\ \left\langle \ddot{S}_{[0.4\ \ 0.5]}, \ddot{S}_{[0.3\ \ 0.5]}\right\rangle\end{matrix}\right)$	$\left(\begin{matrix}\left\langle \dot{S}_{[0.5\ \ 0.5]}, \dot{S}_{[0.5\ \ 0.5]}\right\rangle, \\ \left\langle \ddot{S}_{[0.4\ \ 0.5]}, \ddot{S}_{[0.3\ \ 0.5]}\right\rangle\end{matrix}\right)$	$\left(\begin{matrix}\left\langle \dot{S}_{[0.6\ \ 0.7]}, \dot{S}_{[0.1\ \ 0.3]}\right\rangle, \\ \left\langle \ddot{S}_{[0.4\ \ 0.5]}, \ddot{S}_{[0.3\ \ 0.5]}\right\rangle\end{matrix}\right)$
T_{LQ}	$\left(\begin{matrix}\left\langle \dot{S}_{[0.6\ \ 0.7]}, \dot{S}_{[0.1\ \ 0.3]}\right\rangle, \\ \left\langle \ddot{S}_{[0.4\ \ 0.5]}, \ddot{S}_{[0.1\ \ 0.3]}\right\rangle\end{matrix}\right)$	$\left(\begin{matrix}\left\langle \dot{S}_{[0.3\ \ 0.4]}, \dot{S}_{[0.2\ \ 0.4]}\right\rangle, \\ \left\langle \ddot{S}_{[0.4\ \ 0.5]}, \ddot{S}_{[0.3\ \ 0.5]}\right\rangle\end{matrix}\right)$	$\left(\begin{matrix}\left\langle \dot{S}_{[0.6\ \ 0.7]}, \dot{S}_{[0.1\ \ 0.3]}\right\rangle, \\ \left\langle \ddot{S}_{[0.4\ \ 0.5]}, \ddot{S}_{[0.3\ \ 0.5]}\right\rangle\end{matrix}\right)$

Table 1

Table number 2

General linguistic Interval valued intuitionistic fuzzy soft expert Analysis matrix $\tilde{M}^{(2)}$provided by Expert or decision makes E_1, E_2, E_3.about the attributes A_1 , A_2 , A_3, of all the teams T_{IU} , T_{PZ} , T_{LQ}respectively.

	A_1	A_2	A_3
T_{IU}	$\left(\left\langle \dot{S}_{[0.7\ \ 0.8]}, \dot{S}_{[0.1\ \ 0.2]} \right\rangle, \left\langle \ddot{S}_{[0.5\ \ 0.7]}, \ddot{S}_{[0.1\ \ 0.3]} \right\rangle \right)$	$\left(\left\langle \dot{S}_{[0.6\ \ 0.7]}, \dot{S}_{[0.1\ \ 0.3]} \right\rangle, \left\langle \ddot{S}_{[0.4\ \ 0.5]}, \ddot{S}_{[0.3\ \ 0.5]} \right\rangle \right)$	$\left(\left\langle \dot{S}_{[0.6\ ,\ 0.7]}, \dot{S}_{[0.1\ ,\ 0.3]} \right\rangle, \left\langle \ddot{S}_{[0.4\ \ 0.5]}, \ddot{S}_{[0.3\ ,\ 0.5]} \right\rangle \right)$
T_{PZ}	$\left(\left\langle \dot{S}_{[0.4\ \ 0.5]}, \dot{S}_{[0.3\ \ 0.4]} \right\rangle, \left\langle \ddot{S}_{[0.5\ \ 0.7]}, \ddot{S}_{[0.1\ \ 0.3]} \right\rangle \right)$	$\left(\left\langle \dot{S}_{[0.6\ \ 0.7]}, \dot{S}_{[0.1\ \ 0.3]} \right\rangle, \left\langle \ddot{S}_{[0.4\ \ 0.5]}, \ddot{S}_{[0.3\ \ 0.5]} \right\rangle \right)$	$\left(\left\langle \dot{S}_{[0.5\ ,\ 0.5]}, \dot{S}_{[0.5\ ,\ 0.5]} \right\rangle, \left\langle \ddot{S}_{[0.4\ \ 0.5]}, \ddot{S}_{[0.3\ ,\ 0.5]} \right\rangle \right)$
T_{LQ}	$\left(\left\langle \dot{S}_{[0.5\ \ 0.5]}, \dot{S}_{[0.5\ \ 0.5]} \right\rangle, \left\langle \ddot{S}_{[0.5\ \ 0.7]}, \ddot{S}_{[0.1\ \ 0.3]} \right\rangle \right)$	$\left(\left\langle \dot{S}_{[0.7\ \ 0.8]}, \dot{S}_{[0.1\ \ 0.2]} \right\rangle, \left\langle \ddot{S}_{[0.4\ \ 0.5]}, \ddot{S}_{[0.3\ \ 0.5]} \right\rangle \right)$	$\left(\left\langle \dot{S}_{[0.6\ ,\ 0.7]}, \dot{S}_{[0.1\ ,\ 0.3]} \right\rangle, \left\langle \ddot{S}_{[0.4\ ,\ 0.5]}, \ddot{S}_{[0.3\ ,\ 0.5]} \right\rangle \right)$

Table 2

Table 3

General linguistic Interval valued intuitionistic fuzzy soft expert Analysis matrix $\tilde{M}^{(3)}$provided by Expert or decision makes E_1, E_2, E_3.about the attributes A_1 , A_2 , A_3, of all the teams T_{IU} , T_{PZ} , T_{LQ}respectively.

	A_1	A_2	A_3
T_{IU}	$\left(\left\langle \dot{S}_{[0.6\ \ 0.7]}, \dot{S}_{[0.1\ \ 0.3]} \right\rangle, \left\langle \ddot{S}_{[0.5\ \ 0.7]}, \ddot{S}_{[0.1\ \ 0.3]} \right\rangle \right)$	$\left(\left\langle \dot{S}_{[0.7\ ,\ 0.8]}, \dot{S}_{[0.1\ \ 0.2]} \right\rangle, \left\langle \ddot{S}_{[0.5\ ,\ 0.7]}, \ddot{S}_{[0.1\ \ 0.3]} \right\rangle \right)$	$\left(\left\langle \dot{S}_{[0.5\ \ 0.5]}, \dot{S}_{[0.5\ \ 0.5]} \right\rangle, \left\langle \ddot{S}_{[0.7\ \ 0.9]}, \ddot{S}_{[0\ \ 0.1]} \right\rangle \right.$
T_{PZ}	$\left(\left\langle \dot{S}_{[0.7\ \ 0.8]}, \dot{S}_{[0.1\ \ 0.2]} \right\rangle, \left\langle \ddot{S}_{[0.5\ \ 0.7]}, \ddot{S}_{[0.1\ \ 0.3]} \right\rangle \right)$	$\left(\left\langle \dot{S}_{[0.6\ ,\ 0.7]}, \dot{S}_{[0.1\ \ 0.3]} \right\rangle, \left\langle \ddot{S}_{[0.5\ ,\ 0.7]}, \ddot{S}_{[0.1\ \ 0.3]} \right\rangle \right)$	$\left(\left\langle \dot{S}_{[0.6\ \ 0.7]}, \dot{S}_{[0.1\ \ 0.3]} \right\rangle, \left\langle \ddot{S}_{[0.7\ \ 0.9]}, \ddot{S}_{[0\ \ 0.1]} \right\rangle \right.$
T_{LQ}	$\left(\left\langle \dot{S}_{[0.5\ \ 0.5]}, \dot{S}_{[0.5\ \ 0.5]} \right\rangle, \left\langle \ddot{S}_{[0.5\ \ 0.7]}, \ddot{S}_{[0.1\ \ 0.3]} \right\rangle \right)$	$\left(\left\langle \dot{S}_{[0.5\ ,\ 0.5]}, \dot{S}_{[0.5\ \ 0.5]} \right\rangle, \left\langle \ddot{S}_{[0.5\ \ 0.7]}, \ddot{S}_{[0.1\ \ 0.3]} \right\rangle \right)$	$\left(\left\langle \dot{S}_{[0.7\ \ 0.8]}, \dot{S}_{[0.1\ \ 0.2]} \right\rangle, \left\langle \ddot{S}_{[0.7\ \ 0.9]}, \ddot{S}_{[0\ \ 0.1]} \right\rangle \right.$

Table 3

By using

$$GLIVIFSEWA(\hat{S}_1,\hat{S}_2,\hat{S}_3,....\hat{S}_n)=\begin{pmatrix}\left\langle \begin{matrix}\dot{S}_{\mathfrak{t}}\left[1-\prod_{i=1}^{n}\left(1-\frac{\check{\xi}_{m(i)}}{\mathfrak{t}}\right)^{\varpi i},1-\prod_{i=1}^{n}\left(1-\frac{\check{\xi}'_{m(i)}}{\mathfrak{t}}\right)^{\varpi i}\right], \\ \dot{S}_{\mathfrak{t}}\left[\prod_{i=1}^{n}\left(\frac{\tilde{\Psi}_{(m(i))}}{\mathfrak{t}}\right)^{\varpi i},\prod_{i=1}^{n}\left(\frac{\tilde{\Psi}'_{m(i)}}{\mathfrak{t}}\right)^{\varpi i}\right]\end{matrix}\right\rangle, \\ \left\langle \begin{matrix}\ddot{S}_{\mathfrak{t}'}\left[1-\prod_{i=1}^{n}\left(1-\frac{\check{\mu}_{m(i)}}{\mathfrak{t}'}\right)^{\varpi i},1-\prod_{i=1}^{n}\left(1-\frac{\check{\mu}'_{m(i)}}{\mathfrak{t}'}\right)^{\varpi i}\right], \\ \ddot{S}_{\mathfrak{t}'}\left[\prod_{i=1}^{n}\left(\frac{\nu_{(m(i))}}{\mathfrak{t}}\right)^{\varpi i},\prod_{i=1}^{n}\left(\frac{\nu'_{m(i)}}{\mathfrak{t}}\right)^{\varpi i}\right]\end{matrix}\right\rangle\end{pmatrix}$$

Here we calculate the evaluated values given by all the experts E_1,E_2 and E_3 , giving their prefrences or opinons about the attribute A_1 of team T_{IU}.

$$=\begin{pmatrix}\left\langle \begin{matrix}\dot{S}_{6\left[1-\left(1-\frac{0.6}{6}\right)^{0.25}\left(1-\frac{0.7}{6}\right)^{0.5}\left(1-\frac{0.6}{6}\right)^{0.25},1-\left(1-\frac{0.7}{6}\right)^{0.25}\left(1-\frac{0.8}{6}\right)^{0.5}\left(1-\frac{0.7}{6}\right)^{0.25}\right]}, \\ \dot{S}_{6\left[\left(\frac{0.1}{6}\right)^{0.25}\left(\frac{0.1}{6}\right)^{0.5}\left(\frac{0.1}{6}\right)^{0.25},\left(\frac{0.3}{6}\right)^{0.25}\left(\frac{0.3}{6}\right)^{0.5}\left(\frac{0.3}{6}\right)^{0.25}\right]}\end{matrix}\right\rangle, \\ \left\langle \begin{matrix}\ddot{S}_{4\left[1-\left(1-\frac{0.4}{4}\right)^{0.25}\left(1-\frac{0.5}{4}\right)^{0.5}\left(1-\frac{0.5}{4}\right)^{0.25},1-\left(1-\frac{0.5}{4}\right)^{0.25}\left(1-\frac{0.7}{4}\right)^{0.5}\left(1-\frac{0.7}{4}\right)^{0.25}\right]}, \\ \ddot{S}_{4\left[\left(\frac{0.3}{4}\right)^{0.25}\left(\frac{0.1}{4}\right)^{0.5}\left(\frac{0.1}{4}\right)^{0.25},\left(\frac{0.5}{4}\right)^{0.25}\left(\frac{0.3}{4}\right)^{0.5}\left(\frac{0.3}{4}\right)^{0.25}\right]}\end{matrix}\right\rangle,\end{pmatrix}$$

$$=\begin{pmatrix}\left\langle\dot{S}_{6[1-0.89163,\ 1-0.87496]},\dot{S}_{6[0.01667,\ 0.10574]}\right\rangle, \\ \left\langle\ddot{S}_{4[0.011881,\ 0.162774]},\ddot{S}_{4[0.0329,\ 0.08522]}\right\rangle\end{pmatrix}$$

$$=\begin{pmatrix}\left\langle\dot{S}_{[0.65022,\ 0.75]},\dot{S}_{[0.10002,\ 0.6344]}\right\rangle, \\ \left\langle\ddot{S}_{[0.47524,\ 0.651096]},\ddot{S}_{4[0.1316,\ 0.34087]}\right\rangle\end{pmatrix}$$

Consider Team T_{IU}, Attribute A_2, Experts E_1, E_2, E_3

$$= \begin{pmatrix} \left\langle \begin{array}{c} \dot{S}6_{\left[1-\left(1-\frac{0.3}{6}\right)^{0.25}\left(1-\frac{0.6}{6}\right)^{0.5}\left(1-\frac{0.7}{6}\right)^{0.25},\ 1-\left(1-\frac{0.4}{6}\right)^{0.25}\left(1-\frac{0.7}{6}\right)^{0.5}\left(1-\frac{0.8}{6}\right)^{0.25}\right]}, \\ \dot{S}6_{\left[\left(\frac{0.2}{6}\right)^{0.25}\left(\frac{0.1}{6}\right)^{0.5}\left(\frac{0.1}{6}\right)^{0.25},\ \left(\frac{0.4}{6}\right)^{0.25}\left(\frac{0.3}{6}\right)^{0.5}\left(\frac{0.2}{6}\right)^{0.25}\right]} \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \ddot{S}4_{\left[1-\left(1-\frac{0.4}{4}\right)^{0.25}\left(1-\frac{0.4}{4}\right)^{0.5}\left(1-\frac{0.5}{4}\right)^{0.25},\ 1-\left(1-\frac{0.5}{4}\right)^{0.25}\left(1-\frac{0.5}{4}\right)^{0.5}\left(1-\frac{0.7}{4}\right)^{0.25}\right]}, \\ \ddot{S}4_{\left[\left(\frac{0.3}{44}\right)^{0.25}\left(\frac{0.3}{4}\right)^{0.5}\left(\frac{0.1}{4}\right)^{0.25},\ \left(\frac{0.5}{4}\right)^{0.25}\left(\frac{0.3}{4}\right)^{0.5}\left(\frac{0.3}{4}\right)^{0.25}\right]} \end{array} \right\rangle, \end{pmatrix}$$

$$= \begin{pmatrix} \left\langle \dot{S}_{[0.55203,\ 0.6521]}, \dot{S}6_{[0.118921,\ 0.2913]} \right\rangle, \\ \left\langle \ddot{S}_{[0.4253,\ 0.55111]}, \ddot{S}4_{[0.22795,\ 0.44006]} \right\rangle \end{pmatrix}$$

Here we calculate the evaluated values given by all the experts E_1,E_2 and E_3 , giving their prefrences or opinons about the attribute A_2 of team T_{IU}.

$$= \begin{pmatrix} \left\langle \begin{array}{c} \dot{S}6_{\left[1-\left(1-\frac{0.7}{6}\right)^{0.25}\left(1-\frac{0.6}{6}\right)^{0.5}\left(1-\frac{0.5}{6}\right)^{0.25},1-\left(1-\frac{0.8}{6}\right)^{0.25}\left(1-\frac{0.7}{6}\right)^{0.5}\left(1-\frac{0.5}{6}\right)^{0.25}\right]}, \\ \dot{S}6_{\left[\left(\frac{0.1}{6}\right)^{0.25}\left(\frac{0.1}{6}\right)^{0.5}\left(\frac{0.5}{6}\right)^{0.25},\left(\frac{0.2}{6}\right)^{0.25}\left(\frac{0.3}{6}\right)^{0.5}\left(\frac{0.5}{6}\right)^{0.25}\right]} \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \ddot{S}4_{\left[1-\left(1-\frac{0.5}{4}\right)^{0.25}\left(1-\frac{0.4}{4}\right)^{0.5}\left(1-\frac{0.7}{4}\right)^{0.25},\ 1-\left(1-\frac{0.7}{4}\right)^{0.25}\left(1-\frac{0.5}{4}\right)^{0.5}\left(1-\frac{0.9}{4}\right)^{0.25}\right]}, \\ \ddot{S}4_{\left[\left(\frac{0.5}{4}\right)^{0.25}\left(\frac{0.3}{4}\right)^{0.5}\left(\frac{0.05}{4}\right)^{0.25},\left(\frac{0.7}{4}\right)^{0.25}\left(\frac{0.5}{4}\right)^{0.5}\left(\frac{0.1}{4}\right)^{0.25}\right]} \end{array} \right\rangle, \end{pmatrix}$$

$$= \begin{pmatrix} \left\langle \dot{S}_{[0.6005,\ 0.67611]}, \dot{S}6_{[0.1495,\ 0.20534]} \right\rangle, \\ \left\langle \ddot{S}_{[0.5022,\ 0.6542]}, \ddot{S}4_{[0.21779,\ 0.36371]} \right\rangle \end{pmatrix}$$

Here we calculate the evaluated values given by all the experts E_1,E_2 and E_3 , giving their prefrences or opinons about the attribute A_1 of team T_{PZ}.

$$= \left(\begin{array}{c} \left\langle \begin{array}{c} \dot{S}6_{\left[1-\left(1-\frac{0.7}{6}\right)^{0.25}\left(1-\frac{0.4}{6}\right)^{0.5}\left(1-\frac{0.7}{6}\right)^{0.25},\ 1-\left(1-\frac{0.8}{6}\right)^{0.25}\left(1-\frac{0.5}{6}\right)^{0.5}\left(1-\frac{0.8}{6}\right)^{0.25}\right]}, \\ \dot{S}6_{\left[\left(\frac{0.1}{6}\right)^{0.25}\left(\frac{0.3}{6}\right)^{0.5}\left(\frac{0.1}{6}\right)^{0.25},\ \left(\frac{0.2}{6}\right)^{0.25}\left(\frac{0.4}{6}\right)^{0.5}\left(\frac{0.2}{6}\right)^{0.25}\right]} \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \ddot{S}4_{\left[1-\left(1-\frac{0.4}{4}\right)^{0.25}\left(1-\frac{0.5}{4}\right)^{0.5}\left(1-\frac{0.5}{4}\right)^{0.25},\ 1-\left(1-\frac{0.5}{4}\right)^{0.25}\left(1-\frac{0.7}{4}\right)^{0.5}\left(1-\frac{0.7}{4}\right)^{0.25}\right]}, \\ \ddot{S}4_{\left[\left(\frac{0.3}{44}\right)^{0.25}\left(\frac{0.1}{4}\right)^{0.5}\left(\frac{0.1}{4}\right)^{0.25},\ \left(\frac{0.5}{4}\right)^{0.25}\left(\frac{0.3}{4}\right)^{0.5}\left(\frac{0.3}{4}\right)^{0.25}\right]} \end{array} \right\rangle, \end{array} \right)$$

$$= \left(\begin{array}{c} \left\langle \dot{S}_{[0.5521,\ 0.6521]}, \dot{S}6_{[0.17321,\ 0.2828]} \right\rangle, \\ \left\langle \ddot{S}_{[0.4753,\ 0.651098]}, \ddot{S}4_{[0.13161,\ 0.34087]} \right\rangle \end{array} \right)$$

Here we calculate the evaluated values given by all the experts E_1,E_2 and E_3 , giving their prefrences or opinons about the attribute A_3 of team T_{PZ}

$$= \left(\begin{array}{c} \left\langle \begin{array}{c} \dot{S}6_{\left[1-\left(1-\frac{0.5}{6}\right)^{0.25}\left(1-\frac{0.6}{6}\right)^{0.5}\left(1-\frac{0.6}{6}\right)^{0.25},\ 1-\left(1-\frac{0.5}{6}\right)^{0.25}\left(1-\frac{0.7}{6}\right)^{0.5}\left(1-\frac{0.7}{6}\right)^{0.25}\right]}, \\ \dot{S}6_{\left[\left(\frac{0.5}{6}\right)^{0.25}\left(\frac{0.1}{6}\right)^{0.5}\left(\frac{0.1}{6}\right)^{0.25},\ \left(\frac{0.5}{6}\right)^{0.25}\left(\frac{0.3}{6}\right)^{0.5}\left(\frac{0.3}{6}\right)^{0.25}\right]} \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \ddot{S}4_{\left[1-\left(1-\frac{0.4}{4}\right)^{0.25}\left(1-\frac{0.4}{4}\right)^{0.5}\left(1-\frac{0.5}{4}\right)^{0.25},\ 1-\left(1-\frac{0.5}{4}\right)^{0.25}\left(1-\frac{0.5}{4}\right)^{0.5}\left(1-\frac{0.7}{4}\right)^{0.25}\right]}, \\ \ddot{S}4_{\left[\left(\frac{0.3}{4}\right)^{0.25}\left(\frac{0.3}{4}\right)^{0.5}\left(\frac{0.1}{4}\right)^{0.25},\ \left(\frac{0.5}{4}\right)^{0.25}\left(\frac{0.5}{4}\right)^{0.5}\left(\frac{0.3}{4}\right)^{0.25}\right]} \end{array} \right\rangle, \end{array} \right)$$

$$= \left(\begin{array}{c} \left\langle \dot{S}_{[0.5752,\ 0.65069]}, \dot{S}6_{[0.1495,\ 0.340897]} \right\rangle, \\ \left\langle \ddot{S}_{[0.4253,\ 0.55111]}, \ddot{S}4_{[0.22795,\ 0.4400559]} \right\rangle \end{array} \right)$$

Here we calculate the evaluated values given by all the experts E_1,E_2 and E_3 , giving their prefrences or opinons about the attribute A_3 of team T_{PZ}.

$$= \begin{pmatrix} \left\langle \begin{array}{c} \dot{S}6_{\left[1-\left(1-\frac{0.6}{6}\right)^{0.25}\left(1-\frac{0.5}{6}\right)^{0.5}\left(1-\frac{0.7}{6}\right)^{0.25}, \; 1-\left(1-\frac{0.7}{6}\right)^{0.25}\left(1-\frac{0.5}{6}\right)^{0.5}\left(1-\frac{0.7}{6}\right)^{0.25}\right]}, \\ \dot{S}6_{\left[\left(\frac{0.1}{6}\right)^{0.25}\left(\frac{0.5}{6}\right)^{0.5}\left(\frac{0.1}{6}\right)^{0.25}, \; \left(\frac{0.3}{6}\right)^{0.25}\left(\frac{0.5}{6}\right)^{0.5}\left(\frac{0.3}{6}\right)^{0.25}\right]} \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \ddot{S}4_{\left[1-\left(1-\frac{0.4}{4}\right)^{0.25}\left(1-\frac{0.4}{4}\right)^{0.5}\left(1-\frac{0.7}{4}\right)^{0.25}, \; 1-\left(1-\frac{0.5}{4}\right)^{0.25}\left(1-\frac{0.5}{4}\right)^{0.5}\left(1-\frac{0.9}{4}\right)^{0.25}\right]}, \\ \ddot{S}4_{\left[\left(\frac{0.3}{4}\right)^{0.25}\left(\frac{0.1}{4}\right)^{0.5}\left(\frac{0.5}{4}\right)^{0.25}, \; \left(\frac{0.5}{4}\right)^{0.25}\left(\frac{0.5}{4}\right)^{0.5}\left(\frac{0.1}{4}\right)^{0.25}\right]} \end{array} \right\rangle, \end{pmatrix}$$

$$= \begin{pmatrix} \left\langle \dot{S}_{[0.5764, \; 0.60093]}, \dot{S}6_{[0.22361, \; 0.3873]} \right\rangle, \\ \left\langle \ddot{S}_{[0.4775, \; 0.604596]}, \ddot{S}4_{[0.1917, \; 0.3344]} \right\rangle \end{pmatrix}$$

Here we calculate the evaluated values given by all the experts E_1,E_2 and E_3 , giving their prefrences or opinons about the attribute A_1 of team T_{LQ}.

$$= \begin{pmatrix} \left\langle \begin{array}{c} \dot{S}6_{\left[1-\left(1-\frac{0.6}{6}\right)^{0.25}\left(1-\frac{0.5}{6}\right)^{0.5}\left(1-\frac{0.5}{6}\right)^{0.25}, \; 1-\left(1-\frac{0.7}{6}\right)^{0.25}\left(1-\frac{0.5}{6}\right)^{0.5}\left(1-\frac{0.5}{6}\right)^{0.25}\right]}, \\ \dot{S}6_{\left[\left(\frac{0.1}{6}\right)^{0.25}\left(\frac{0.5}{6}\right)^{0.5}\left(\frac{0.5}{6}\right)^{0.25}, \; \left(\frac{0.3}{6}\right)^{0.25}\left(\frac{0.5}{6}\right)^{0.5}\left(\frac{0.5}{6}\right)^{0.25}\right]} \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \ddot{S}4_{\left[1-\left(1-\frac{0.4}{4}\right)^{0.25}\left(1-\frac{0.5}{4}\right)^{0.5}\left(1-\frac{0.5}{4}\right)^{0.25}, \; 1-\left(1-\frac{0.5}{4}\right)^{0.25}\left(1-\frac{0.7}{4}\right)^{0.5}\left(1-\frac{0.7}{4}\right)^{0.25}\right]}, \\ \ddot{S}4_{\left[\left(\frac{0.1}{44}\right)^{0.25}\left(\frac{0.1}{4}\right)^{0.5}\left(\frac{0.1}{4}\right)^{0.25}, \; \left(\frac{0.3}{4}\right)^{0.25}\left(\frac{0.3}{4}\right)^{0.5}\left(\frac{0.3}{4}\right)^{0.25}\right]} \end{array} \right\rangle \end{pmatrix}$$

$$= \begin{pmatrix} \left\langle \dot{S}_{[0.5252, \; 0.55069]}, \dot{S}6_{[0.3344, \; 0.4400558]} \right\rangle, \\ \left\langle \ddot{S}_{[0.4753, \; 0.6511]}, \ddot{S}4_{[0.1, \; 0.3]} \right\rangle \end{pmatrix}$$

Here we calculate the evaluated values given by all the experts E_1,E_2 and E_3 , giving their prefrences or opinons about the attribute A_2 of team T_{LQ}.

$$= \begin{pmatrix} \left\langle \begin{array}{c} \dot{S}6_{\left[1-\left(1-\frac{0.3}{6}\right)^{0.25}\left(1-\frac{0.7}{6}\right)^{0.5}\left(1-\frac{0.5}{6}\right)^{0.25},\ 1-\left(1-\frac{0.4}{6}\right)^{0.25}\left(1-\frac{0.8}{6}\right)^{0.5}\left(1-\frac{0.5}{6}\right)^{0.25}\right]}, \\ \dot{S}6_{\left[\left(\frac{0.2}{6}\right)^{0.25}\left(\frac{0.1}{6}\right)^{0.5}\left(\frac{0.5}{6}\right)^{0.25},\ \left(\frac{0.4}{6}\right)^{0.25}\left(\frac{0.2}{6}\right)^{0.5}\left(\frac{0.5}{6}\right)^{0.25}\right]} \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \ddot{S}4_{\left[1-\left(1-\frac{0.4}{4}\right)^{0.25}\left(1-\frac{0.4}{4}\right)^{0.5}\left(1-\frac{0.5}{4}\right)^{0.25},\ 1-\left(1-\frac{0.5}{4}\right)^{0.25}\left(1-\frac{0.5}{4}\right)^{0.5}\left(1-\frac{0.7}{4}\right)^{0.25}\right]}, \\ \ddot{S}4_{\left[\left(\frac{0.3}{4}\right)^{0.25}\left(\frac{0.3}{4}\right)^{0.5}\left(\frac{0.1}{4}\right)^{0.25},\ \left(\frac{0.5}{4}\right)^{0.25}\left(\frac{0.5}{4}\right)^{0.5}\left(\frac{0.3}{4}\right)^{0.25}\right]} \end{array} \right\rangle, \end{pmatrix}$$

$$= \begin{pmatrix} \left\langle \dot{S}_{[0.5525,\ 0.62796]}, \dot{S}6_{[0.17783,\ 0.29907]} \right\rangle, \\ \left\langle \ddot{S}_{[0.4253,\ 0.55110]}, \ddot{S}4_{[0.22795,\ 0.440056]} \right\rangle \end{pmatrix}$$

Here we calculate the evaluated values given by all the experts E_1,E_2 and E_3 , giving their prefrences or opinons about the attribute A_3 of team T_{LQ}.

$$= \begin{pmatrix} \left\langle \begin{array}{c} \dot{S}6_{\left[1-\left(1-\frac{0.6}{6}\right)^{0.25}\left(1-\frac{0.6}{6}\right)^{0.5}\left(1-\frac{0.7}{6}\right)^{0.25},\ 1-\left(1-\frac{0.7}{6}\right)^{0.25}\left(1-\frac{0.7}{6}\right)^{0.5}\left(1-\frac{0.8}{6}\right)^{0.25}\right]}, \\ \dot{S}6_{\left[\left(\frac{0.1}{6}\right)^{0.25}\left(\frac{0.1}{6}\right)^{0.5}\left(\frac{0.1}{6}\right)^{0.25},\ \left(\frac{0.3}{6}\right)^{0.25}\left(\frac{0.3}{6}\right)^{0.5}\left(\frac{0.2}{6}\right)^{0.25}\right]} \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \ddot{S}4_{\left[1-\left(1-\frac{0.4}{4}\right)^{0.25}\left(1-\frac{0.4}{4}\right)^{0.5}\left(1-\frac{0.7}{4}\right)^{0.25},\ 1-\left(1-\frac{0.5}{4}\right)^{0.25}\left(1-\frac{0.5}{4}\right)^{0.5}\left(1-\frac{0.9}{4}\right)^{0.25}\right]}, \\ \ddot{S}4_{\left[\left(\frac{0.3}{4}\right)^{0.25}\left(\frac{0.3}{4}\right)^{0.5}\left(\frac{0.05}{4}\right)^{0.25},\ \left(\frac{0.5}{4}\right)^{0.25}\left(\frac{0.5}{4}\right)^{0.5}\left(\frac{0.25}{4}\right)^{0.25}\right]} \end{array} \right\rangle, \end{pmatrix}$$

$$= \begin{pmatrix} \left\langle \dot{S}_{[0.6252,\ 0.7252]}, \dot{S}6_{[0.1,\ 0.2711]} \right\rangle, \\ \left\langle \ddot{S}_{[0.4775,\ 0.6064]}, \ddot{S}4_{[0.1917,\ 0.42045]} \right\rangle \end{pmatrix}$$

Table 4

Collective General linguistic Interval valued intuitionistic fuzzy soft expert decision matrix $\tilde{M}$, by using GLIVIFSEWA operators.

	A_1	A_2	A_3
T_{IU}	$\left(\begin{array}{c}\left\langle \dot{S}_{[0.65022 \quad 0.75]},\ \dot{S}_{[0.10002 \quad 0.6344]}\right\rangle, \\ \left\langle \ddot{S}_{[0.4752 \quad 0.6511]},\ \ddot{S}_{[0.1316 \quad 0.3409]}\right\rangle\end{array}\right)$	$\left(\begin{array}{c}\left\langle \dot{S}_{[0.5520 \quad 0.6521]},\ \dot{S}_{[0.1189 \quad 0.2913]}\right\rangle, \\ \left\langle \ddot{S}_{[0.4253 \quad 0.5511]},\ \ddot{S}_{[0.2280 \quad 0.4401]}\right\rangle\end{array}\right)$	$\left(\begin{array}{c}\left\langle \dot{S}_{[0.6005 \quad 0.6761]},\ \dot{S}_{[0.1495 \quad 0.2053]}\right\rangle, \\ \left\langle \ddot{S}_{[0.5022 \quad 0.6542]},\ \ddot{S}_{[0.21779 \quad 0.3637]}\right\rangle\end{array}\right)$
T_{PZ}	$\left(\begin{array}{c}\left\langle \dot{S}_{[0.5521 \quad 0.6521]},\ \dot{S}_{[0.1732 \quad 0.2828]}\right\rangle, \\ \left\langle \ddot{S}_{[0.4753 \quad 0.6511]},\ \ddot{S}_{[0.1316 \quad 0.3409]}\right\rangle\end{array}\right)$	$\left(\begin{array}{c}\left\langle \dot{S}_{[0.5752 \quad 0.6507]},\ \dot{S}_{[0.1495 \quad 0.34096]}\right\rangle, \\ \left\langle \ddot{S}_{[0.4253 \quad 0.5511]},\ \ddot{S}_{[0.2280 \quad 0.4401]}\right\rangle\end{array}\right)$	$\left(\begin{array}{c}\left\langle \dot{S}_{[0.5756 \quad 0.6009]},\ \dot{S}_{[0.2236 \quad 0.3873]}\right\rangle, \\ \left\langle \ddot{S}_{[0.4775 \quad 0.6046]},\ \ddot{S}_{[0.1917 \quad 0.3344]}\right\rangle\end{array}\right)$
T_{LQ}	$\left(\begin{array}{c}\left\langle \dot{S}_{[0.5252 \quad 0.5507]},\ \dot{S}_{[0.3344 \quad 0.4401]}\right\rangle, \\ \left\langle \ddot{S}_{[0.4753 \quad 0.6511]},\ \ddot{S}_{[0.1 \quad 0.3]}\right\rangle\end{array}\right)$	$\left(\begin{array}{c}\left\langle \dot{S}_{[0.5525 \quad 0.6280]},\ \dot{S}_{[0.1778 \quad 0.2991]}\right\rangle, \\ \left\langle \ddot{S}_{[0.4253 \quad 0.5511]},\ \ddot{S}_{[0.2280 \quad 0.4401]}\right\rangle\end{array}\right)$	$\left(\begin{array}{c}\left\langle \dot{S}_{[0.6252 \quad 0.7252]},\ \dot{S}_{[0.1 \quad 0.2711]}\right\rangle, \\ \left\langle \ddot{S}_{[0.4775 \quad 0.6046]},\ \ddot{S}_{[0.1917 \quad 0.4205]}\right\rangle\end{array}\right)$

Here we calculate the evaluated values given by all the expert E_1 , giving their prefrences or opinons about the attribute $A_1,\ A_2$, and A_3 of team T_{IU}.

$$= \left(\begin{array}{c} \left\langle \begin{array}{c} \dot{S}6_{\left[\begin{array}{c} 1-\left(1-\frac{0.6502}{6}\right)^{0.3}\left(1-\frac{0.552}{6}\right)^{0.3}\left(1-\frac{0.6005}{6}\right)^{0.4}, \\ 1-\left(1-\frac{0.75}{6}\right)^{0.3}\left(1-\frac{0.6521}{6}\right)^{0.3}\left(1-\frac{0.6761}{6}\right)^{0.4} \end{array}\right]}, \\ \dot{S}6_{\left[\begin{array}{c} \left(\frac{0.1000}{6}\right)^{0.3}\left(\frac{0.1189}{6}\right)^{0.3}\left(\frac{0.1495}{6}\right)^{0.4}, \\ \left(\frac{0.6344}{6}\right)^{0.3}\left(\frac{0.2913}{6}\right)^{0.3}\left(\frac{0.2053}{6}\right)^{0.4} \end{array}\right]} \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \ddot{S}4_{\left[\begin{array}{c} 1-\left(1-\frac{0.4752}{4}\right)^{0.3}\left(1-\frac{0.4253}{4}\right)^{0.3}\left(1-\frac{0.5022}{4}\right)^{0.4}, \\ 1-\left(1-\frac{0.6511}{4}\right)^{0.3}\left(1-\frac{0.5511}{4}\right)^{0.3}\left(1-\frac{0.6542}{4}\right)^{0.4} \end{array}\right]} \\ ,\ddot{S}4_{\left[\begin{array}{c} \left(\frac{0.1316}{44}\right)^{0.3}\left(\frac{0.2280}{4}\right)^{0.3}\left(\frac{0.2178}{4}\right)^{0.4}, \\ \left(\frac{0.3409}{4}\right)^{0.3}\left(\frac{0.4401}{4}\right)^{0.3}\left(\frac{0.3637}{4}\right)^{0.4} \end{array}\right]} \end{array} \right\rangle, \end{array} \right)$$

$$= \left(\left\langle \dot{S}_{[0.6010,\ 0.69122]}, \dot{S}6_{[0.1237,\ 0.3199]}\right\rangle, \left\langle \ddot{S}_{[0.4712,\ 0.6227]}, \ddot{S}4_{[0.1898,\ 0.3777]}\right\rangle\right)$$

Here we calculate the evaluated values given by all the expert E_1 , giving their prefrences or opinons about the attribute $A_1,\ A_2$, and A_3 of team T_{pz}.

$$= \left(\begin{array}{c} \left\langle \begin{array}{c} \dot{S}6_{\left[\begin{array}{c} 1-\left(1-\frac{0.5521}{6}\right)^{0.3}\left(1-\frac{0.5752}{6}\right)^{0.3}\left(1-\frac{0.5756}{6}\right)^{0.4}, \\ 1-\left(1-\frac{0.6521}{6}\right)^{0.3}\left(1-\frac{0.6507}{6}\right)^{0.3}\left(1-\frac{0.6009}{6}\right)^{0.4} \end{array} \right]}, \\ \dot{S}6_{\left[\begin{array}{c} \left(\frac{0.1732}{6}\right)^{0.3}\left(\frac{0.1495}{6}\right)^{0.3}\left(\frac{0.2236}{6}\right)^{0.4}, \\ \left(\frac{0.2828}{6}\right)^{0.3}\left(\frac{0.3409}{6}\right)^{0.3}\left(\frac{0.3873}{6}\right)^{0.4} \end{array} \right]} \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \ddot{S}_4{}_{\left[\begin{array}{c} 1-\left(1-\frac{0.4753}{4}\right)^{0.3}\left(1-\frac{0.4253}{4}\right)^{0.3}\left(1-\frac{0.4775}{4}\right)^{0.4}, \\ 1-\left(1-\frac{0.6511}{4}\right)^{0.3}\left(1-\frac{0.5511}{4}\right)^{0.3}\left(1-\frac{0.6046}{4}\right)^{0.4} \end{array} \right]}, \\ \ddot{S}_4{}_{\left[\begin{array}{c} \left(\frac{0.1316}{44}\right)^{0.3}\left(\frac{0.2280}{4}\right)^{0.3}\left(\frac{0.1917}{4}\right)^{0.4}, \\ \left(\frac{0.3409}{4}\right)^{0.3}\left(\frac{0.4401}{4}\right)^{0.3}\left(\frac{0.3344}{4}\right)^{0.4} \end{array} \right]} \end{array} \right\rangle, \end{array} \right)$$

$$= \left(\begin{array}{c} \left\langle \dot{S}_{[0.56884,\ 0.6313]}, \dot{S}6_{[0.1237,\ 0.3199]} \right\rangle, \\ \left\langle \ddot{S}_{[0.1835,\ 0.3392]}, \ddot{S}_4{}_{[0.4613,\ 0.60247]} \right\rangle \end{array} \right)$$

Here we calculate the evaluated values given by all the expert E_1 , giving their prefrences or opinons about the attribute A_1, A_2, and A_3 of teamT_{LQ}.

$$= \left(\begin{array}{c} \left\langle \begin{array}{c} \dot{S}6_{\left[1-\left(1-\frac{0.5252}{6}\right)^{0.3}\left(1-\frac{0.5525}{6}\right)^{0.3}\left(1-\frac{0.6252}{6}\right)^{0.4},\ 1-\left(1-\frac{0.5507}{6}\right)^{0.3}\left(1-\frac{0.628}{6}\right)^{0.3}\left(1-\frac{0.7252}{6}\right)^{0.4}\right]}, \\ \dot{S}6_{\left[\left(\frac{0.3344}{6}\right)^{0.3}\left(\frac{0.1778}{6}\right)^{0.3}\left(\frac{0.1}{6}\right)^{0.4},\ \left(\frac{0.4401}{6}\right)^{0.3}\left(\frac{0.2991}{6}\right)^{0.3}\left(\frac{0.2711}{6}\right)^{0.4}\right]} \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \ddot{S}_4{}_{\left[1-\left(1-\frac{0.4753}{4}\right)^{0.3}\left(1-\frac{0.4253}{4}\right)^{0.3}\left(1-\frac{0.4475}{4}\right)^{0.4},\ 1-\left(1-\frac{0.6511}{4}\right)^{0.3}\left(1-\frac{0.5511}{4}\right)^{0.3}\left(1-\frac{0.6046}{4}\right)^{0.4}\right]}, \\ \ddot{S}_4{}_{\left[\left(\frac{0.1}{44}\right)^{0.3}\left(\frac{0.2280}{4}\right)^{0.3}\left(\frac{0.1917}{4}\right)^{0.4},\ \left(\frac{0.3}{4}\right)^{0.3}\left(\frac{0.4401}{4}\right)^{0.3}\left(\frac{0.4205}{4}\right)^{0.4}\right]} \end{array} \right\rangle, \end{array} \right)$$

$$= \left(\begin{array}{c} \left\langle \dot{S}_{[0.5735,\ 0.6442]}, \dot{S}6_{[0.1707,\ 0.3229]} \right\rangle, \\ \left\langle \ddot{S}_{[0.4613,\ 0.6067]}, \ddot{S}_4{}_{[0.1661,\ 0.3852]} \right\rangle \end{array} \right)$$

Hence

$$\begin{array}{ll} \mathrm{T}_{IU} & \left(\begin{array}{c} \left\langle \dot{S}_{[0.6010 \quad 0.6912]}, \dot{S}_{[0.1237 \quad 0.3199]} \right\rangle, \\ \left\langle \ddot{S}_{[0.4712 \quad 0.6227]}, \ddot{S}_{[0.1898 \quad 0.3777]} \right\rangle \end{array} \right) \\ \mathrm{T}_{PZ} & \left(\begin{array}{c} \left\langle \dot{S}_{[0.5684 \quad 0.6313]}, \dot{S}_{[0.1835 \quad 0.3392]} \right\rangle, \\ \left\langle \ddot{S}_{[0.4613 \quad 0.6027]}, \ddot{S}_{[0.1803 \quad 0.3652]} \right\rangle \end{array} \right) \\ \mathrm{T}_{LQ} & \left(\begin{array}{c} \left\langle \dot{S}_{[0.5735 \quad 0.6442]}, \dot{S}_{[0.1707 \quad 0.3229]} \right\rangle, \\ \left\langle \ddot{S}_{[0.4613 \quad 0.6067]}, \ddot{S}_{[0.1661 \quad 0.3852]} \right\rangle \end{array} \right) \end{array}$$

Since for comparison of any two GLIVIFSESs we use score function so,
By using **??**

$$\tilde{\sigma}\left(\hat{S}\right) = \tilde{\sigma}_{\left[\left(\frac{\text{ţ}+\breve{\xi}+\breve{\xi}'-\tilde{\Psi}-\tilde{\Psi}'}{2\text{ţ}}\right)\times\left(\frac{\text{ţ}+\breve{\mu}+\breve{\mu}'-\nu-\nu'}{2\text{ţ}'}\right)\right]}$$

$$\tilde{\sigma}\left(S_{T_{IU}}\right) = \tilde{\sigma}_{\left[\left(\frac{6+0.6010+0.6912-0.1237-0.3199}{3(6)}\right)\times\left(\frac{4+0.4712+0.6227-0.1898-0.3777}{3(4)}\right)\right]}$$

$$= \tilde{\sigma}_{\left[\left(\frac{6.8486}{18}\right)\times\left(\frac{4.5264}{12}\right)\right]}$$

$$= \tilde{\sigma}_{[0.3805]\times[0.3772]}$$

$$\tilde{\sigma}\left(S_{T_{IU}}\right) = \tilde{\sigma}_{[0.1435]}$$

$$\tilde{\sigma}\left(S_{T_{PZ}}\right) = \tilde{\sigma}_{\left[\left(\frac{6+0.5684+0.6313-0.1835-0.3392}{3(6)}\right)\times\left(\frac{4+0.4613+0.6027-0.1803-0.3652}{3(4)}\right)\right]}$$

$$= \tilde{\sigma}_{[0.37094]\times[0.3765]}$$

$$\tilde{\sigma}\left(S_{T_{IU}}\right) = \tilde{\sigma}_{[0.1397]}$$

$$\tilde{\sigma}\left(S_{T_{LQ}}\right) = \tilde{\sigma}_{\left[\left(\frac{6+0.5735+0.6442-0.1707-0.3229}{2(6)}\right)\times\left(\frac{4+0.4613+0.6067-0.1661-0.3852}{2(4)}\right)\right]}$$

$$= \tilde{\sigma}_{\left[\frac{6.7241}{18}\right]\times\left[\frac{4.5167}{12}\right]}$$

$$= \tilde{\sigma}_{[0.1406]}$$

$$\tilde{\sigma}\left(S_{T_{IU}}\right) = \tilde{\sigma}_{[0.1435]}$$

$$\tilde{\sigma}\left(S_{T_{PZ}}\right) = \tilde{\sigma}_{[0.1397]}$$

$$\tilde{\sigma}\left(S_{T_{LQ}}\right) = \tilde{\sigma}_{[0.1406]}$$

Step 5: Rank all the Teams (alternatives) T_{ij} (T_{IU} , T_{PZ} , T_{LQ}) according to score values $\tilde{\sigma}(S_{T_{ij}})$ of the overall General Linguistic interval valued fuzzy soft expert preference values as $T_{IU} > T_{LQ} > T_{PZ}$.

Hence Islamabad United Team Performed the best.

In the above illustrated example we can use GLIVIFESEOWA operater, GLIVIFSEWG operater and GLIVIFSEOWG operaters by following the same steps.

Chapter 6

Conclusion and Future Work

In this thesis, we have explored the decision-analysis problems under the GLIVIFSESs. Firstly, this thesis has represented the proper definition of General Linguistic Interval Valued Intuitionistic Fuzzy Soft Expert (GLIVIFSESs) and then described some basic operations and then operational laws defined on them. One of central features of GLIVIFSESs is that it associates the advantages of 2-DLIFVs and IVFSESs in a particular formulation. So, we are able to characterize ambiguous or fuzzy knowledge / information in a more genuine and practical manner. GLIVIFSESs have the benefit over the current theories in that it gives better results for each interval separately. This thesis has also developed a comparative method for GLIVIFSESs by using the score and accuracy functions.

Moreover, we established numerous aggregational operators such that, the GLIVIFSEWA operator, GLIVIFSEOWA operator, GLIVIFSEWG operator, GLIVIFSEOWG operator for accumulating GLIVIFSESs knowledge / information attained from several sources. Additionally, on the basis of these operators, A decision analysis has been formulated in this thesis, which is based on the knowledge submitted by the experts, which is embodied in the form of GLIVIFSESs. In decision analysis defined structure that is GLIVIFSESs is a flexible and more accurate for dealing with vagueness, obscurity, and uncertainty than fuzzy sets. We have also presented a numerical example to illustrate how the decision-analysis procedure is implemented in practice and to provide evidence to support the validity of the newly developed approach. Furthermore, we have included a comparison study between some of the newly developed strategies and existing procedures to illustrate how they compare.

It will also be examined in the upcoming work or research how GLIVIFSESs are applied in various fields. Moreover, we will take into account the comparative measure among the experts or decision makers in order to examine the group decision analysis problem with GLIVIFSE knowledge / information. Further, we intend to investigate the distance and entropy measures related to this structure.

Bibliography

[1] Ali, M. I., Feng, F., Liu, X., Min, W. K., & Shabir, M. (2009). On some new operations in soft set theory. Computers & Mathematics with Applications, 57(9), 1547-1553.

[2] Agarwal, M., Hanmandlu, M., & Biswas, K. K. (2011, June). Generalized intuitionistic fuzzy soft set and its application in practical medical diagnosis problem. In 2011 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE 2011) (pp. 2972-2978). IEEE.

[3] Alkhazaleh, S., Salleh, A. R., & Hassan, N. (2011). Fuzzy parameterized interval-valued fuzzy soft set. Applied Mathematical Sciences, 5(67), 3335-3346.

[4] Alkhazaleh, S., Salleh, A. R., & Hassan, N. (2011). Possibility fuzzy soft set. Advances in Decision Sciences, 2011.

[5] Alkhazaleh, S., & Salleh, A. R. (2012). Generalised interval-valued fuzzy soft set. Journal of Applied Mathematics, 2012.

[6] Atanassov, K. T, (1986) Intuitionistic fuzzy sets, Fuzzy Sets, and Systems, 20(1),87-96.

[7] Atanassov, K. T, G. Gargov,(1989), Interval-valued intuitionistic fuzzy sets, Fuzzy Sets and Systems, 31(3), 343-349.

[8] Atanassov, K. T. (1994). New operations defined over the intuitionistic fuzzy sets. Fuzzy sets and Systems, 61(2), 137-142

[9] Atanassov, K. T. (1999). Intuitionistic fuzzy sets. In Intuitionistic fuzzy sets (pp. 1-137). Physica, Heidelberg.

[10] Burillo, P., & Bustince, H. (1996). Entropy on intuitionistic fuzzy sets and on interval-valued fuzzy sets. Fuzzy sets and systems, 78(3), 305-316.

[11] Cantor, G. (2014). Set theory. Informatika, 2(3073), 2k.

[12] Cagman, N., Enginoglu, S., & Citak, F. (2011). Fuzzy soft set theory and its applications. Iranian journal of fuzzy systems, 8(3), 137-147.

[13] Cabrerizo, F. J., Ureña, R., Pedrycz, W., & Herrera-Viedma, E. (2014). Building consensus in group decision making with an allocation of information granularity. Fuzzy Sets and Systems, 255, 115-127.

[14] Cabrerizo, F. J., Morente-Molinera, J. A., Pedrycz, W., Taghavi, A., & Herrera-Viedma, E. (2018). Granulating linguistic information in decision making under consensus and consistency. Expert Systems with Applications, 99, 83-92

[15] Chen, D., Tsang, E. C. C., Yeung, D. S., & Wang, X. (2005). The parameterization reduction of soft sets and its applications. Computers & Mathematics with Applications, 49(5-6), 757-763.

[16] Chen, Z., Liu, P., & Pei, Z. (2015). An approach to multiple attribute group decision making based on linguistic intuitionistic fuzzy numbers. International Journal of Computational Intelligence Systems, 8(4), 747-760.

[17] De, S. K., Biswas, R., & Roy, A. R. (2001). An application of intuitionistic fuzzy sets in medical diagnosis. Fuzzy sets and Systems, 117(2), 209-213.

[18] Dengfeng, L., & Chuntian, C. (2002). New similarity measures of intuitionistic fuzzy sets and application to pattern recognitions. Pattern recognition letters, 23(1-3), 221-225.

[19] Dubois, D., & Prade, H. (1998). Possibility theory: qualitative and quantitative aspects. In Quantified representation of uncertainty and imprecision (pp. 169-226). Springer, Dordrecht.

[20] Ferreirós, J. (2008). Labyrinth of thought: A history of set theory and its role in modern mathematics. Springer Science & Business Media.

[21] Guo, K., & Xu, H. (2019). Knowledge measure for intuitionistic fuzzy sets with attitude towards non-specificity. International Journal of Machine Learning and Cybernetics, 10(7), 1657-1669.

[22] Herrera F., Martínez,L. , (2001). A model based on linguistic 2-tuples for dealing with multi granular hierarchical linguistic contexts in multi-expert decision-making. IEEE Transactions Systems Man Cybernetics-Part B (Cybernetics) 31, 227–234.

[23] He, Y., Chen, H., Zhou, L., Liu, J., & Tao, Z. (2014). Intuitionistic fuzzy geometric interaction averaging operators and their application to multi-criteria decision making. Information Sciences, 259, 142-159.

[24] Jiang, Y., Tang, Y., Chen, Q., Liu, H., & Tang, J. (2010). Interval-valued intuitionistic fuzzy soft sets and their properties. Computers & Mathematics with Applications, 60(3), 906-918.

[25] Jun, Y. B., Kim, C. S., & Yang, K. O. (2012). Cubic sets. Ann. Fuzzy Math. Inform, 4(1), 83-98.

[26] Ju, D., Ju, Y., & Wang, A. (2019). Multi-attribute group decision making based on power generalized Heronian mean operator under hesitant fuzzy linguistic environment. Soft Computing, 23(11), 3823-3842.

[27] Li, C. C., Dong, Y., Herrera, F., Herrera-Viedma, E., & Martínez, L. (2017). Personalized individual semantics in computing with words for supporting linguistic group decision making. An application on consensus reaching. Information Fusion, 33, 29-40.

[28] Li, D. F. (2005). Multiattribute decision making models and methods using intuitionistic fuzzy sets. Journal of computer and System Sciences, 70(1), 73-85.

[29] Liu, P. (2012). An approach to group decision making based on 2-dimension uncertain linguistic information. Technological and Economic Development of Economy, 18(3), 424-437.

[30] Liu, P., & Wang, Y. (2016). The aggregation operators based on the 2-dimension uncertain linguistic information and their application to decision making. International Journal of Machine Learning and Cybernetics, 7(6), 1057-1074.

[31] Liu. P, Yu X (2014) 2-Dimension uncertain linguistic power generalized weighted aggregation operator and its application in multiple attribute group decision making. Knowl-Based Syst 57:69–80.

[32] Liu, P., & Teng, F. (2016). An extended TODIM method for multiple attribute group decision-making based on 2-dimension uncertain linguistic Variable. Complexity, 21(5), 20-30.

[33] Liu, P., & Qin, X. (2017). Maclaurin symmetric mean operators of linguistic intuitionistic fuzzy numbers and their application to multiple-attribute decision-making. Journal of Experimental & Theoretical Artificial Intelligence, 29(6), 1173-1202.

[34] Liu P, Teng F (2018) Multiple attribute decision-making method based on 2-dimension uncertain linguistic density generalized hybrid weighted averaging operator. Soft Comput 22:797–810.

[35] Li, Y., Wang, Y., & Liu, P. (2016). Multiple attribute group decision-making methods based on trapezoidal fuzzy two-dimension linguistic power generalized aggregation operators. Soft Computing, 20(7), 2689-2704.

[36] Maji, P.K., Roy, A.R. and Biswas, R. (2001) Fuzzy Soft Sets. Journal of Fuzzy Mathematics, 9, 589-602.

[37] Maji, P. K., Roy, A. R., & Biswas, R. (2002). An application of soft sets in a decision making problem. Computers & Mathematics with Applications, 44(8-9), 1077-1083.

[38] Maji, P.K., Roy, A.R. and Biswas, R. (2003) Soft Set Theory. Computers & Mathematics with Applications, 54, 555-562.

[39] Majumdar, P., & Samanta, S. K. (2010). Generalised fuzzy soft sets. Computers & Mathematics with Applications, 59(4), 1425-1432.

[40] Molodtsov, D. (1999). Soft set theory—first results. Computers & mathematics with applications, 37(4-5), 19-31.

[41] Mukherjee, A., & Sarkar, S. (2014). Similarity measures for interval-valued intuitionistic fuzzy soft sets and its application in medical diagnosis problem. New Trends in Mathematical Sciences, 2(3), 159-165.

[42] Pal, N. R., & Pal, S. K. (1991). Entropy: A new definition and its applications. IEEE transactions on systems, man, and cybernetics, 21(5), 1260-1270.

[43] Portilla, M. I., Burillo, P., & Eraso, M. L. (2000). Properties of the fuzzy composition based on aggregation operators. Fuzzy sets and systems, 110(2), 217-226.5.

[44] Sezgin, A., & Atagün, A. O. (2011). On operations of soft sets. Computers & Mathematics with Applications, 61(5), 1457-1467.

[45] Sezgin, A., & Atagün, A. O. (2011). Soft groups and normalistic soft groups. Computers & Mathematics with Applications, 62(2), 685-698.

[46] Roy, R. and Maji, P.K. (2007) A Fuzzy Soft Set Theoretic Approach to Decision Making Problems. Journal of Computational and Applied Mathematics, 203, 412-418.

[47] Verma, R. K., & Sharma, B. D. (2011). Intuitionistic fuzzy sets: Some new results. Notes on intuitionistic fuzzy sets, 17(3), 1-10.

[48] Verma R, Sharma BD (2012) On generalized intuitionistic fuzzy divergence (relative information) and their properties. J Uncertain Syst 6:308–320.

[49] Verma R, Sharma BD (2013a) Intuitionistic fuzzy Jensen-Re´nyi divergence: applications to multiple-attribute decision making. Informatica 37:399–409.

[50] Verma R, Sharma BD (2013b) Exponential entropy on intuitionistic fuzzy sets. Kybernetika 49:114–127.

[51] Verma R, Sharma BD (2013c) Some new results on intuitionistic fuzzy sets. Proc Jangjeon Math Soc 16:101–114

[52] Verma R, Sharma BD (2014a) On intuitionistic fuzzy entropy of order-a. Adv Fuzzy Syst.

[53] Verma R, Sharma BD (2014b) A new measure of inaccuracy with its application to multi-criteria decision making under intuitionistic fuzzy environment. J Intell Fuzzy Syst 27:1811–1824.

[54] Verma R, Sharma BD (2015a) R-norm entropy on intuitionistic fuzzy sets. J Intell Fuzzy Syst 28:327–335.

[55] Verma R, Sharma BD (2015b) Intuitionistic fuzzy Einstein prioritized weighted average operators and their application to multiple attribute group decision making. Appl Math Inf Sci 9:3095–3107

[56] Verma, R., & Merigó, J. M. (2020). Multiple attribute group decision making based on 2-dimension linguistic intuitionistic fuzzy aggregation operators. Soft Computing, 24(22), 17377-17400.

[57] Verma, R. (2020). On aggregation operators for linguistic trapezoidal fuzzy intuitionistic fuzzy sets and their application to multiple attribute group decision making. Journal of Intelligent & Fuzzy Systems, 38(3), 2907-2950.

[58] Wei, P., & Ye, J. (2010, November). Improved intuitionistic fuzzy cross-entropy and its application to pattern recognitions. In 2010 IEEE International Conference on Intelligent Systems and Knowledge Engineering (pp. 114-116). IEEE.

[59] Xu, Z. (2009). Multi-period multi-attribute group decision-making under linguistic assessments. International journal of general systems, 38(8), 823-850.

[60] Yager, R. R. (1993). Families of OWA operators. Fuzzy sets and systems, 59(2), 125-148.

[61] Ye, J. (2009). Multicriteria fuzzy decision-making method based on a novel accuracy function under interval-valued intuitionistic fuzzy environment. Expert systems with Applications, 36(3), 6899-6902.

[62] Yu X, Xu Z, Liu S, Chen Q (2012) Multicriteria decision making with2-dimension linguistic aggregation techniques. Int J Intell Syst 27:539–562.

[63] Zadeh, L. A. (1965). Fuzzy sets. Information and control, 8(3), 338-353.

[64] Zadeh, L. A., Klir, G. J., & Yuan, B. (1996). Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers (Vol. 6). World Scientific.

[65] Zadeh, L.A. ,(1975a). The concept of a linguistic variable and its application to approximate reasoning-Information Science 8,43–80.

[66] Zadeh, L.A. , (1975b). The concept of a linguistic variable and its application to approximate reasoning—II. Information Science, 8(1975b),301–357

[67] Zhang, H. (2014). Linguistic intuitionistic fuzzy sets and application in MAGDM. Journal of Applied Mathematics, 2014.

[68] Zhu WD, Yang SL, Zhou GZ (2009) An approach to group decision making based on 2-dimension linguistic assessment information.Syst Eng 27:113–118

[69] Zimmermann, H. J. (2010). Fuzzy set theory. Wiley interdisciplinary reviews: computational statistics, 2(3), 317-332